EXERCISES FOR LECTURES 2 AND 3

(1) Show that $C_3 = \{e, x, x^2\}$ has a degree three representation over F^3 with

$$\rho(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } \rho(x^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(2) Show the above representation is not irreducible by finding a one dimensional subrepresentation. Can you find more than one of these one-dimensional subrepresentations?

(3) Let $G = C_n = \{e, x, \dots, x^{n-1}\}$ and let $V = \mathbb{C}$ (a one dimensional vector space over $\mathbb{C}$). Let $\zeta = \exp(2i\pi/n)$. Given $j \in \mathbb{Z}$ show that $\rho_j : G \longrightarrow GL(V)$ defined by $\rho_j(x^i) = \zeta^{ij}$ for $i = 0, \dots, n-1$ is a representation of G. For which j and k do we have $(\rho_j, V) \cong (\rho_k, V)$?

(4) Suppose that $\rho : G \longrightarrow GL(n, F)$ is a degree n representation. Define $\rho^* : G \longrightarrow GL(n, F)$ by the rule $\rho^*(g) = \rho(g^{-1})^T$ (where the T stands for transpose matrix). Show that ρ^* also defines a degree n representation of G.

(5) Let ρ be the representation of C_3 defined in Question 1. Is ρ^* isomorphic to ρ? In Question 2, do we ever have that $(\rho_j^*, V) \cong (\rho_k, V)$ for some j and k?

(6) Suppose the group G acts on the set X (in the sense of Year 3). Define a vector space $F[X]$ to be the set of F-valued functions on X i.e. functions $f : X \longrightarrow F$
   (a) Show that $F[X]$ is an F-vector space of dimension $|X|$.
   (b) Show that if $g \in G$ there is an endomorphism of $F[X]$ defined by sending $f \in F[X]$ to the function we call $g \cdot f$ which sends x to $f(g^{-1} \cdot x)$.
   (c) Deduce that $F[X]$ can be considered as a representation of G of degree $|X|$.
   (d) Is $F[X]$ irreducible? If so, prove it! If not, find a proper subrepresentation!

Long aside on generators and relations - very important! In the second lecture we chose to present D_3 in terms of *generators and relations*,

$$D_3 = \langle a, b : a^3 = b^2 = e, bab^{-1} = a^2 \rangle.$$

This can be done for many groups G and leads to a presentation:

$$G = \langle x_1, \dots, x_r : s_\alpha(x_1, \dots, x_r) = e, \alpha = 1, \dots, \ell \rangle,$$

with the generators being the x_i's and the relations the s_α's. We call strings of generators *words* and two words are equivalent if they can be brought into the same form by repeatedly using the relations.

For the example of D_3 there are two generators for the group, a and b, and there are three relations $a^3 = e$, $b^2 = e$ and $bab^{-1} = a^2$ in addition to the trivial relations $aa^{-1} = e$ and $ae = ea = a$ (and similarly for b) which are assumed. (The relation $bab^{-1} = a^2$ can also be written as $bab^{-1}a^{-2} = e$, or better still as $baba = e$ since $b^{-1} = b$ – because $b^2 = e$ – and $a^{-2} = a$ – because $a^3 = e$. This makes it have the form $\langle a, b : a^3 = b^2 = baba = e\rangle$ and so $s_1(a, b) = a^3, s_2(a, b) = b^2, s_3(a, b) = baba$ in this example.) We can always move any word into the an equivalent form a^ib^j with $0 \le i \le 2$ and $0 \le j \le 1$. For instance

$$b^3a^2babab \equiv ba^2babab \equiv ba^2b \equiv a^2bab \equiv a^4 \equiv a$$

where the symbol $\equiv$ means 'equivalent', and at each step we have used some form of the relations (first $b^2 = e$, then $baba = e$, then $ba = a^2b$, then $bab = a^2$, then $a^3 = e$).

(7) The dihedral group D_4 is defined by

$$D_4 = \langle a, b : a^4 = b^2 = e, bab^{-1} = a^{-1}\rangle.$$

(a) Show that each element of D_4 can be put in the form a^ib^j for suitable i and j.

(b) Write out the conjugacy classes of D_4 in terms of the elements a^ib^j that you found.

(8) The dihedral group D_n is defined by

$$D_n = \langle a, b : a^n = b^2 = e, bab^{-1} = a^{-1}\rangle.$$

(a) Show that each element of D_n can be put in the form a^ib^j for suitable i and j.

(b) Write out the conjugacy classes of D_n in terms of the elements a^ib^j that you found.

(9) The symmetric group S_3 is defined by

$$S_3 = \langle a, b : a^2 = b^2 = e, aba = bab\rangle.$$

(a) List six different elements of S_3 in terms of a and b.

(b) How would you interpret a and b in terms of permutations of $\{1, 2, 3\}$?

Given two groups G and H we may define a mapping $\rho : G \longrightarrow H$ by specifying the images of the generators $\rho(x_i) = X_i \in H$ and setting

$$\rho(x_{i_1}x_{i_2}\cdots x_{i_t}) = \rho(x_{i_1})\rho(x_{i_2})\cdots\rho(x_{i_t}).$$

To check that this map is a well-defined group homomorphism we need that $\rho(s_\alpha(x_1, \ldots, x_r)) = e_H$ for each relation defining G.

This applies in particular to the case $H = GL(V)$ or $H = GL(n, F)$ for any $n \ge 0$ and so can be used with representations. So if we are given a presentation of a group

$$G = \langle x_1, \ldots, x_r : s_\alpha(x_1, \ldots, x_r) = e, \alpha = 1, \ldots, \ell\rangle$$

we can construct a representation (ρ, F^n) simply by taking some n-by-n invertible matrices X_i (which will be the $\rho(x_i)$), checking that each $s_\alpha(X_1, \ldots, X_r)$ is the identity matrix and then setting

$$\rho(x_{i_1} \cdots x_{i_t}) = X_{i_1} \cdots X_{i_t}$$

for any word $x_{i_1} \cdots x_{i_t}$.

(10) Show that checking $\rho(s_\alpha(x_1, \ldots, x_r)) = e_H$ is the same as checking $s_\alpha(X_1, \ldots, X_r) = e_H$.

(11) Verify that $\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ defines a representation of the dihedral group D_8 defined above.

(12) Verify that Examples 2.4' and 2.7 in the lectures do indeed define representations. Show that they are isomorphic.

(13) Using the presentation above, find four non-isomorphic degree 1 representations of D_4.

(14) Verify that $\rho(a) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\rho(b) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ defines a degree two representation of the symmetric group S_3 defined above. Show this is isomorphic to the representation W that appears at the end of Example 3.6 in the lectures. Prove that this representation is irreducible.