The University of Edinburgh

College of Science and Engineering

Mathematics 4 Honours

U03420 Representation Theory

Tuesday, 8th May 2007

2:30pm – 4:30pm

Chairman of Examiners – Professor D C Heggie

External Examiner – Professor P H Kropholler

Credit will be given for the best THREE answers

Only calculators with at most a one or two line display may be used in this examination.

This examination will be marked 'anonymously'. Please ensure that you have entered your name where instructed then seal the flap in order to conceal it.

(1) (a) Explain what it means to call a representation irreducible. **1**

(b) State Maschke's theorem. **4**

(c) State and prove Schur's lemma. **6**

(d) Deduce that every irreducible complex representation of a finite abelian group is one-dimensional. **7**

(e) Let $G = \langle x : x^5 = e \rangle$ and let $F_2 = \{0, 1\}$ be the field with two elements. Show that the regular representation $F_2[G]$ decomposes as the direct sum of two irreducible subrepresentations. **7**

(2) Let G be the group with the presentation

$$G = \langle a, b : a^8 = e, a^4 = b^2, b^{-1}ab = a^{-1} \rangle.$$

(a) Let $\omega = \exp(2\pi\sqrt{-1}/8) \in \mathbb{C}$. Prove that for an integer r with $0 \le r \le 7$ there is a representation $\rho^{(r)} : G \longrightarrow GL(2, \mathbb{C})$ with

$$\rho^{(r)}(a) = \begin{pmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{pmatrix}, \qquad \rho^{(r)}(b) = \begin{pmatrix} 0 & 1 \\ (-1)^r & 0 \end{pmatrix}.$$

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(b) By using (a) (or otherwise), prove that $|G| = 16$. **5**

(c) G has character table

	C_e	C_{a^4}	C_a	C_{a^2}	C_{a^3}	C_b	C_{ab}
	1	1	2	2	2	A	4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	B	C
χ_3	1	1	-1	1	-1	1	-1
χ_4	D	1	-1	1	-1	-1	1
χ_5	2	-2	$\omega + \omega^{-1}$	0	$\omega^3 + \omega^{-3}$	0	0
χ_6	2	2	0	E	0	0	0
χ_7	2	-2	$\omega^3 + \omega^{-3}$	0	$\omega + \omega^{-1}$	0	0

Complete the character table by determining A, B, C, D, E. **7**

(d) For each of $r = 1, 2, 4$ write the character $\chi_{\rho^{(r)}}$ as a sum of the irreducible characters $\chi_1, \ldots, \chi_7$. **5**

(e) Let G' be the derived subgroup of G. Describe G/G' as a (product of) cyclic group(s). **4**

(3) (a) Let $H \le G$ be a subgroup and let W be a complex representation of H. Define $\mathrm{Ind}_H^G W$, the representation of G induced from W. Prove that

$$\dim_{\mathbb{C}} \mathrm{Ind}_H^G W = [G : H] \dim_{\mathbb{C}} W.$$

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(b) Let $G = S_4$ and let $H = \langle (12), (34) \rangle$. Determine the character of $\mathrm{Ind}_H^G W$ where W is the trivial representation of H. (You may use without proof the formula for the character of an induced representation, but you should state it clearly.) **12**

(c) How many distinct isotypic components does $\mathrm{Ind}_H^G W$ have in (b)? **2**

(4) (a) "Character theory makes representation theory easy." Discuss. **15**

(b) Give an example (without proof) of two non-isomorphic finite groups that have the same character table. **2**

(c) Let V be a complex representation of G. The *tricharacter* of V is the function

$$\chi_V^{tri} : G \times G \times G \longrightarrow \mathbb{C}$$

defined by $\chi_V^{tri}(g_1, g_2, g_3) = \chi_V(g_1 g_2 g_3)$ for $g_1, g_2, g_3 \in G$.
By considering the regular representation $\mathbb{C}[G]$ (or otherwise), show that the multiplication table of G can be recovered from the knowledge of the tricharacters of G. **8**

[End of Paper]