EXERCISES FOR LECTURE 8 (AND 9)

(1) Here are two different irreducible characters of the group $G = S_4$, with the conjugacy class representatives and sizes listed on the top two rows.

	e	(12)	(123)	$(12)(34)$	(1234)
size	1	6	8	3	6
χ_1	A	B	-1	2	0
χ_2	3	1	C	D	1

Use the orthogonality relations to work out A, B, C, D.

(2) Let $G = S_5$ and recall that the conjugacy classes of G are labelled by cycle type.

(a) Find a representative for each conjugacy class of S_5.

(b) How many elements are there in each conjugacy class. (When you add these up you should get 60. It might help you use to use the orbit-stabiliser theorem which states that $|Cl(x)| = |G|/|C_G(x)|$ where $C_G(x) = \{g \in G : gxg^{-1} = x\}$ is the centraliser of x in G.)

(c) Let G act naturally on $V = \mathbb{C}^5$ (by permuting the co-ordinates). What is the character of V? (Remember by Proposition 7.2 you only need to say what it does to each of your conjugacy class representatives.)

(d) Calculate $< \chi_V : \chi_V >$ and deduce that χ_V is not irreducible.

(e) By considering characters show that there is one copy, T, of the trivial representation in V.

(f) By Maschke's theorem $V = T \oplus W$ for some W. Use characters to show that W is irreducible.

(3) Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of a group G and suppose that

$$\chi = \sum_{i=1}^{k} d_i \chi_i$$

is a character of G. What can you say about the d_i in each of the cases $\langle \chi, \chi \rangle = 1, 2, 3, 4$?

(4) Here are two representations of $D_4 = \langle a, b : a^4 = b^2 = e, bab^{-1} = a^{-1} \rangle$:

$$\rho^{(1)}(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho^{(1)}(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\rho^{(2)}(a) = \begin{pmatrix} 83 & 130 \\ -53 & -83 \end{pmatrix}, \rho^{(2)}(b) = \begin{pmatrix} 43 & 66 \\ -28 & -43 \end{pmatrix}.$$

Using characters, decide whether these representations are isomorphic.

(5) Recall that A_4 is the subgroup of S_4 consisting of even permutations. It can be presented as $A_4 = \langle s, t : s^3 = t^2 = e, sts = ts^{-1}t \rangle$ (think of s and t being (123) and (12)(34)).

(a) Find representatives for the conjugacy classes of A_4. What is the size of each conjugacy class.

(b) How many non-isomorphic 1-dimensional representations does A_4 have. Calculate their characters.

(c) By using one of the results of the lecture deduce that A_4 only has one other irreducible representation, V, and that V has degree 3.

(d) Using the orthogonality relations, calculate χ_V.

(e) Well done! You've just calculated the "character table" of A_4.

(6) Using similar arguments to the previous question, calculate all of the irreducible characters of D_4 (the symmetry group of the square – 8 elements), remembering that $D_4 = \langle a, b : a^4 = b^2 = e, bab^{-1} = a^{-1} \rangle$.

(7) Similarly, calculate all the irreducible characters of $Q = \langle a, b : a^4 = e, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$. Do you notice anything funny compated with the previous question? Are Q and D_4 isomorphic?

(8) Suppose that a group G acts on the finite set X (not a representation, just a set).

(a) Let $\mathbb{C}[X]$ be a (complex) vector space of dimension $|X|$ with basis $(v_x : x \in X)$. Then G acts on V by the rule

$$\rho_V(g)(\sum_{x \in X} \lambda_x v_x) = \sum_{x \in X} \lambda_x v_{g \cdot x}.$$

Check that this defines a representation of G of degree $|X|$. It is called a permutation representation.

(b) Confirm to yourself that if we took $X = G$ with the action being left multiplication, then this procedure would just produce the regular representation again.

(c) Prove that for all $g \in G$ we have $\chi_{\mathbb{C}[X]}(g) = |\{x \in X : g \cdot x = x\}|$.