Solutions for Exercises for Lectures 2 & 3.

1. Let $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $X^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $X^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Now $\rho(x^i) = X^i \quad 0 \le i \le 2$ and we get

$$\begin{aligned} \rho(x^i . x^j) = \rho(x^{i+j}) &= \rho(x^k) \qquad \begin{pmatrix} k \equiv i+j \ (3) \\ 0 \le k \le 2 \end{pmatrix} \\ &= X^k \\ &= X^i X^j \qquad (\text{since } X^3 = I_d) \\ &= \rho(x^i)\rho(x^j) \end{aligned}$$

so it's a group homomorphism i.e. a rep$^{\underline{n}}$.

2. $e_1 + e_2 + e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is fixed by these matrices

i.e $X\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \therefore \quad X^i\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\therefore \ \mathbb{C}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a 1-D subrep$^{\underline{n}}$.

Now the answer on whether there are more 1-D rep$^{\underline{ns}}$ depends on which field we work over:

suppose Fv is stable under X (and hence X^i).

Then $Xv = \lambda v$ for some $\lambda \in F$ i.e v is an e-vector and λ an e-value.

$$\det(tI - X) = \begin{vmatrix} t & -1 & 0 \\ 0 & t & -1 \\ -1 & 0 & t \end{vmatrix} = t(t^2) + 1(-1) + 0$$
$$= t^3 - 1$$
$$= (t-1)(t^2+t+1)$$

Now we have the e-value 1 and we found $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to be an e-vector. Now the existence of other e-vectors depends on whether t^2+t+1 factorises over F

e.g. it doesn't factorise over $\mathbb{R}$ and so there are no other 1-D subrep^{ns}.

e.g. over $\mathbb{C}$ $\quad t^2+t+1 = (t-\omega)(t-\omega^2)$ where $\omega = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$ and so there are 2 more 1-D rep^{ns} spanned by
$$\begin{pmatrix}1\\ \omega\\ \omega^2\end{pmatrix} \text{ and } \begin{pmatrix}1\\ \omega^2\\ \omega\end{pmatrix} \text{ respectively.}$$

3. $\rho_j(x^m x^{m'}) = \rho_j(x^{m+m'}) = \rho_j(x^k)$ (where $k \equiv m+m'(n)$, $0 \le k \le n-1$)
$$= \rho^{jk}$$
$$= \rho^{j(m+m')} \quad \text{since } \rho^n = 1$$
$$= \rho_j(x^m)\rho_j(x^{m'})$$

So we get a rep^{n}.

Obviously $\rho_j \cong \rho_{tn+j}$ $\quad \forall t \in \mathbb{Z}$ since $\rho^{tn+j} = \rho^j$.

Now we claim that $\rho_0, \ldots, \rho_{n-1}$ are all non-isomorphic

This is easy: suppose $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ is an isom. between ρ_i and ρ_j. Then we have $\forall m$
$$\psi(\rho_i(x^m).\lambda) = \rho_j(x^m).\psi(\lambda) \quad \forall \lambda \in \mathbb{C}.$$
$$\psi(\rho^{im}\lambda) \qquad\qquad \rho^{jm}\psi(\lambda)$$
$$\rho^{im}\psi(\lambda) \quad \text{(by } \mathbb{C}\text{-linearity)}$$

Thus if $\psi(\lambda) \neq 0$ we get $\rho^{im} = \rho^{jm}$ $\forall m$

so that $\rho^i = \rho^j$ ie $i = j$ (since $0 \leq i, j \leq n-1$)

This is a contradiction, so $\psi(\lambda) = 0$ $\forall \lambda \in \mathbb{C}$ and so $\psi = 0$ i.e ψ is not invertible, i.e ψ is <u>not</u> an isom.

(4). We have to check that ρ^* is a group homomorphism.

$$\rho^*(g_1 g_2) = \rho((g_1 g_2)^{-1})^T = \rho(g_2^{-1} g_1^{-1})^T$$

$$\overset{\rho \text{ is a homom.}}{=} (\rho(g_2^{-1})\,\rho(g_1^{-1}))^T$$

$$= \rho(g_1^{-1})^T \rho(g_2^{-1})^T$$

$$= \rho^*(g_1)\,\rho^*(g_2).$$

Thus ρ^* is a rep$^{\underline{n}}$, and it has the same degree as ρ as it consists of matrices of the same size.

(5) $\rho^*(x) = \rho(x^{-1})^T = \rho(x^2)^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \rho(x)$, so the rep^{ns} are the same.

For qn 2 : $\rho_j^*(x^k) = \rho_j(x^{n-k})^T = \rho^{j(n-k)}$.
$$= \rho^{(n-j)k} \quad \text{(since } \rho^n = 1)$$
$$= \rho_{n-j}^*(x^k)$$

So $\rho_j^* \cong \rho_{n-j}$.

(6) (a) $F[X]$ has a basis $(e_x : x \in X)$ where $e_x \in F[X]$ is the function $e_x(y) = \delta_{xy} = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$

To see this : (i) $f \in F[X]$, then $f = \sum_{x\in X} f(x) e_x$

since $\left(\sum_{x\in X} f(x) e_x\right)(y) = \sum_{x \in X} f(x) e_x(y)$

$$= f(y)$$

Thus $(e_x : x \in X)$ spans.

(ii) Suppose that $\sum_{x\in X} \lambda_x e_x = 0 \quad \lambda_x \in F \ \forall x$

then $0 = \left(\sum_{x\in X} \lambda_x e_x\right)(y) = \sum_{x\in X} \lambda_x e_x(y) = \lambda_y \quad \forall y \in X$

i.e $(e_x : x \in X)$ linearly independent set.

(b) We need to check that $f \longmapsto g \cdot f$ is a linear mapping

i.e $\quad g \cdot (\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 (g \cdot f_1) + \lambda_2 (g \cdot f_2) \quad \forall \lambda_1, \lambda_2 \in F$
$f_1, f_2 \in F[X]$

Well let $x \in X$

$$\left(g \cdot (\lambda_1 f_1 + \lambda_2 f_2)\right)(x) = (\lambda_1 f_1 + \lambda_2 f_2)(g^{-1}x)$$
$$= \lambda_1 (f_1(g^{-1}x)) + \lambda_2 (f_2(g^{-1}x))$$
$$= \lambda_1 ((g \cdot f_1)(x)) + \lambda_2 (g \cdot f_2(x))$$
$$= (\lambda_1 (g \cdot f_1) + \lambda_2 (g \cdot f_2))(x)$$

so the maps are equal!

(c) We need to show that

$$\rho : G \longrightarrow GL(V) \qquad V = F[X]$$
$$g \longmapsto (f \longmapsto g\cdot f) \qquad f \in V$$

is a group homomorphism:

($f \mapsto g\cdot f$ belongs to $GL(V)$ since its inverse is $f \mapsto g^{-1}\cdot f$)

$$\rho(gh)(f) = (gh)\cdot f \qquad \rho(g)\rho(h)(f) = g\cdot(h\cdot f)$$

ie we must show that $(gh)\cdot f = g\cdot(h\cdot f)$

Let $x \in X$
$$\begin{aligned}((gh)\cdot f)(x) &= f((gh)^{-1}x) \\ &= f((h^{-1}g^{-1})x) \qquad G \text{ action on } X. \\ &= f(h^{-1}(g^{-1}x)) \\ &= (h\cdot f)(g^{-1}x) \\ &= (g\cdot(h\cdot f))(x).\end{aligned}$$

(d) It's not irreducible unless X is a single element, because

$f = \sum_{x\in X} e_x$ is fixed by G : $(g\cdot f) = \sum_{x\in X} g\cdot e_x \overset{(*)}{=} \sum_{x\in X} e_{gx} = \sum_{x\in X} e_x = f$

we use $(g\cdot e_x)(y) = e_x(g^{-1}y) = \begin{cases} 1 & x = g^{-1}y \\ 0 & x \neq g^{-1}y \end{cases}$

$$= \begin{cases} 1 & gx = y \\ 0 & gx \neq y \end{cases} = e_{gx}(y)$$

So Ff is a 1-D subrep$^{\underline{n}}$.

(7) (a) To start with we have arbitrary words of the form

$$a^{i_1} b^{i_2} a^{i_3} b^{i_4} \dots$$

but we notice that the relation $bab^{-1} = a^{-1}$ can be rewritten $bab^{-1} = a^3$ and then $ba = a^3 b$. This allows to ~~swap~~ move b's from the left to the right

ie $\quad ba^2 = \underline{ba}\, a = a^3 \underline{b\, a} = a^3 a^3 b$

So all elements of D_8 can actually be written as

$a^i b^j \qquad i, j \in \mathbb{Z}$.

Now we can choose $0 \leq i \leq 3$, $0 \leq j \leq 1$ since $a^4 = b^2 = e$ so we can write every element in the form

$a^i b^j \quad$ with $\quad 0 \leq i \leq 3, \; 0 \leq j \leq 1$.

(b) (we haven't actually proved that the 8 elements are all different in $\langle a, b : a^4 = b^2 = e,\ bab^{-1} = a^3 \rangle$; that will follow from Q.11 — at the moment we'll just assume it)

$$e, \ \{a, a^3\}, \ \{a^2\}, \ \{b, a^2 b\}, \ \{ab, a^3 b\}.$$

(by calculation.

e.g $\{a, a^3\}$ are conjugate because of the relation $bab^{-1} = a^3$

e.g ab, $a^3 b$ are conjugate since $b(ab)b^{-1} = ba$
$= a^3 b$.)

(8) (a) Exactly the same argument as 7(a) gives every element in $\{a^i b^j : 0 \leq i \leq n-1, 0 \leq j \leq 1\}$.

(b) The same arguments give : n even

$\{e\}$, $\{a^{n/2}\}$, $\{a^i, a^{n-i}\}_{1\le i<\frac{n}{2}}$ $\{b, a^2b, a^4b, \ldots a^{n-2}b\}$

$\{ab, a^3b, a^5b, \ldots a^{n-1}b\}$

n odd $\{e\}$, $\{a^i, a^{n-i}\}_{1\le i<\frac{n}{2}}$, $\{b, ab, a^2b, \ldots, a^{n-1}b\}$.

(9) (a) e, a, b, ab, ba, aba. (Can you convince yourself these are really different? See Q.14)

(b) $a=(12)$, $b=(23)$. Then

$(12)^2=(23)^2=e$ $\underset{(13)}{\underset{\parallel}{(12)(23)(12)}} = \underset{(13).}{\underset{\parallel}{(23)(12)(23)}}$

(10) Write $s_\alpha(x_1 \ldots, x_r)$ as some word $x_1^{i_1} \ldots x_r^{i_r} x_1^{j_1} \ldots x_r^{j_r} \ldots$

Then $\rho(s_\alpha(x_1 \ldots, x_r)) = \rho(x_1^{i_1} \ldots x_r^{i_r} x_1^{j_1} \ldots x_r^{j_r} \ldots)$

$= \rho(x_1)^{i_1} \ldots \rho(x_r)^{i_r} \rho(x_1)^{j_1} \ldots \rho(x_r)^{j_r} \ldots$ by def^n

$= X_1^{i_1} \ldots X_r^{i_r} X_1^{j_1} \ldots X_r^{j_r} \ldots$

$= s_\alpha(X_1, \ldots X_r)$

$\therefore \rho(s_\alpha(x_1 \ldots x_r)) = e_H \Leftrightarrow s_\alpha(X_1 \ldots X_r) = e_H$

(11) We need to check that $\rho(a^4) = Id_2 = \rho(b^2)$ and that $\rho(bab^{-1}) = \rho(a^{-1})$; then it follows from the analysis preceding Q.10 that we have a homomorphism from $\langle a, b : a^4 = b^2 = e, bab^{-1} = a^{-1} \rangle$ to $GL_2(F)$ i.e a rep$^{\underline{n}}$.

$\rho(a^4) = \rho(a)^4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (by calculation)

$$\rho(b^2) = \rho(b)^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{, obviously}$$

$$\rho(bab^{-1}) = \rho(b)\rho(a)\rho(b)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \rho(a)^3 = \rho(a)^{-1} = \rho(a^{-1})$$

~~(12)~~ (Aside: this also proves that ~~each of~~ the elements $a^i b^j$ ($0 \le i \le 3$, $0 \le j \le 1$) are all distinct since they are all sent to different matrices — that fully finishes off (7).)

(12) Here's 2.7 : $a^i \xmapsto{\sigma} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^i$, $b^j \xmapsto{\sigma} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^j$

we need to check that the relations $a^3 = b^2 = e$, $bab^{-1} = a^{-1}$ are satisfied by these matrices:

$$\sigma(a^3) = \sigma(a)^3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma(b^2) = \sigma(b)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma(bab^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}^{-1} = \sigma(a^{-1})$$

so we do get a repⁿ.

Here's 2.4′ : $a \xmapsto{\rho'} \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$, $b \xmapsto{\rho'} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Again, we have to check the relations:

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ (because the matrix represents rotⁿ by $2\pi/3$)

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}^{-1}$$

To prove they are isomorphic we think a little :

$B_1 = \rho'(b)$ is already diagonalised and so we have $B = \sigma(b)$ we could ask how we get

$$X B_1 X^{-1} = B$$

$$\text{or} \quad B_1 = X^{-1} B X$$

and we must answer that the two columns of X must be e-vectors for B with e-value -1 and 1 respectively. A quick calculation finds that the -1 e-vector for B is $\begin{pmatrix}\alpha\\-\alpha\end{pmatrix}$, 1 e-vector is $\begin{pmatrix}\beta\\\beta\end{pmatrix}$.

So we'll let $X = \begin{pmatrix}\alpha & \beta\\ -\alpha & \beta\end{pmatrix}$ and then we'll get $XB_1X^{-1} = B$

We have to determine α, β so that $X\rho'(a)X^{-1} = \sigma(a)$.

Rewriting this gives $X\rho'(a) = \sigma(a)X$ and so we need to find α, β s.t.

$\cancel{RHS \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix} = }$

$$X\rho'(a)\begin{pmatrix}1\\0\end{pmatrix} = \sigma(a)X\begin{pmatrix}1\\0\end{pmatrix} \quad \& \quad X\rho'(a)\begin{pmatrix}0\\1\end{pmatrix} = \sigma(a)X\begin{pmatrix}0\\1\end{pmatrix}$$

i.e. $\begin{pmatrix}-\frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta\\ \frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta\end{pmatrix} = \begin{pmatrix}\alpha\\ 2\alpha\end{pmatrix}$ and $\begin{pmatrix}-\frac{\sqrt{3}}{2}\alpha - \frac{1}{2}\beta\\ \frac{\sqrt{3}}{2}\alpha - \frac{1}{2}\beta\end{pmatrix} = \begin{pmatrix}-\beta\\ 0\end{pmatrix}$

It's easy to check that $\sqrt{3}\alpha = \beta$ solves these eq^n.

Therefore

$$\mathbb{C}^2 \xrightarrow{\begin{pmatrix}1 & \sqrt{3}\\ -1 & \sqrt{3}\end{pmatrix}} \mathbb{C}^2$$

gives an isomorphism between $(\rho', \mathbb{C}^2)$ and $(\sigma, \mathbb{C}^2)$

(13) $D_4 = \langle a, b : a^4 = b^2 = e \quad bab^{-1} = a^{-1} \rangle$

Any 1-D rep^n (over $\mathbb{C}$) must ~~~~ be a group homom.

$$\rho : D_4 \longrightarrow GL(1, \mathbb{C}) = \mathbb{C}^\times$$

and so it is determined by 2 non-zero complex nos $\rho(a)$, $\rho(b)$. These cannot be arbitrary because they must satisfy the relations

$$\rho(a)^4 = 1$$
$$\rho(b)^2 = 1$$
$$\rho(b)\rho(a)\rho(b)^{-1} = \rho(a)^{-1}$$

These three cond^{ns} force $\rho(a) = \pm 1$, $\rho(b) = \pm 1$ as the four different possibilities. (In fact there are four 1-D rep^{ns} as long as we are not in a field of characteristic 2, when there is only 1.)

(14) $S_3 = \langle a, b : a^2 = b^2 = e \quad aba = bab \rangle$

Let $\rho(a) =: A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $\rho(b) =: B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. We must check that $A^2 = B^2 = Id_2$, $ABA = BAB$. This is an easy calculation.

(N.B. This shows that e, a, b, ab, ba, aba are distinct elements since ~~~~ all the matrices are distinct. This finishes off Q. 9(a) completely.).

In 9(b) we pointed out that $a \leftrightarrow (12)$ $b \leftrightarrow (23)$
Now look at the matrices for W given for (12), (23) in 3.6. They are exactly the above matrices!

Now suppose for a contradiction that this representation is reducible. Since this has degree 2 we must be able to find a subspace of dim^n 1 stable under the S_3 action. Suppose $\mathbb{C}v$ is such a subspace. Then we must have $\rho(a)(v) = \lambda v$ and $\rho(b)(v) = \mu v$ i.e v is an e-vector for both A and B. We'll calculate the e-vectors now:

$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ has e-values ± 1; e-vectors $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has value -1

e-vectors $\beta \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has value 1

$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ has e-values ± 1; e-vectors $\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ value -1

e-vectors $\delta \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ value 1

All of these e-vectors are different, so the only possibility for v is 0 i.e there is no 1-D S_3 stable subspace i.e. W is irreducible. $\square$