Representation Theory

1.

(a) Explain what it means to call a representation irreducible. **1**

(b) State Maschke's theorem. **4**

(c) State and prove Schur's lemma. **6**

(d) Deduce that every irreducible complex representation of a finite abelian group is one-dimensional. **7**

(e) Let $G = \langle x : x^5 = e\rangle$ and let $F_2 = \{0, 1\}$ be the field with two elements. Show that the regular representation $F_2[G]$ decomposes as the direct sum of two irreducible subrepresentations. **7**

Comment: *(a), (b), (c)(2 for Schur's lemma, 4 for proof) and (d) are pure bookwork. Finding the trivial representation is similar to bookwork (2 marks), finding W can be done using homework techniques (2 marks); only showing that W is irreducible is unseen (3 marks).*

Solution: (a) A representation is irreducible if it has no proper subrepresentation, i.e. no proper G-invariant subspaces.

(b) Maschke's theorem states that if $|G|^{-1} \in F$ then if V is a finite dimensional representation of G and U is a subrepresentation of V, then there exists a G-invariant complement to U.

(c) Schur's Lemma: Let V be a finite dimensional irreducible complex representation of G. Then every G-homomorphism $\phi : V \longrightarrow V$ is a scalar.

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of ϕ and let E_λ be the corresponding (non-zero) eigenspace so that $E_\lambda = \{v \in V : \phi(v) = \lambda v\}$. We claim that E_λ is G-invariant. Let $x \in E_\lambda$ and $g \in G$. Then $\phi(g \cdot x) = g \cdot \phi(x) = g \cdot (\lambda x) = \lambda(g \cdot x)$ so that $g \cdot x \in E_\lambda$. Since E_λ is non-zero and V is irreducible we deduce that $E_\lambda = V$.

(d) Let V be an irreducible representation of G (abelian) over $\mathbb{C}$. Consider the homomorphism $\phi_g = \rho_V(g) : V \longrightarrow V$ for some $g \in G$. We claim ϕ_g is a G-homomorphism. To see this let $h \in G$. Then

$$\phi_g(h \cdot v) = g \cdot (h \cdot v) = (gh) \cdot v = (hg) \cdot v = h \cdot (g \cdot v) = h \cdot \phi_g(v).$$

Thus ϕ_g is a G-homomorphism and hence a scalar by Schur's lemma. This is true for all $g \in G$. Thus any line in V is G-invariant and so, since V is irreducible, it must be that V is one dimensional.

(e) We know that $F_2[G]$ has a basis v_{x^i} with $0 \leq i \leq 4$ and, just as in lectures, $v_e + v_x + \cdots + v_{x^4}$ is a G-invariant vector which produces $U = sp(v_e + \cdots + v_{x^4})$, a one dimensional subrepresentation.

Either by inspection (similar to lectures) or by applying the constructive part of Maschke's theorem, we see that $W = sp(v_{x^j} + v_{x^{j+1}} : 0 \leq j \leq 3)$ is a G-invariant complement to U in $F_2[G]$. There are several ways to see that W is irreducible. Brute force will do it quite quickly. But I suggest observing that $|GL_2(F_2)| = 6$ and $|GL_3(F_2)| = 7 \times 6 \times 4$ and so neither group contains an element of order 5, which means that there are no non-trivial two or three dimensional representations of G over F_2 (and clearly the only one dimensional representation is the trivial one). Since W is non-trivial it follows that W is irreducible.

Thus $F_2[G] = U \oplus W$.

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2. Let G be the group with the presentation

$$G = \langle a, b : a^8 = e, a^4 = b^2, b^{-1}ab = a^{-1} \rangle.$$

(a) Let $\omega = \exp(2\pi\sqrt{-1}/8) \in \mathbb{C}$. Prove that for an integer r with $0 \le r \le 7$ there is a representation $\rho^{(r)} : G \longrightarrow GL(2, \mathbb{C})$ with

$$\rho^{(r)}(a) = \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{pmatrix}, \qquad \rho^{(r)}(b) = \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}.$$

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(b) By using (a) (or otherwise), prove that $|G| = 16$. **5**

(c) G has character table

	C_e	C_{a^4}	C_a	C_{a^2}	C_{a^3}	C_b	C_{ab}
	1	1	2	2	2	A	4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	B	C
χ_3	1	1	-1	1	-1	1	-1
χ_4	D	1	-1	1	-1	-1	1
χ_5	2	-2	$\omega + \omega^{-1}$	0	$\omega^3 + \omega^{-3}$	0	0
χ_6	2	2	0	E	0	0	0
χ_7	2	-2	$\omega^3 + \omega^{-3}$	0	$\omega + \omega^{-1}$	0	0

Complete the character table by determining A, B, C, D, E. **7**

(d) For each of $r = 1, 2, 4$ write the character $\chi_{\rho^{(r)}}$ as a sum of the irreducible characters $\chi_1, \ldots, \chi_7$. **5**

(e) Let G' be the derived subgroup of G. Describe G/G' as a (product of) cyclic group(s). **4**

Comment: *(a) is just like homework, (b) is just like homework (3 marks for bounding $|G|$ with 16; 2 for getting equality), (c) is a standard type of exercise (marks for $A, (BandC), D, E$ are $1, 3, 1, 2$), (d) is a standard exercise (2 marks for $r = 2$, 3 for $r = 1$ and 4 together), (e) is unseen, but the derived subgroup was discussed in exercises.*

Solution: (a)

$$\rho^{(r)}(a^8) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho^{(r)}(a^4) = \begin{pmatrix} \omega^{4r} & 0 \\ 0 & \omega^{-4r} \end{pmatrix} = \begin{pmatrix} (-1)^r & 0 \\ 0 & (-1)^r \end{pmatrix} = \rho^{(r)}(b^2),$$

$$\rho^{(r)}(b^{-1}ab) = \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix} = \begin{pmatrix} \omega^{-r} & 0 \\ 0 & 0 \end{pmatrix} = \rho^{(r)}(a^{-1}).$$

Therefore we get a representation of G.

(b) We use the relation $b^{-1}ab = a^{-1}$ to write any element of G in the form a^ib^j. Now $a^8 = e$ and $a^4 = b^2$ allow us to assume that $0 \le i \le 7$ and $0 \le j \le 1$. Thus $|G| \le 16$. Finally, to get exactly 16 elements we observe that

$$\rho^{(1)}(a^ib^j) = \begin{cases} \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix} & j = 0 \\ \begin{pmatrix} 0 & -\omega^i \\ \omega^{-i} & 0 \end{pmatrix} & j = 1 \end{cases}.$$

Since these are all distinct matrices each a^ib^j must be distinct. Thus $|G| = 16$.

(c) $A = 4$ since $|G| = 16$. $D = 1$ since $1^2 + 1^2 + 1^2 + D^2 + 2^2 + 2^2 + 2^2 = 16$. There are many ways to calculate B and C. Perhaps the quickest is to observe that χ_2 is a linear character and so we must have that $\chi_2(b^2) = B^2 = \chi_2(a^4) = 1$ so that $B = \pm 1$. Also $C = \chi_2(ab) = \chi_2(a)\chi_2(b) = B$. We deduce that $B = C = -1$ since $\chi_2 \neq \chi_1$. Finally we calculate

$$\langle \chi_1 | \chi_6 \rangle = \frac{1}{16}(2 + 2 + 0 + 2E + 0 + 0 + 0) = 0$$

and so $E = -2$.

(d) Obviously $\chi_{\rho^{(2)}} = \chi_6$ (when we note that $\omega^2 = \sqrt{-1}$ and $\omega^{-2} = -\sqrt{-1}$). Now $\chi_{\rho^{(1)}}(a) = \omega + \omega^{-1}$ and so we must have $\chi_{\rho^{(1)}} = \chi_5$. Finally $\chi_{\rho^{(4)}}(a) = -2$ and so the only possibility is $\chi_{\rho^{(4)}} = \chi_3 + \chi_4$.

(e) G/G' is abelian and from the character table has four 1-dimensional representations. Therefore it's either C_4 or $C_2 \times C_2$. But C_4 has irreducibles with imaginary characters and so we get that G/G' is $C_2 \times C_2$.

3.

(a) Let $H \le G$ be a subgroup and let W be a complex representation of H. Define $\mathrm{Ind}_H^G W$, the representation of G induced from W. Prove that

$$\dim_{\mathbb{C}} \mathrm{Ind}_H^G W = [G : H] \dim_{\mathbb{C}} W.$$

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(b) Let $G = S_4$ and let $H = \langle (12), (34) \rangle$. Determine the character of $\mathrm{Ind}_H^G W$ where W is the trivial representation of H. (You may use without proof the formula for the character of an induced representation, but you should state it clearly.) **12**

(c) How many distinct isotypic components does $\mathrm{Ind}_H^G W$ have in (b)? **2**

Comment: *(a) is bookwork (3 marks for definition, 8 for dimension), (b) is similar (but not the same) to questions we will do in homework (3 marks for character, 1 for conjugacy classes, 3+2+2+1 for calculation of character), (c) is an easy application of (b).*

Unfortunately, we didn't have time for the tensor product. This makes the induced representation a little trickier than it should be!

Solution: (a) $Ind_H^G W = \{f : G \longrightarrow W : h \cdot f(gh) = f(g) \text{ for all } h \in H, g \in G\}$. The action of G on $Ind_H^G W$ is given by $(g \cdot f)(g') = f(g^{-1}g')$.

Let $r = |G : H|$. We will set up a linear isomorphism between $Ind_H^G W$ and $\oplus_{i=1}^r W$. Once we've done this the result will follow.

Let $g_1, \ldots, g_r$ be coset representatives so that $G = \cup_{i=1}^r g_i H$. Define

$$\Psi : Ind_H^G W \longrightarrow \bigoplus_{i=1}^r W \text{ and } \Phi : \bigoplus_{i=1}^r W \longrightarrow Ind_H^G W$$

by the rules $\Psi(f) = (f(g_1), \ldots, f(g_r))$ and $\Phi((w_1, \ldots, w_r))(g_j h) = h^{-1} \cdot w_j$. We need to check that Φ really has image in $Ind_H^G W$:

$$h_2 \cdot \Phi((w_1, \ldots, w_r))(g_j h_1 h_2) = h_2 \cdot ((h_1 h_2^{-1}) \cdot w_j) = h_1 \cdot w_j = \Phi((w_1, \ldots, w_r))(g_j h_1).$$

Now

$$\Psi\Phi((w_1, \ldots, w_r)) = (\Phi((w_1, \ldots, w_r))(g_1), \ldots, \Phi((w_1, \ldots, w_r))(g_r)) = (w_1, \ldots, w_r),$$

whilst

$$\Phi\Psi(f)(g_j h) = h^{-1} f(g_j) = f(g_j h).$$

(b) Let $V = Ind_H^G W$. The character formula states

$$\chi_V(g) = \frac{1}{|H|} \sum_{x \in G} \dot{\chi}_W(x^{-1} g x) = \frac{1}{|H|} |\{x \in G : x^{-1} g x \in H\}|$$

since

$$\dot{\chi}_W(y) = \begin{cases} \chi_W(y) = 1 & y \in H \\ 0 & y \notin H \end{cases}.$$

We have to work with the conjugacy classes in S_4: they have representatives $e, (12), (123), (1234), (12)(34)$ and have respective sizes $1, 6, 8, 6, 3$.

Now $x^{-1}(12)x \in H$ if and only if $x^{-1}(12)x = (12)$ or (34) if and only if $x = e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)$. Thus $\chi_V((12)) = 1/4 \times 8 = 2$.

Since H contains no 3-cycles or 4-cycles we have $x^{-1}(123)x \notin H$ and $x^{-1}(1234)x \notin H$. Thus $\chi_V((123)) = \chi_V((1234)) = 0$.

Similar to the above calculation we see that $x^{-1}(12)(34)x \in H$ if and only if x belongs to the above list. Thus $\chi_V((12)(34)) = 2$.

Of course, $\chi_V(e) = 1/4 \times 24 = 6$.

(c) $\langle \chi_V : \chi_v \rangle = \frac{1}{24}(1 \cdot 6^2 + 6 \cdot 2^2 + 3 \cdot 2^2) = 3$. It follows that V must be the sum of three distinct irreducible representations and thus it has 3 isotypic components.

4.

(a) "Character theory makes representation theory easy." Discuss. **15**

(b) Give an example (without proof) of two finite groups that have the same character table. **2**

(c) Let V be a complex representation of G. The *tricharacter* of V is the function

$$\chi_V^{tri} : G \times G \times G \longrightarrow \mathbb{C}$$

defined by $\chi_V^{tri}(g_1, g_2, g_3) = \chi_V(g_1 g_2 g_3)$ for $g_1, g_2, g_3 \in G$.

By considering the regular representation $\mathbb{C}[G]$ (or otherwise), show that the multiplication table of G can be recovered from the knowledge of the tricharacters of G. **8**

Comment: *(a) is unseen, but they have been warned there will be an essay question of this type. (b) is standard. (c) is unseen (2 marks for a strategy, 2 marks for character of regular representation, 4 marks for carrying calculations through).*

Solution: (a) I'll write that soon!

(b) Q_8 and D_8.

(c) We need to use the tricharacter to deduce what $g_1 g_2$ is for any $g_1, g_2 \in G$. We know that

$$\chi_{\mathbb{C}[G]}(g) = \begin{cases} |G| & g = e \\ 0 & g \neq e \end{cases}.$$

We can find e easily: it is the unique element which doesn't change $\chi_{\mathbb{C}[G]}^{tri}(g, g', -)$ as we vary $g, g' \in G$. Now

$$\chi_{\mathbb{C}[G]}^{tri}(g_1, g_2, h) = \begin{cases} |G| & h = (g_1 g_2)^{-1} \\ 0 & h \neq (g_1 g_2)^{-1} \end{cases}.$$

Thus we can find out $(g_1 g_2)^{-1}$. To finish we point out that we can get h from knowledge of h^{-1}. This comes from

$$\chi_{\mathbb{C}[G]}^{tri}(e, h^{-1}, g) = \begin{cases} |G| & g = h \\ 0 & g \neq h \end{cases}.$$
