"Character Theory makes Representation Theory straightforward".

Working in characteristic zero over an algebraically closed field, the principal problem in the representation theory of finite groups is to describe how representations decompose into irreducible representations (by Maschke's theorem, this is always possible). In the abstract, this is a difficult problem: we are dealing with (potentially) large vector spaces; linear transformations of these vector spaces for each group element (and these all depend on a choice of basis); ~~matr~~ subspaces fixed by these matrices. As any smallish example could show, this is difficult to calculate in practice.

So, in an ideal world, we would like to have another way to test for irreducible subrepresentations. This way should not be expensive in demanding a lot of vector space data; it should not depend on a choice of basis; it should detect subrepresentations. Incredibly we are in an ideal world. The character of a representation does this. It is defined as follows: given a representation (ρ, V) of G we define

$$\chi_V : G \longrightarrow \mathbb{C}, \quad \chi_V(g) = \mathrm{Tr}\, \rho(g).$$

So this produces a number instead of a matrix, and since $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$ for invertible matrices

A, we also immediately that this does not depend on the choice of basis we make to evaluate the trace. Moreover, we also see that χ_V is a class function i.e constant on conjugacy classes. In fact, on studying more closely the set of class functions from $G \to \mathbb{C}$ we see that it is a vector space under addition and that the characters of irreducible representations form a basis (the regular representation plays a crucial role here). The <u>crucial</u> fact then is that the space of class functions is an inner product space with $\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \psi(g)$. and that the characters of irreducible representations form an orthonormal basis (a more tricky theorem). This has many wonderful consequences

- any character χ_V decomposes as a sum $\sum n_i \underline{\chi_i}$ where the χ_i are characters of irreducible rep^{ns} I_i (basis property): ~~The~~ this reflects $V = \oplus I_i^{n_i}$. Then $n_i = \langle \chi_V, \chi_i \rangle$ so this solves our original problem.
- $\chi_V = \chi_W \Rightarrow V \cong W$ an immediate consequence of the above
- ~~...~~ V is irreducible $\iff \langle \chi_V, \chi_V \rangle = 1$: another immediate consequence of the above (so we have an easy way to test whether a representation is

irreducible)

- knowing some of the irreducible characters (and many, e.g. the trivial character, are easy to find) allows us to find other irreducible characters by orthogonality and basic arithmetic properties of characters (e.g. they are a sum of roots of unity).

So, many questions become easy: we can decompose arbitrary representations when we know all irreducible characters; we can start to find irreducible characters (even if we never quite know what the representations are explicitly).

However, calculating characters can be hard and start to involve arithmetic (= number theoretic) problems and in these problems we may often get stuck: we've thrown away all the nice extra structure that representations have and reduced just to numbers. Moreover, non-isomorphic groups can have identical character tables (e.g. D_4 and Q_8): so character theory doesn't distinguish groups. All this means that character theory is incredibly useful for complex representation theory, but we have to be more sophisticated when we work over other fields, fields where representation theory makes good sense and is very interesting.