Exercises for 10-13.

1) χ is a linear character. So $\chi(g) = \rho(g)$ where $\rho : G \to \mathbb{C}^*$ is a homomorphism.
Thus
$$\chi(xyx^{-1}y^{-1}) = \rho(xyx^{-1}y^{-1}) = \rho(x)\rho(y)\rho(x)^{-1}\rho(y)^{-1} = 1$$
since $\rho(x), \rho(y) \in \mathbb{C}^*$.

2) G' is normal: let $g \in G'$. Then
$$g = x_1 y_1 x_1^{-1} y_1^{-1} \cdot x_2 y_2 x_2^{-1} y_2^{-1} \cdots x_r y_r x_r^{-1} y_r^{-1} \text{ for some}$$
$x_1, \ldots, x_r, y_1, \ldots, y_r \in G$.

Now let $h \in G$. Then
$$\begin{aligned} h x_i y_i x_i^{-1} y_i^{-1} h^{-1} &= h x_i h^{-1} h y_i h^{-1} h x_i^{-1} h^{-1} h y_i^{-1} h^{-1} \\ &= h x_i h^{-1} h y_i h^{-1} (h x_i h^{-1})^{-1} (h y_i h^{-1})^{-1} \in G' \end{aligned}$$
Thus
$$\begin{aligned} hgh^{-1} &= h x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \cdots x_r y_r x_r^{-1} y_r^{-1} h^{-1} \\ &= (h x_1 y_1 x_1^{-1} y_1^{-1} h^{-1})(h x_2 y_2 x_2^{-1} y_2^{-1} h^{-1}) \cdots (h x_r y_r x_r^{-1} y_r^{-1} h^{-1}) \\ &\in G' \end{aligned}$$

$\therefore G' \trianglelefteq G$.

Now take gG' and hG' for $g, h \in G$.

Then
$$\begin{aligned} (gG')(hG') = (hG')(gG') &\iff ghG' = hgG' \\ &\iff (gh)^{-1} hg \in G' \\ &\iff h^{-1} g^{-1} hg \in G'. \end{aligned}$$

Since it's clear that $h^{-1}g^{-1}hg \in G'$ we get $(gG')(hG') = (hG')(gG')$ and so G/G' is abelian.

3) Let χ be a linear character. Then $G' \subseteq \ker \chi$ by (1) $\therefore$ χ is also a linear character of G/G'.

4) So we have linear characters of $G \longrightarrow$ linear characters of G/G'.

On the other hand any linear character of G/G' produces a linear character of G by lifting. These operations are inverse to one another so that

$$\#\text{linear characters of } G = \#\text{linear characters of } G/G'.$$

But G/G' is abelian so that all irreducible characters are 1-D (i.e. linear). But the number of irreducible characters of G/G' = # conjugacy classes $= |G/G'|$ since G/G' is abelian.

5) We know from calculations on previous sheets that there are 4 linear characters of D_4, 3 of A_4 and 2 of S_4.

$\therefore$ For D_4, $\left|D_4/D_4'\right| = 4$ i.e $|D_4'| = 2$. We find that

$$bab^{-1}a^{-1} = a^{-1}a^{-1} = a^{-2} = a^2 \in D_4' \text{ and so}$$

$$D_4' = \{e, a^2\}.$$

For A_4, $\left|A_4/A_4'\right| = 3$ i.e $|A_4'| = 4$. The elements of A_4 which are sent to 1 by linear characters are e, $(12)(34)$, $(13)(24)$ and $(14)(23)$ only (see Ex. for 889 Q.5). We deduce that $A_4' \subseteq \{e, (12)(34), (13)(24), (14)(23)\}$ by Q.1 and so this must be an equality since $|A_4'| = 4$.

For S_4 we see that $|S_4'| = 12$. Now the elements which are sent to 1 by the linear character

$$\pi \longmapsto \text{sign}(\pi)$$

are precisely the elements of A_4. Thus by Q1. $S_4' \subseteq A_4$. But both groups have order 12 and so we have $S_4' = A_4$. (a ~~[illegible]~~ way to see that $S_4' = A_4$ is that $(132) = (12)(23)(12)^{-1}(23)^{-1}$ and so $(132) \in S_4'$. As $S_4' \trianglelefteq S_4$ all 3-cycles are in S_4' *)

6) a) Clearly $(\lambda\rho)(g)$ is linear since it is a scalar multiple of $\rho(g)$. We need to check that $(\lambda\rho)$ is a group homomorphism.

$$(\lambda\rho)(gh) = \lambda(gh)\rho(gh) = \lambda(g)\lambda(h)\rho(g)\rho(h)$$
$$= \lambda(g)\rho(g)\lambda(h)\rho(h) \quad \text{since } \lambda(g), \lambda(h) \in \mathbb{C}$$
$$= (\lambda\rho)(g)(\lambda\rho)(h).$$

$\therefore$ $\lambda\rho$ is a rep$^{\underline{n}}$.

Now $\chi_{\lambda\rho}(g) = \mathrm{Tr}((\lambda\rho)(g)) = \mathrm{Tr}(\lambda(g)\rho(g))$
$$= \lambda(g)\,\mathrm{Tr}\,\rho(g) \quad \lambda(g) \in \mathbb{C}$$
$$= \lambda(g)\chi_\rho(g)$$
$$= (\lambda\chi_\rho)(g) \quad \text{(definition of } \lambda\chi_\rho\text{)}.$$

$\therefore$ $\chi_{\lambda\rho} = \lambda\chi_\rho$.

(b) $\langle \chi_{\lambda\rho}, \chi_{\lambda\rho} \rangle = \langle \lambda\chi_\rho, \lambda\chi_\rho \rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\lambda(g)}\,\overline{\chi_\rho(g)}\,\lambda(g)\chi_\rho(g)$

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* Now there are 8 3-cycles and so S_4' contains at least 9 elements and so is either A_4 or S_4. But every element is in A_4 and so we get $S_4' = A_4$

$$= \frac{1}{|G|} \sum_{g \in G} |\lambda(g)|^2 \overline{\chi_\rho(g)} \chi_\rho(g). \quad (*)$$

But $\lambda(g)$ is a root of unity in $\mathbb{C}$ (since $\lambda(g)^n = 1$ if $g^n = e$) and so $|\lambda(g)|^2 = 1$. Thus

$$* = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\rho(g)} \chi_\rho(g) = \langle \chi_\rho, \chi_\rho \rangle$$

i.e. $\langle \chi_{\lambda\rho}, \chi_{\lambda\rho} \rangle = \langle \chi_\rho, \chi_\rho \rangle$

Thus $\chi_{\lambda\rho}$ is irred $\iff \langle \chi_{\lambda\rho}, \chi_{\lambda\rho} \rangle = 1$

$$\iff \langle \chi_\rho, \chi_\rho \rangle = 1$$

$$\iff \chi_\rho \text{ is irred.}$$

(c) ...

7) D_4 and Q have same character table and so we'll get the same answer:

	e	a	a^2	b	ab
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

Normal subgps are int^{ns} of kernels of χ_i's:

$\ker \chi_1 = G$

$\ker \chi_2 = \{e\} \cup \{a^2\} \cup \{b, a^2b\} = \overset{G=D_4}{\langle a^2 \rangle \times \langle b \rangle}$ or $\overset{G=Q}{\langle b \rangle}$

$\ker \chi_3 = \{e\} \cup \{a, a^3\} \cup \{a^2\} = \langle a \rangle$

$\ker \chi_4 = \{e\} \cup \{a^2\} \cup \{ab, a^3b\} = \overset{G=D_4}{\langle a^2 \rangle \times \langle ab \rangle}$ or $\overset{G=Q}{\langle ab \rangle}$

$\ker \chi_5 = \{e\}$

$\therefore$ normal subgps are

G, $\{e, a^2, b, a^2b\}$, $\{e, a, a^2, a^3\}$, $\{e, a^2, ab, a^3b\}$

and $\{e, a^2\}$ $(= \ker \chi_2 \cap \ker \chi_3)$

For A_4 we have

	e	(123)	(132)	$(13)(24)$
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4	3	0	0	-1

So we get $\operatorname{Ker} \chi_1 = G$, $\operatorname{Ker} \chi_2 = \{e, (13)(24), (12)(34), (14)(23)\}$
$\operatorname{Ker} \chi_3 = \operatorname{Ker} \chi_2$, $\operatorname{Ker} \chi_4 = \{e\}$.

So these are the only normal subgroups.

8) By (3) we know there are two linear characters of S_4 (since $S_4' = A_4$ and $|S_4/S_4'| = 2$). We know these already!

χ_1 — the trivial character

χ_2 — corresponding to $\pi \in S_4 \longmapsto \operatorname{sign}(\pi)$.

We argue as in previous exercises (particularly Q2 + 8 for Ex. Sheet 8 + 9)

	e	(12)	(123)	$(12)(34)$	(1234)
$\chi_{\mathbb{C}^4}$	4	2	1	0	0

and so $\langle \chi_{\mathbb{C}^4}, \chi_1 \rangle = 1$ and $\chi_3 := \chi_{\mathbb{C}^4} - \chi_1$ is irreducible

χ_3	3	1	0	-1	-1

So far we have

	e	(12)	(123)	$(12)(34)$	(1234)
	1	6	8	3	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1

Now $\chi_2\chi_3$ is a character by 6 ~~3~~ and it looks like

3	-1	0	-1	1

so this is a new irreducible (by 6(3)) character, say χ_4. There are 5 conjugacy classes so we will ~~stay to~~ have to find χ_5.

χ_5	A	B	C	D	E

Now $1^2 + 1^2 + 3^2 + 3^2 + A^2 = 24$ i.e. $A = 2$.

$24\langle \chi_5, \chi_1 \rangle = 2 + 6B + 8C + 3D + 6E = 0$ (1)

$24\langle \chi_5, \chi_2 \rangle = 2 - 6B + 8C + 3D - 6E = 0$ (2)

$24\langle \chi_5, \chi_3 \rangle = 6 + 6B + 0 - 3D - 6E = 0$ (3)

$24\langle \chi_5, \chi_4 \rangle = 6 - 6B + 0 - 3D + 6E = 0$ (4)

$24\langle \chi_5, \chi_5 \rangle = 4 + 6|B|^2 + 8|C|^2 + 3|D|^2 + 6|E|^2 = 24$ (5)

(1) + (2) gives $4 + 16C + 6D = 0$
(3) + (4) gives $12 - 6D = 0$ i.e $D = 2$ and so $C = -1$.

Now (5) gives $4 + 6|B|^2 + 8 + 12 + 6|E|^2 = 24$

i.e $B = E = 0$

$\therefore$

	e	(12)	(123)	$(12)(34)$	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1
χ_5	2	0	-1	2	0

9) $|G|\langle \chi_4, \chi_5 \rangle = 3E - A + 3 - 6 = 0$ (1)

$|G|\langle \chi_1, \chi_5 \rangle = 3 - A + 0 - 3 + 6 = 0$ $\therefore A = 6$

$|G|\langle \chi_1, \chi_2 \rangle = 1 - A + B + 3 - 6 = 0$ $\therefore B = 8$

$\therefore |G| = \Sigma$(size of conj. classes) $= 1 + 6 + 8 + 3 + 6 = 24$

Then $24 = 1^2 + 1^2 + C^2 + E^2 + 3^2$ (~~squar~~ $|G| = \sum_j (\dim I_j)^2$)

$\therefore C^2 + E^2 = 13.$

the only possibilities are 2 and 3. From (1) we see that $3E - 6 + 3 - 6 = 0$ i.e $E = 3$ and hence $C = 2$. Only D remains.

$24\langle \chi_1, \chi_3 \rangle = C + 0 - B + 3D + 0 = 0$

i.e $2 + 0 - 8 + 3D + 0 = 0$ i.e $D = 2$.

(so we actually get S_4 char. table)

(b) $\operatorname{Ker} \chi_1 = G$

$\operatorname{Ker} \chi_2 = \{e\} \cup Cl(ts) \cup Cl(s^2)$,

$\operatorname{Ker} \chi_3 = \{e\} \cup \operatorname{Cl}(s^2)$
$\operatorname{Ker} \chi_4 = \{e\} = \operatorname{Ker} \chi_5$.

So we have all the normal subgroups

$\{e\}$, $\{e\} \cup \operatorname{Cl}(s^2)$, $\{e\} \cup \operatorname{Cl}(s^2) \cup \operatorname{Cl}(ts)$, G.

The centre $\longleftrightarrow$ elements in conjugacy class by themselves
$= \{e\}$ ~~by~~ (since $A, B > 1$)

G' corresponds to linear characters (there are 2),
corresponds to $\{e\} \cup \operatorname{Cl}(s^2) \cup \operatorname{Cl}(ts)$ (as this has
$1 + 8 + 3 = 12$ elements).

c) $\left| G/G' \right| = 2 \quad \therefore \quad G/G' \cong C_2$.

d) Both χ_4 and χ_5 have trivial kernel.

10) Calculation of χ_E.

Basis for E is $e_{(i,j)} - e_{(j,i)} : i < j$.

e fixes all these vectors $\therefore \chi_E(e) = 10$

(123) fixes $e_{(4,5)} - e_{(5,4)}$ and moves all other vectors about $\therefore \chi_E((123)) =$

$(12)(34)$ doesn't fix anything; sends $e_{(1,2)} - e_{(2,1)}$ to $-(e_{(1,2)} - e_{(2,1)})$
and $e_{(3,4)} - e_{(4,3)}$ to $-(e_{(3,4)} - e_{(4,3)})$
and moves rest around $\therefore \chi_E((12)(34)) = -2$

(12345) + (13245) moves everything around
$\therefore \chi_E((12345)) = 0 = \chi_E((13245))$

$$\langle \chi_E, \chi_1 \rangle = \frac{1}{60}(1.10.1 + 20.1.1 + 15.1.-2 + 12.1.0 + 12.1.0)$$

$$= \frac{1}{60}(10+20-30) = 0$$

$$\langle \chi_E, \chi_2 \rangle = \frac{1}{60}(1.10.4 + 20.1.1 + 15.0.-2 + 12.0.-1 + 12.0.-1)$$

$$= 1$$

$$\langle \chi_E, \chi_3 \rangle = \frac{1}{60}(1.10.5 + 20.1.-1 + 15.-2.1 + 12.0.0 + 12.0.0)$$

$$= \frac{1}{60}(50-20-30) = 0$$

$$\langle \chi', \chi' \rangle = \frac{1}{60}(1.6.6 + 20.0.0 + 15.-2.-2 + 12.1.1 + 12.1.1)$$

$$= \frac{1}{60}(36+60+12+12) = 2.$$