EXERCISES FOR LECTURES 10-13

A *linear character* is a fancy name for a character of degree 1. This means that χ is the character of representation $\rho : G \longrightarrow GL(1, \mathbb{C})$ and so by definition $\chi(g) = Tr(\rho(g)) = \rho(g)$. So a linear character is exactly the same thing as a degree 1 representation.

(1) Let χ be a linear character. Prove that $\chi(xyx^{-1}y^{-1}) = 1$ for all $x, y \in G$.

(2) Let G' be the *derived subgroup* of G, i.e. the subgroup of G generated by all elements of the $xyx^{-1}y^{-1} \in G$. Show that G' is a normal subgroup and that G/G' is abelian.

(3) Deduce from the above two questions that any linear character is actually the lift of a character of the abelian group G/G'.

(4) Deduce that the number of linear characters of G equals the order of G/G'.

(5) Calculate the derived subgroup of D_4, A_4 and S_4.

(6) Let λ be a linear character of G and let ρ be any representation of G.
   - (a) Prove that $(\lambda\rho)(g) = \lambda(g)\rho(g)$ is a representation of G with character $\lambda\chi_\rho$.
   - (b) Prove that χ_ρ is irreducible if and only $\lambda\chi_\rho$ is irreducible.
   - (c) Meditate on the fact that this process may allow you to produce new irreducible characters from old ones.

(7) Find the normal subgroups of Q, A_4 and D_4 by using their character tables (you calculated these in (5), (6) and (7) on the Exercises 9).

(8) Calculate the character table for S_4 as follows:
   - find the irreducible representations of degree 1 ((3) should help you);
   - calculate the character of the natural degree 4 representation of S_4 and use it to get an irreducible character of degree 3;
   - use (6) to produce another degree 3 character;
   - use the orthogonality relations to get anything else you need.

(9) Here is (most of) the character table of the group $G = \langle s, t : s^4 = t^2 = (st)^3 = e\rangle$

	$\{e\}$	$\{t\}$	$\{ts\}$	$\{s^2\}$	$\{s\}$
	1	A	B	3	6
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	C	0	-1	D	0
χ_4	E	1	0	-1	-1
χ_5	3	-1	0	-1	1

(a) Complete the character table by calculating A, B, C, D, E.

(b) Determine all of the normal subgroups of G. Specify both the commutator subgroup and also the centre of G.

(c) What is the standard form for G/G'?

(d) Does G have a faithful representation (i.e. a representation with trivial kernel)?

(10) Check the inner product calculations we made while working out the character table of A_5.