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1994 Russian Acad. Sci. Izv. Math. 42 371

(http://iopscience.iop.org/1468-4810/42/2/A06)

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Сер. Матем. Том 57 (1993), № 2

Russian Acad. Sci. Izv. Math.
Vol. 42 (1994), No. 2

SEMISTABLE 3-FOLD FLIPS

UDC 512.7

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Dedicated to Professor I. R. Shafarevich on the occasion of his seventieth birthday.

ABSTRACT. The existence of 3-fold semistable flips is proved, and the complete list of 3-fold semistable singularities is given.

This paper grew out of a course of lectures given at Moscow State University in the spring of 1985, the early period of "perestroïka". The course was designed not only to expose Tsunoda's result [23], Theorem 1, which has been treated very similarly to his presentation in spite of independence of our proof, but also to find a new approach to Kulikov's theorem ([12], Theorem 1). That time we envisaged semistable singularities that are singularities of a relative minimal model for a surface semistable degeneration as rather special. Tempora mutantur et nos mutamur in illis. Now we will present an opposite opinion.

We use the terminology and results of [7], [26], and [27], which are independent of the existence of flips. In §§1-3 we consider only the analytic case. Applications to the algebraic case are given in §§4-5. All varieties (spaces) are complex (analytic) and normal. We assume that the reader is aware of the following notation:

$f^{-1}C$ for a cycle C, its proper inverse image with respect to a birational (bimeromorphic) morphism f;

$\mathbb{F}_n$, the rational scroll with a section having the self-intersection number $-n$;

$\operatorname{ind}_p K$, the index of $p \in X$;

K, a canonical divisor on a variety (space) X;

K_Y, a canonical divisor on a variety (space) Y; $\overline{\mathrm{NE}}(X/Z; V)$, the relative Kleiman-Mori cone;

$1/r(a_1, \dots, a_d)$, the type of a quotient singularity;

$\rho(X/Z; V)$, the relative Picard number;

n-curve means a nonsingular rational curve C on a surface S with the self-intersection number $C^2 = n$;

$\sigma(X, p)$, the $\mathbb{Q}$-factorial defect at a point $p \in X$ ([5], 1.12, [18], 3.4);

$V_1(r, a; n)$, $V_2(r, a)$, V_3, the moderate singularities ([6], 1.1).

1. SEMISTABILITY: INDUCTIVE APPROACH

All the notions below with the adjective "semistable" generalize the similar notions for a surface semistable degeneration. They have a different flavor in [11], C3: *main but not generic.*

1.1. Conventions, notation, and definitions. In the sequel, we assume that X is an *analytic 3-fold*, i.e., a normal complex analytic space of dimension 3, *with a canonical*

1991 *Mathematics Subject Classification.* Primary 14E05, 14E35, 14J30.

1064-5632/94 $1.00 + $.25 per page

divisor K and *only terminal singularities.* A *semistable divisor* is a reduced effective divisor $D = \sum D_i$ such that

(1.1.1) All prime components D_i of D are normal $\mathbb{Q}$-Cartier divisors, and

(1.1.2) locally there is a resolution $g\colon Y \to X$ with a reduced divisor g^*D whose prime components are nonsingular and cross normally.

We assume also that g is projective in the following weak sense:

(1.1.3) g can be decomposed into a product $g = g_1 \circ \cdots \circ g_N$ of *locally projective* morphisms g_j, i.e., projective in a neighborhood of each fiber, and such that

(1.1.4) for the partial resolutions $G_j = g_1 \circ \cdots \circ g_j\colon Y_j \to X$, Y_j is an analytic 3-fold with only terminal singularities and all prime components of G_j^*D's are normal $\mathbb{Q}$-Cartier divisors.

The singularities of X that belong to D will be referred to as *semistable for* D.

A contraction $f\colon X \to Z$ is (*numerically*) *semistable for* D if D is semistable and a (*numerical*) *fiber with respect to* f. The last means that D is linearly (numerically) trivial near (respectively, on) each fiber of f. Thus, in the linearly trivial case and in a neighborhood of any fiber of f over $f(D)$, D is a scheme fiber for a morphism on a nonsingular curve passing through f. If in addition K is nef with respect to the contraction f, it will be called a *minimal* (*numerically*) *semistable model for* D.

According to (1.1.2), every point $p \in D$ has a resolution $g\colon Y \to X$ defined locally at p, which is semistable for g^*D by the Contraction Theorem (cf. Lemma 1.4 below). Partial resolutions g_j and their compositions G_j are semistable too for G_j^*D's. The minimal number $i(X, p, D)$ of prime divisors $E_i \subset Y_j$, exceptional for g_j's and with $g_jE_i = \text{pt.}$, needed for such resolutions g/p will be called the *depth in* p *for* D. For a compact analytic subset $W \subseteq D$, we define the *depth* as

$$i(X, W, D) = \sum_{p\in W} i(X, p, D).$$

Soon we check that it is finite (see Corollary 1.5). Moreover, it is independent of the choice of D (see Corollary 4.6). The definition implies also that the depth is not less than the difficulty in a neighborhood of W ([25], 2.15).

Fix a (numerically) semistable contraction $f\colon X \to Z$ and a compact analytic subset $V \subseteq f(D)$. Since $f^{-1}V \subseteq D$, we can also introduce the *depth of* X *over* V *for* D as $i(X/V, D) = i(X, f^{-1}V, D)$.

1.2. **Example.**

(1.2.1) If X is nonsingular near D, and all D_i's are nonsingular and cross normally, then D is semistable and $i(X, p, D) = 0$ for any $p \in D$. The converse holds when every point $p \in D$ is $\mathbb{Q}$-factorial, i.e., $\sigma(X, p) = 0$. Moreover, we will check that every point $p \in D$ with $i(X, p, D) = 0$ has a small semistable resolution by $\sigma(X, p)$ curves $\mathbb{CP}^1$ (see (1.3.6) below). The next example illustrates this.

(1.2.2)(Mori) Let X and D be as in (1.2.1) above, and let f be an extremal blow-down with respect to K, i.e., f is bimeromorphic, negative with respect to K, and $\rho(X/Z; V) = 1$. Then $i(Z, V, D) \leq 1$ with equality only in the following two cases:

> f is a contraction of type [14], 3.3.5 with a divisorial exceptional locus $D_1 = \mathbb{CP}^2$ such that a unique divisor D_i, say D_2, intersects it, and $C = D_1 \cap D_2 = \mathbb{CP}^1$ is a nonsingular conic on D_1 with self-intersection numbers $C^2_{D_1} = 4$ and $C^2_{D_2} = -4$ on D_1 and D_2, respectively (Figure 1(a));

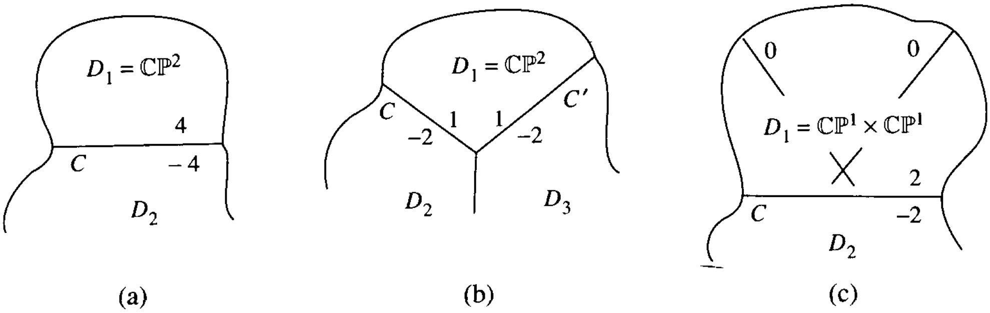

FIGURE 1

> f is a contraction of type [14], 3.3.5 with a divisorial exceptional locus D_1 such that only two divisors D_i, say D_2 and D_3, intersect it, $C = D_1 \cap D_2 = \mathbb{CP}^1$, and $C' = D_1 \cap D_3 = \mathbb{CP}^1$ are lines on D_1 with self-intersection numbers $C^2_{D_1} = C'^2_{D_1} = 1$ and $C^2_{D_2} = C'^2_{D_3} = -2$, $D_1 \cap D_2 \cap D_3$ is a triple point of normal crossing (Figure 1(b)).

Note also that $i(Z, V, D) = 0$ and Z or, equivalently, $f(D)$ has a singular point $p \in V$ only in the following case:

> f is a contraction of type [14], 3.3.3 with a divisorial exceptional locus $D_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and only one divisor D_2 intersecting it, after an appropriate renumbering; $C = D_1 \cap D_2 = \mathbb{CP}^1$ is a nonsingular rational curve of bidegree $(1, 1)$ on D_1 with self-intersection numbers $C^2_{D_1} = 2$ and $C^2_{D_2} = -2$ on D_1 and D_2, respectively (Figure 1(c)). So, $p = g(D_1)$ is an isolated singularity on Z and is analytically equivalent to $x^2 + y^2 + z^2 + u^2 = 0$ with the origin $(0, 0, 0, 0)$ as p ([14], 3.4.3), whereas $f(D) = f(D_2)$ with a singularity $x^2 + y^2 + z^2 = 0$ at p. It has two Atiyah small resolutions, which contract families of generators on $D_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$.

All these facts are easily derived from Mori's classification of extremal contractions [14], 3.3. The last contraction does not stay extremal when we consider it over a small neighborhood of p. Indeed, $\overline{\mathrm{NE}}(X/Z; p)$ has two extremal rays corresponding to contractions of generators.

(1.2.3)(*Danilov, Barlow*) Let $p \in X$ be a terminal quotient singularity of type $1/r(a, -a, 1)$ with a coprime to $r = \mathrm{ind}_p K$, the index of p ([22], 5). Denote by D_1 and D_2 the quotients of hyperplanes $x = 0$ and $y = 0$, respectively. Then p is semistable for $D = D_1 + D_2$ with $i(X, p, D) = r - 1$ and will be referred to as of type $V_2(r, a)$ due to Kawamata ([6], 1.1.2). Such a singularity has an economical projective resolution $g: Y \to X$, which is semistable for g^*D. *Economical* means that all discrepancies a_i for exceptional divisors E_i of g belong to the interval $(0, 1)$ or, equivalently, have the form $a_i = n_i/r$ with integer $0 < n_i < r$. In particular, all E_i's lie over p. Moreover, $n_i = i$ and $1 \le i < r$, after an appropriate renumbering of E_i's.

(1.2.4)(*Kawamata*) A *moderate singularity* $V_1(r, a; n)$ with positive integers r, a, and n such that $(r, a) = 1$ is the quotient of the hypersurface

$$xy + z^r + w^n = 0$$

for the quotient singularity of type $1/r(a, -a, 1, 0)$, whereas D is the quotient of its hyperplane section $w = 0$ ([6], 1.1.1) and p corresponds to the origin $(0, 0, 0, 0)$. Note that, for $n = 1$, we obtain again the quotient singularity $1/r(a, -a, 1)$ but with a different type of D. Two irreducible components of D pass through a singularity of type $V_2(r, a)$, and only one does so for $V_1(r, a; n)$. The singularities appearing in the two cases with $i(Z, V, D) = 1$ in (1.2.2) above are analytically equivalent to $V_1(2, 1; 1)$ and $V_2(2, 1)$, respectively.

It turns out that up to a $\mathbb{Q}$-factorialization we have nothing else. But first we establish the following half-inductive and half-explicit description of semistable singularities.

1.3. Theorem on semistable singularities. *Let $p \in D$ be a semistable singularity, and let $d = \#\{D_i | p \in D_i\}$. Then*

(1.3.1) $1 \le d \le 3$.

(1.3.2) For $d = 3$, p is nonsingular on X and a triple point of D, or of type V_3 due to Kawamata ([6], 1.1.3), i.e., D_i's passing through p are nonsingular and cross normally in p (Figure 2(a), after an appropriate renumbering of D_i's).

(1.3.3) For $d = 2$, p is $\mathbb{Q}$-factorial and locally of type $V_2(r, a)$ with $r = \mathrm{ind}_p K = i(X, p, D) + 1$ (Figure 2(b), after an appropriate renumbering of D_i's; cf. example (1.2.3)).

(1.3.4) For $d = 2$ and 3, p is $\mathbb{Q}$-factorial.

(1.3.5) For $d = 1$; there exists a semistable $\mathbb{Q}$-factorialization, i.e., a (possibly nonprojective) small blow-up $g: Y \to X$, which is semistable for $g^*D = g^{-1}D$ and with only $\mathbb{Q}$-factorial points on Y over p (Figure 2(c), after an appropriate renumbering of D_i's; cf. the last example in (1.2.2)). In this case, $i(X, p, D) = i(Y/p, g^*D)$. The $\mathbb{Q}$-factorialization is identical if and only if p is $\mathbb{Q}$-factorial.

(1.3.6) $\mathbb{Q}$-factorial p is nonsingular if and only if p is Gorenstein on X, i.e., $\mathrm{ind}_p K = 1$. In this case, D_i's passing through p are nonsingular, cross normally near p and $i(X, p, D) = 0$. In addition, for $d = 1$, p is Gorenstein if and only if it is canonical on D, or if and only if $i(X, p, D) = 0$.

(1.3.7) For $\mathbb{Q}$-factorial and singular p, there exists a projective divisorial blow-up $g: Y \to X$ having an irreducible exceptional divisor G over p, which is semistable for $g^*D = g^{-1}D + G$, with $\mathbb{Q}$-factorial Y/p, and is extremal with respect to K (Figures 2(d) and 2(e), respectively, for $d = 1$ and 2, after an appropriate renumbering of D_i's; cf. the first two examples in (1.2.2)). The last means that Y has only terminal singularities, $\rho(Y/X; p) = 1$, and K is numerically negative with respect to g. Moreover, there exists a blow-up g such that $i(X, p, D) = i(Y/p, g^*D) + 1$. In this case, g is minimal on its nonexceptional divisors $g^{-1}D_i$, i.e., the curves $G \cap g^{-1}D_i$ are exceptional but are not of the first kind on the minimal resolutions of $g^{-1}D_i$'s, and the discrepancy a of G is $0 < a < 1$ (cf. example (1.2.3) for $d = 2$). So, birationally G appears in any resolution of p and gives a contribution to the depth as well as to the difficulty of X over p.

In §4 we find out more, in particular, that Y/p in (1.3.7) has at most three singularities that are $\mathbb{Q}$-factorial again (see Corollary 4.5). This will imply that, in a neighborhood of D, X can be resolved in two stages: first, by making a $\mathbb{Q}$-factorialization of all singularities on D; second, by applying (1.3.7) in consecutive order to the remaining singularities $/D$. However, according to the last theorem, to resolve a semistable singularity with $d = 1$ we must use $\mathbb{Q}$-factorializations and divisorial blow-ups from (1.3.7) one after an other. Such extractions are trivially extended to the whole range (X in Theorem 1.3) but as a rule are projective only for (1.3.7). We remark that the $\mathbb{Q}$-factorialization in (1.3.5) is not uniquely defined, but

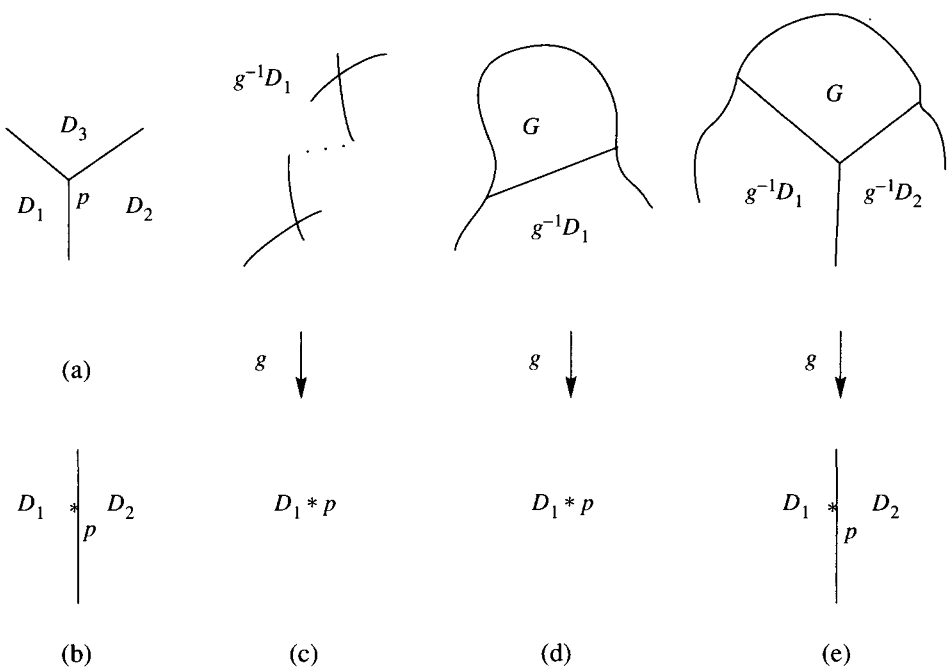

FIGURE 2

is defined up to so-called flops [9] discussed later (see Corollary 3.7). The following result somewhat clarifies the above statements and enables us to move on.

1.4. Lemma. *D is Cartier with normal crossings, and $K+D$ is divisorially log terminal, with log discrepancies equal to discrepancies of K for the exceptional divisors in g^*D when g is semistable.*

Proof. According to definition (1.1.2), D is numerically Cartier. So, if we mimic a proof of the Contraction Theorem, we obtain that D is Cartier. Again (1.1.2) locally implies the log canonical property of $K+D$ with required log discrepancies. By our assumption, $K+D$ is terminal outside D.

So, let $p \in X$ and $d \geq 1$. Take a resolution g from (1.1.2). Obviously, we can assume that g is nontrivial over p. But the exceptional locus of g can be non-pure-divisorial and, much worse, without the property of normal crossing for $g^{-1}D + \sum E_i$, where the E_i's are exceptional divisors for g. However, by (1.1.2),

$$g^*D = g^{-1}D + \sum_{E_i/D} E_i$$

is reduced with nonsingular and normally crossing prime components. Now we check that $g^{-1}D$ has no triple points or double curves over p and outside E_i's. Indeed, let a point

$$q \in g^{-1}D_1 \cap g^{-1}D_2 \cap g^{-1}D_3,$$

after an appropriate renumbering of D_i's, be over p. Then the fiber $f^{-1}p$ contains a curve passing through q, but not lying on $g^{-1}D_1$, because $f^{-1}p$ is nontrivial and connected. Since D_1 is $\mathbb{Q}$-Cartier, there is an exceptional divisor $E_i/D_1/D$ with $(E_i \cdot C) < 0$. So, E_i passes through q, contradicting the normal crossing property of g^*D. In the remaining case when the curve

$$C = g^{-1}D_1 \cap g^{-1}D_2$$

lies over p and does not lie on E_i's, C is exceptional over p with log discrepancy 0 for

$$(K + D_1 + D_2)|_{D_1} = K_{D_1} + (D_2)_{D_1},$$

where $(D_2)_{D_1}$ is a reduced divisor passing through p, and with multiplicity 1 in the subboundary

$$g^*(K_{D_1} + (D_2)_{D_1}) = g^*(K + D)|_{g^{-1}D_1} = (K_Y + g^*D - \sum a_i E_i)|_{g^{-1}D_1},$$

where the a_i's are the discrepancies of K and $a_i > 0$ under our assumption. By the previous fact, C intersects only two surfaces $g^{-1}D_i$, namely, for $i = 1$ and 2. Hence, near C, for the subboundary

$$\left.\left(\sum_{j\neq 1} g^{-1}D_j + \sum_{E_i/D} E_i - \sum a_i E_i\right)\right|_{g^{-1}D_1}$$

of $g^*(K_{D_1} + (D_2)_{D_1})$, C is a complete locus of log canonical singularities. But $(D_2)_{D_1}$ passes through p, and all components of the above subboundary with negative multiplicities are contracted by g in contradiction with [27], 5.7.

Therefore, $g^{-1}D$ has no triple points, and double curves over p lie on E_i's. Besides, by the definition and monotonicity ([27], 1.3.3),

$$g^*(K + D) = K_Y + g^*D - \sum a_i E_i$$

is divisorially log terminal in the sense of [27], 3.1, because the effective part of the subboundary is supported in g^*D. This allows us, possibly after an additional resolution over a neighborhood of p, to make the exceptional locus g pure divisorial. (A new g may be nonsemistable.) Thus, we establish that locally $K + D$ is divisorially log terminal. By [27], 3.8, this implies that D has normal crossing, and the exceptional divisors with 0 log discrepancies lie over triple points (1.3.2) and double curves of D. Hence $K + D$ is divisorially log terminal, and a required resolution may be done (by B58 loaded with blow-ups) over isolated points of X, where D does not have normal crossing in the usual nonsingular sense, i.e., where X or D_i is singular. ■

A contraction or a minimal model is numerically semistable when it is semistable. By the Contraction Theorem, the inverse holds when $-K$ is nef and big with respect to f, in particular, when K is numerically nonpositive with respect to f and f is a *blow-down*, i.e., bimeromorphic.

Proof of (1.3.1-4) *and for* $d \geq 2$ (1.3.6-7). The normal crossing property of D implies (1.3.1-2). According to Lemma 1.4 and [27], 3.2.3, for $d = 2$ and near p, the restriction

$$(K + D)|_{D_1} = (K + D_1 + D_2)|_{D_1} = K_{D_1} + (D_2)_{D_1}$$

is purely log terminal, and the different $(D_2)_{D_1} = D_1 \cap D_2$ is irreducible and nonsingular. In particular, if $\operatorname{ind}_p K = 1$, then D_1, D_2, and X are nonsingular by [27], 3.9.2 and 3.7, respectively. In general, using the covering trick we obtain that p has type $1/r(a, -a, 1)$ with $r = \operatorname{ind}_p K$. But D is Cartier, and so, near p, may be given as a quotient of two planes $x = 0$ and $y = 0$. This in view of Example (1.2.3) completes the proof. The required blow-up g in (1.3.7) extracts an exceptional divisor with minimal (log) discrepancy $1/r$ for K (for $K + D$). ■

It is much more difficult to prove the rest of Theorem 1.3, consisting of (1.3.5-7) for $d = 1$, and it will be done in §§2-3. According to Mori's classification [15], it is easy to see that these singularities are of type (1) ([22], [5]). Indeed, they have an invariant hyperplane section with a Du Val singularity. But not all of (1) are semistable (cf. Corollary 4.7 below). Now we remark that $i(X, p, D) \geq 1$ only when $d = 2$ and p is singular on X, or $d = 1$ and p is singular on D (more precisely, on the unique component D_i passing through p). Indeed, for $d = 1$, by [27], 3.7, p is nonsingular on X and $i(X, p, D) = 0$, whenever p is nonsingular on D (D_i). Hence p with $i(X, p, D) \geq 1$ form a discrete subset of D. This gives

1.5. Corollary. *$i(X, W, D)$ and $i(X/V, D)$ are finite.*

Hence we can try to prove the rest of Theorem 1.3 by induction on $i(X, p, D)$. If $i(X, p, D) = 0$, then by definition the point $p \in D$ locally has a resolution $g: Y \to X$ semistable for g^*D with decomposition as in (1.1.3), where all g_j's have at most 1-dimensional fibers. If g_N is small and $N \geq 2$, the composition $g_{N-1} \circ g_N$ is again locally projective with at most 1-dimensional fibers. Hence we may replace the last decomposition of g by a new one with $g_{N-1} := g_{N-1} \circ g_N$ and $N := N - 1$. So, we can assume that $g_N: Y_N \to Y_{N-1}$ is not small when g is not small. But locally/ Y_{N-1} we can replace g_N by a minimal model. Indeed, the model exists according to [14], 3.3, because of fibers of g_N are at most 1-dimensional. Since Y_{N-1} has only terminal singularities, such models are small/ Y_{N-1}. This defines a new locally projective and small morphism $g_N: Y_N \to Y_{N-1}$. Moreover, the new Y_N is nonsingular, and the new G_N satisfies (1.1.4) by (1.2.2) and (1.3.4). Therefore, after a finite number of such replacements we find a small resolution g that is semistable for g^*D. According to (1.3.4), g is nontrivial only for the interesting case when $d = 1$. So, Theorem 1.3 holds when $i(X, p, D) = 0$. Now we can assume that

(I1). *Theorem 1.3 holds when $i(X, p, D) \leq n$ for a fixed $n \geq 0$ and any choice of X, D, and $p \in D$.*

We must check Theorem 1.3 or the rest of it when $i(X, p, D) = n+1$. According to (1.1.2-3), after shrinking X to a neighborhood of p, we have a locally projective resolution $g_N: Y_N \to Y_{N-1}$ and a semistable boundary $G_N^*D = g^*D$ on $Y_N = Y$ with normal crossings of components, (1.1.2). But by definition, g_N is semistable.

As above, it is natural to apply Mori's theory to the morphism g_N, locally over points

$$q \in G_{N-1}^{-1}p \subset g_N(G_N^*D) = G_{N-1}^*D \subset Y_{N-1}.$$

Obviously, the boundary G_N^*D is Cartier, numerically 0 with respect to g_N, and this holds after flipping modifications in extremal rays of $\overline{\mathrm{NE}}(Y_N/Y_{N-1}; q)$ negative for K_{Y_N}. So, we may consider these modifications in the log category as modifications negative for $K_{Y_N} + G_N^*D$. It is well known that after such modifications Y_N and $K_{Y_N} + G_N^*D$ remain, respectively, terminal and divisorially log terminal ([27], 1.12). The *strictly terminal* or log terminal property, which includes the $\mathbb{Q}$-factorial property of Y_N and the projectivity of g_N over q, remains true also, because it holds for the original g_N. Hence, by [27], 3.8, and Lemma 1.4, G_N^*D always satisfies condition (1.1.1), i.e., all prime components of G_N^*D remain normal $\mathbb{Q}$-Cartier and cross normally. Moreover, we prove later that g_N^*D remains semistable, and *this is the point of the paper.* This includes also a proof of existence for the required flips, which will be referred to as *semistable flips.*

To give a somewhat more general statement, we consider the following situation. Let $f: X \to Z$ be a contraction numerically semistable for D, as above, and projective. Now let $g: X \to Y$ be a contraction of an extremal ray $R \subset \overline{\mathrm{NE}}(X/Z; V)$ that is negative with respect to K or, equivalently, to $K + D$. We assume also that g

is a blow-down, i.e., bimeromorphic. We remark that g is a projective contraction to Y that is semistable for D, and not only numerically. In this situation we must replace X/Z by X^+/Z over a neighborhood of V. This makes a *flip of* g (or *in the corresponding extremal ray* R) *with respect to* K (or $K+D$) that is a bimeromorphic transform t of g into g^+ over Z, i.e., a commutative diagram

$$\begin{array}{ccccc} X & & \overset{t}{\dashrightarrow} & & X^+, \\ & \searrow^{g} & & {}^{g^+}\swarrow & \\ & & Y & & \\ & {}_{f}\searrow & \downarrow & \swarrow_{f^+} & \\ & & Z & & \end{array}$$

where g^+ is a small blow-down numerically positive with respect to K_{X^+} (or $K_{X^+} + t(D)$). The notion of a flip here is slightly more general than the usual one ([7], 5-1-10). It means that we do not care whether g is small. But we assume only that g^+ is small, K_{X^+} is $\mathbb{Q}$-Cartier and ample/ Y because X^+ is a 3-fold. The terminal property of X^+ and the uniqueness of the flip are well known. And we know at least that $t(D)$ is Cartier satisfying (1.1.1), and it is numerically or linearly trivial on or near each fiber of f^+ if the same holds respectively for f (see Lemma 1.4 and [27], 1.12, 3.8; cf. [5], 10.4-5). The most difficult part of the paper consists in construction of such flips and a discussion of their properties, the main of which is preservation of semistability. Note that, by [25], 2.17, such flips terminate.

Denote by E the exceptional locus of g. We recall that g is *divisorial* when E contains a divisorial component over a neighborhood of V. Otherwise g is *small*. Since g is extremal, if we assume that X is $\mathbb{Q}$*-factorial over* V, i.e., over neighborhoods of V, then, over a neighborhood of V and in the divisorial case, E is an irreducible divisor, and $t = g$ is a holomorphic contraction. Otherwise we have an ordinary flip with small g.

1.6. Theorem on semistable blow-down. *Let* g *be divisorial and, moreover,* $D_1 \subseteq E$ *over a neighborhood of* V, *after an appropriate renumbering of* D_i*'s. Then, over a neighborhood of* V, $E = D_1$ *and again* Y *has only terminal singularities,* $g(D)$ *is semistable on* Y, Y/Z *is numerically semistable for* $g(D)$ *and*

$$i(Y/V, g(D)) \leq i(X/V, D) + 1$$

with equality only if $E = D_1$ *is contracted to a singular and* $\mathbb{Q}$*-factorial point* $g(D_1) \in g(D)/V$ *of index* > 1, X *is* $\mathbb{Q}$*-factorial over this point* $g(D_1)$, g *is minimal in the sense of* (1.3.7), *and the discrepancy of* K_Y *in* D_1 *is less than* 1. *Moreover,*

$$i(Y/V, g(D)) \leq i(X/V, D)$$

when $g(D_1)$ *is a curve, with equality if and only if* X *is nonsingular on* E *over a neighborhood of* V, *which gives one of the contractions in Example* (1.2.2).

1.7. Theorem on semistable flip. *Let* $\dim E \cap D \leq 1$ *over a neighborhood of* V. *Then* $\dim E \cap D = 1$ *over a neighborhood of* V, *and there exists a flip* $t\colon X \dashrightarrow X^+/Z$ *of* g *such that* X^+ *has again only terminal singularities,* g^+ *is semistable for* $t(D)$, f^+ *is numerically semistable for* $t(D)$, *and*

$$i(X^+/V, t(D)) \leq i(X/V, D)$$

with equality only if Y *is* $\mathbb{Q}$*-factorial at every point of* $g(E)/V$, g *is divisorial,* E *is pure divisorial, and* $t = g$ *is a holomorphic contraction to a curve that is not contained in* $g(D)$. *In particular, for small* g,

$$i(X^+/V, t(D)) \leq i(X/V, D) - 1.$$

In fact, equality holds only when X is Gorenstein in E/V (see (4.10.4)).

We may treat the blow-down g in Theorem 1.6 as divisorial in the semistable sense, and g in Theorem 1.7 as small in the semistable sense even when g is not small in the usual sense.

We will prove Theorems 1.6-7 simultaneously with Theorem 1.3 by induction. However, first we check that, in Theorem 1.6, $E = D_1$ over a neighborhood of V. Indeed, D_1 is $\mathbb{Q}$-Cartier by (1.1.1) and negative for g, hence exceptional for g. Therefore, D_1 contains all curves sweeping the exceptional locus E (by the extremal property of g) and coincides with E. Moreover, this implies Theorem 1.6 except for the assertion about equality.

Suppose now that $i(X/V, D) = 0$. Then, according to Lemma 1.4 and 1.3.6), for $d \geq 2$, both D and $K+D$ are Cartier. Hence g is divisorial. This is easily derived from (1.3.5-6) and Mori's classification [14], 3.3 (cf. also [1]). If g has a surface in a fiber over a neighborhood of V, then it coincides with D_1 after an appropriate renumbering of D_i's. But $E = D_1$ is contracted to a point by g. Since g is semistable, there exists at least one component D_i intersecting D_1. The adjunction formula ([27], 3.1 and 3.9) gives a Cartier divisor

$$(K+D)|_{D_1} = K_{D_1} + \left(\sum_{i\neq 1} D_i\right)_{D_1}$$

negative on D_1 with a reduced curve $(\sum_{i\neq 1} D_i)_{D_1} \neq 0$ as the different. The curve lies in a nonsingular part of D_1 by (1.3.3). Hence by [27], 3.2.3, D_1 is a del Pezzo surface with only canonical singularities, and the Fano index of D_1 is greater than 1, because g is extremal. So, D_1 is $\mathbb{CP}^2$ or a quadric Q. In the first case, we have the same two possibilities for $i(Y/V, g(D)) = 1$ as in (1.2.2), which obviously satisfy (1.3.7). We treat the case of the nonsingular quadric Q similarly. If Q is singular, it is a quadric cone with a vertex p, and then a unique surface D_2, after an additional renumbering of D_i's, intersects D_1. By (1.3.5-6) for $i(X, p, D) = 0$, there is a small resolution $h\colon W \to X$. But $h^{-1}D_1$ is nonsingular, and again by adjunction,

$$K_{h^{-1}D_1} + \left(\sum_{i\neq 1} h^{-1}D_i\right)_{h^{-1}D_1} = (K_W + h^{-1}D)|_{h^{-1}D_1}$$

is numerically trivial on a curve $h^{-1}p$ that does not intersect the different

$$\left(\sum_{i\neq 1} h^{-1}D_i\right)_{h^{-1}D_1} = h^{-1}D_1 \cap h^{-1}D_2$$

isomorphic to $\mathbb{CP}^1$, the base of the cone. Therefore, h induces a minimal resolution, on D_1 and $h^{-1}D_1$ is a rational scroll $\mathbb{F}_2$ with a (-2)-curve $h^{-1}p$ as the negative section, the proper inverse images of generators as fibers, and with the different as another section. According to Nakano's criterion [16], we can contract the scroll to a curve along fibers. Indeed, in a neighborhood of D_1, D_1 and K are linearly equivalent to $-D_2$ and D_1. So, in a neighborhood of $h^{-1}D_1$, $h^{-1}D_1$ and K_W are linearly equivalent to $-h^{-1}D_2$ and $h^{-1}D_1$. The contraction of the scroll g' gives a decomposition $g \circ h = h' \circ g'$ (Figure 3), where h' is a small semistable resolution of a point $g(D_1)$ on Y. Hence in this case $i(Y/V, g(D)) = 0$ and $g(D_1)$ is a non-$\mathbb{Q}$-factorial point with $d = 1$.

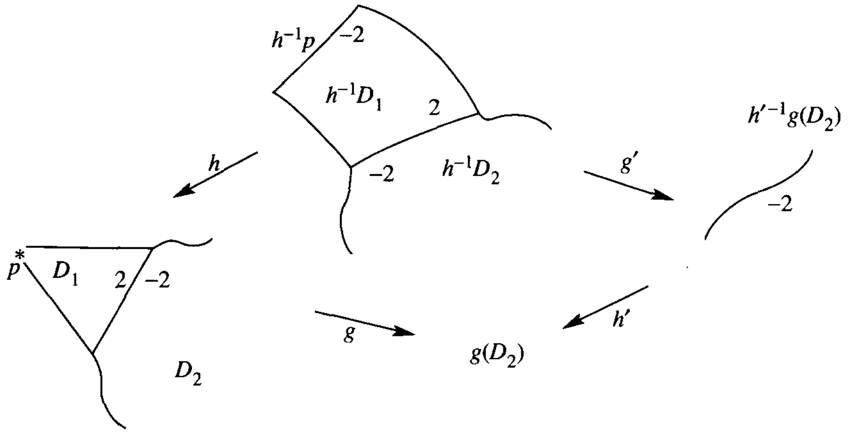

FIGURE 3

Now consider the case when all exceptional fibers, i.e., fibers over $g(E)$, are 1-dimensional. For a small resolution h, as above, the composition $g \circ h$ is projective over any point p in $g(E)/V$. I contend that the (log) canonical model coincides with the (log) minimal model of $g \circ h$ and, moreover, coincides with Y. So, g itself gives a flip. Indeed, all fibers of $g \circ h$ are at most 1-dimensional, and, for 1-dimensional fibers of $g \circ h$, $K_W + h^{-1}D$ is negative at least on one curve and nonpositive on others. Hence by (1.2.2) a minimal model is obtained by successive divisorial contractions to curves and coincides with Y, more precisely, within a neighborhood of p. The model has a trivial semistable resolution near any such p and $i(Y/V, g(D)) = 0$. In addition, h is the identity on $E = D_1/V$. This concludes the proof of Theorems 1.6 and 1.7 for $i(X/V, D) = 0$.

Now we can assume that

(I2). *Theorem* 1.6 *holds when* $i(X/V, D) \le n$ *for fixed* $n \ge 0$ *and any choice of* $f, g, D,$ *and* V.

(I3). *Theorem* 1.7 *holds when* $i(X/V, D) \le n$ *for fixed* $n \ge 0$ *and any choice of* $f, g, D,$ *and* V.

We must check Theorems 1.6 and 1.7 for $i(X/V, D) = n + 1$. However, we begin with

2. THE WEAK INDUCTION STEP FOR THEOREMS 1.6 AND 1.7

2.1. Proposition. *Let* $f, g, D,$ *and* V *be as* $i(X/V, D) \le n + 1$ *in Theorem* 1.7, *and suppose Theorem* 1.3 *holds for all points of* X. *Then Theorem* 1.7 *holds for* g, *and Theorem* 1.3 *holds for all points of* X^+ *with a new boundary* $t(D)$.

Here and in the sequel "Theorem 1.3 holds" means that every point $p \in D$ has a semistable resolution decomposed into successive $\mathbb{Q}$-factorializations (1.3.5) and divisorial blow-ups from (1.3.7).

First, we replace f by g. Thus, we assume

(2.1.1) $f = g$ *is extremal*, i.e., $\rho(X/Z; V) = 1$.

Since the flip is unique, our statement is local over Z, and we can replace V by a point $p \in f(E) \cap V$. We can also assume, after a $\mathbb{Q}$-factorialization of X over p, that

(2.1.2) X is *locally $\mathbb{Q}$-factorial*, i.e., every point of X is $\mathbb{Q}$-factorial.

However, after both of the last changes, the extremal property of f may be lost. But the required flip coincides with the canonical model X^{can}/Z of f, for which it is enough to construct a minimal one $X^{\min}/Z$. The fibers of f are at most 1-dimensional, and $\dim E \cap D \leq 1$ by the assumption in Theorem 1.7 and our construction. So, the required modification of X/Z is as in Proposition 2.1. Hence we need to check only the extremal case. Indeed, all the assumptions remain after such flips, and the flips terminate by a decrease of the Picard number $\rho(X/Z; p)$ and then by a decrease of the depth $i(X/p, D)$ (cf. the difficulty in [25], 2.17). By [27], 1.12, $X^{\min}/Z$, as in the previous modifications of X/Z, is projective and semi-stable/ Z, and even strictly terminal/ p. Since K is numerically negative on generic fibers of E over $f(E)$, $X^{\min}/Z$ and $/X^{\mathrm{can}}$ are small. Hence $X^{\min}$ is locally $\mathbb{Q}$-factorial and $X^{\min}/X^{\mathrm{can}} = X^+$ gives a required $\mathbb{Q}$-factorialization. Therefore, Theorem 1.7 holds for f and Theorem 1.3 holds for X^+ with the boundary $t(D)$. In addition we remark that $i(X^+/p, D) = i(X/p, D)$ only if all extremal modifications are divisorial contractions and $X^{\min} = X^{\mathrm{can}} = Z$, because the fibers of f are connected and numerically nonpositive with respect to K.

Thus, we can assume $V = \{p\}$ and (2.1.1-2). In particular, a fiber $C = f^{-1}p$ is an irreducible curve. Possibly after shrinking to a neighborhood of p, we can assume

(2.1.3) $E \cap D = C$. This implies that $\dim E \cap D = 1$. The shrinking also allows us to assume

(2.1.4) The singularities of X and D and triple points of D belong to C. All D_i's and double curves of D intersect C. Moreover, $C \subset D_1$ after an appropriate renumbering of D_i's.

The last holds because $p \in f(D)$ and f is semistable for D, and so $C \subset D$. By (I1) and (I3), we may restrict ourselves to the condition

(2.1.5) $i(X/p, D) = n + 1 \geq 1$.

Since $f^{-1}p = C$ is a curve, property (2.1.2) implies that X is *strictly* (log) terminal over a neighborhood of $p \in Z$, i.e., projective and $\mathbb{Q}$-factorial over such a neighborhood. Hence $X^{\min} = X^+/Z$ when $E = C$ (cf. [27], 1.5.5-7 and 1.7). This is because $\sigma(Z, p) = 1$. Otherwise, f is divisorial, $X^{\min} = X^+ = Z$, and p is $\mathbb{Q}$-factorial. So, we can *construct the flip* X^+ as a minimal model $X^{\min}/Z$. We will do it according to Mori's theory, starting from some semistable model $\tilde{f}: \tilde{X} \to Z$ for $\tilde{D} = \tilde{f}^* f(D)$. Because all flips will be semistable for flipped $\tilde{D}$, we can apply (I2-3) when the depth is not higher than n. As was explained before Theorems 1.6-7, we have termination for such modifications. We can also obtain this using the inductive statement (I3) for depth of small contractions. Of course, $\tilde{X}$ must be projective/ Z. Moreover, *in this section we consider only semistable and projective modifications of* X/Z. Indeed, they will be constructed by projective blow-ups from (1.3.7), and subsequent flips or their inverses in the case of small contractions. This projectivity is local/ p and follows from the projectivity of the composite of projective blow-ups and the local projectivity of its composite with f ([17], 1.3).

The construction of a required starting model $\tilde{X}/Z$, as well as the proof, is on the whole very combinatorial. We classify the possible cases by two natural invariants: the number a of *components* D_i *intersecting* C and the number b of *singularities of* X *on* C. So, *case a.b.* means that C intersects a components D_i and contains b singular points of X.

2.2. Lemma. *All possible pairs* (a, b) *are* $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, *and* $(3, 1)$ (*Figures* 4(a), (b), (c), (d), *and* (e), *respectively*), *more precisely, after an appropriate renumbering of* D_i*'s,* (2.2.1) The curve C is exceptional of the first kind on a minimal resolution of D_1.

(2.2.2) In cases 2.1 and 2.2 the double curve $D_1 \cap D_2$ is irreducible, does not coincide with C, and intersects it in one point, which is singular on X and D_1.

(2.2.3) In case 3.1, C coincides with the double curve $D_1 \cap D_2$ and has one triple point $D_1 \cap D_2 \cap D_3$, which is not singular on X.

Proof. First of all note that $b \geq 1$ by (1.3.6) in (I1) and (2.1.5).

If $a = 1$, i.e., $D = D_1$ is irreducible, then by adjunction ([27], 3.2.3 and 3.9)

$$K_{D_1} = (K + D)|_{D_1}$$

is log terminal and numerically negative on C. So, (2.2.1) holds in this case, and by (2.1.3) C is contracted on D_1 to a log terminal point. The graph of exceptional curves, for any resolution of such singularity, is a tree. Hence $b \leq 2$.

Now we may consider the case when $a = 2$. Then $C \not\subset D_2$. Otherwise $C = D_1 \cap D_2$, C is exceptional on both surfaces D_1 and D_2, contradicting the numerical semistability of f: $(D_1 + D_2 \cdot C) = (D \cdot C) = 0$. Hence we have again a log terminal and negative (on C) adjunction

$$K_{D_1} + D_{2|D_1} = (K + D_1 + D_2)|_{D_1} = (K + D)|_{D_1},$$

where the boundary $D_2|_{D_1}$ is reduced and $\not\supset C$. Then, as above, C satisfies (2.2.1) and $b \leq 2$. Moreover, by [27], 5.7, the double curve is irreducible and intersects C, by (2.1.4), at least in one point, which is singular. Otherwise we have a contradiction

$$(K \cdot C) = (K + DC) = (K_{D_1} + D_2|_{D_1} \cdot C) \geq 0.$$

In the remaining case $a \geq 3$, and, by [27], 3.16, $C \subset D_i$ for at most two values of i. So, we may assume that $C \subset D_2$ if we have two such values, and $C \not\subset D_i$ for

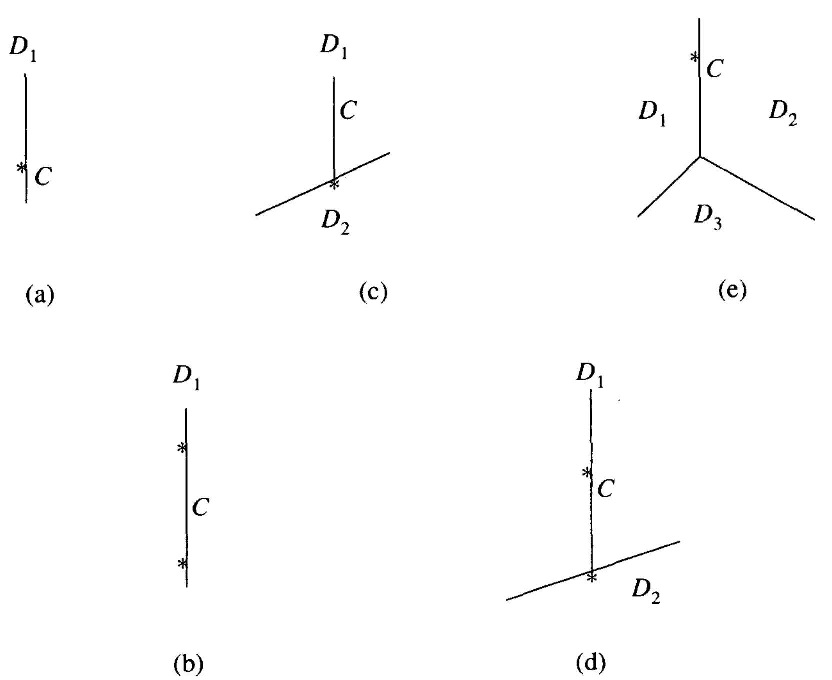

FIGURE 4

$i \neq 1$ and 2. But if $C \not\subset D_2$, then by the normal crossing property of D we have a log terminal and negative (on C) adjunction

$$K_{D_1} + \sum_{i \neq 1} D_i|_{D_1} = \left(K + \sum D_i\right)|_{D_1} = (K+D)|_{D_1}$$

with reduced boundary

$$\sum_{i \neq 1} D_i|_{D_1},$$

which does not contain C. The components $D_i|_{D_1}$ have a common point on C. This is a double point of the last boundary. As above, C is exceptional on D_1 of the first kind and

$$(K \cdot C) = \left(K_{D_1} + \sum_{i \neq 1} D_i|_{D_1} \cdot C\right) \geq -1 + (a-1) \geq 1,$$

contradicting our assumption. Hence $C \subset D_2$, $C = D_1 \cap D_2$ is a double curve with

$$(K + D_1 + D_2 \cdot C) = (K_{D_1} + C \cdot C) = \deg\left(K_C + \sum \frac{n_j - 1}{n_j} p_j\right) \geq -2$$

by [27], 3.9, where the n_j are the indices of $p_j \in C$ on D_1. Since $(K + D \cdot C) < 0$, it follows that $a = 3$, and D has only one triple point and only one singular point of X. Let $g\colon Y \to X$ be its blow-up as in (1.3.7) (Figure 5(c)). But $g^{-1}C$ is exceptional and has no singularities on both surfaces $g^{-1}D_1$ and $g^{-1}D_2$. So, by [12], 2.1,

$$(g^{-1}D_1 \cdot g^{-1}C) = (g^{-1}D_2 \cdot g^{-1}C) = -1.$$

This implies (2.2.1), because $C = \mathbb{CP}^1$. ■

We start a check from the last and related cases. For this we need only the following fact. Let $g\colon Y \to X$ be a blow-up of a $\mathbb{Q}$-factorial singularity as in (1.3.7). Put

$$g^*(K + D) = K_Y + eG + g^{-1}D.$$

2.3. Lemma. *The coefficient e is a rational number in $(0, 1)$ and $K_Y + eG + g^{-1}D$ is log terminal.*

Proof. It is easy to see that $1 - e$ is a log discrepancy of $K + D$ in G. This implies the statement, by Lemma 1.4 and by (1.3.7). ■

Proof of Proposition 2.1 *for cases* 1.1, 2.1, *and* 3.1. (Note that in cases 2.1 and 3.1 f is small, since it is semistable, and $(D_2 \cdot C)$ and $(D_3 \cdot C) > 0$, respectively.) According to the assumption, a unique singular point has a blow-up $g\colon Y \to X$ as in (1.3.7) (Figures 5(a), (b), and (c) for cases 1.1, 2.1, and 3.1, respectively). This extraction corresponds to an extremal ray, say R_1, in $\overline{\mathrm{NE}}(Y/Z\,;p)$. Since $\rho(Y/Z\,;p) = 2$, we have one more extremal, say R_2. But by Lemma 2.3,

$$g^*(K + D) = K_Y + eG + g^{-1}D$$

is log terminal. According to the construction, this divisor is numerically trivial on G and negative on $g^{-1}C$. So, $|R_2| = g^{-1}C$, because $(G \cdot R_2) > 0$. In particular, R_2 defines a small contraction.

I contend that R_2 is numerically nonpositive for $K_Y + g^*D = K_Y + g^{-1}D + G$, and is numerically trivial if and only if $g^{-1}C$ has no singularities of Y. In case 3.1, by Lemma 1.4, Y is nonsingular on $g^{-1}C$ and we have

$$\begin{aligned}(K_Y + G + g^{-1}D \cdot g^{-1}C) &= (K_{g^{-1}D_1} + g^{-1}C + G|_{g^{-1}D_1} + g^{-1}D_3|_{g^{-1}D_1} \cdot g^{-1}C) \\ &= \deg(K_{g^{-1}C}) + 2 = 0.\end{aligned}$$

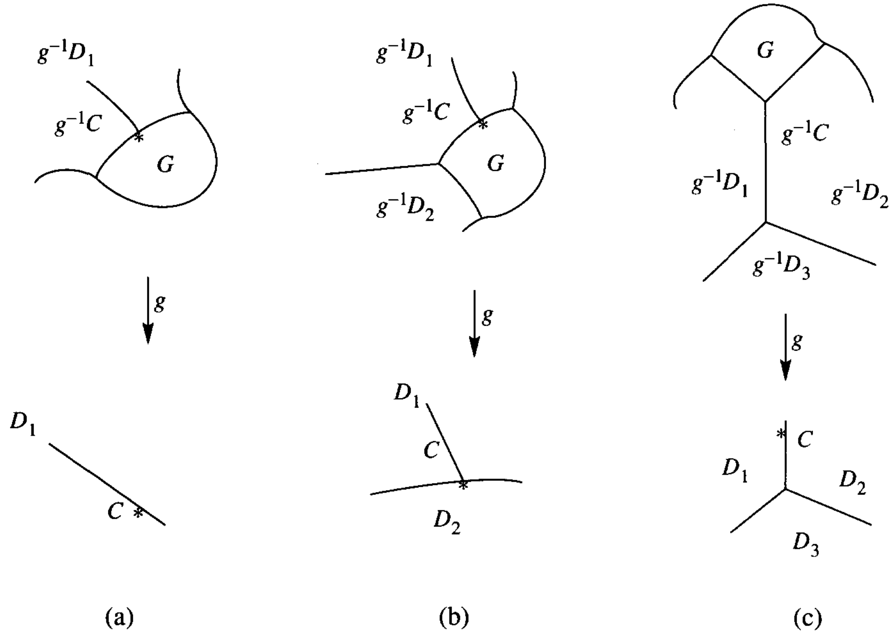

FIGURE 5

In cases 1.1 and 2.1, C does not belong to the boundary

$$\sum_{i\neq 1} D_i|_{D_1}$$

of the log terminal divisor

$$K_{D_1} + \sum_{i\neq 1} D_i|_{D_1} = (K+D)|_{D_1}.$$

The last is negative on the curve C, which is contracted on D_1 to a pure log terminal point with a reduced boundary (possibly empty). According to the classification of such singularities, C intersects normally only one exceptional curve on any resolution of the singular point of D_1 on C, and has no common points with the proper inverse image of the boundary

$$\sum_{i\neq 1} D_i|_{D_1}.$$

Like C, the curve $g^{-1}C$ is exceptional of the first kind because g is minimal in the sense of (1.3.7). Hence, $(K_Y + g^*D \cdot R_2) \leq 0$, $K_Y + g^*D$ is nonpositive for $f \circ g$ and negative on G. Therefore, a minimal model $X^{\min}/Z$ of f or $f \circ g$ will be small/Z (cf. [27], 1.5.6).

By the construction, (2.1.5), and (1.3.7), $i(Y/p, g^*D) = n$. Hence, if

$$(K_Y + g^*D \cdot R_2) < 0,$$

then, according to our inductive hypothesis, there exists a flip $\tau\colon Y \dashrightarrow Y^+$ in R_2. Moreover, Y^+/Z is semistable for $D^+ = \tau(g^*D)$ with $i(Y^+/p, D^+) \leq n-1$. Note also, that, like Y, Y^+ has only one exceptional prime divisor $G^+ = \tau(G)$ over p. Now according to the above, we start with $\widetilde{X} = Y^+/Z$. Using (I2-3) we obtain a minimal model $X^{\min}/Z$, which is semistable for the bimeromorphic transform $D^{\min}$ of D^+, and

$$i(X^{\min}/p, D^{\min}) \leq i(Y^+/p, D^+) + 1 = n.$$

As was explained earlier, $X^{\min}/Z$ is a flip of f. This implies our statement in this case.

In the remaining case,

$$(K_Y + g^*D \cdot R_2) = 0$$

and Y is nonsingular near $g^{-1}C$. Then instead of a flip we have Atiyah's flop $\tau\colon Y \dashrightarrow Y^+$ in R_2, since $g^{-1}C = \mathbb{CP}^1$ with the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. More precisely, if we consider the flop with the boundary configuration, in cases 1.1 and 2.1 we get (Figure 6, where $D_1^+ = \tau(g^{-1}D_1)$) a flop of type I ([12], 4.2 and Figure 4), and (Figure 7, where $D_i^+ = \tau(g^{-1}D_i)$) a flop of type II in case 3.1 ([12], 4.3 and Figure 5). This modification does not affect the normal crossing property of g^*D. So, Y^+/Z is semistable for $D^+ = \tau(g^*D)$, and $G^+ = \tau(G)$ is a unique exceptional prime divisor of Y^+ over p. But $i(Y^+/p, D^+) = i(Y/p, g^*D) = n$, because the flop is symmetric! Starting again with $\widetilde{X} = Y^+$ we proceed with construction of a minimal model $X^{\min}/Z$. Now note that $(G \cdot R_2) > 0$ and $G^+ \cdot C^+) < 0$ for a flopped curve C^+. Hence $C^+ \subset G^+$ and $(K_{Y^+} \cdot C^+) = 0$. So, the first modification of Y^+/Z is not a divisorial contraction to a point, and we can proceed as above. Eventually, we obtain the flip $X^+ = X^{\min}/Z$, which is semistable for the bimeromorphic transform $t(D) = D^{\min}$. By (I2), if the first modification $Y^+ \dashrightarrow Y^{++}$ contracts G^+,

$$i(X^+/p, t(D)) \le i(Y^{++}/p, D^{++}) \le i(Y^+/p, D^+) = n.$$

Otherwise, by (I3),

$$i(X^+/p, t(D)) \le i(Y^{++}/p, D^{++}) + 1 \le i(Y^+/p, D^+) + 1 = n + 1$$

with equality only if the exceptional locus E of f is divisorial, $Y^+ \dashrightarrow Y^{++}$ contracts the modification E^+ to a curve, and then G^{++} is contracted to the point p. (This case, like some others below, is impossible; see (4.10.4).) ■

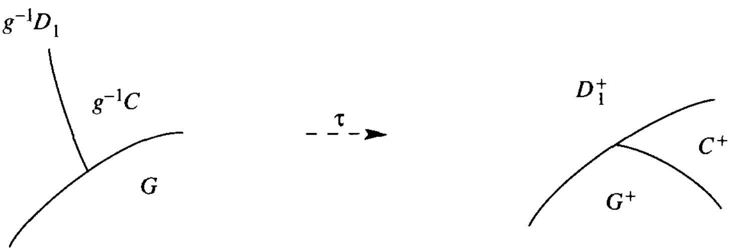

FIGURE 6

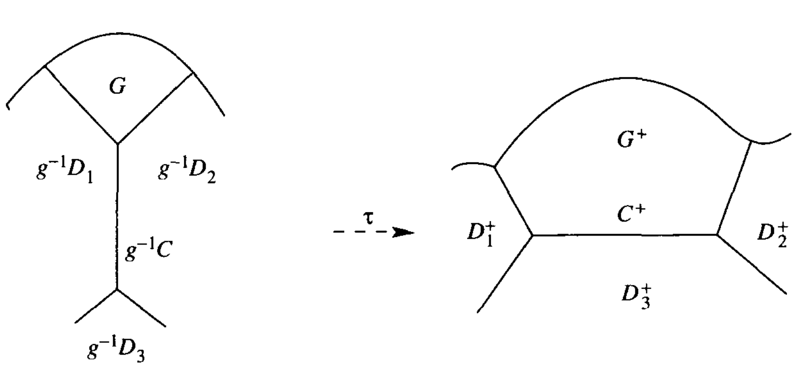

FIGURE 7

The remaining cases are more serious and need a more powerful prelude. For both of these cases 1.2 and 2.2, X *has two singular points* $x, y \in C \subset D_1$ and *they are non-Gorenstein*, i.e., $\operatorname{ind}_x K$ and $\operatorname{ind}_y K \geq 2$. (Note that in case 2.2 f is small; cf. the beginning of the proof above.) In case 2.2 we can assume also that $x \notin D_2$, but $y \in D_2$. However, this time we are obliged to make successive extractions of both singularities to remove them from C. Indeed, as above, take a partial resolution $g: Y \to X$ of one of the singularities. Then again $\rho(Y/Z; p) = 2$ and $|R_2| = g^{-1}C$. But now $g^{-1}C$ can have two singularities of Y, and possibly $(K_Y + g^*D \cdot R_2) > 0$. This last reflects essentially the fact that a sum of two quantities from $(0, 1)$ can be greater than 1. Hence we cannot perform the previous trick—flip-flop of $g^{-1}C$—and we have committed ourselves to proceed with resolution of singularities on $g^{-1}C$ according to (1.3.7) under the assumption of Proposition 2.1. We can do it, because the new singularities, i.e., singularities of Y on the exceptional locus G, belong to $G \cap g^{-1}D_1$ and have type (1.3.3). So, we have

2.4. Construction. *The composition* $c: X^{r,s} \to X$ *of blow-ups from* (1.3.7) *gives a projective extraction with exceptional surfaces* $E_1, \dots, E_r$ *and* $F_1, \dots, F_s$ *over* x *and* y, *respectively, and such that* (*Figure* 8(a)):

(2.4.1) r and $s \geq 1$.

(2.4.2) $X^{r,s}$ and $D_1^{r,s}$ have no singularities on $C^{r,s}$.

(2.4.3) The surfaces $E_r, \dots, E_1$ (respectively, $F_s, \dots, F_1$) are successively blown down to points of images of $C^{r,s}$ over x (respectively, y).

(2.4.4) The contraction c is semistable for $c^*D = D^{r,s} + \sum E_i + \sum F_j$ with

$$i(X^{r,s}/p, c^*D) = i(X/p, D) - r - s = n + 1 - r - s,$$

and $K_{X^{r,s}} + D^{r,s} + \sum E_i + \sum F_j$ is divisorially log terminal by Lemma 1.4.

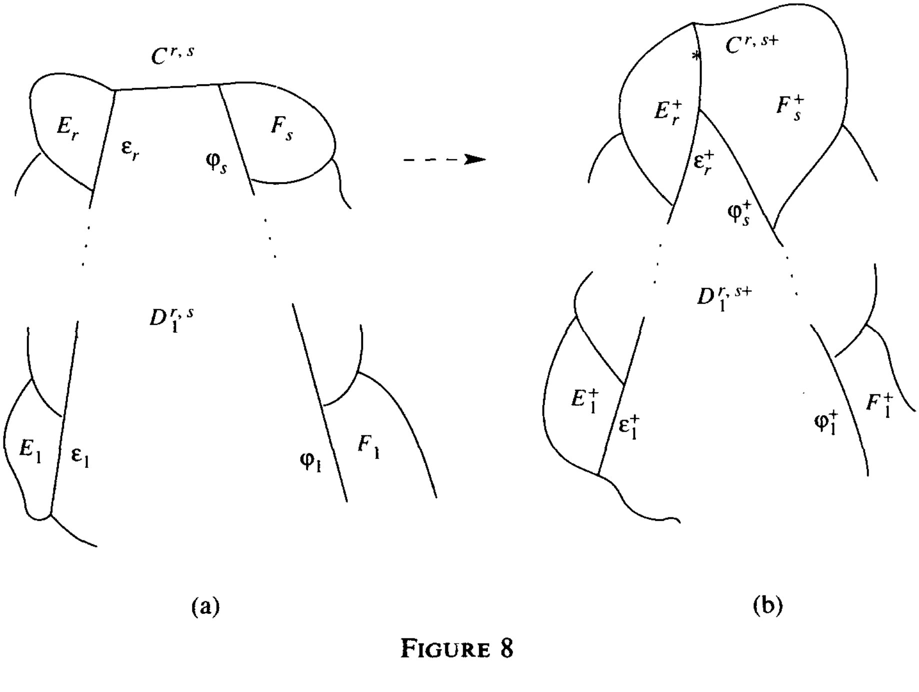

FIGURE 8

(2.4.5) Curves $\varepsilon_i = D_1^{r,s} \cap E_i = \mathbb{CP}^1$, $C^{r,s} = \mathbb{CP}^1$, and $\varphi_j = D_1^{r,s} \cap F_j = \mathbb{CP}^1$ (as well as $\varphi_0 = D_1^{r,s} \cap D_2^{r,s}$ in case 2.2) form a chain $\varepsilon_1, \dots, \varepsilon_r, C^{r,s}, \varphi_s, \dots, \varphi_1$ (respectively, $\varepsilon_1, \dots, \varepsilon_r, C^{r,s}, \varphi_s, \dots, \varphi_1, \varphi_0$), which is blown down on $D_1^{r,s}$ to a log terminal point.

(2.4.6) The curves ε_i and φ_j, $j \neq 0$, are exceptional but not of the first kind on the minimal resolution of $D_1^{r,s}$.

(2.4.7) $(\bigcup_{i>k} E_i) \cup C^{r,s} \cup (\bigcup_{j>l} F_j)$ is exceptional for $k+l \geq 1$, $r \geq k \geq 0$, $s \geq l \geq 0$ (and even for $k+l \geq 0$ in case 2.2). Moreover, the curves on this locus generate a face of $\overline{\mathrm{NE}}(X^{r,s}/Z; p)$ of dimension $1+r+s-k-l$.

(2.4.8) $X^{r,s}$ and $D_1^{r,s}$ have no singularities on ε_i and φ_j with $i \neq 1, r$ and $j \neq 1, s$ (and even for $j \neq s$ in case 2.2), respectively. Moreover, ε_r and φ_s have at most one singularity whenever $r \geq 2$ and $s \geq 2$, respectively.

Here and in the sequel, objects with superscripts r, s mean their proper inverse images looking back from X to $X^{r,s}$. As was remarked earlier, the composition c is projective because its components are projective. According to [27], 3.8, and to the classification of the surface log terminal singularities, (2.4.4) implies (2.4.5) and (2.4.8). So, we need to check only (2.4.7). The following generalization of Lemma 2.3 will help us in this.

2.5. Lemma. *The log divisor*

$$c^*(K+D) = K_{X^{r,s}} + D^{r,s} + \sum e_i E_i + \sum f_j F_j$$

is divisorially log terminal with rational e_i *and* $f_j \in (0, 1)$. *It is numerically trivial on curves/*x, y *and is negative on* $C^{r,s}$.

Proof. Note that $-e_i$ and $-f_j$ are discrepancies of E_i and F_j, respectively, for the log divisor $K+D$. So, by Lemma 1.4, the corresponding discrepancies $1-e_i$ and $1-f_j$ for K are positive, and $e_i, f_j < 1$. On the other hand, $D_1^{r,s}$ is normal, and its intersections with E_i, F_j (as well as with $D_2^{r,s}$ in case 2.2) are generic normal. Hence

$$c^*(K+D)|_{D_1^{r,s}} = K_{D_1^{r,s}} + \sum e_i \varepsilon_i + \sum f_j \varphi_j = (c|_{D_1^{r,s}})^* K_{D_1} \ (+D_1 \cap D_2 \text{ in case 2.2})$$

(where $f_0 = 1$ in case 2.2), and $-e_i$ and $-f_j$ are also discrepancies of ε_i and φ_j, respectively, for the log divisor K_{D_1} $(+D_1 \cap D_2$ in case 2.2). Then, according to (2.4.6) and Lemma 2.3, all e_i, $f_j > 0$, and $c^*(K+D)$ is divisorially log terminal by monotonicity ([27], 1.3.3) and (2.4.4). Numerical properties of $c^*(K+D)$ follow from those of $K+D$. ∎

By this lemma, if r or $s \geq 1$ (cf. (2.4.1)), there is only one extremal ray $R \subset \overline{\mathrm{NE}}(X^{r,s}/Z; p)$ negative with respect to $c^*(K+D)$ and with $|R| = C^{r,s}$ (cf. the arguments in the corresponding part of the proof of Proposition 2.1 above). This implies (2.4.7) for $k = s$ and $l = r$. In general, we can use the same reasoning applied to a composition $c\colon X^{k,l} \to X$ of extractions with exceptional surfaces $E_1, \dots, E_k$ and $F_1, \dots, F_l$ over x and y, respectively, i.e., for $r := k$ and $s := l$.

We remark that the surface interpretation of coefficients f_i (in the proof of Lemma 2.5) implies the monotonic property $f_1 \geq f_2 \geq \cdots \geq f_s$ in case 2.2. In general, this property holds if we assume the partial resolution c to be *smallest*, i.e., satisfying (2.4.4).

2.6. Lemma. $e_1 \geq e_2 \geq \cdots \geq e_r$ *and* $f_1 \geq f_2 \geq \cdots \geq f_s$.

Proof. It is enough to prove that $f_1 \geq f_2 \geq \cdots \geq f_s$. The same arguments are valid for the e_i's.

Suppose that

$$f_1 \geq f_2 \geq \cdots \geq f_l \quad \text{but} \quad f_l < f_{l+1}.$$

Then consider a composition $c \colon Y \to X$ of blow-ups/y with exceptional surfaces $F_1, \ldots, F_{l+1}$. The log terminal divisor

$$c^*(K+D) = K_Y + c^{-1}D + \sum_{j=1}^{l+1} f_j F_j$$

has the same coefficients f_j and is numerically trivial with respect to c. According to the construction, $(F_{l+1} \cdot C) < 0$ for the curves $C \subset F_{l+1}$, $(F_{l+1} \cdot \varphi_l) > 0$ for the curve $\varphi_l = c^{-1}D_1 \cap F_l$. So, there is an extremal ray $R \subset \overline{\mathrm{NE}}(Y/X; y)$ with $(F_{l+1} \cdot R) > 0$, and $|R| \subseteq F_l$. In addition, $(F_j \cdot R) \geq 0$ for $0 \leq j < l$ (where $F_0 = c^{-1}D_2$ in case 2.2). For $l \geq 2$, using [12], 2.1, and (2.4.6), (2.4.8) we can check that

$$(c^{-1}D_1 \cdot \varphi_l) = (\varphi_l)^2_{F_l} = -2 - (\varphi_l)^2_{c^{-1}D_1} \geq 0,$$

which implies that $(c^{-1}D \cdot R) \geq 0$ and that

$$\begin{aligned}\sum_{j=1}^{l+1}(1-f_j)F_j &= (1-f_l)\sum_{j=1}^{l+1}F_j - \sum_{j=1}^{l+1}(f_j - f_l)F_j \\ &= (1-f_l)c^*D - (1-f_l)c^{-1}D - \sum_{j=1}^{l+1}(f_j - f_l)F_j\end{aligned}$$

is negative on R. The same holds for $l = 1$. Indeed, when $l = 1$ and φ_1 has at most one singular point of Y, the above arguments are valid after a resolution. More exactly, in this case we can check that φ_1 is movable on F_1, for example, in a linear sense. The same is always true except in one case, namely, when in a neighborhood of y, D is irreducible, $l = 1$, φ_1 has two singular points of Y, and φ_1 is a (-2)-curve on the minimal resolution of $c^{-1}D$. Then, by the arguments in the proof of Lemma 2.5 and by the classification of the surface log terminal singularities [4], y is log terminal but is not a canonical singularity on D with graph of type $\mathbb{D}_t$, $t \geq 3$ (Figure 9(a) and (b)), or of exceptional types $\mathbb{E}_6$, $\mathbb{E}_7$ (Figure 9(c)), and φ_1 corresponds to the vertex at which three segments are joined. In addition, $-f_j$'s are discrepancies of $\varphi_j = c^{-1}D \cap F_j$, $j = 1, 2$. But this is impossible, because $-f_1$, the discrepancy at the vertex, is equal to or less than the nearest three discrepancies. In the case when φ_1 on the minimal resolution intersects two (-2)-curves and a chain of $(-p_i)$-curves with $(-p_1)$-curve intersecting φ_1, we have discrepancies $-f_1/2$ for the first two curves and $-f_1$ for the $(-p_1)$-curve (Figure 9(a)). In the other cases we have a finite set of opportunities and can make a direct check in each of them (Figure 9(b-c), where the fractions are discrepancies).

Therefore, R is always negative with respect to

$$K_Y + c^{-1}D + \sum_{j=1}^{l+1} F_j = c^*(K+D) + \sum_{j=1}^{l+1}(1-f_j)F_j.$$

The corresponding blow-down $Y \to Y'$ is defined over X and does not contract F_l to a point. The last holds because F_{l+1} is contractible to a point. According to our assumption, c is the smallest:

$$i(Y/y, c^*D) = i(X, y, D) - l - 1 \quad \text{and} \quad < n - l.$$

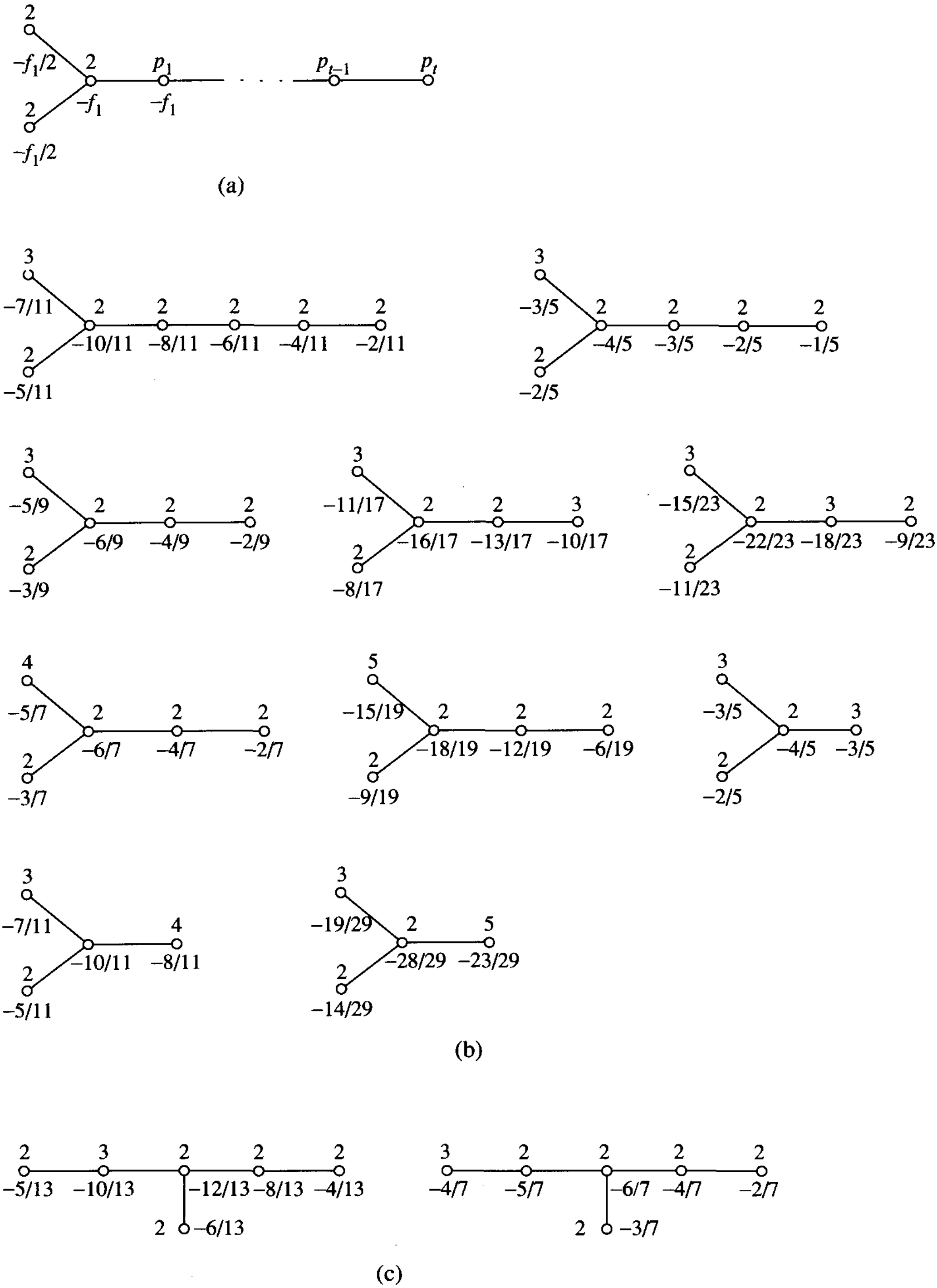

FIGURE 9

So, we can apply (I2-3) to R and to the subsequent extremal rays/y. By (2.1.2) and [27], 1.5.7, eventually we obtain $X = X^{\min}$ with

$$i(X, y, D) \leq i(X, y, D) - 1,$$

which gives a contradiction. ■

As we know, $X^{r,s}$ is projective $/Z$ and we can apply the cone theorem. In particular, by (2.4.7) with $k = r$ and $l = s$, we have an extremal ray $R_c \subset \overline{\mathrm{NE}}(X^{r,s}/Z\,;p)$ with $|R_c| = C^{r,s}$.

2.7. Construction. *There is a flip* $X^{r,s} \dashrightarrow X^{r,s+}$ *in* R_c (*Figure* 8(a-b)), *and it satisfies the following conditions*:

(2.7.1) r and $s \geq 1$.

(2.7.2) $X^{r,s+}$ is semistable for $D^{r,s+} + \sum E_i^+ + \sum F_j^+$ with

$$i(X^{r,s+}/p\,, D^{r,s+} + \sum E_i^+ + \sum F_j^+) = i(X^{r,s}/p\,, c^*D) + 1$$
$$= i(X/p\,, D) - r - s + 1 = n + 2 - r - s.$$

(2.7.3) $X^{r,s+}$ has only terminal singularities, and

$$K_{X^{r,s+}} + D^{r,s+} + \sum E_i^+ + \sum F_j^+$$

is divisorially log terminal, negative on $C^{r,s+}$.

(2.7.4) Curves $\varepsilon_i^+ = D_1^{r,s+} \cap E_i^+ = \mathbb{CP}^1$ and $\varphi_j^+ = D_1^{r,s+} \cap F_j^+ = \mathbb{CP}^1$ (as well as $\varphi_0^+ = D_1^{r,s+} \cap D_2^{r,s+}$ in case 2.2) form a chain $\varepsilon_1^+, \dots, \varepsilon_r^+, \varphi_s^+, \dots, \varphi_1^+$ (respectively, $\varepsilon_1^+, \dots, \varepsilon_r^+, \varphi_s^+, \dots, \varphi_1^+, \varphi_0^+$), which is blown down on $D_1^{r,s+}$ to a log terminal point.

(2.7.5) $(\bigcup_{i>k} E_i^+) \cup (\bigcup_{j>l} F_j^+)$ is exceptional for any $k + l \geq 1$, $r \geq k \geq 0$, $s \geq l \geq 0$ (and even for $k + l \geq 0$ in case 2.2). Moreover, $C^{r,s+}$ and the curves on this locus generate a face of $\overline{\mathrm{NE}}(X^{r,s+}/Z\,;p)$ of dimension $1 + r + s - k - l$.

(2.7.6) $K_{X^{r,s+}} + D^{r,s+} + \sum e_i E_i^+ + \sum f_i F_j^+$ is divisorially log terminal, negative on ε_r^+ and φ_s^+, nonpositive on all curves of $X^{r,s+}/p$ except $C^{r,s+}$, and positive on $C^{r,s+}$.

(2.7.7) $X^{r,s+}$ and $D_1^{r,s+}$ have no singularities on ε_i^+ and φ_j^+ with $i \neq 1$, r and $j \neq 1, s$ (and even for all $j \neq s$ in case 2.2), respectively. Moreover, for $r \geq 2$ and $s \geq 2$, respectively (and even $s \geq 1$ in case 2.2), ε_r^+ and φ_s^+ have at most one singularity, and have no singularities whenever they are curves of the first kind on the minimal resolution of $D_1^{r,s+}$.

(2.7.8) $X^{r,s+}$ and $D_1^{r,s+}$ have exactly one singular point of type $V_2(2, 1)$ on $C^{r,s+} = E_r^+ \cap F_s^+$.

Lemma 2.5 and the adjunction formula ([27], 3.1)

$$c^*(K + D)|_{D_1^{r,s}} = K_{D_1^{r,s}} + \sum e_i \varepsilon_i + \sum f_j \varphi_j$$

imply that $C^{r,s}$ is an exceptional curve of the first kind (cf. (2.4.6)). So, the required flip is a composition of a monoidal transformation in $C^{r,s}$, Atiyah's flop, and Mori's blow-down ([14], 3.3.5) (Figure 10). More precisely, the monoidal transformation gives a rational scroll $\mathbb{F}_1/C^{r,s}$ such that its negative section is a (-1)-curve in the intersection with proper transform of $D_1^{r,s}$. Then Atiyah's flop transforms it into $\mathbb{CP}^2$ with normal bundle $\mathcal{O}_{\mathbb{CP}^2}(-2)$. The flop with boundary configuration coincides with Kulikov's one of type II ([12], 4.3, Figure 5) (cf. Figure 7). Mori's blow-down contracts $\mathbb{CP}^2$ to a quotient singularity of type $1/2(1, -1, 1)$ ([14], 3.4.3) lying on $C^{r,s+}$. The rational scroll has multiplicity 1 in c^*D. So, Mori's blow-down gives a semistable resolution of this singularity, which proves (2.7.2). Other properties of

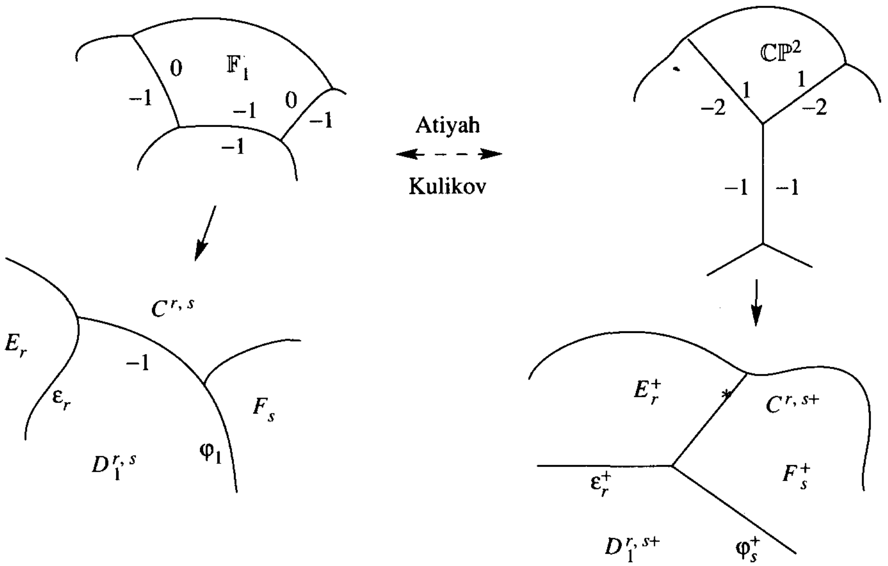

FIGURE 10

the flip follow from Construction 2.4 and Lemmas 1.4, 2.5. According to (2.7.3), the flip is an antiflip with respect to

$$K_{X^{r,s+}} + D^{r,s+} + \sum E_i^+ + \sum F_j^+.$$

Let R_c^+ be an extremal ray in $\overline{\mathrm{NE}}(X^{r,s+}/Z\,;p)$ generated by the flipped curve $C^{r,s+}$. By (2.7.5) with $k = r$ and $l = s-1$, curves on F_s^+ generate a face of $\overline{\mathrm{NE}}(X^{r,s+}/Z\,;p)$ of dimension 2. It contains the extremal ray R_c^+. So, we have one more extremal ray R_f on this face, and the face is generated by R_c^+ and R_f. By (2.7.6), it is really extremal and, like φ_s^+, is negative with respect to the log divisor

$$K_{X^{r,s+}} + D^{r,s+} + \sum e_i E_i^+ + \sum f_i F_j^+.$$

We have the inclusion $|R_f| \subseteq F_s^+$, and

$$(E_i^+ \cdot R_f) = (F^+ \cdot R_f) = 0 \quad \text{for } i \neq r \text{ and } j \neq s,\, s-1, \text{ respectively.}$$

It is easy to see also that $(E_r^+ \cdot R_c^+) < 0$ and $(F_{s-1}^+ \cdot R_c^+) = 0$. But $(E_r^+ \cdot \varphi_s^+) > 0$, $(F_{s-1}^+ \cdot \varphi_s^+) > 0$. Of course, F_{s-1}^+ and the corresponding formulas are meaningful for $s \geq 2$, or for $s = 1$ in case 2.2 if we take $F_0^+ = D_2^{r,s+}$. Hence, R_f satisfies also the following numerical properties:

$$(E_r^+ \cdot R_f) > 0 \quad \text{and} \quad (F_{s-1}^+ \cdot R_f) > 0.$$

Note that, since the flip $X^{r,s} \dashrightarrow X^{r,s+}$ contracts on $D_1^{r,s}$ the exceptional curve of the first kind $C^{r,s}$, the curve ε_r^+ or φ_s^+ cannot satisfy (2.4.6) and, moreover, this holds almost always.

2.8 Lemma. *Suppose that the exceptional curves ε_i^+ and φ_j^+, $j \geq 1$, are not of the first kind on the minimal resolution of $D_1^{r,s+}$. Then*:

(2.8.1) For $e_r \geq f_s$, $s = 1$, and this is possible only in case 1.2.

(2.8.2) For $e_r \leq f_s$, $r = 1$.

Proof. We check the first statement (2.8.1). The same argument proves (2.8.2). So, let $e_r \geq f_s$ and $s \geq 2$ in case 1.2. Then, by Lemma 2.6 and (2.7.1), $f_s \leq f_{s-1}$, where $r, s \geq 1$ and $s \geq 2$ in case 1.2. We assume that $f_{s-1} = f_0 := 1$ when $s = 1$ in case 2.2. This is the multiplicity of $F_0 := D_2^{r,1}$ in D (cf. (2.4.5), (2.7.4), and Lemma 2.5). I contend that $(\varphi_s')^2_{D'} \leq -3$ on the minimal resolution $g\colon D' \to D_1^{r,s+}$. Indeed,

$$g^*\left(K_{D_1^{r,s+}} + \sum e_i\varepsilon_i^+ + \sum_{j\geq 0} f_j\varphi_j^+\right) = K_{D'} + E' + \sum e_i\varepsilon_i' + \sum_{j\geq 0} f_j\varphi_j',$$

where E' is an effective divisor on D'. So, for $(\varphi_s')^2_{D'} = -2$, $(K_{D'} \cdot \varphi_s') = 0$ and we get a contradiction

$$\begin{aligned} 0 \leq e_r + f_{s-1} - 2f_s &\leq \left(K_{D'} + E' + \sum e_i\varepsilon_i' + \sum_{j\geq 0} f_j\varphi_j' \cdot \varphi_s'\right) \\ &= \left(K_{D_1^{r,s+}} + \sum e_i\varepsilon_i^+ + \sum_{j\geq 0} f_j\varphi_j^+ \cdot \varphi_s^+\right) \\ &= \left(K_{X^{r,s+}} + D^{r,s+} + \sum e_iE_i^+ + \sum f_jF_j^+ \cdot \varphi_s^+\right) \end{aligned}$$

with (2.7.6). If $D_1^{r,s+}$ does not have singularities on φ_s^+, using [12], 2.1, as in the proof of Lemma 2.6, we get the inequality $(\varphi_s^+)^2_{F_s^+} \geq 1$. Otherwise, by (2.7.7), φ_s^+ has at most one such singularity, and, using now [12], 2.1 after a partial resolution (1.3.7), we get the inequality $(\varphi_s')^2_{F'} \geq 0$ on the minimal resolution $F' \to F_s^+$. In any case, φ_s^+ is movable on F_s^+, and $(D_1^{r,s+} \cdot R_f) \geq 0$. But, according to the semistability of $X^{r,s+}/Z$,

$$\left(D^{r,s+} + \sum E_i^+ + \sum F_j^+ \cdot R_f\right) = 0,$$

whence

$$\left(\sum E_i^+ + \sum_{j\geq 0} F_j^+ \cdot R_f\right) \leq 0.$$

Therefore, by the above numerical properties of R_f,

$$\left(\sum(1-e_i)E_i^+ + \sum(1-f_j)F_j^+ \cdot R_f\right) \leq 0,$$

because $1 - e_r \leq 1 - f_s$ and $1 - f_s \geq 1 - f_{s-1}$. So,

$$\begin{aligned} &\left(K_{X^{r,s+}} + D^{r,s+} + \sum E_i^+ + \sum F_j^+ \cdot R_f\right) \\ &= \left(K_{X^{r,s+}} + D^{r,s+} + \sum e_iE_i^+ + \sum f_jF_j^+ \cdot R_f\right) \\ &\quad + \left(\sum(1-e_i)E_i^+ + \sum(1-f_j)F_j^+ \cdot R_f\right) \\ &< 0. \end{aligned}$$

By the adjunction formula,

$$(K_{F_s^+} + \varphi_s^+ + (E_r^+ + F_{s-1}^+)|_{F_s^+} \cdot C_f) < 0$$

for any curve $C_f \in R_f$. (As we know, $C_f \subset |R_f| \subset F_s^+$.) This contradicts the connectedness of the locus of log canonical singularities for

$$K_{F_s^+} + \varphi_s^+ + (E_r^+ + F_{s-1}^+)|_{F_s^+}$$

in a neighborhood of C_f ([27], 5.7) when R_f is of a flipping type. So, R_f is of a divisorial type, and the corresponding blow-down contracts $F_s^+ = |R_f|$ to a curve, because $C^{r,s+} \notin R_f$. This defines a ruling with generic fiber $C_f = \mathbb{CP}^1$ such that

$$(K_{F_s^+} + \varphi_s^+ + (E_r^+ + F_{s-1}^+)|_{F_s^+} \cdot C_f) \geq -2 + 1 + 1 = 0$$

(cf. continuation of the proof of Proposition 2.1 below). This contradicts the above. ■

In particular, (2.8.1) is *possible only in case* 1.2. If we permute the singularities x, y in this case, we obtain

2.9. Corollary. *Under the assumptions of Lemma* 2.8 *and after an appropriate choice of* x *in case* 1.2, *we have* $r = 1$ *and* $e_1 \leq f_s$.

However, according to the construction (and (2.7.4)), it is possible that one (and only one) curve ε_r^+ or φ_s^+ is the exceptional curve of the first kind on the minimal resolution of $D_1^{r,s+}$. Suppose first that $s \geq 2$ in case 1.2, and φ_s^+ is such a curve. By (2.7.7), $D_1^{r,s+}$ and $X^{r,s+}$ do not have singularities on φ_s^+. Using again [12], 2.1, we can check that (Figure 11(a))

$$(D_1^{r,s+} \cdot \varphi_s^+) = (\varphi_s^+)^2_{F_s^+} = -2 - (\varphi_s^+)^2_{D_1^{r,s+}} = -1 < 0.$$

Since R_c^+ is positive with respect to $D_1^{r,s+}$, this implies that R_f is negative with respect to $D_1^{r,s+}$, and $|R_f| = \varphi_s^+$. In this case Atiyah's flop $X^{r,s+} \dashrightarrow X^{r,s++}$ gives us a modification $X^{r,s++}/Z$ (Figure 11(b)) having only terminal singularities and semistable for $D^{r,s++} + \sum E_i^{++} + \sum F_j^{++}$ with

$$\begin{aligned} i&\left(X^{r,s++}/p, D^{r,s++} + \sum E_i^{++} + \sum F_j^{++}\right) \\ &= i\left(X^{r,s+}/p, D^{r,s+} + \sum E_i^{+} + \sum F_j^{+}\right) \\ &= i(X/p, D) - r - s + 1 = n + 2 - r - s. \end{aligned}$$

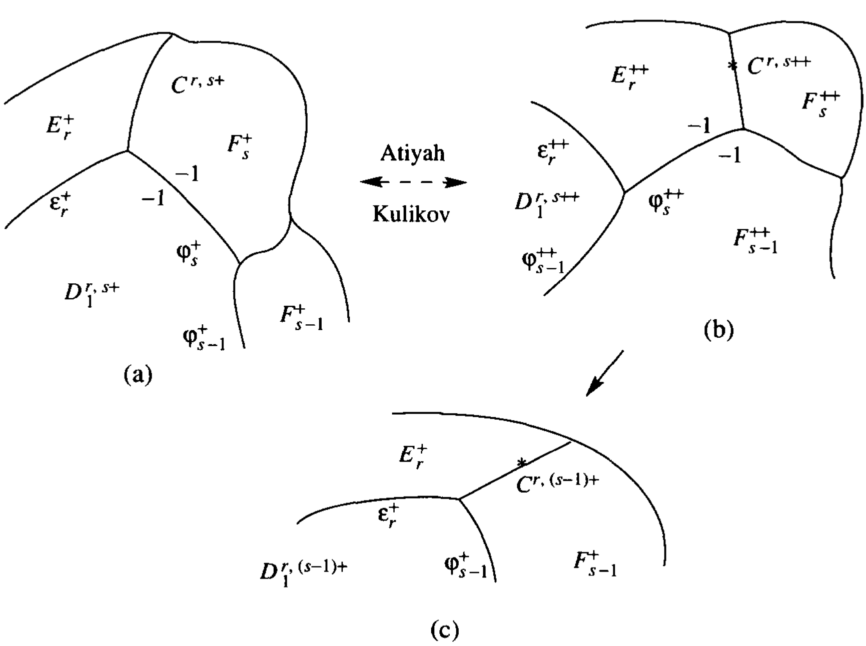

FIGURE 11

The curves on $\varphi_s^{++} \cup F_s^{++}$ generate a two-dimensional face of $\overline{\mathrm{NE}}(X^{r,s++}/Z\,;p)$. One of the extremal rays of this face is generated by the flopped curve $\varphi_s^{++} = E_r^{++} \cap F_{s-1}^{++}$. The other ray is generated by the curves on F_s^{++} and is negative for

$$K_{X^{r,s++}} + D^{r,s++} + \sum E_i^{++} + \sum F_j^{++}.$$

The last assertion for $C^{r,s++} \subset F_s^{++}$ follows from (2.7.3). (Like $C^{r,s+}$, $C^{r,s++}$ has only one terminal singularity of $X^{r,s++}$.) Hence we have an elementary blow-down $X^{r,s++} \to X^{r,(s-1)+}$ over Z, which contracts F_s^{++} to a point (Figure 11(c)). We can also construct $X^{r,(s-1)+}$ as a flip of $X^{r,s}$ for the blow-down of $C^{r,s} \cup F_s$ and with respect to

$$K_{X^{r,s}} + D^{r,s} + \sum e_i E_i + \sum f_j F_j,$$

or as a similar flip in the curve $C^{r,s-1}$ after the blow-down $X^{r,s} \to X^{r,s-1}$ of F_s. Again $X^{r,(s-1)+}$ with $D^{r,(s-1)+}$, E_i^+, and F_j^+ satisfies the above properties (2.7.2-7) if we take $r := r$ and $s := s-1$. Indeed, we get (2.7.3) and (2.7.6) from the construction and the last explanations, respectively, because

$$K_{X^{r,s++}} + D^{r,s++} + \sum E_i^{++} + \sum F_j^{++}$$

is numerically trivial on the flopped curve φ_s^{++} and

$$K_{X^{r,s-1}} + D^{r,s-1} + \sum e_i E_i + \sum_{j=1}^{s-1} f_j F_j$$

is negative on $C^{r,s-1}$. (The blow-down of F_s^{++} is a small resolution from (1.3.7).) The depth in (2.7.2) is $\leq n+2-r-s$, and the inequality contradicts (2.4.4) by (I3), since $r \geq 1$, and we have a flop $X^{r,s+} \dashrightarrow X^{r,s}$ in $C^{r,s+}$. The remaining properties follow directly from 2.7. However, we must replace (2.7.1) and (2.7.8) by the new versions

(2.7.1) $r \geq 1$, $s \geq 1$ *in case* 1.2. So, it is possible that $s = 0$ but only in case 2.2.

(2.7.8) $X^{r,s+}$ *and* $D_1^{r,s+}$ *have exactly one singular point of type* V_2 (see Example (1.2.3)) *on* $C^{r,s+}$.

The last follows from the construction, or from [12], 2.1 after a partial resolution (1.3.7) of the singularity on $C^{r,s+}$. Indeed, $C^{r,s+} = |R_c^+|$ and this is an exceptional curve of the first kind on the minimal resolutions of E_r^+ and F_s^+ (cf. Figure 11(c)). (The singularity has type V_2 by the proven part of Theorem 1.3.)

The same construction works when $r \geq 2$ and ε_r^+ is an exceptional curve of the first kind on the minimal resolution of $D_1^{r,s+}$. (In case 2.2 with $s = 0$ we take, as above, $F_0^+ = D_2^{r,0+}$.) In particular, we define an extremal ray $R_e \subset \overline{\mathrm{NE}}(X^{r,s+}/Z\,;p)$ similar to R_f. This means that R_c^+ and R_e generate a two-dimensional face of $\overline{\mathrm{NE}}(X^{r,s+}/Z\,;p)$ corresponding to curves on E_r^+. So, after the above modifications and a permutation of singularities x, y on C, we obtain:

If ε_i^+ or φ_j^+ is an exceptional curve of the first kind on the minimal resolution of $D_1^{r,s+}$, then $r = 1$ and ε_1^+ is such a curve.

Otherwise, the exceptional curves ε_i^+ and φ_j^+, $j \geq 1$, are not of the first kind on the minimal resolution of $D_1^{r,s+}$. In this case Lemma 2.8 and Corollary 2.9 work even for $s = 0$ in case 2.2. Eventually, after an appropriate choice of x in case 1.2, we have the final version

(2.7.1) $r = 1$ *and* $s \geq 1$ *in case* 1.2.

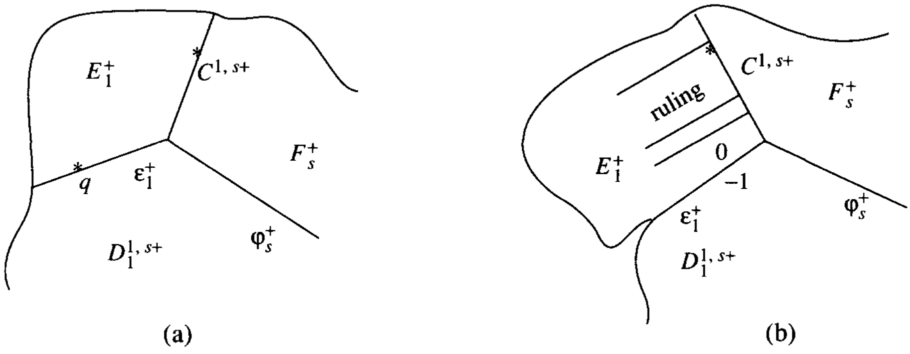

FIGURE 12

(2.8.2) $e_1 \leq f_s$ *if* ε_1^+ *is not an exceptional curve of the first kind on the minimal resolution of* $D_1^{r,s+}$. (As above, $f_0 = 1$.)

Proof of Proposition 2.1 *for cases* 1.2 *and* 2.2. First we check that

$$(2.10) \qquad \left(K_{X^{1,s+}} + D^{1,s+} + E_1^+ + \sum F_j^+ \cdot R_e\right) < 0$$

(cf. the proof of Lemma 2.8). Next we consider the case when ε_1^+ is an exceptional curve of the first kind on the minimal resolution of $D_1^{1,s+}$ and the latter really has singularities on ε_1^+. Then, by (2.7.1) in the final form, (2.7.4), and the classification of surface log terminal singularities, ε_1^+ has exactly one such singularity q (Figure 12(a)). Using [12], 2.1 after a partial resolution (1.3.7) of q, we obtain that ε_1^+ is also an exceptional curve of the first kind on the minimal resolution of E_1^+. So, ε_1^+ is exceptional on E_1^+, $D_1^{1,s+}$ is negative on ε_1^+, and $|R_e| = \varepsilon_1^+$. In addition, by [27], 3.9,

$$\left(K_{X^{1,s+}} + D^{1,s+} + E_1^+ + \sum F_j^+ \cdot \varepsilon_1^+\right) = \left(K_{D_1^{1,s+}} + \varepsilon_1^+ + \sum_{j\geq 0} \varphi_j^+ \cdot \varepsilon_1^+\right)$$
$$= \deg\left(K_{\varepsilon_1^+} + (\varepsilon_1^+ \cap \varphi_s^+) + \frac{m-1}{m} q\right) = -\frac{1}{m} < 0,$$

where m is the index of $K_{D_1^{1,s+}}$ in q.

Now we consider the case when $D_1^{1,s+}$ does not have singularities on ε_1^+, and ε_1^+ is an exceptional curve of the first kind on $D_1^{1,s+}$ (Figure 12(b)). Then we have a ruling on E_1^+ with fiber ε_1^+. So, $(D_1^{1,s+} \cdot \varepsilon_1^+) = (D_1^{1,s+} \cdot R_e) = 0$ and $\varepsilon_1^+ \in |R_e|$. Using the above arguments we obtain again

$$\left(K_{X^{1,s+}} + D^{1,s+} + E_1^+ + \sum F_j^+ \cdot \varepsilon_1^+\right) = -1 < 0.$$

In the remaining cases ε_1^+ is not an exceptional curve of the first kind on the minimal resolution of $D_1^{1,s+}$, and $e_1 \leq f_s$ by (2.8.2) in the last version. So, to prove (2.10), we can use arguments similar to those in the proof of Lemma 2.8 as soon as we check that $(D_1^{1,s+} \cdot R_e) \geq 0$. Indeed, $D_1^{1,s+}$ has at most $\zeta \leq 2$ singular points on ε_1^+, because $s \geq 1$ in case 1.2. (Moreover, $\zeta \leq 1$ in case 2.2.) Again by [12], 2.1, we obtain that ε_1' will be an $(m-1-\zeta)$-curve and a $(-m)$-curve on the minimal resolutions $E_1' \to E_1^+$ and $D' \to D_1^{1,s+}$, respectively. We now assume that $m \geq 2$. Hence ε_1^+ will be movable on E_1^+ and $(D_1^{1,s+} \cdot R_e) \geq 0$ for the case when $\zeta = 2$ and $m = 2$. I contend that the latter is impossible, i.e. $m \geq 3$ when $\zeta = 2$.

Indeed, suppose that $\zeta = 2$ and $m = 2$. In particular, $(\varepsilon_1')^2_{D'} = -2$, and $(K_{D'} \cdot \varepsilon_1') = 0$. We denote by q_i, $i = 1, 2$, the singularities of $D_1^{1,s+}$ on ε_1^+. As in the proof of Lemma 2.8, we get a contradiction with (2.7.6):

$$0 \le f_s + 2(\tfrac{1}{2}e_1) - 2e_1 \le \left(K_{D'} + E' + e_1\varepsilon_1' + \sum_{j\ge 0} f_j\varphi_j' \cdot \varepsilon_1'\right)$$

$$= \left(K_{D_1^{1,s+}} + e_1\varepsilon_1^+ + \sum_{j\ge 0} f_j\varphi_j^+ \cdot \varepsilon_1^+\right) = \left(K_{X^{1,s+}} + D^{1,s+} + e_1E_1^+ + \sum f_jF_j^+ \cdot \varepsilon_1^+\right),$$

where E' is an effective divisor on D'. We have only to check that the multiplicity of E' in a curve C_i'/q_i $(i = 1, 2)$ intersecting (normally) ε_1' is at least $e_1/2$. Since the resolution g is minimal, $K_{D'}$ is nef/ q_i, whereas C_i' is a $(-n_i)$-curve with $n_i \ge 2$. Hence it is enough to consider the case when both singularities q_i are resolved only by C_i''s; these intersect ε_1' normally ([27], 3.5). Then

$$E' = \frac{n_1 - 2 + e_1}{n_1}C_1' + \frac{n_2 - 2 + e_1}{n_2}C_2',$$

and the required multiplicities satisfy

$$\frac{n_i - 2 + e_1}{n_i} \ge \frac{(n_i - 2)e_1 + e_1}{n_i} = \frac{(n_i - 1)}{n_i}e_1 \ge \frac{e_1}{2},$$

because $n_i \ge 2$, and $0 < e_1 < 1$ (see Lemma 2.5).

Thus, (2.10) holds, and by (2.7.1-2) and by inductive assumptions (I2-3) we have a semistable modification $X^{1,s+} \dashrightarrow X^{1,s++}$ in R_e when $s \ge 1$. In this case we start with $\widetilde{X} = X^{1,s++}/Z$.

If the modification $X^{1,s+} \dashrightarrow X^{1,s++}$ is a flip, then $i(X^{1,s++}/p, D^{1,s++} + E_1^{++} + \sum F_j^{++}) \le i(X^{1,s+}/p, D^{1,s+} + E_1^+ + \sum F_j^+) - 1 = n - s$. All subsequent modifications exist, because we have only $s+1$ irreducible surfaces E_1^{++} and F_j^{++}/p. As above, we get the required modification $X^+ = X^{\min}/Z$, semistable for $D^+ = D^{\min}$, and with $i(X^+/p, D^+) \le n+1$. Indeed, if we get a modification X^{mod}/Z with $i(X^{\mathrm{mod}}/p, D^{\mathrm{mod}}) = n+1$, then all modifications on the way to X^{mod} were divisorial, and the surfaces E_1^{++} and F_j^{++} were blown down to points. Note that $(F_s^+ \cdot R_e) > 0$, and the flipped curve belongs to F_s^{++}. So, the fiber of $D^{1,s++}/p$ consists of the curves ε_1^{++} (except for the case when $\zeta = 1$ and $\varepsilon_1^+ = |R_e|$ is an exceptional curve of the first kind on the minimal resolution of $D_1^{1,s+}$) and φ_j^{++}, $j \ge 1$ (but case 2.2 and $F_1^{++} \cap D_2^{1,s++}$ are impossible, as we shall see later). These curves will be contracted to a point on D^{mod}. Since X^{mod}/Z is semistable for D^{mod}, this means that $X^{\mathrm{mod}} = Z = X^{\min} = X^+$. This is possible only in case 1.2 with $\dim E = 2$, and when the last modification is a blow-down of modified $G = E_1^{++}$ or F_j^{++} to the point p. By (2.1.3), f contracts E on a curve $f(E)$, that is not contained in $f(D)$. This completes the proof of Theorem 1.7 in the case under consideration, and, respectively, Theorem 1.3 when $i(X^+/p, D^+) = n+1$. Indeed, then the last modification $X^{\mathrm{mod}-} \to Z$ is a blow-down of $G^{\mathrm{mod}-}$ to the point p, semistable for $D^{\mathrm{mod}-} + G^{\mathrm{mod}-}$ with $i(X^{\mathrm{mod}-}/p, D^{\mathrm{mod}-} + G^{\mathrm{mod}-}) = n$. So, in this case Theorem 1.3 holds for X^+ by (I2). Otherwise, $i(X^+/p, D^+) \le n$ and Theorem 1.3 holds by (I1).

Now we consider the case when again $s \ge 1$ and R_e defines a divisorial blow-down $X^{1,s+} \to X^{1,s++}$ of E_1^+. By (I2), $X^{1,s++}/Z$ is semistable for $D^{1,s++} + \sum F_j$ with $i(X^{1,s++}/p, D^{1,s++} + \sum F_j^{++}) \le i(X^{1,s+}/p, D^{1,s+} + E_1^+ + \sum F_j^+) = n+1-s$.

Since $|R_c^+| = C^{1,s+} \subset E_1^+$, the blow-down contracts E_1^+ to a curve. A generic fiber C_e of the corresponding ruling on E_1^+ intersects $C^{1,s+}$ but does not intersect ε_1^+. Indeed, C_e is a 0-curve on E_1^+ (nonsingular on C_e), and $(F_s^+ \cdot C_f) = 1$ (cf. the contradiction at the end of the proof of Lemma 2.8). So, $(D_1^{1,s+} \cdot R_e) = 0$ and $\varepsilon_1^+ \in |R_e|$. (In fact, this is possible only when $\zeta = 0$ and ε_1^+ is an exceptional curve of the first kind on $D_1^{1,s+}$; Figure 12(b).) Therefore, this time we have only s irreducible surfaces F_j^{++}/p and curves φ_j^{++} (as well as $F_1^{++} \cap D_2^{1,s++}$ in case 2.2) on $D_1^{1,s++}/p$. Hence, all subsequent modifications exist and we can proceed as above.

Our concluding case is very special: $r = 1$ and $s = 0$. By the final (2.7.1), it is possible only in case 2.2. (In particular, $e_1 < f_0 = 1$ automatically.) So, $D_1^{1,0+}$ has at most one singularity on ε_1^+. By construction, E_1^+ is the fiber of $X^{1,0+}/p$. But according to (2.7.5), the curves of E_1^+ generate a 2-dimensional face of $\overline{\mathrm{NE}}(X^{1,0+}/Z\,;p)$. Moreover, it is generated by the extremal rays R_e and R_c^+. Hence

$$K_{X^{1,0+}} + D^{1,0+} + E_1^+$$

is negative on $X^{1,0+}/p$, and, by (2.7.6), there exists a rational number a such that $0 \le e_1 < a < 1$ and

$$K_{X^{1,0+}} + D^{1,0+} + aE_1^+$$

is numerically trivial on $C^{1,0+}$ and negative on all other irreducible curves$/p$. This time we move one step back, i.e., we take a blow-up (1.3.7) in the singular point $q' \in C^{1,0+}$ of $X^{1,0+}$ (see the last version of (2.7.8) and Figures 13(a-b)), and then we take Atiyah's flop in the modification of $C^{1,0+}$ (Figures 13(b-c)). So we obtain $X^{\mathrm{mod}}/Z = X^{2,0+}$ or $X^{1,1+}$ semistable for $D^{\mathrm{mod}} + E_1^{\mathrm{mod}} + G$ with

$$i(X^{\mathrm{mod}}/p\,,\, D^{\mathrm{mod}} + E_1^{\mathrm{mod}} + G) = n$$

(see (2.7.2)), where $D^{\mathrm{mod}} = D_1^{\mathrm{mod}} + D_2^{\mathrm{mod}} = D_1^{2,0+} + D_2^{2,0+}$ or $D^{1,1+} = D_1^{1,1+} + D_2^{1,1+}$,

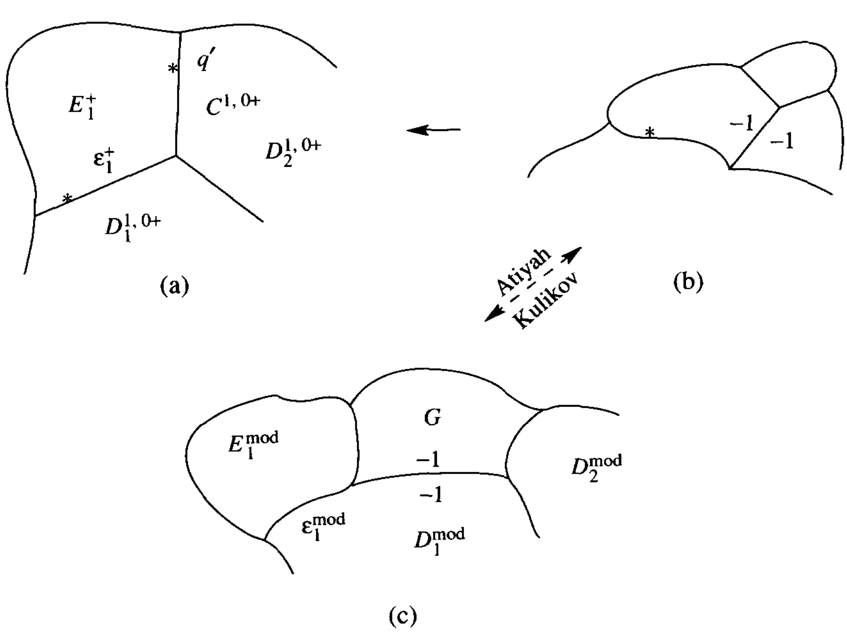

FIGURE 13

respectively, and $G = E_2^+$ or F_1^+. I contend that

$$K_{X^{\mathrm{mod}}} + D^{\mathrm{mod}} + aE_1^{\mathrm{mod}} + aG$$

is numerically trivial on G and negative on all irreducible curves/p outside G. Since a flip in a curve numerically trivial with respect to some divisor preserves the intersections with the curve and the blow-up in q' is extremal, we have a rational number a' such that

$$K_{X^{\mathrm{mod}}} + D^{\mathrm{mod}} + aE_1^{\mathrm{mod}} + a'G$$

satisfies the required properties. Moreover, $a = a'$ is the coefficient of the different on $D_1^{1,0+} \cap D_2^{1,0+} \subset D_2^{1,0+}$ at the point to which $C^{1,0+}$ is contracted on $D_2^{1,0+}$ (Figure 13(a)). But now $\varepsilon_1^{\mathrm{mod}}$ is a $(-m)$-curve on the minimal resolution of D_1^{mod}, where $m \geq 2$, and it has at most one singularity on $\varepsilon_1^{\mathrm{mod}}$. Using again [12], 2.1, we find that $\varepsilon_1^{\mathrm{mod}}$ is an $(m-2)$-curve on the minimal resolution of E_1^{mod} when E_1^{mod} and D_1^{mod} have a singularity on it, and $\varepsilon_1^{\mathrm{mod}}$ is an $(m-1)$-curve otherwise. So, $\varepsilon_1^{\mathrm{mod}}$ is movable on E_1^{mod} with a fixed point. Therefore, we have an extremal ray $R' \subset \overline{\mathrm{NE}}(X^{\mathrm{mod}}/Z\,;p)$ such that

$$(K_{X^{\mathrm{mod}}} + D^{\mathrm{mod}} + aE_1^{\mathrm{mod}} + aG \cdot R') < 0,$$

$(D^{\mathrm{mod}} \cdot R') \geq 0$. This implies that (cf. the proof of Lemma 2.8)

$$\begin{aligned}(K_{X^{\mathrm{mod}}} &+ D^{\mathrm{mod}} + E_1^{\mathrm{mod}} + G \cdot R')\\ &= (K_{X^{\mathrm{mod}}} + D^{\mathrm{mod}} + aE_1^{\mathrm{mod}} + aG \cdot R') + (1-a)(E_1^{\mathrm{mod}} + G \cdot R')\\ &= (K_{X^{\mathrm{mod}}} + D^{\mathrm{mod}} + aE_1^{\mathrm{mod}} + aG \cdot R') + (1-a)(-D^{\mathrm{mod}} \cdot R') < 0,\end{aligned}$$

because $D^{\mathrm{mod}} + E_1^{\mathrm{mod}} + G$ is numerically trivial/p. Note also that if R' is of a divisorial type, then the corresponding blow-down $X^{\mathrm{mod}} \to Y/Z$ contracts E_1^{mod} to a curve, since $E_1^{\mathrm{mod}} \cap G \notin R'$. But this is also impossible. Indeed, $E_1^{\mathrm{mod}} \cap D^{\mathrm{mod}} = \varepsilon_1^{\mathrm{mod}} \notin R'$, and we have the same contradiction as in the proof of Lemma 2.8. Hence R' is of a flipping type and, by (I3), its flip $X^{\mathrm{mod}} \dashrightarrow X^{\mathrm{mod}+}$ exists. The new modification $X^{\mathrm{mod}+}$ is semistable for $D^{\mathrm{mod}+} + E_1^{\mathrm{mod}+} + G^+$ with

$$i(X^{\mathrm{mod}+}/p\,, D^{\mathrm{mod}+} + E_1^{\mathrm{mod}+} + G^+) \leq i(X^{\mathrm{mod}}/p\,, D^{\mathrm{mod}} + E_1^{\mathrm{mod}} + G) - 1 = n - 1.$$

Note also that $(G \cdot C') \leq 0$ (in fact < 0) for irreducible curves $C' \subset G$. Otherwise, by semistability, $(D^{\mathrm{mod}} \cdot C')$ or $(E^{\mathrm{mod}} \cdot C') < 0$, and $C' = D_1^{\mathrm{mod}} \cap G$, $D_2^{\mathrm{mod}} \cap G$, or $E_1^{\mathrm{mod}} \cap G$. These curves are exceptional on D_1^{mod}, D_2^{mod}, and E_1^{mod}, respectively, and $(G \cdot C') < 0$ for them. But $(G \cdot \varepsilon_1^{\mathrm{mod}}) > 0$. Therefore, by the cone theorem we can assume that $(G \cdot R') > 0$. So, for the flipped curves $C' \subset |R'|^+$, $(G^+ \cdot C') < 0$, and $|R'|^+ \subset G^+$. This implies that again the fiber of $D^{\mathrm{mod}+}/p$ belongs to $E_1^{\mathrm{mod}+} \cup G^+$. (Actually, by connectedness ([27], 5.7) and the arguments in the proof of Lemma 2.8, one can show that $|R'|$ does not intersect D_1^{mod} and D^{mod}. So, the flip $X^{\mathrm{mod}} \dashrightarrow X^{\mathrm{mod}+}$ does not touch D^{mod}, nor the fiber of D^{mod}/p.) We take $\widetilde{X} = X^{\mathrm{mod}+}$ and proceed as above. Since $E = C$ in case 2.2, this is a flipping case in Proposition 2.1. Hence one of the subsequent modifications of $\widetilde{X} = X^{\mathrm{mod}+}$ should not be that of a divisorial blow-down to a point, and $i(X^+/p\,, D^+) \leq n$. ∎

Now we are ready to prove, partially, Theorem 1.6.

2.11. Proposition. *Let f, g, D, and V be as in Theorem* 1.6, $i(X/V\,, D) \leq n+1$, $g(D_1) \neq \mathrm{pt.}$, *and suppose Theorem* 1.3 *holds for all points of X. Then Theorem* 1.6 *holds for g, and Theorem* 1.3 *holds for Y with boundary $g(D)$.*

We know that $E = D_1$. Following the statement of Proposition 2.1, we can add the following assumptions:

(2.11.1) $f = g$ *is extremal*, and $V = \{p\}$.

(2.11.2) X *is locally* $\mathbb{Q}$*-factorial.*

In the reduction to this case we must use Proposition 2.1 for flipping contractions. Moreover, after a flip we obtain the required statements by (I1-3) because all subsequent depths of modified X/Z will be at most n (cf. (2.12) below and its proof). Note that in this case the image curve $g(D_1) \subset g(D)$ contains a point/V that is not locally $\mathbb{Q}$-factorial. The same arguments, namely (I1) and (I3), work after a divisorial contraction to a curve. Thus, we may restrict ourselves to the following conditions:

(2.11.3) $E = D_1$, and *the fiber of* D_1/p *consists of one irreducible curve* C. In particular, $C \subset D_1$, and $C \in |R|$.

After possibly shrinking a neighborhood of C, this implies

(2.11.4) *The singularities of* X *and* D *and the triple points of* D *belong to* C. *All* D_i*'s and double curves of* D *intersect* C. In particular, p *is a unique possible singularity of* Z.

(2.11.5) $i(X/p, D) = n + 1 \geq 1$. In particular, X *has a singular point*/p.

Proof of Proposition 2.11. So, $Y = Z$ is $\mathbb{Q}$-factorial with only terminal singularities, and, by the contraction theorem, it is semistable for $g(D) = f(D)$. As we know, $E = D_1$. Directly from the definition we see also that

$$i(Z, p, f(D)) \leq i(X/p, D) \leq n + 1.$$

But we must check a little more, namely, the inequality

$$i(Z, p, f(D)) \leq n, \tag{2.12}$$

which implies Theorem 1.3 by (I1). It is obvious for nonsingular p.

Otherwise, p is a $\mathbb{Q}$-factorial singularity of X and $f(D_i)$ with $i \neq 1$. Since $(D_1 \cdot C) < 0$, and f is semistable for D, we have one more irreducible component of D, say D_2, such that $(D_2 {\cdot} C) > 0$. In particular, C intersects D_2 and $p \in f(D_2)$. So, $d = \sharp\{D_i | i \neq 1\} = \sharp\{f(D_i) | f(D_i)$ is an irreducible component of $f(D)$ through $p\} \geq 1$. I contend that actually $d = 1$. Indeed, if $d \geq 2$ at p, one can check that $D = D_1 + D_2 + D_3$ for an appropriate renumbering of D_i's. Moreover, in contradiction with (2.11.5), X, $C = D_1 \cap D_3$, and the D_i's are *nonsingular*, whereas C is a 0-curve and a (-1)-curve on D_1 and D_3, respectively (cf. Figure 12(b)). To prove this, one must first check that C is numerically equivalent to C_f, a generic fiber of the ruling on D_1 induced by f (cf. Figure 14(a)). Further arguments are carried out as below (cf. (2.13)).

So, we consider later only the case with $d = 1$ in p or, equivalently, $D = D_1 + D_2$. By Lemma 1.4 and [27], 3.8, the double curve $C' = D_1 \cap D_2$ is normal, whence C'

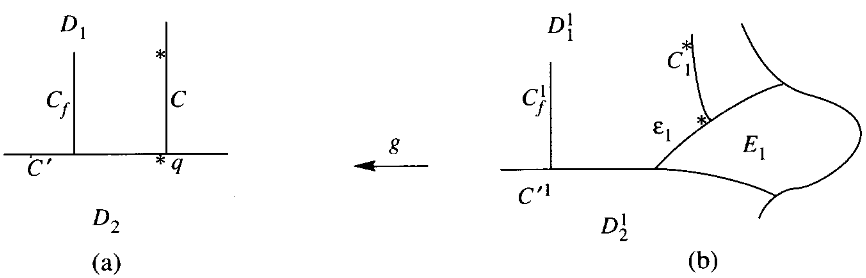

FIGURE 14

is nonsingular. Thus, $C \not\subset C'$ by (2.11.4). This means also that the components of C' are not contained in the fibers of the ruling on D_1 induced by f, and they intersect C. This is possible only when C' is irreducible as D_2 (cf. the proof of Lemma 2.8; Figure 14(a)). According to Lemma 1.4 and [27], 3.8, $f(D_2)$ is normal. So, f induces an isomorphism $D_2 \to f(D_2)$. Hence C intersects C' in a single point q that is *singular* on D_i's, X, and maps to p.

We make a blow-up $g: X^1 \to X$ at q as in (1.3.7), with an exceptional divisor E_1 (Figures 14(a-b)). Let $R_1 \subset \overline{\mathrm{NE}}(X^1/Z; p)$ be the corresponding extremal ray. It is generated by the intersection curve $\varepsilon_1 = D_1^1 \cap E_1 = \mathbb{CP}^1$, which is contracted to the point q, log terminal for

$$K_{D_1} + C' = (K + D)|_{D_1}$$

on D_1 (cf. (2.4.5) in case 2.2). Then, by the adjunction formula and [27], 3.9,

$$0 > (K_{X^1} + D^1 + E_1 \cdot \varepsilon_1) = (K_{D_1^1} + C'^1 + \varepsilon_1 \cdot \varepsilon_1)$$

$$= \deg\left(K_{\varepsilon_1} + (\varepsilon_1 \cap C'^1) + \sum \frac{m_i - 1}{m_i} q_i\right) = -1 + \sum \frac{m_i - 1}{m_i} \geq -1, \tag{2.13}$$

where the m_i are the indices of $K_{D_1^1}$ at singular points $q_i \in \varepsilon_1$. Moreover, $= -1$ is possible only when D_1^1 is nonsingular on ε_1. (This implies also that D_1^1 has at most one singularity on ε_1.)

Then we can proceed as in the proof of Proposition 2.1 for cases 1–3.1. Since $\rho(X^1/Z; p) = 2$, we have one more extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z; p)$. Moreover, $|R_2| = C^1$, because $(E_1 \cdot C^1) > 0$, $(E_1 \cdot \varepsilon_1) < 0$, and $(E_1 \cdot C_f^1) = 0$ for a generic fiber C_f^1 of the ruling on D_1^1 induced by $f \circ g$.

Now I contend that

$$(K_{X^1} + D^1 + E_1 \cdot R_2) < 0 \tag{2.14}$$

or, equivalently,

$$(K_{D_1^1} + C'^1 + \varepsilon_1 \cdot C^1)_{D_1^1} < 0. \tag{2.15}$$

Indeed, C_f is numerically equivalent to $a\varepsilon_1 + bC^1$ with integers $a, b \geq 1$. So,

$$1 = (C'^1 \cdot C_f)_{D_1^1} = a + b(C'^1 \cdot C^1)_{D_1^1},$$

whence $a = 1$, and $(C'^1 \cdot C^1)_{D_1^1} = 0$, i.e., C'^1 does not intersect C^1 (Figure 14(b)). Similarly,

$$-1 = (K_{D_1^1} + C'^1 + \varepsilon_1 \cdot C_f)_{D_1^1} = (K_{D_1^1} + C'^1 + \varepsilon_1 \cdot \varepsilon_1)_{D_1^1} + b(K_{D_1^1} + C'^1 + \varepsilon_1 \cdot C^1)_{D_1^1},$$

whence, by (2.13), we obtain (2.15), except for the case when D_1^1 is nonsingular on ε_1, and

$$(K_{D_1^1} + C'^1 + \varepsilon_1 \cdot C^1)_{D_1^1} = 0.$$

But C^1 intersects ε_1. So, in the last case C^1 is an exceptional curve of the first kind on the minimal resolution of D_1^1. Moreover, D_1^1 may have only canonical (Du Val) singularities on C^1, which is impossible by (2.11.2) and (1.3.6). Hence, D_1^1 is nonsingular, and C^1 crosses ε_1 normally. Then

$$0 = (C^1 \cdot C_f)_{D_1^1} = 1 + b(C^1)^2_{D_1^1} = 1 - b$$

and

$$0 = (\varepsilon_1 \cdot C_f)_{D_1^1} = (\varepsilon_1)^2_{D_1^1} + b,$$

whence $b = 1$, and ε_1 is an exceptional curve of the first kind on D_1^1. This contradicts the fact that q is singular (or that g is minimal in the sense of (1.3.7)).

Thus, we have proved (2.14). By (1.3.7) and (2.11.5), X^1/Z is semistable for $g^*D = D^1 + E_1$ with

$$i(X^1/p, g^*D) = i(X/p, D) - 1 = n.$$

So, according to (I3), we have a flip $X^1 \dashrightarrow X^{1+}$ in R_2. Moreover, X^{1+}/Z is semistable too for $D^{1+} + E_1^+ = (g^*D)^+$ with

$$i(X^{1+}/p, D^{1+} + E_1^+) \leq i(X^1/p, g^*D) - 1 = n - 1.$$

The proof of (2.12) can be completed by applying inductive assumptions (I2-3) to $\widetilde{X} = X^{1+}/Z$, because X^{1+} has only one exceptional divisor/p (namely, E_1^+). Note also that p is $\mathbb{Q}$-factorial, whence $X^{\min}/Z = Z$ (cf. [27], 1.5.7). ■

3. Induction steps

3.1. **Induction step for Theorem 1.3.** We will check *Theorem* 1.3 *for a semistable singularity with* $i(X, p, D) \leq n + 1$.

By definition, there exists a resolution $g = g_1 \circ \cdots \circ g_N$, semistable for g^*D, with $\leq n + 1$ prime divisors $E_i \subset Y_j$ exceptional for the components $g_j \colon Y_j \to Y_{j-1}$ ($Y_0 := X$) and such that $g_j E_i = \mathrm{pt}$. (see 1.1). Any partial resolution $G_m = g_1 \circ \cdots \circ g_m \colon Y_m \to X$ is semistable too for $G_m^* D$ and with

$$i(Y_m/p, G_j^* D) + \delta(Y_m/X) \leq n + 1,$$

where $\delta(Y_m/X)$ is the number of prime divisors E_i, exceptional for g_j's with $j \leq m$ and such that $g_j E_i = \mathrm{pt}$. ($G_j E_i = p$).

By (1.1.2), Theorem 1.3 holds for Y_N (i.e., for any point on G_N^*). So, we can use induction on N. This means that *it is enough to check Theorem* 1.3 *for* Y_{j-1} *when it holds for* Y_j. However, g_j may be nonprojective, even over a neighborhood of $V = G_j^{-1} p$. But Theorem 1.3 is local, and g_j is locally projective by (1.1.3). Therefore, we may restrict ourselves to a *partial resolution* $g \colon Y := Y_j \to X := Y_{j-1}$ such that

(3.1.1) *g is projective/p.*

(3.1.2) *Theorem* 1.3 *holds for* Y.

(3.1.3) *Y/X is semistable for g^*D with*

$$i(Y/p, g^*D) + \delta(Y/X) \leq n + 1.$$

Here, after possibly shrinking a neighborhood of p, $\delta(Y/X)$ is the number of exceptional divisors/p. Note, that all such divisors lie in g^*D. Thus, we must *check Theorem* 1.3 *for* $p \in X$.

Suppose first that K_Y or, equivalently, $K_Y + g^*D$ is nef/p. Then g is small, because p is a terminal singularity (cf. [27], 1.5.7). By (3.1.3), it is a partial $\mathbb{Q}$-factorialization of X (nontrivial only when $d = 1$) with $i(Y/p, g^*D) \geq i(X, p, D)$. Then Theorem 1.3 for p follows from that for Y/p.

Otherwise, K_Y or, equivalently, $K_Y + g^*D$ is not nef/p, and we must apply Mori's theory to g. Namely, by the cone theorem, we have an extremal ray $R \subset \overline{\mathrm{NE}}(Y/X; p)$, negative with respect to K_Y and $K_Y + g^*D$. Let $f \colon Y \to Z/X$ be the corresponding contraction with the exceptional locus E. Like Y/X, it is semistable for g^*D and bimeromorphic. So, it corresponds to the one in Theorem 1.6 or 1.7. If $i(Y/p, g^*D) \leq n$, we can apply (I1-3). Namely, then there exists a modification $Y \dashrightarrow Y^+/X$ in R, where Y^+/X is again semistable for $D^+ = (g^*D)^+$ with $i(Y^+/p, D^+) \le n+1$. By (I2), equality in the last relation is possible only when $E = E_i$ is a component of g^*D, and the modification $Y \dashrightarrow Y^+/X$ coincides with the contraction $f\colon Y \to Z/X$. Moreover, $f(E_i) = \text{pt.}/p$ is $\mathbb{Q}$-factorial of index > 1, f is minimal in the sense of (1.3.7), and the discrepancy of K_X in E is less than 1, i.e., even in this case Theorem 1.3 holds for $f(E_i) \in Y^+ = Z$ and for Y^+. So, in any case, Y^+/X satisfies (3.1-3), and we can replace Y by it. Indeed, by (3.1.3), $i(Y/p, g^*D) \le n+1$, and equality holds only when $f(E) \ne \text{pt}$. But then we can apply Propositions 2.1 and 2.11 instead of (I1-3). (Note that after one flip or divisorial blow-down of a component of g^*D to a curve, we simplify the situation: we can replace $\le n+1$ in (3.1.3) by $\le n$. Hence later (I1-3) will be enough.) The termination of modifications leads us to the above case, when $K_Y + g^*D$ is nef/p, which completes the proof of Theorem 1.3 for p.

Note that (1.3.6) in 3.1 follows from (1.3.5) and (1.3.7) by Lemma 2.5 and its proof. ■

As a corollary of this, as well as Propositions 2.1, 2.11, we obtain

3.2. **Induction step for Theorem 1.7.** Theorem 1.7 holds when $i(X/V, D) \le n+1$.

3.3. **Induction step for Theorem 1.6 in the case of blow-downs to a curve.** Theorem 1.6 holds when $g(D_1)$ is a curve, and $i(X/V, D) \le n+1$.

3.4. **Induction step for Theorem 1.6 in the case of blow-downs to a point.** We will check here *Theorem* 1.6 *when* $g(D_1) = \text{pt.}$ *and* $i(X/V, D) \le n+1$.

As we know, $E = D_1$. According to our assumption, $g(D_1) = \text{pt.} \in g(D)/V$, $g(D)$ is semistable on Y, and Y/Z is at least numerically semistable for $g(D)$ (cf. [25], 2.9). Directly from the definition we see that $i(Y/V, g(D)) \le i(X/V, D) + 1$, and equality is possible only due to $E = D_1$. So, we must investigate when equality holds, and we can do it locally/pt.$= g(D_1)$. Thus we assume that

(3.4.1) $f = g$ *is extremal,* and $V = \{p = g(D_1) = g(E)\}$.

Indeed, D_1, K_Y, and $K_Y + g^*D$ are negative/p. Hence we have again an extremal ray $R \subset \overline{\text{NE}}(Y/Z; p)$, negative with respect to D_1, K_Y, and $K_Y + g^*D$. If the corresponding contraction differs from $f = g\colon X \to Z = Y$, it will be small (flipping) or a divisorial blow-down to a curve. Then by 3.2 and 3.3, respectively, we have a modification $X \dashrightarrow X^+/Z$ such that X^+/Z is again a semistable partial resolution with $i(X^+/p, D^+) \le i(X/p, D) - \delta$, where $\delta = 0$ or 1, and $= 1$ if the modification is flipping and we have one exceptional prime divisor/p on X^+ (namely, D_1^+). So, as above, after similar steps we obtain a small partial resolution $X^{\min}/Z$ that is semistable for $D^{\min}$ with $i(X^{\min}/p, D^{\min}) \le i(X/p, D) \le n+1$ and $i(Z, p, f(D)) \ne i(X/p, D) + 1$. Therefore, in the remaining cases we assume that the contraction $f = g\colon X \to Z = Y$ corresponds to R and is extremal/p. We can restrict ourselves also to

(3.4.2) $i(X/p, D) = n+1$, and $i(Z, p, f(D)) = n+2$.

So, $i(Z, p, f(D)) = i(X/p, D) + 1$, and *we must check the required properties of* f *from Proposition* 1.6. Namely,

(3.4.3) p *is* $\mathbb{Q}$*-factorial,* and X *is* $\mathbb{Q}$*-factorial/*p.

(3.4.4) f *is minimal in the sense of* (1.3.7).

(3.4.5) p *has index* > 1, and *the discrepancy of* K_Z *in* D_1 *is* < 1.

Since f is extremal, the second part of (3.4.3) follows from the first. But if p is not $\mathbb{Q}$-factorial, then there exists an (effective and even prime) Weil divisor $S \subset Z$

in a neighborhood of p, such that S is not $\mathbb{Q}$-Cartier. Therefore, by the contraction theorem, its proper inverse image $f^{-1}S \subset X$ is not $\mathbb{Q}$-Cartier because f is extremal. Moreover, according to (1.3.4), $f^{-1}S$ is not $\mathbb{Q}$-Cartier only at a finite set of points $p_i \in D_1$ in a neighborhood of which $D = D_1$ is irreducible. By (3.4.2) and 3.1, they have a semistable $\mathbb{Q}$-factorization. But we need something different.

3.5. Lemma. *Under the assumption of Theorem* 1.3, *let* $d = 1$, *and suppose that* p *has a* $\mathbb{Q}$*-factorialization* $g: Y \to X$ *semistable for* $g^*D = g^{-1}D$ *with* $i(Y/p, g^*D) = i(X, p, D)$. *Then for any Weil divisor* $S \subset X$, *there exists a partial* $\mathbb{Q}$*-factorialization* $h: Z \to X$ *semistable for* $h^*D = h^{-1}D$ *with* $i(Z/p, h^*D) = i(X, p, D)$ *and ample with respect to* $h^*S = h^{-1}S$.

Such a $\mathbb{Q}$-factorialization is *trivial* (i.e., an isomorphism) if and only if S is $\mathbb{Q}$-Cartier.

Proof. The required partial $\mathbb{Q}$-factorialization is a flip of id_X with respect to S ([27], §1). It is unique, and it exists by [5], 6.1 (cf. [27], 2.7). It must be semistable and have a certain depth/p, which does not affect these general results. But it is also a flip of a complete $\mathbb{Q}$-factorization g that can be reconstructed from the last. First, we can replace S by an effective Weil divisor in a neighborhood of p. Second, g is small (nontrivial when S is not $\mathbb{Q}$-Cartier), and g^*S is $\mathbb{Q}$-Cartier. Third, if g^*S is nef/p, then, by the contraction theorem, we can contract curves $C \subset Y/p$ with $(g^*S \cdot C) = 0$, which gives the required partial $\mathbb{Q}$-factorialization.

Otherwise, $(g^*S \cdot C) < 0$ for some irreducible curve/p. Since p is $\mathbb{Q}$-Gorenstein, $(K_Y \cdot C) = (K_Y + g^*D \cdot C) = 0$. Again we have a flip in C with respect to g^*S that can be considered as a log-terminal flip with respect to $K_Y + g^*D + \varepsilon g^*S$ (or $K_Y + \varepsilon g^*S$), where $0 < \varepsilon << 1$. Note that the extremal rays of $\overline{\mathrm{NE}}(Y/X; p)$ are in $1-1$ correspondence with the irreducible curves Y/p, and they belong to a prime divisor g^*D because D is Cartier. So, we have the termination of such flips ([27], 4.1), and we must check only that such flips $Y \dashrightarrow Y^+/X$ are semistable for $D^+ = g^*D$ with $i(Y^+/p, D^+) = i(Y/p, g^*D) = i(X, p, D)$.

Thus, it is enough to consider the case when g is extremal or, equivalently, C is the fiber of Y/p. According to Kollár ([9], 2.4), Y and Y^+ have the same analytic singularities p_i and p_i^+/p, respectively. This means that there exists a 1-1 correspondence $p_i \leftrightarrow p_i^+$, among them such that a neighborhood U_i of $p_i \in Y$ is isomorphic to a neighborhood U_i^+ of $p_i^+ \in Y^+$. I contend that in our situation we have a little more: $(U_i, g^*D) \cong (U_i^+, D^+)$, i.e., the isomorphism $U_i \to U_i^+$ transforms $g^*D|_{U_i}$ into $D^+|_{U^+}$. This follows from the proof of [9], 2.4, whereas $D = (u = 0)/G$ in the notation of [9], pp. 17–18, for the right side (!). Indeed, in the first case, due to Kollár, D is invariant under the induced involution. In the second case, $s = f_X^* u$ is invariant under tGt^{-1}, and the above isomorphism transforms $g^*D|_{U_i} = (s = 0)/G$ into $D^+|_{U_i^+} = (s = 0)/tGt^{-1}$. ■

3.6. *Remark*. According to Corollary 4.7 with $\sigma = 1$, only the second case can occur at the end of the last proof. Moreover, if Y is nonsingular, then Y^+ is also nonsingular, and the corresponding flip-flop can be done with the help of Reid's pagoda [21].

One can also prove Lemma 3.5 using the ideas of §2.

3.7. Corollary. *The* $\mathbb{Q}$*-factorialization in* (1.3.5) *is defined up to a flop.*

Thus we have a partial $\mathbb{Q}$-factorialization $g: Y \to X$ such that g is semistable and is ample with respect to the proper inverse image $(f \circ g)^{-1}S$. Hence $f \circ g: Y \to Z$

is projective, after possibly shrinking a neighborhood of p ([17], 1.3). Moreover, Y/Z is semistable for $(f\circ g)^*D$ with

$$i(Y/p, (f\circ g)^*D) = i(X, p, D) = n+1.$$

Thus, we can proceed as above. Let $R \subset \overline{\mathrm{NE}}(Y/Z; p)$ be an extremal ray negative with respect to K_Y and $K_Y + (f\circ g)^*D$. It defines a semistable and bimeromorphic contraction $h\colon Y \to$ something$/Z$. If h is not a divisorial contraction to a point, we can apply 3.2-3. Appeal to (I2-3) is then made, and we obtain a contradiction with (3.4.2): $i(X, p, f(D)) \le n+1$. So, h contracts $g^{-1}D_1$ to a point. This is also impossible when g is nontrivial because D_1 is $\mathbb{Q}$-Cartier, and its fibers belong to $g^{-1}D_1$. By Lemma 3.5, this means that $f^{-1}S$ and S are $\mathbb{Q}$-Cartier, which completes the proof of (3.4.3).

3.8. Now we prove (3.4.4). More exactly, we prove that *if f is not minimal in the sense of* (1.3.7), *then* $i(Z, p, f(D)) \le n+1$, which contradicts (3.4.2).

Thus, suppose that f *is not minimal*, i.e., there exists a double curve of D on D_1, say $C = D_1 \cap D_2$ after an appropriate renumbering of D_i's, such that C is an exceptional curve of the first kind on the minimal resolution of D_2. Note that $d = \sharp\{D_i | i \ne 1\} = \sharp\{f(D_i) | f(D_i)$ is an irreducible component of $f(D)$ through $p\} = 1$ or 2, because by (3.4.2) p is singular, and $1 \le d \le 2$ by (1.3.1-2). Moreover, we have the following two opportunities, respectively after possibly shrinking a neighborhood of p and renumbering D_i).

(3.8.1) $D = D_1 + D_2$, and $C = D_1 \cap D_2 = \mathbb{CP}^1$ *has at most two singular points of* D_2, *as well as of* D_1, *and* X (Figures 15-16(a)).

(3.8.2) $D = D_1 + D_2 + D_3$, and $C = D_1 \cap D_2 = \mathbb{CP}^1$ *has at most one singular point of* D_2, D_1, *and* X.

This easily follows from Lemma 1.4 and the classification of surface log terminal singularities (cf. 2.4). By (1.3.3), the singularities of D_2 on C coincide with these of D_1 and X. The curve $C = D_1 \cap D_2$ is *irreducible*, since f is extremal, and C, like D_2, is ample on D_1.

The subsequent considerations will run case by case, distinguished, as in the proof of Proposition 2.1, by two natural invariants: the number $a = d+1$ of components D_i and the number b of singularities of D_2, as well as those of D_1 and X, on C. So, case $a.b$ means that D has a components D_i, and C has b singular points of D_2, as well as those of D_1 and X. By the way, cases 2.0 and 3.0-1 will be excluded even before an estimate of $i(Z, p, f(D))$ is found.

We begin with the situation of (3.8.1), where $a = 2$. The cases below are ordered according to $b = 0, 1, 2$.

Case 2.0. D_2, as well as X and D_1, is nonsingular on C. So, C is an exceptional curve of the first kind on D_2, and, by [12], 2.1,

$$C^2_{D_1} = -C^2_{D_2} = 1 \quad \text{and} \quad (K_{D_1}\cdot C) = -3,$$

i.e., C is a 1-curve on D_1. Since it is numerically positive on D_1, and D_1 is normal, it follows that $D_1 = \mathbb{CP}^2$ with a (very ample) line C. So, D_1 does not have singularities of X and normally crosses D_2. But this contradicts $i(X/p, D) = n+1 \ge 1$ in (3.4.2).

Case 2.1. One singular point x of D_2, as well as of D_1 and X (Figure 15(a)), lies on C. By 3.1 and (3.4.2), we can make a blow-up $g\colon X^1 \to X$ at x as in (1.3.7), with an exceptional surface E_1 (Figures 15(a-b)). Then X^1, D_1^1, D_2^1, E_1

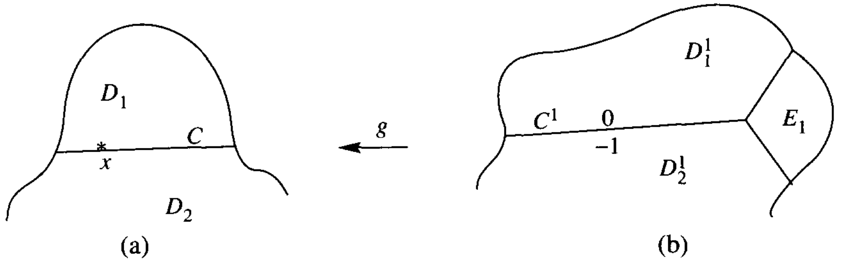

FIGURE 15

do not have singularities on C^1, and, by the minimal property of g, C^1 is an exceptional curve of the first kind on D_2^1. In particular, $(D_1^1.C^1) = -1$. The blow-up corresponds to an extremal ray $R_1 \subset \overline{\mathrm{NE}}(X^1/Z\,; p)$. Since $\rho(X^1/Z\,; p) = 2$, we have one more extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z\,; p)$ (cf. the proof of Proposition 2.1 for cases 1-3.1). Moreover, R_2 is generated by C^1 because $(D_2^1 \cdot C^1) = 0$ ([12], 2.1), while $(D_2^1 \cdot R^1) > 0$, and $C^1 = D_1^1 \cap D_2^1$. Hence

$$\begin{aligned}(K_{X^1} + D^1 + E_1 \cdot C^1) &= (K_{D_1^1} + C^1 + (E_1 \cap D_1^1) \cdot C^1) \\ &= \deg(K_{C^1} + (E_1 \cap D_1^1 \cap D_2^1)) = -2 + 1 < 0,\end{aligned}$$

whence

$$(K_{X^1} + D^1 + E_1 \cdot R_2) < 0.$$

The above implies also that $|R_2| = D_1^1$, and the corresponding contraction $X^1 \to X^{1+}/Z$ transforms D_1^1 to a curve (cf. Figure 12(b)). But by construction, X^1/Z is semistable for $g^*D = D^1 + E_1$ with $i(X^1/p, g^*D) = n$. By (I2), the modification X^{1+}/Z is semistable for $D^{1+} + E_1^+$ with $i(X^{1+}/p, D^{1+} + E_1^+) \le n$. However, now X^{1+} has only one prime surface/p (namely, E_1^+, the image of E_1). Therefore, we can proceed further as in the proof of Proposition 2.1. By (I2-3), we finally obtain $Z = X^{\min}/Z$, which is semistable for $f(D) = D^{\min}$ with $i(Z, p, f(D)) \le n+1$. (In fact, the next modification will be the last, and it will be a divisorial contraction of E_1^+ to the point p.)

Case 2.2. On C we have two singular points x and y of D_2 as well as of D_1 and X (Figure 16(a)). As above, we can make a simultaneous blow-up $g\colon X^{1,1} \to X$ at x and y, with exceptional surfaces E_1 and F_1, respectively (Figures 16(a-b); cf. Construction 2.4). Then $X^{1,1}$, $D_1^{1,1}$, $D_2^{1,1}$, E_1, and F_1 do not have singularities on $C^{1,1}$, and, by the minimal property of g, $C^{1,1}$ is an exceptional curve of the first kind on $D_2^{1,1}$. Again by [12], 2.1, $C^{1,1}$ is exceptional on $D_1^{1,1}$. In particular, $(D_1^{1,1} \cdot C^{1,1}) = (D_2^{1,1} \cdot C^{1,1}) = -1 < 0$. Lemma 2.5 implies (cf. (2.4.7)) that $C^{1,1}$ generates an extremal ray $R_c \subset \overline{\mathrm{NE}}(X^{1,1}/Z\,; p)$ with $|R_c| = C^{1,1}$. Hence we can make Atiyah's flop $X^{1,1} \dashrightarrow X^{1,1+}/Z$ in R_c (Figures 16(b-c); cf. Figure 7). Again $X^{1,1+}/Z$ is semistable for $D^{1,1+} + E_1^+ + F_1^+$ with

$$i(X^{1,1+}/p, D^{1,1+} + E_1^+ + F_1^+) = i(X^{1,1}/p, D^{1,1} + E_1 + F_1) = n - 1.$$

After this we can proceed as above. But $Z = X^{\min}/Z$, $f(D) = D^{\min}$, and $i(Z, p, f(D)) \le n+1$. Indeed, otherwise Z is obtained from $X^{1,1+}$ by three successive blow-downs of the surfaces $D_1^{1,1+}$, E_1^+, and F_1^+ to points that are minimal in the sense of (1.3.7). Since $(E_1^+ \cdot C^{1,1+}) = (F_1^+ \cdot C^{1,1+}) = -1 < 0$, $D_1^{1,1+}$ must go first, and we can accept the order E_1^+ before F_1^+, possibly after a permutation of the corresponding singularities x and y. So, the curves $\varphi_1^+ = D_1^{1,1+} \cap F_1^+ = \mathbb{CP}^1$,

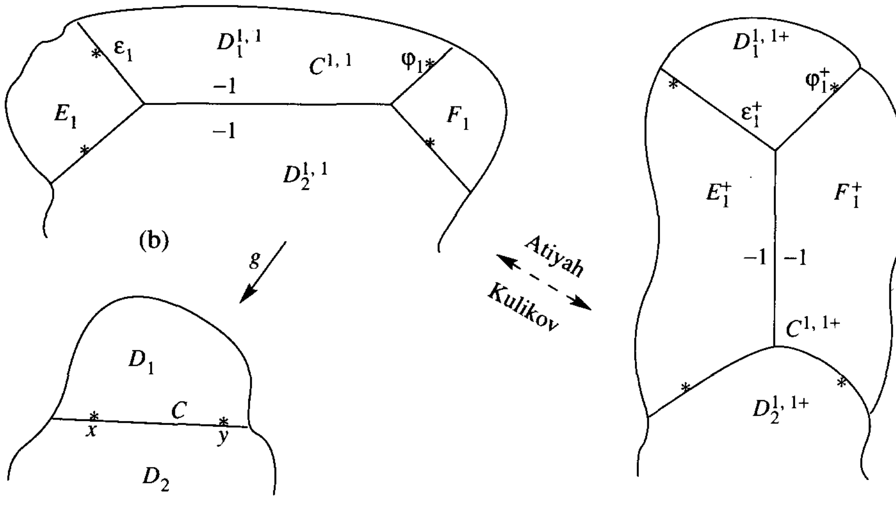

FIGURE 16

$C^{1,1+} \cup \varphi_1^+$, and $\varphi_1 = D_1^{1,1} \cap F_1 = \mathbb{CP}^1$ are exceptional. But like $D^{1,1}$, φ_1 is ample on F_1 by the extremal property of its blow-down, which gives a contradiction (cf. [27], 8.10), i.e., $i(Z, p, f(D)) \leq n+1$.

In the situation (3.8.2) both cases 3.0-1 will be excluded, because then f is *not extremal*, which contradicts (3.4.1) (and leads to the required inequality). For this it is enough to check that C is not ample on D_1. (Cf. the first two cases in the proof of Proposition 2.1 for the case 1-2.2; Figures 12(a-b).)

Case 3.0. By [12], 2.1, C is a 0-curve on D_1, or $(C \cdot C)_{D_1} = 0$.

Case 3.1. By [12], 2.1, after a partial resolution (1.3.7) of a unique singular point of D_2 on C, C is an exceptional curve of the first kind on the minimal resolution of D_1. This implies that C is exceptional on D_1 or $(C \cdot C)_{D_1} < 0$.

This completes the proof of (3.4.4).

3.9. Finally, we prove (3.4.5). Again $d = \sharp\{D_i | i \neq 1\} = \sharp\{f(D_i) | f(D_i)$ is an irreducible component of $f(D)$ through $p\} = 1$ or 2, because by (3.4.2) p is singular. Moreover, if $d = 2$, by (1.3.3) p is a singularity of type $V_2(r, a)$ with index $r \geq 2$. It is known also that the discrepancy of K_Z in D_1 is $1/r$. (Cf. [27], 3.9 and the Appendix, and arguments in the proof of Lemma 2.3.) So, we may assume later that $d = 1$. This means, after possibly shrinking a neighborhood of p and renumbering D_i, that

(3.9.1) $D = D_1 + D_2$, and $C = D_1 \cap D_2 = \mathbb{CP}^1$ *has at most three singular points of* D_2, *as well as of* D_1 *and* X (Figures 18-20 (a) below).

The curve $C = D_1 \cap D_2$ is *irreducible*, since f is extremal. In (3.4.5), the first statement follows from the second. To prove the second statement we must check only that p *is not a canonical* (*Du Val*) *singularity of* $f(D_2)$. Indeed, let a be a discrepancy of K_Z in D_1. Then as in Lemma 2.3 we obtain

$$f^*(K_Z + f(D)) = K + eD_1 + D_2,$$

where $0 \leq e = 1 - a < 1$, and $-e$ is the discrepancy of K_{D_2} in C. This time $e \geq 0$ because C is not an exceptional curve of the first kind on the minimal resolution of D_2. Thus, we must exclude the case when $e = 0$, or p is canonical on $f(D_2)$. More exactly, we *derive a contradiction with* (3.4.3), *which means the existence of a small semistable resolution* (*$\mathbb{Q}$-factorialization*) *of* p *on* Z *when* p *is a canonical singularity of* $f(D_2)$ (cf. (1.3.6) for $d = 1$). So, we suppose that

(3.9.2) *p is a canonical singularity of $f(D_2)$*. In particular, C is a *(-2)-curve on the minimal resolution of D_2*. Moreover, on C, *D_2 has only canonical singularities with graphs* $\mathbb{A}_r$, i.e., the exceptional curves γ_i of the minimal resolution of any such singularity are (-2)-curves, cross normally, and form a chain $\gamma_1, \dots, \gamma_r$. For an appropriate renumbering of γ_i, *only γ_1 crosses the proper inverse image of C on the resolution, and crosses normally. The discrepancies of C and γ_i for $K_{f(D)}$ are* 0.

Subsequent considerations will run case by case, distinguished by one natural invariant: the number b of singularities of D_2, as well as that of D_1 and X, on C. So, case b means that C has b singular points of D_2, as well as of D_1 and X. By the way, case 0 will be excluded even before the construction starts. By (3.9.1), $b = 0, 1, 2$, or 3.

Case 0. D_2, as well as X and D_1 is nonsingular on C. In addition, D_1 normally crosses D_2. So, C is an exceptional (-2)-curve on D_2, and, by [12], 2.1,

$$C_{D_1}^2 = -C_{D_2}^2 = 2 \quad \text{and} \quad (K_{D_1} \cdot C) = -4,$$

i.e., C is a 2-curve on D_1. Since it is numerically positive on D_1, and D_1 is normal, it follows that D_1 is an irreducible quadric with a (very ample) conic section C. By (3.4.2), D_1 has one ordinary quadratic singularity q, i.e., it is a quadratic cone with vertex q. But this is impossible by (3.4.2) and (1.3.6), because then $i(X, q, D) = i(X/p, D) = 0$. This contradicts the relation $i(X/p, D) = n + 1 \geq 1$ in (3.4.2) and, in fact, the property (3.4.3), by (1.3.6). (The small resolution of $p \in X$ was constructed in the proof of Theorem 1.6 for the case $i(X/V, D) = 0$; see §1.)

In the next cases we encounter singular points x of D_2, as well as those of X and D_1, belonging to C (Figure 17(a)). Then by (1.3.3) and (3.9.2), x has type $V_2(r+1, r)$ on X with $r \geq 1$ (see Example (1.2.3) and [6], 1.1.2) when x on D_2 is a canonical singularity with graph $\mathbb{A}_r$. In particular, x is a log terminal singularity of D_1 having type $1/(r+1)(1, 1)$, i.e., it has index $r+1$ and graph $\mathbb{A}_1$. So, it can be resolved by one exceptional curve ε_1, which is a $(-r-1)$-curve. This implies the following fact for a blow-up $g\colon X^1 \to X$ of x as in (1.3.7), with an exceptional surface E_1 (Figures 17(a-b)):

(3.9.3) *In a neighborhood of $\varepsilon_1 = D_1^1 \cap E_1$, X, D_1^1, D_2^1, and E_1 are nonsingular, and D^1 has only normal crossings*. Moreover, ε_1 *is a $(-r-1)$-curve on D_1^1 with* $r \geq 1$.

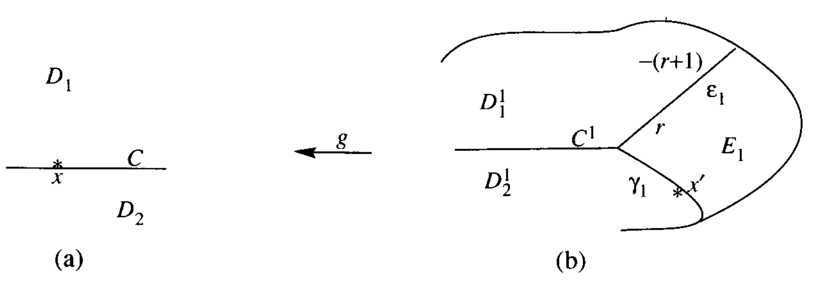

FIGURE 17

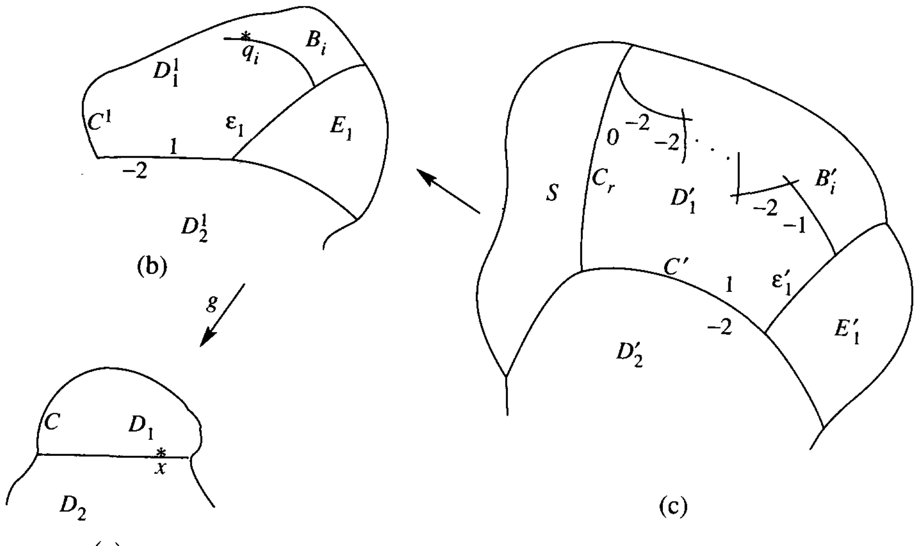

FIGURE 18

We remark that $\gamma_1 = D_2^1 \cap E_1$ is a proper bimeromorphic transform of γ_1 from (3.9.2). It has at most one singularity x' of D_2, as well as those of X^1 and E_1, which is similar to x (Figure 17(b)). Namely, it exists when $r \geq 2$, and has type $V_2(r, r-1)$. So, one can also prove (3.9.3) by induction on r. It can also be proved that ε_1 is an r-curve on E_1, and E_1 is a cone over a rational normal curve of degree r with a hyperplane section ε_1 (cf. cases 2.0 and 0 above). Now we continue our considerations.

Case 1. One singular point x of D_2, as well as that of D_1 and X, lies on C (Figure 18(a)). By 3.1 and (3.4.2), we can make a blow-up $g: X^1 \to X$ at x as in (1.3.7), with an exceptional surface E_1 (Figures 18(a-b)). Then X^1, D_1^1, D_2^1, E_1 do not have singularities on $C^1 = D_1^1 \cap D_2^1$, as well as on $\varepsilon_1 = D_1^1 \cap E_1$, and, by the minimal property of g, C^1 is a (-2)-curve on D_2^1. In particular, $(D_1^1 \cdot C^1) = -2$, and by [12], 2.1, C^1 is a 1-curve on D_1^1. The blow-up corresponds to an extremal ray $R_1 \subset \overline{\mathrm{NE}}(X^1/Z; p)$. Since $\rho(X^1/Z; p) = 2$, we have one more extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z; p)$ (cf. the proof of Proposition 2.1 for cases 1-3.1). Moreover, $(D_2^1 \cdot R_2) = 0$, and $|R_2|$ does not intersect C^1. Indeed, otherwise C^1, like D_2^1, is ample on D_1^1 because $(D_2^1 \cdot R^1) > 0$. Then as in case 2.0, $D_1^1 = \mathbb{CP}^2$ with a (very ample) line C^1, which is impossible, since ε_1 is exceptional by (3.9.3).

Thus, $(D_2^1 \cdot R_2) = 0$, whence $(D_1^1 + E_1 \cdot R_2) = 0$. But $(E_1 \cdot R_2) > 0$ because $(E_1 \cdot R_1) < 0$ and $(E_1 \cdot C^1) > 0$. So, $(D_1^1 \cdot R_2) < 0$ and $|R_2| \subset D_1^1$. Since $\overline{\mathrm{NE}}(X^1/Z; p)$ is generated by R_1 and R_2, and $(D_2^1 \cdot R_1) > 0$, it follows that $|R_2| = \bigcup B_i$ is a curve whose irreducible components B_i are exactly the irreducible curves on D_1^1 not intersecting C^1. Note that the B_i's intersect ε_1. So, a semiample 1-curve C^1 determines a birational contraction $c: D_1^1 \to \mathbb{CP}^2$ with the exceptional locus $|R_2|$. But the singularities q_i of D_1^1 belong to $|R_2|$. By (3.9.3), they do not belong to ε_1. I contend that on each B_i there is at most one singularity q_i, and it is canonical. Indeed, as in the proof of Lemma 2.5, we see that

$$(f \circ g)^*(K_Z + f(D)) = K_{X^1} + D_2^1,$$

because the discrepancies of D_1^1 and E_1 for $K_Z + f(D)$ coincide with those of C

and γ_1 for $K_{f(D)}$, respectively, and so all of them are 0 by (3.9.2). Hence

$$(K_{D_1^1} + C^1 + \varepsilon_1 \cdot B_i) = (K_{X^1} + D^1 + E_1 \cdot B_i) = (K_{X^1} + D_2^1 \cdot B_i) = 0.$$

This implies that the B_i's are disjoint curves that are exceptional of the first kind on the minimal resolution $D_1' \to D_1^1$. Moreover, the singularities q_i of D_1^1 are canonical, and there is at most one on each B_i. (Their graphs are $\mathbb{A}_*$, and the exceptional curves of D_1'/q_i are (-2)-curves combining a chain with the proper inverse images B_i' and ε_1'; see Figures 18(b-c).) So, the composition $D_1' \to D_1^1 \to \mathbb{CP}^2$ can be decomposed into monoidal transformations with nonsingular points, one of which, $c(B_i)$, belongs to $c(\varepsilon_1)\backslash c(C^1)$. Since

$$(C^1 \cdot \varepsilon_1)_{D_1^1} = 1,$$

$c(\varepsilon_1)$, like $c(C^1)$, is a line on $\mathbb{CP}^2$. The pencil of lines through $c(B_i)$ induces a rule $D_1' \to C^1$ on D_1' such that its generic fiber C_r intersects $C' = C^1$ normally and does not intersect ε_1' (Figure 18(c)).

By (3.4.2) and (1.3.6) in 3.1, we can extend $D_1' \to D_1^1$ to a small semistable resolution (or a $\mathbb{Q}$-factorialization) $X' \to X_1^1$ (possibly nonprojective/p). Since $D_1' + D_2'$ and $K_{X'} + D_2'$ are linearly trivial in a neighborhood of C_r, we can apply Nakano's criterion. This implies that the generic fiber C_r of D_1'/C^1 is cut by a surface S defined over a neighborhood of $p \in Z$. Let $h \colon X' \to Z$ be the canonical projection. Then if $h(S)$ is $\mathbb{Q}$-Cartier, we have $h^*h(S) = S + dD_1' + eE_1'$ with positive rational numbers d, e. In addition, $h^*h(S)$ is numerically trivial over Z, which is impossible for positive d on C_r. So, $h(S)$ is not $\mathbb{Q}$-Cartier, which contradicts (3.4.3).

Case 2. This time, two singular points x and y of D_2, as well as of D_1 and X, lie on C (Figure 19(a)). As above, we can make a simultaneous blow-up $g \colon X^{1,1} \to X$ at x and y, with exceptional surfaces E_1 and F_1, respectively (Figures 19(a-b), cf. Construction 2.4). The points x and y are canonical on D_2 with graphs $\mathbb{A}_r$ and $\mathbb{A}_s$, respectively. We can assume that $r \le s$. Again $C^{1,1} = D_1^{1,1} \cap D_2^{1,1}$ is a

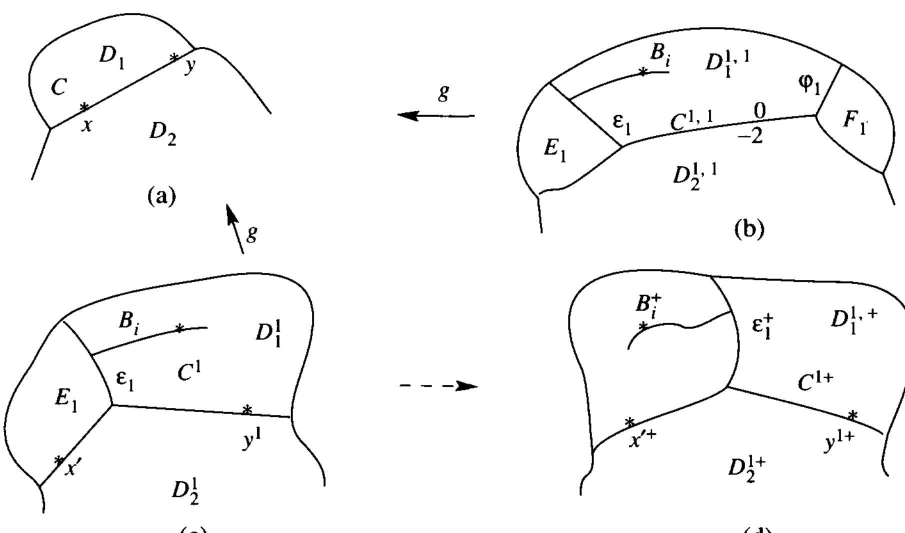

FIGURE 19

(-2)-curve on $D_2^{1,1}$. Hence $(D_1^{1,1} \cdot C^{1,1}) = -2$, and by [12], 2.1, $C^{1,1}$ is a 0-curve on $D_1^{1,1}$. This defines a ruling on $D_1^{1,1}$ with two disjoint and exceptional sections $\varepsilon_1 = D_1^{1,1} \cap E_1$ and $\varphi_1 = D_1^{1,1} \cap F_1$. So, the ruling is not minimal, i.e., it has a nonirreducible fiber. Since $D_1^{1,1}$ is nonsingular on these sections, there exist curves $B_i \subset D_1^{1,1}$ in such fibers that intersect ε_1 and do not intersect φ_1. Now we consider a blow-up $g: X^1 \to X$ only at x, with exceptional surfaces E_1 (Figures 19(a-c), cf. case 1). I contend that B_i's, the proper bimeromorphic transforms of the above B_i's, generate a second extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z; p)$. Moreover, $(D_2^1 \cdot R_2) = 0$, and $|R_2| = \bigcup B_i$ does not intersect C^1. Indeed, by construction, $(D_2^1 \cdot B_i) = 0$. So, $(D_2^1 \cdot R_2) \leq 0$. But $C^1 = D_1^1 \cap E_1$ is a 0-curve on the minimal resolution of D_1^1 and is movable on D_1^1 with $(C^1 \cdot C^1)_{D_1^1} = (D_2^1 \cdot C^1) > 0$. Hence $(D_2^1 \cdot R_2) = 0$ is the only possibility, and R_2 is of a flipping type with $|R_2| = \bigcup B_i$. Note that the B_i's intersect ε_1.

Then as in case 1 we check that

$$(K_{D_1^1} + C^1 + \varepsilon_1 \cdot B_i) = (K_{X^1} + D^1 + E_1 \cdot B_i) = (K_{X^1} + D_2^1 \cdot B_i) = 0,$$

and at most one singularity q_i lies on each B_i, and it is canonical on D_1^1. Now we can make a flop $X^1 \dashrightarrow X^{1+}$ in R_2 (Figures 19(c-d)). It will be symmetric and can be constructed as follows. First, we consider a small semistable resolution (or a $\mathbb{Q}$-factorialization) $X' \to X_1^1$ (possibly nonprojective/p). Then we make Atiyah's flops in B_i''s, the proper bimeromorphic transforms of B_i's, after which the curves intersecting B_i' in the last resolution become exceptional curves of the first kind intersecting ε_1 normally, etc. Each elementary flop with the boundary in this procedure coincides with Kulikov's flop of type I ([12], 4.2, Figure 4) (cf. Figure 6). Finally, we contract the curves B_i and the resolved ones on D_1^1. Then we contract the flopped curves that do not intersect ε_1 on D_2^1. This concludes the construction. Note that X^{1+}/Z is again semistable for D^{1+} with

$$i(X^{1+}/p, D^{1+} + E_1^+) = i(X^1/p, D^1 + E_1) = i(X/p, D) - 1 = n$$

and with two exceptional divisors D_1^{1+}, E_1^+/p. So, by (3.4.2), Z is obtained by two successive divisorial contractions of D_1^{1+} and E_1^+ to points (in this order since $K_{X^{1+}} + D^{1+} + E_1^+$ are numerically trivial on the flopped curves). So, we interchange the roles of D_1^1 and E_1. Moreover, by the demonstration of (3.9.3), x', the new point x, has the smaller index $r-1$. Hence we reduce case 2 to case 1 by induction.

Case 3. Finally, the three singular points x, y, and z of D_2, as well as those of D_1 and X, lie on C (Figure 20(a)). Points x, y, and z are canonical on D_2 with graphs $\mathbb{A}_r$, $\mathbb{A}_s$, and $\mathbb{A}_t$, respectively, and we can assume that $r \leq s \leq t$. First, we consider the case when $s = 1$. Then $r = s = 1 \leq t$. As above, we can make a blow-up $g: X^1 \to X$ at z, with exceptional surfaces E_1 (Figures 20(a-b)). By (3.9.3), X^1, D_1^1, D_2^1, E_1 are nonsingular on $\varepsilon_1 = D_1^1 \cap E_1$, and D_1^1 has ordinary quadratic singularities at x and y on $C^1 = D_1^1 \cap D_2^1$, which are identified with the corresponding points on X. Now (3.9.2) and [12], 2.1 imply that C^1 is an exceptional curve of the first kind on the minimal resolution of D_1^1. Hence C^1 is a double irreducible fiber of a ruling $D_1^1 \to \varepsilon'$, i.e., its generic fiber intersects ε_1 in two points. So, it defines a double covering $\varepsilon_1 \to \varepsilon'$ of Riemann spheres with two branch points $q = C^1 \cap \varepsilon_1$ and $q' \in \varepsilon_1$. Since D_2^1 is positive on E_1 and numerically trivial on C^1, the second extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z; p)$ is generated by C^1, and the ruling D_1^1/ε' is defined by the corresponding contraction. In particular, $K_{D_1^1} + \varepsilon_1$

is numerically equivalent to

$$K_{D_1^1} + C^1 + \varepsilon_1 = (K_{X^1} + D^1 + E_1)|_{D_1^1}$$

and numerically trivial with respect to the ruling. Moreover, its nonirreducible fibers that do not intersect q' have the form $B_i \cup B_i'$, where B_i, B_i' are irreducible curves with at most one canonical singularity on D_1^1 at the unique intersection point $B_i \cap B_i'$ (Figure 20(b)). If such a fiber contains q', then D_1^1 is nonsingular on B_{i_0}, B_{i_0}', and B_{i_0}, B_{i_0}' intersect normally in one point q' (Figure 20(b)). Both curves in both cases are exceptional of the first kind on the minimal resolution of D_1^1 and intersect ε_1 normally in one point (cf. case 1). Make the flop $X^1 \dashrightarrow X^{1+}$ in the curves B_i' (including B_{i_0}'). This defines a meromorphic modification, and X^{1+} is nonprojective/Z (cf. case 2). The flop contracts B_i''s and converts the ruling into a minimal one $D_1^{1+} \to \varepsilon'$. Moreover, D_1^{1+} has a singularity only in fibers corresponding to q and q'. Of course, D_1^{1+} has the above two singularities x and y on C^{1+}. The same holds for the fiber containing q' when it has singularities on D_1^{1+}.

If $(\varepsilon_1^+)^2_{D_1^{1+}} \geq -1$, then by [12], 2.1, $(\varepsilon_1^+)^2_{E_1^+} \leq 0$. However, by the same reasoning and (3.9.3), $(\varepsilon_1)^2_{E_1} > 0$. Hence replacing the flop by a partial one, we obtain $(\varepsilon_1^+)^2_{E_1^+} = 0$, which gives a ruling on E_1^+ not intersecting D_1^{1+}. The last assertion, by Nakano's criterion, leads to a contradiction as in case 1. So, we can assume that $(\varepsilon_1^+)^2_{D_1^{1+}} = -m \leq -2$.

The fiber C^{1+} can be modified into a nonsingular one in the following manner. Make blow-ups of x and y, then contract C^{1+} and one of the blown up curves. We can proceed similarly with the fiber containing q' when it has singularities on D_1^{1+}. This leads to a nonsingular and minimal ruling $D_1^{\min} = \mathbb{F}_l \to \varepsilon'$ with a nonsingular double section $\varepsilon_1^{\min}$ such that $(\varepsilon_1^{\min})^2 = -m+2$ or $-m+4 \leq 2$ when D_1^{1+} has

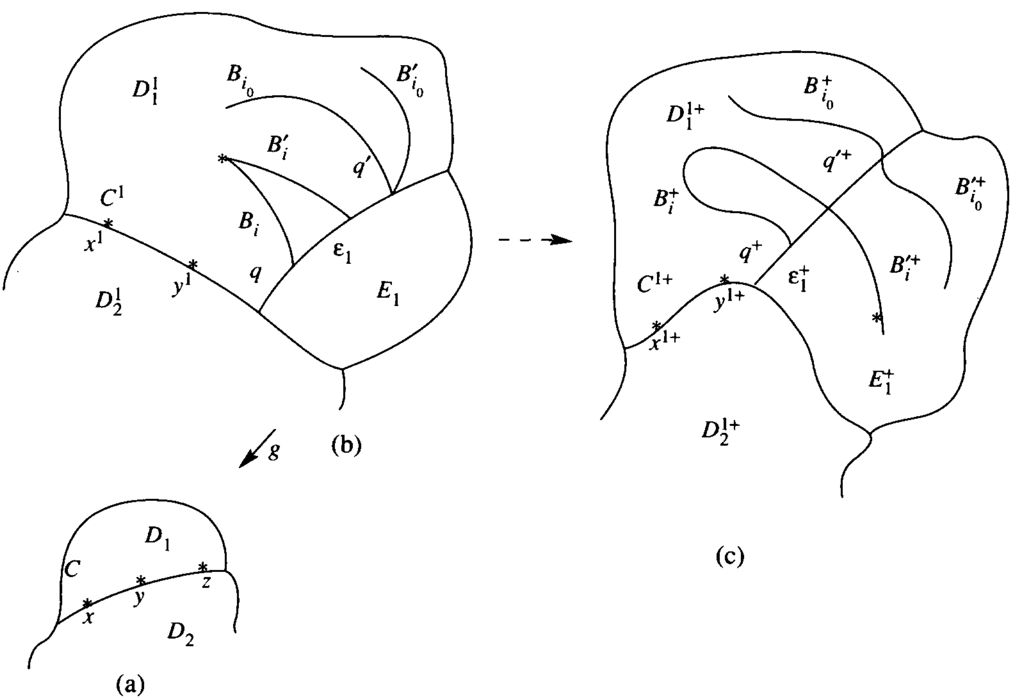

FIGURE 20

singularities only on C^{1+} or also on the fiber passing through q', respectively. So, the double section $\varepsilon_1^{\min}$ is linearly equivalent to $2b_l + \sigma s_l$, where b_l is a negative section with self-intersection $-l$, and s_l is a fiber. Therefore, $(\varepsilon^{\min})^2 \equiv 0 \pmod 4$ from which it is ≤ 0. It is well known also that $\sigma \geq 2l$, which then implies that $(\varepsilon_1^{\min})^2 = 0$, $l = 0$, and $\varepsilon_1^{\min}$ is linearly equivalent to $2b_0$. This is impossible because $\varepsilon_1^{\min}$, like ε_1, is irreducible, which concludes the proof for $s = 1$.

By the classification of canonical singularities, in the remaining cases $r = 1$, $s = 2 \leq t \leq 4$. Again we can make a blow-up $g\colon X^1 \to X$ at z (cf. Figures 20 (a-b)). However, this time the curve C^1 is exceptional on D_1^1. Its contraction $D_1^1 \to D'$ defines a del Pezzo surface D', nonsingular on the image ε' of ε_1. Indeed, C^1 is an exceptional curve of the fist kind on the minimal resolution of D_1^1, and x, y are resolved on it by (-2), (-3)-curves respectively. So, successive contractions of C^1, (-2) and (-3) curves after the resolution of x and y give D' and its nonsingularity on ε'. This shows also that C^1 is contracted to a cuspidal singularity q on ε'. As we know,

$$(f \circ g)^*(K_Z + f(D)) = K_{X^1} + D_2^1,$$

and it is linearly equivalent, over a neighborhood of $p \in Z$, to $K_{X^1} + D_2^1 + D^1 + E_1$. Hence, by adjunction, $K_{D_1^1} + 2C^1 + \varepsilon_1$ is numerically trivial on D_1^1, from which $K_{D'} + \varepsilon'$ is numerically trivial on D'. On the other hand, D', like D_1^1, has only log terminal singularities and, by [27], 8.10, ε' is ample on D_1^1 like C on D_1. I claim that $K_{D'} + \varepsilon'$ is linearly trivial on D_1^1, i.e., D' is del Pezzo with a Cartier anticanonical divisor ε' and with only canonical singularities. Otherwise, D' has noncanonical singularities, and on the minimal resolution $h\colon D'' \to D'$ a log divisor

$$h^*(K_{D'} + \varepsilon') = K_{D''} + \varepsilon'' + \sum c_i C_i$$

is numerically trivial, where ε'' is a proper bimeromorphic transform of ε', the C_i's are exceptional for h, $1 > c_i \geq 0$ are rational, and at least one $c_i > 0$. Now ε'' is semiample and numerically trivial only on C_i's. So, if $E \subset D''$ is an exceptional curve of the first kind, then it intersects ε'' normally in one point and does not intersect C_i's with $c_i > 0$. After contraction of E we get the same situation. Finally, we can assume that D'' is minimal, i.e., has no exceptional curves of the first kind. Since $K_{D''}$ is positive on C_i's with $c_i > 0$ and negative on a generic curve on D'', by the classification of complex algebraic surfaces, D'' is a rational scroll $\mathbb{F}_l$ with $l \geq 3$ and a numerically trivial log divisor

$$K_{\mathbb{F}_l} + \varepsilon'' + ((l-2)/l)b_l$$

which is impossible for a fiber s_l.

Thus, D' is del Pezzo with a Cartier anticanonical divisor ε' and with only canonical singularities. Its degree is

$$1 \leq (\varepsilon')^2_{D'} = (\varepsilon_1)^2_{D_1^1} + 1 + 1 + 4 = (-t-1) + 6 = 5 - t \leq 3.$$

Now we replace $g\colon X^1 \to X$ by its composition with a small semistable resolution (or a $\mathbb{Q}$-factorialization) of the singular points of D_1^1 outside C^1. According to the above, such points are canonical on D_1^1 as on D', and we have the resolution by (3.4.2) and (1.3.6) in 3.1. Then D' will be a nonsingular generalized del Pezzo surface with an anticanonical divisor ε'. "Generalized" means that ε' is semiample and numerically trivial only on blown curves. Suppose we have an exceptional curve of the first kind $B_i' \subset D'$. Then it intersects ε' normally and in one point; in particular, outside the cusp q. So, its proper bimeromorphic transform $B_i \subset D_1^1$ is again an exceptional curve of the first kind, intersecting ε' normally and in one point outside C^1. As above, we can make a flop in B_i. Any such transformation increases $(\varepsilon_1)^2_{D_1^1}$ and $(\varepsilon')^2_{D_1^1}$ by 1. After t such flops we obtain $(\varepsilon_1)^2_{D_1^1} = -1$ and $(\varepsilon_1)^2_{E_1} = 0$, which leads, as above, to a contradiction with (3.4.3). Hence, we have at most $t-1$ such flops. They correspond to contractions of B_i''s and preserve the situation except the degree, which will be at most $(\varepsilon')^2_{D'} = 5 - t + (t-1) = 4$ after such modifications. Thus we can assume that the modified D' is minimal. By the classification of complex algebraic surfaces, the last is possible only for $D' = \mathbb{CP}^2$, and $\mathbb{CP}^1 \times \mathbb{CP}^1 = \mathbb{F}_0$ or $\mathbb{F}_2$. But in these cases we have the degrees 9 and 8, respectively. This contradiction concludes case 3.

Hence Theorems 1.3, 1.6, and 1.7 are proved. As applications we consider first.

4. Semi-stability: Explicit forms

Suppose $p \in D$ is a semistable point. Then by (1.3.5) we have its small semistable $\mathbb{Q}$-factorialization. This reduces the description to the case when p is $\mathbb{Q}$-factorial. Moreover, by (1.3.6) p is nonsingular or has index $r \geq 2$ and by (1.3.2) $d \leq 2$. We know also that for $d = 2$, such a singularity has type $V_2(r, a)$, i.e., locally is isomorphic to such a singularity (see (1.2.3)). The following covers the remaining cases.

4.1. Theorem on moderation. *Let $p \in D$ be a $\mathbb{Q}$-factorial semistable singularity with $d = 1$ (see 1.3). Then it is isomorphic to a moderate singularity $V_1(r, a; n)$ (see 1.2.4), where $r \geq 2$ is the index of p. Moreover, the divisorial blow-up in (1.3.7) coincides with the weighted blow-up (see [27], case 1 in Kawamata's Appendix) and the discrepancy of G will be $1/r$.*

It follows from [13, Theorem] and from

4.2. Proposition. *Let $p \in D$ be a $\mathbb{Q}$-factorial semistable singularity with $d = 1$ (see 1.3), and $g: Y \to X$ a blow-up from (1.3.7). Then the discrepancy of G will be $1/r$, where $r \geq 2$ is the index of p, and the index at any point q of Y on G is at most r.*

Proof. Thus, we must check that g belongs to the case (IN) in [13]. If $d = 1$ for the point $q \in G$, then G is Cartier in the neighborhood of q. So, its index divides r, like the index of $f^*(K+D) = K_Y + D' + (k/r)G$, where D' is the proper inverse image of D, and $1 \leq k \leq r-1$.

Now suppose that $d \geq 2$ for $q \in G$. Then $d = 2$ and q belongs also to $D' \cap G$. So, q has type $V_2(s, a)$, where s is the index of q. On the other hand, for the next blow-up $f: Z \to Y$, as in (1.3.7) with an exceptional surface F we have $f^*g^*(K+D) = K_Z + D'' + (k/r)G' + (l/r)F$, where D'' and G' are the proper inverse images of D and G, respectively. By Lemmas 2.5–6, $1 \leq l \leq k$. Since $f^*(D' + G) = D'' + G + F$, the multiplicity α of F in G is rational and belongs to $(0, 1)$. Hence $f^*(K_Y + D' + G) = K_Z + D'' + G' + ((l + \alpha(r-k))/r)$, F, and we know that

$$\frac{s-1}{s} = \frac{l+\alpha(r-k)}{r} < \frac{l+r-k}{r}.$$

So, $s < r$ when $l \leq k-1$. Moreover, by [13], 2.1, the discrepancy of G will be $1/r$.

In the remaining cases $1 \leq k = l \leq r-1$, and we show that they are impossible. As in 3.9 we will run case by case, depending on *the number b of singularities of D'; as well as those of G and Y on $C = D' \cap G$*. But first we change notation: now $X^1 := Z$, $X := Y$, $Z := X$, $g := f$, $f := g$, $D_1 := G$, $D_2 := D'$, $D_1^1 = G'$, $D_2^1 = D''$, and $E_1 := F$. By our assumption, $b \geq 1$. Since C is contracted to a log terminal point on D, we have $b = 1, 2$, or 3. Again we will derive a contradiction with the $\mathbb{Q}$-factorial property of p. By the minimal property of g, the curve $C = D_1 \cap D_2$ is exceptional but not of the first kind on the minimal resolution of D_2.

Case 1. We have an extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z; p)$ with $(D_2^1 \cdot R_2) = 0$, and $|R_2|$ does not intersect C^1. Indeed, otherwise C^1, like D_2^1, is ample on D_1^1 because $(D_2^1 \cdot R^1) > 0$. Then D_1^1 is a cone Q over C^1 or $\mathbb{F}_s$. The first case is impossible, since $\varepsilon_1 = D_1^1 \cap E_1$ is exceptional by construction. In the second case ε_1 will be the negative section of $\mathbb{F}_s$, and the singularity $x = q \in C$ has type $V_2(s, s-1)$ (cf. 3.9.3)). But this is possible only when x on D_2 is a canonical singularity with the tree A_{s-1} and C^1 is a (-2)-curve on D_2^1. Hence $k = l = 0$, which contradicts our assumption. This shows also that ε_1 has a unique singularity q of X^1 and, as above, with index dividing r.

Thus $|R_2| = \bigcup B_i$ is a curve whose irreducible components B_i are exactly the irreducible curves on D_1^1 that do not intersect C^1. Note that the B_i's intersect ε_1. A semiample curve C^1 determines a birational contraction $c\colon D_1^1 \to Q$ with the exceptional locus $|R_2|$, and $c(\varepsilon_1)$ is a generator of a cone Q, whereas the vertex of Q corresponds to the singularity $q \in \varepsilon_1$. In addition, at most one singularity q_i lies on each B_i, and with index dividing r. Indeed,

$$(f \circ g)^*(K_Z + f(D)) = K_{X^1} + D_2^1 + \frac{k}{r}(D_1^1 + E_1)$$

and

$$(K_{D_1^1} + C^1 + \varepsilon_1 \cdot B_i) = (K_{X^1} + D^1 + E_1 \cdot B_i) = \left(K_{X^1} + D_2^1 + \frac{k}{r}(D_1^1 + E_1) \cdot B_i\right) = 0,$$

because $(D_1^1 + E_1 \cdot B_i) = -(D_2^1 \cdot B_i) = -(C^1 \cdot B_i)_{D_1^1} = 0$. This implies that the B_i's are disjoint curves that are exceptional of the first kind on the minimal resolution $D_1' \to D_1^1$.

By (1.3.5), after a semistable $\mathbb{Q}$-factorialization of X^1 (possibly nonprojective $/p$) we can assume that all singularities of X^1 on D_1^1 are $\mathbb{Q}$-factorial. By induction on the depth, we know that each singularity of D_1^1 is log terminal of index $t|r$ and with the single log discrepancy $1/t$. (More exactly, such singularities have the type $1/t^2(a, t-a)$ [6, Lemma 1.2].) Now not every curve B_i intersects ε_1 but each B_i belongs to a chain intersecting ε_1. If B_i intersects ε_1 outside the singularity q, then, as in case 1 in 3.9, it will have no singularities on D_1^1 and will be an exceptional curve of the first kind on D_1^1. So, we can make Atiyah's flop in B_i, which contracts B_i on D_1^1. After such transformation the self-intersection of ε_1 on the minimal resolution of D_1^1 decreases by 1.

We can proceed similarly in the case when B_i passes through q. The existence of such flops can now be derived from [9], 2.4 (cf. Lemma 3.5) or by the arguments of §2 (see Remark 3.6). Such a flop is symmetric and semistable, contracts B_i to q on D_1^1, and does not change the index t of $K_{D_1^1} + \varepsilon_1$ in q. Due to the monotonic property of log discrepancies (cf. 2.6), the self-intersection of ε_1 on the minimal resolution of D_1^1 will remain the same or decrease by 1. The last case is possible if B_i has a singularity of the same index t as that of q, and its exceptional divisor with the log discrepancy $1/t$ after the contraction of B_i replaces the exceptional divisor with the same discrepancy for a minimal resolution of q ([27], 3.9). As we know, after contracting all B_i's, ε_1 will be a generator, i.e., its self-intersection number on the minimal resolution of $D_1^1 = Q$ will be 0. Hence, after a partial flopping we obtain the situation when ε_1 is the exceptional curve of the first kind on the minimal resolution of D_1^1. Since ε_1 has exactly one singularity q, by [12], 2.1, ε_1 is also an exceptional curve of the first kind on the minimal resolution of E_1. According to the classification of the log terminal singularities, ε_1 is exceptional on both surfaces D_1^1 and E_1 and we must have the flip in ε_1. The point is that in the nonprojective case we have no theorem about a contraction of ε_1 on X^1 to a point. Nevertheless, we have

4.3. Lemma. *There exists a semistable modification of ε_1 that contains a ruling surface with the generic fiber intersecting only the modification of D_2^1 along its section.*

Proof. We can proceed in the same way as in the case when ε_1 contracts to a point (cf. the proof of Proposition 2.1 in case 3.1). After making a blow-up at q we do Atiyah's flop in the modification of ε_1. The restricted log canonical divisor on the modified blown-up surface F will have the form $K_F + C_1 + C_2 + C_3$, where C_1 is the modified intersection with D_1^1, C_2 is the modified ε_1, and C_3 is the modified intersection with E_1. Hence $K_F + C_1 + C_2 + C_3$ is numerically trivial on C_2 and is negative on C_1 and C_2. In particular, we have an extremal contraction on F, negative with respect to $K_F + C_1 + C_2 + C_3$. If it gives a ruling on F, we get the required modification with a surface F.

Otherwise, we have a divisorial contraction of a curve C' on F. If this curve does not intersect C_2, then, by construction, C' intersects C_1 and C_3, which contradicts the connectedness lemma ([27], 5.7). So, since F is nonsingular near C_2, $C' = C_1$ or C_3. In this case we have a singularity on C' with smaller index and we can assume now that the required modification in C' exists by induction. ■

The surface F can be contracted on the modified X along the generic fiber of the ruling, which gives a contradiction with the $\mathbb{Q}$-factorial property of p. Indeed, this time the r-multiple of the scheme-theoretic image of $K_Z + f(D)$ is linearly trivial and has the form $r(K_{X'} + D')$ in a neighborhood of the generic fiber of the ruling on F, where X' and D' are the modifications of X and D_2^1, respectively. In addition, D' is linearly equivalent to $-F$ near this fiber. So, we can apply Nakano's criterion after the covering trick ([27], 2.4.1 and 2.5). In particular, this implies that in fact $K_{X'} + D'$ is linearly trivial near the generic fiber. ($\mathbb{CP}^1$ has no unramified covers!)

Case 2. Let $x = q$, and let $g: X^1 \to X$ be its blow-up. Then we have an extremal ray $R_2 \subset \overline{\mathrm{NE}}(X^1/Z; p)$ with $(E_1 \cdot R_2) > 0$. Note that by [12], 2.1 and the minimal property of f, $C^1 = D_1^1 \cap D_2^1$ has positive square on D_1^1. So, if $(D_2^1 \cdot R_2) = 0$, the support $|R_2| \subset D_1^1$ does not intersect C^1 and we can derive a contradiction as above, because after the semistable flop in R_2 we get a different order for contractions of D_1^1 and E_1 (cf. case 2 in 3.9).

Suppose now that $(D_2^1 \cdot R_2) > 0$. Then $(D_1^1 + E_1 \cdot R_2) = -(D_2^1 \cdot R_2) < 0$ and

$$\begin{aligned}(K_{X^1} + D^1 + E_1 \cdot R_2) &= \left((f \circ g)^*(K_Z + f(D)) + \frac{r-k}{r}(D_1^1 + D_1) \cdot R_2\right) \\ &= \frac{r-k}{r}(D_1^1 + E_1 \cdot R_2) < 0.\end{aligned}$$

The corresponding contraction is semistable and is not to a point. So, using Theorem 1.6 we can decrease $i(Z, p, f(D))$, which is impossible if we consider the blow-up of p from (1.3.7).

Case 3. In this case the graph of p as a log terminal singularity of $f(D)$ has type $\mathbb{D}_m$ or $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. Moreover, C will correspond to a vertex with tree edges. The above arguments work when C^1 has positive square on D_1^1. So, by the minimal property and [12], 2.1, C is a (-2)-curve on the minimal resolution of D_2. The singularity p is not canonical, and it has two equal smallest discrepancies $-k/r$. A direct check according to their classification [4] (Figure 9) gives only cases of type $\mathbb{D}_m$, whereas y and z are ordinary quadratic singularities on D_2 and D_1. Moreover, since $x = q$ is not canonical on D_2, we have a singularity q' on ε_1 of index $t|r$. C^1 is a fiber of a ruling on D_1^1, corresponding to R_2. This ruling is numerically trivial for $K_{D_1^1} + \varepsilon_1$, and the curve ε_1 is a two-section. By [27], 3.9, ε_1 has exactly one singularity q'. This implies that q' is the second branch point of the double cover $\varepsilon_1 \to \varepsilon'$ induced by the ruling (cf. case 3 in 3.9). Using a $\mathbb{Q}$-factorialization and induction we may assume that all singularities on D_1^1 are of type $1/t^2(a, t-a)$, in particular, with a discrepancy $\leq -1/2$. Then contracting the exceptional components of the fiber C' of the ruling passing through q', we obtain the same situation with the irreducible fiber C'. Since q' is again singular, C' has one more singularity $q'' \in C'$. Now we consider a log divisor $K_{D_1^1} + \varepsilon_1 + aC'$ with maximal $a > 0$ such that it is still log canonical. According to the connectedness theorem ([27], 6.9), q' will not be log terminal for $K_{D_1^1} + \varepsilon_1 + aC'$. If $a = 1$, then q'' will also not be log terminal but log canonical for $K_{D_1^1} + \varepsilon_1 + C'$, because $K_{D_1^1} + \varepsilon_1 + C'$ is numerically trivial on C'. Then, according to restrictions on singularities, q'' is cyclic having the graph $\mathbb{A}_3$ with two (-2)-curves at the ends ([5], 9.6). But such a singularity on D_1^1 is not of type $1/t^2(a, t-a)$ ([6], 3.1).

Suppose now that $a < 1$. Then, using the log terminal blow-up q' and [27], 3.9, we can check that $a = 1/2$, and $K_{D_1^1} + \varepsilon_1 + (1/2)C'$ is pure log terminal and has index 2 at q'', which contradicts the condition that one of its discrepancies is $< -1/2$. ■

4.4. **Remark.** In the last proof and in the following one, an explicit form for terminal singularities and their classification plays an important role. In particular, we use [13], [13] uses [27], Kawamata's Appendix, and the last one uses [15].

Another approach is related to the notion of "n-complement" ([27], 5.1). First, we must exclude case 3 even when $l < k$ for all three singularities on C. This means that $K + D$ and $K_Z + f(D)$ have a 1-complement in a neighborhood of C and p, respectively (see Corollary 4.9 below). Again this was proved by Kawamata [5], 10.9, using Mori's classification; for his proof it is enough that $f(D)$ is Cartier. I give an outline in the case when p has type $\mathbb{D}_m$ with $m \geq 4$. This time $K_Z + f(D)$ has a 2-complement ([27], 5.2.3 and 5.12) because we have a resolution of Z minimal on $f(D)$. We may assume also that it is nonexceptional. Suppose, as above, that the singularities x, y of X on C correspond to the edge (-2)-curves of the graph of $p \in f(D)$. Then the complementary divisor cuts a curve γ on D_1 intersecting C only in z, the third singularity of X on C. The inverse image γ^1 of γ on the blow-up $X^1 \to X$ of z does not intersect C^1. We need to eliminate only the case when $l < k$. Hence C is a $(-n)$-curve with $n \geq 3$ (see Figure 9(a)) and C^1 is semiample on D_1^1. So, γ^1 generates R_2, and the last ray is of a flipping type. By virtue of the minimal property of the blow-up, γ^1, like γ, is an exceptional curve of the first kind on the minimal resolution of D_1. Using induction on depth, we can assume that we know the singularities on D_1^1 (see case 3 in the above proof and Corollary 4.6 below). Then we may check that γ^1 has at most one canonical singularity and $(K_{X^1} + D^1 + E_1 \cdot R_2) \leq 0$. So, we can do a flip-flop and derive a contradiction with the fact that the first blow-up of p extracts C on $f(D)$.

On the second step we must check that p is a singularity of type $1/r^2(a, r-a)$ on $f(D)$. Indeed, after the canonical covering $\widetilde{D} \to f(D)$, we get a cyclic canonical singularity $\tilde{p}$ given by an equation $xy + z^t = 0$ with an invariant action of $\mathbb{Z}_r$. Hence $r|t$, $\tilde{p}$ and p have types $1/t(1, -1)$ and $1/tr(a, t-a)$ on $\widetilde{D}$ and $f(D)$, respectively. The last singularity has a unique exceptional divisor with minimal log discrepancy $1/r$ only for $t = r$. Therefore, p has type $1/r^2(a, r-a)$ on $f(D)$ and $V_1(r, a; n)$ on Z. In particular, this proves that p has type (1) [22] and is moderate.

Proof of Theorem 4.1. By [13], Theorem, p must be a quotient or hyperquotient singularity of type $1/r(a, -a, 0, 1)$ with $\varphi = xy + f(z^r, w)$ and $\operatorname{ord} f = 1$ (see also the last remark). Then we can reduce it to the case with $f = z^r + w^n$. Since in this case we have only one exceptional divisor G with minimal log discrepancy, the blow-up (1.3.7) is Kawamata's weighted blow-up (cf. [13], Theorem 2.3). Indeed, Kawamata's blow-up is projective, semistable, and extremal (cf. Corollary 4.5). So, both extractions can be constructed as the log canonical model for an appropriate resolution and boundary $g^{-1}D + (r-1-\varepsilon)/rG + \sum E_i$ on it, where the E_i are exceptional divisors $\neq G$. ∎

4.5. Corollary on a generalized flower pot. *Let $g\colon Y \to X$ be the blow-up from* (1.3.7) *of a semistable singularity p of type $V_1(r, a; n)$. Then its exceptional divisor G is log terminal del Pezzo with $\rho(G) = 1$, and it has at most three singularities x, y, and z of types $V_2(a, -r)$, $V_2(r-a, -r)$, and $V_1(r, a; n-1)$. More exactly, the respective singularity exists when a, $r-a$, and $n \geq 2$. Furthermore,*

$$i(X, p, D) = (r-1)n$$

independently of D. The difficulty of this point is equal to $r(r-1)/2$ for $n \geq r$, and to $n(2r-n-2)/2$ otherwise.

The semistable resolution is possibly minimal with respect to the number of its exceptional divisors. Then it is economical in the sense of (1.2.3) if and only if $n = 1$ or $r = 1$, or equivalently, if and only if p is the quotient singularity or is Gorenstein semistable.

Proof. See [13], 3.4 and Remark 2.5. ∎

The resolutions of $V_2(a, -r)$ and $V_2(r-a, r)$ play the role of leaves. They are nontrivial for $r \geq 3$. A "flower pot" type of semistable degeneration of an Enriques surface ([19], p. 85) corresponds to $r = 2$ and $a = 1$.

4.6. Corollary on a garland of points. *Let $g\colon Y \to X$ be a $\mathbb{Q}$-factorialization of a semistable point $p \in D$ with $d = 1$ and with index $r \geq 2$. Then its exceptional curves form a chain $C_1, \ldots, C_\sigma$, where $\sigma = \sigma(X, p)$. There are $\sigma + 1$ singular points $p_1 \in C_1$, $p_2 = C_1 \cap C_2, \ldots, p_\sigma = C_{\sigma-1} \cap C_\sigma$, $p_{\sigma+1} \in C_\sigma$ on Y of similar type $V_1(r, a; n_i)$. Moreover, C_i intersects the edge curve of the minimal resolutions of p_i and p_{i+1}, and the types of these edges are opposite. Furthermore,*

$$i(X, p, D) = (r-1)\left(\sum n_i\right)$$

independently of D. The difficulty of this point is equal to the sum of those for the p_i (see Corollary 4.5).

We denote the type of such a singularity of X by $V_1(r, a; n_1, \ldots, n_{\sigma+1}) = V_1(r, a; \overline{n})$, where $\overline{n} = (n_1, \ldots, n_{\sigma+1})$. The numbers in parentheses are invariants of the singularity and are independent of the choice of D. It is a quotient singularity if and only if $\sigma = 0$ and $n_1 = 1$. The singularity p on D has the same analytic description as $1/r^2(a, r-a)$ in [6], 3.1, if we replace L by a wheel with $\sigma + 1$ (-2)-curves. It is analytically equivalent to the quotient singularity of type $1/(\sigma+1)r^2(a, (\sigma+1)r - a)$.

Proof. Since the singularities on Y/p are $\mathbb{Q}$-factorial, we have $\sigma = \sigma(X, p)$ exceptional curves C_i/p. According to the classification of the surface log terminal singularities, these curves form a tree. There exists at least one singularity $p_i \in Y/p$ on some C_i with index $2 \leq t|r$ because otherwise Y is nonsingular/p and p has index 1 by the contraction theorem (cf. (1.3.6)). But $K_{g^{-1}D} = g^*K_D$ is numerically trivial on C_i. Hence C_i is the exceptional curve of the first kind on the minimal resolution of $g^{-1}D$ and it has one more singularity p_j. By the classification of surface log terminal singularities, Y has exactly two singularities on C_i, and other curves C_k can intersect C_i only in them. The required properties of the configuration, for an appropriate renumbering of C_i and p_i, follow from the statement on types of singularities and the location of edge curves on the minimal resolution/p_i. It is enough to check this for $\sigma = 1$. We know that in this case $C = C_1$ has two singularities p_1 and p_2 of type $V_1(t_1, a_1; n_1)$ and $V_1(t_2, a_2; n_2)$, respectively, where $t_i|r$ and is the index of p_i. If t is odd and $t|t_1$, then using a covering of order t over a neighborhood of C and the same configuration properties of a covering curve as above, we can check that $t|t_2$. So, t_1 and t_2 have the same odd divisors and one of the t_i's equals r, say $t_1 = r$, whereas $t_2|r$. Therefore, again by a covering trick, $t_1 = t_2 = r$. Note that after the canonical covering of order t_2, p_2 converts into a canonical singularity (!).

Each singularity p_i has only one exceptional divisor with smallest discrepancy $-(r-1)/r$. By the monotonic property of the discrepancies for the minimal resolution, the curves between them on the minimal resolution of $g^{-1}D$ must be contracted to obtain the minimal resolution of p. Then [6], 3.1 implies the equality $a_1 = a_2 = a$ and the required properties of C passing through p_i. (See also [6], 3.2.) ■

4.7. Corollary on genericness. *The semistable singularities of type $V_1(r, a; n_1, \ldots, n_\sigma)$ are exactly the terminal hyperquotient singularities of type $1/r(a, -a, 1, 0)$ or type (1) in Mori's classification [22] with*

$$f(z^r, w) = \text{unit} \cdot \prod_{i=1}^{\sigma+1} f_i(z^r, w),$$

where the $f_i(z, w)$ are analytic functions having no common factor and with $\operatorname{ord} f_i(z, 0) = 1$ *and* $\operatorname{ord} f_i(0, w) = n_i \geq 1$. *So, semistable singularities form a nonempty open subset of the singularities of given type, or even of given type and given* $\operatorname{ord} f$.

We consider the topology on germs of analytic functions for which the natural maps to k-forms are continuous. The corollary works even for canonical singularities, i.e., when $r = 1$, if we replace the last conditions by an equivalent one: all $\operatorname{ord} f_i = 1$.

Proof. For a semistable singularity $V_1(r, a; n)$, the canoniccal singularity on the canonical cover of D has type $1/(\sigma+1)r^2(a, (\sigma+1)r - a)$. This implies that it is a hyperquotient singularity of type $1/r(a, -a, 1, 0)$ given by an analytic function $xy + f(z^r, w)$ with $\operatorname{ord} f(z, 0) = \sigma + 1$. On the other hand, by Reid-Mori-Shepherd-Barron-Ue ([10], 2.2.7), $f(z, w)$ can be written as a product of $\sigma + 1$ irreducible factors $f_i(z, w)$. They are nonassociated because the singularity is isolated. Moreover, $\operatorname{ord} f_i(z, 0) = 1$, and this gives the required factorization.

Conversely, if we have a hyperquotient singularity of type $1/r(a, -a, 1, 0)$ with f under consideration, then it is semistable for a divisor D that is the quotient of the hyperplane $w = 0$. Indeed, the factorization of f gives a $\mathbb{Q}$-factorialization of a semistable singularity such as in Corollary 4.6 (cf. the proof of [6], 1.3, and [10], 2.2.8). This reduces the proof to the case when $f = f_i$ with $\operatorname{ord} f(z, 0) = 1$ and $\operatorname{ord} f(0, w) = n_i$. The last singularity is analytically equivalent to $V_1(r, a; n_i)$.

Now suppose that $\operatorname{ord} f(z, w) = \sigma + 1$ and $f_{\sigma+1}$ is its homogeneous form of order $\sigma + 1$. Then we can modify the coefficients of f slightly, and only in $f_{\sigma+1}$ that $\operatorname{ord} f(z, 0) = \sigma + 1$ and

$$f_{\sigma+1}(z, w) = \text{const} \cdot \prod_{i=1}^{\sigma+1} (z - a_i w)$$

with distinct $a_i \in \mathbb{C}$. Thus, f possesses the required factorization. ■

4.8. Corollary. *Let p be a 3-fold terminal point of index r. Then for any integer l, $1 \le l \le r - 1$, there is an exceptional divisor/p with discrepancy l/r.*

Proof. First, this holds for the quotient singularities, in particular, for $V_1(r, a; 1)$. Second, Corollaries 4.5-6 imply this for $V_1(r, a; n)$ and any semistable singularity. Since semistable singularities (or even the quotient singularities $V_1(r, a; n)$) form a nonempty open subset of the singularities of given type, the corollary holds for the main types (1) in Mori's classification. Types (3), (5), and (6) ([22]) are included in Kawamata's Appendix ([27], Appendix: cases 2, 3, and 5). In the remaining cases (2) and (4), $r = 4$ and 3, respectively. Again according to Kawamata, we can assume that $2 \le l \le r - 1$. Moreover, in case (4) the generic singularity is the hyperquotient singularity of type $1/3(1, 2, 2, 0)$ given by the function $w^2 + x^3 + y^3 + z^3$, and we can find the required divisor using the weighted blow-up ([27], Appendix: Case 4). In case (2) the generic singularity has type $1/4(1, 3, 2, 1)$ with the function $x^2 + y^2 + z + w^2$, which is the quotient singularity $1/4(1, 3, 1)$. The last possesses the required exceptional divisor. ■

In the semistable case we can find a good elephant even when the contraction is not extremal and its fibers are not irreducible (cf. [11], 1.7).

4.9. Corollary. *Let $f\colon X \to Y/Z$ be a semistable contraction that is negative/V with respect to K and does not contract divisors of D/V. We assume also that V is a fiber of a projective morphism, for example, $V = \mathrm{pt}$. Then $K + D$ and K have a 1-complement over a neighborhood of V, canonical for K.*

Of course, such contractions must be generic in the main type ([11], C3). A complement here belongs to a linear system $|-K + f^*H|$, where H is a hyperplane section of Y over a neighborhood of V.

Proof. Thus, the statement is local and we can assume that $V = \mathrm{pt}$. For simplicity, we assume also that f is extremal. In general, we can glue a lower complement.

After an appropriate renumbering of D_i's, every connected exceptional locus C of f on D/V belongs to a component D_1. Moreover, if $C \subset D_2$, then $(D_1 \cdot C)$ and $(D_2 \cdot C) < 0$. So, $C = D_1 \cap D_2$, and D_3 intersects C. In this case C has a singular point of X, and the required complement passes through it. According to the minimal property of its resolution (see (1.3.7)), it is enough to construct a 1-complement on D_1 for $(K + D)|_{D_1}$.

If $C \not\subset D_i$ for $i \ne 1$ but C intersects D_2 in a point p, then p is singular on X, and all irreducible curves C_i of C have only p as a common point. A 1-complement can be constructed as above. Note that at most one curve C_i possesses a singularity of index ≥ 2 different from p. Such a curve belongs to the complement.

In the remaining cases C does not intersect D_i for $i \ne 1$. If D_1 has only canonical singularities on C, then C is irreducible with at most one canonical singularity, and we can find a 1-complement as above. (We can even take a trivial one in this case.) If D_1 has only one noncanonical singularity p on C, then all irreducible components of C have only p as a common point. This time we can choose the 1-complement that also intersects C only in p. In the remaining cases C contains a chain $C_1, \dots, C_n$, $n \geq 1$, of curves such that $p_1 \in C_1$, $p_2 = C_1 \cap C_2, \dots, p_n = C_{n-1} \cap C_n$, $p_{n+1} \in C_n$ are all noncanonical points of D_1 on C. Moreover, the C_i 's are exceptional curves of the first kind on the minimal resolution of D_1 and they intersect the edge curve of resolutions/p_i (cf. garlands in Corollary 4.6). The proof uses the classification of surface log terminal singularities, an explicit form of p_i 's, and the fact that the C_i are contracted to the log terminal singularities. Other components of C pass through only one of the p_i 's. Then we can construct a 1-complement passing only through C_i. ■

4.10. **Remarks.** We can characterize some types of 3-fold terminal singularities in terms of their discrepancies. Suppose that $p \in X$ is such a singularity of index r and of the main type (1) in Mori's classification. Then

(4.10.1) For $r \geq 2$, $\sigma(X, p) \leq \sharp$ {exceptional divisors/p with discrepancy $1/r\} - 1$, and = holds if and only if p is semistable for an appropriate D. In particular, p is analytic $\mathbb{Q}$-factorial if there is a unique exceptional divisor with discrepancy $1/r$.

For $r = 1$, the same holds if we drop -1 in the first statement and replace "a unique" by "no" (see (4.10.2)).

(4.10.2) p is the quotient singularity if and only if the number of exceptional divisors over p with discrepancies ≤ 1 (something like difficulty) is equal to $r - 1$ or, equivalently, there is a unique exceptional divisor/p with each discrepancy l/r, $1 \leq l \leq r-1$, but there is none with discrepancy 1, or, equivalently, there is no exceptional divisor/p with discrepancy 1. So, we have a gap for such discrepancy. In a proof, for $r = 1$, we would have to use an unpublished result of Markushevich. This shows also that 3-fold terminal quotient singularities are rigid.

(4.10.3) So, if p is not a quotient singularity, then there is an exceptional divisor/p with discrepancy l/r for any integer $l \geq 1$ (cf. Corollary 4.8).

Do these assertions hold for any type of terminal singularities?

An explicit form of semistable singularities allows us to improve some of the above statements. For example,

(4.10.4) In Theorem 1.7 equality holds only when X is Gorenstein (and even nonsingular) on E/V (if X is $\mathbb{Q}$-factorial/V, respectively) (cf. Theorem 1.6). Indeed, as in §2 it is enough to consider the cases 1.1-2 under restrictions (2.1.1-4). Then the contraction of C on D again gives a singularity of type $1/r^2(a, r-a)$, which is impossible by a direct check using [6], 3.1 in case 1.2. In case 1.1 equality can hold only when $g^{-1}C$ intersects G in a nonsingular point. Thus, C intersects a non-2-curve on the minimal resolution of the singularity. Again this is impossible by the classification [6], 3.1.

In the conclusion of this section we consider the algebraic case. If $f\colon X \to Z$ and $D \subset X$ are algebraic (in particular, X and Z are algebraic), then, for projective f, the extremal contraction $g\colon X \to Y$ defined by the ray $R \subset \overline{\mathrm{NE}}(X/Z; f(D))$ is also algebraic. The modification X^+/Z in R will again be algebraic and projective/Z. In the algebraic case the extremal rays $R \subset \overline{\mathrm{NE}}(X/Z; f(D))$ correspond to $R \subset \overline{\mathrm{NE}}(X/Z)$ with support intersecting D, and their modifications correspond to those of X/Z in a Zariski neighborhood of D. So, we can always take $W = D$ and $V = f(D)$, and define

$$i(X, D) = \sum_{p \in D} i(X, p, D)$$

and $i(X/Z, D) = i(X, D)$. (D is compact in Zariski's topology!)

But if p is algebraic, we can construct its $\mathbb{Q}$-factorialization in the algebraic sense. So, we reduce the problem of classification of such points to the case when p is $\mathbb{Q}$-factorial in the algebraic sense. We can introduce the notion of a semistable singularity in the algebraic category and introduce the *algebraic depth* $i^{\mathrm{alg}}(X, p, D)$ at p for D as the minimal number of the same prime divisors for algebraic semistable resolutions. Obviously, any algebraic semistable singularity is analytic and

$$i(X, p, D) \leq i^{\mathrm{alg}}(X, p, D).$$

The converse also holds.

4.11. Comparison Theorem. *Let $p \in D \subset X$ be a point on a complex algebraic variety with a prime divisor D. Then p is semistable with respect to D in the algebraic sense if and only if it is semistable in the analytic sense. Moreover, if p has index $r \geq 2$, then the first blow-up gives an exceptional curve with minimal discrepancy $-(r-1)/r$ on D. If p has index 1, we can construct such blow-ups with the first blow-up of any exceptional curve$/p$ for D with discrepancy 0 in case $\mathbb{A}_\sigma$ and the central component in other cases.*

Proof. It is enough to check that we can resolve p by subsequent divisorial blow-ups of points that are semistable but not necessarily extremal. However, they are projective in the following sense: each of them blows up a prime divisor G and is negative with respect to $-G$. Note that by semistability this holds when the inverse image of D cuts an ample curve on G. In addition, we will cancel our assumption that p is semistable in the algebraic sense. By (1.3.1-4), we may assume that $d = 1$. Then it is enough to construct a semistable blow-up $g\colon Y \to X$, minimal in the sense of (1.3.7), i.e., with an irreducible intersecting curve $C = G \cap g^{-1}D$ that is not an exceptional curve of the first kind on the minimal resolution of D. Indeed, then C is semiample on G by [12], 2.1, and we can contract the curves numerically trivial for C. This gives the required first blow-up and leads us from the analytic category back to the algebraic one. The rest can be done by induction on the minimal number of such blow-ups for a resolution of p. By σ, as above, we denote the analytic $\sigma(X, p)$.

We begin with the case when p is Gorenstein. The required resolution is related to Reid's pagoda [21] when $\sigma = 1$. Moreover, on each step the new semistable singularities have $\sigma = 1$. So, we can proceed by induction on σ. Then we consider a small semistable partial resolution $X^1 \to X$ with irreducible exceptional curve C/p and a single Gorenstein singularity q of X^1 on C. Moreover, we assume by induction that a first divisorial blow-up $X^2 \to X^1$ of q blows up similarly the exceptional curve $\varepsilon = D^2 \cap E$, where E is the exceptional divisor of this blow-up, and C^2 intersects ε in a nonsingular point. We find it for canonical singularities p of type $\mathbb{A}_\sigma$, because we can interchange the order of such blow-ups. For other types we must blow up the single curve intersecting the central one.

Now we can make a blow-up $X^3 \to X^2$ of the curve C^2 with an exceptional divisor F. Then $F = \mathbb{F}_1$ with section $C^3 = D^3 \cap F$ (a 1-curve) and fiber $E^3 \cap F$. So, the negative section s_1 coincides with the support of an extremal ray in $\overline{\mathrm{NE}}(X^3/Z; p)$. Since $(K_{X^3} + D^3 + E^3 + F \cdot s_1) = 0$, we can do Atiyah's flop in s_1, after which F is contractible to a point of type $V_2(2, 1)$. Then we must contract the irreducible curves that are (-2)-curves on the minimal resolution of the modified E^3 numerically trivial for D^3 and have t canonical singularities of types $\mathbb{A}_{m_j}$, $1 \leq j \leq t$, with $m = \sum m_j \leq \sigma - 1$. The last holds by the induction assumption. The curves are contracted to canonical singularities of types $\mathbb{A}_*$, $\mathbb{D}_{**}$ or $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ with $*, **$ or $6, 7, 8 \leq \sigma$, respectively. Moreover, if one of them has the same type as p on D, then a unique curve is contracted, $t = 1$, $m_1 = \sigma - 1$, and by induction we can construct a required resolution. In exceptional cases a proof uses the corresponding nontrivial complement of $K + D$ in p. Otherwise we get the following decrease of types:

$$\mathbb{E}_8 \to \mathbb{E}_7 \to \mathbb{E}_6 \to \mathbb{D}_{**} \to \mathbb{A}_{*}.$$

(It means also the decrease of the subscripts.) Again we can construct a required resolution by induction.

In particular, if p has type $\mathbb{A}_\sigma$ on D, then we obtain only the first type $\mathbb{A}_*$. In this case, using the modifications from case 2 in 3.9 we can find the first blow-up for any exceptional curve of D/p with discrepancy 0.

If $r \geq 2$, by Corollary 4.7 we can use Kawamata's blow-ups ([27], Appendix: Case 1). We have exactly $\sigma + 1$ such blow-ups corresponding to each exceptional curve of D/p with the minimal discrepancy $-(r-1)/r$. A direct check shows that $-G$ is positive for the blow-ups, and we have at most one singularity with $d = 1$ on G. It has type $V_1(r, a; n_1 - 1, \dots, n_{\sigma+1} - 1)$ (cf. Corollary 4.12). ∎

The last blow-up is extremal in the algebraic sense when p is $\mathbb{Q}$-factorial in this sense. This implies

4.12. Corollary. *Theorem* 1.3 *and Corollary* 4.6 *hold in the algebraic case if we understand them in the algebraic sense. For $r = 1$,* (1.3.7) *holds with $a = 0$, and G does not contribute to the difficulty of X/p. For $r \geq 2$, divisorial blow-ups in* (1.3.7) *coincide with weighted blow-ups, and the discrepancy of G will be $1/r$. In addition, if $r \geq 2$, Y has at most three semistable singularities on G of types $V_2(a + ir, -r)$, $V_2((\sigma+1-i)r - a, -r)$, and $V_1(r, a; n_1 - 1, \dots, n_{\sigma+1} - 1)$, whereas $0 \leq i \leq \sigma$, p has type $V_1(r, a; n_1, \dots, n_{\sigma+1})$, and we skip $n_j - 1$ for $n_j = 1$.*

Note that we have $\sigma + 1$ Kawamata blow-ups depending on i ([27], Appendix: Case 1).

4.13. Corollary. *If p is an algebraic semistable and $\mathbb{Q}$-factorial singularity with $d = 1$ and of type $V_1(r, a; n_1, \dots, n_{\sigma+1})$, then*

$$i^{\mathrm{alg}}(X, p, D) = i(X/p, D) - N + \sum n_i = r(\sum n_i) - N,$$

where $N = \max\{n_i\}$.

4.14. Corollary on genericness. *Algebraic semistable singularities of index $r \geq 2$ are exactly algebraic terminal hyperquotient singularities of type $1/r(a, -a, 1, 0)$ or type* (1) *in Mori's classification* [22] *with*

$$f(z^r, w) = \mathrm{unit} \cdot \prod_{i=1}^{s+1} f_i(z^r, w),$$

where s is the algebraic σ, and $f_i(z, w)$ are nonassociated irreducible polynomial functions, in a neighborhood of the origin, satisfying 4.7. *Thus, semistable singularities and $\mathbb{Q}$-factorial ones among them form open nonempty subsets in their type, and both subsets are dense in the analytic type.*

4.15. Corollary. *Theorems* 1.6 *and* 1.7 *hold in the algebraic case except for the statement that $g(D_1) \in g(D)/V$ is of index > 1 in* 1.6. *Moreover, we can replace $i(-,-)$ by $i^{\mathrm{alg}}(-,-)$.*

The most difficult to prove is the last statement, especially when g is a small contraction. For this, we can use

4.16. Corollary. *Algebraic version of Corollary* 4.9.

Do the above (and below, in §5) statements hold in positive characteristic? The analytic category can then be replaced by Moishezon's category of varieties and morphisms. In particular, we can define in it such notions as a semistable divisor, singularity, resolution, etc. The point is that the resolution in (1.3.7) is projective and algebraic for algebraic X, but the $\mathbb{Q}$-factorialization required in (1.3.5) may be only Moishezon.

5. Semistable models

5.1. Theorem on a minimal semistable model. *Let f be projective and (numerically) semistable for D. Then over a neighborhood of V (of $f(D)$ in the algebraic case) there exists a modification (algebraic in the algebraic case) of f that is a nontrivial fiber space of Fano/Z or a projective and minimal (numerically) semistable model $g: Y \to Z$ for the modification of D. More exactly, we have the second model if and only if $\kappa(X/Z) \geq 0$, and its singularities in both cases are semistable with respect to the modification of D.*

Does this hold if we replace the projective property by the proper one?

Proof. This follows directly from Mori's theory and Theorems 1.6-7. The last statements are derived from the Abundance Conjecture, well known for $\dim(X/Z) \leq 2$. ∎

This theorem implies several important results. Using the semistable reduction theorem ([8]) and (1.3.6) we get Brieskorn-Tyurina's simultaneous resolutions ([2], [24], and [21]). By the same arguments, Lemma 3.5, and the covering trick ([27], 2.5), we obtain also [5], 4.1. Note that Kawamata [5] does things in the reverse order: he uses the simultaneous resolution to prove [5], 4.1, and then [5], 10.1 and 10.1′, which are special cases of Theorem 5.1. Theorem 5.1 implies also the existence of some flips ([27], 2.6). The same special case is Tsunoda's theorem ([23], Theorem 1) with S-degeneration having only algebraic $\mathbb{Q}$-factorial singularities and S-resolutions given by blow-ups as in (1.3.7).

A minimal semistable model in Theorem 5.1 possesses an essential property of Mori's type models: it is projective/Z, but may have rather difficult analytic singularities. By this I mean that they may not be $\mathbb{Q}$-factorial in the analytic sense. Even if we start from X with $\mathbb{Q}$-factorial singularities, we may obtain $\mathbb{Q}$-factorial Y/Z, but not locally in the analytic sense. By (1.3.1) and (1.3.4), the latter may occur only for singularities with $d = 1$. But using (1.3.5) we can replace Y/Z by its analytic $\mathbb{Q}$-factorialization, whereas we will lose the projectivity of Y/Z. The last model has only analytic $\mathbb{Q}$-factorial semistable singularities and will be referred to as Kulikov's model. By (1.3.6), it has only non-Gorenstein singularities. In particular, when such a model Y is Gorenstein, i.e., with Gorenstein canonical divisor K_Y, it will be nonsingular, as will be the irreducible components of the modification of D, and D will have normal crossings.

5.2. Corollary on Kulikov's models. *In Theorem 5.1 we can replace a projective minimal model $g: Y \to Z$ by a proper semistable model for D with singularities only of types $V_2(r, a)$ and $V_1(r, a; n)$ of index $r \geq 2$. We have only singularities $V_1(r, a; n)$ if the original D has no triple points.*

The last can be checked for each modification in Theorems 1.6-7 and is essentially related to the connectedness lemma ([27], 5.7).

This implies Kawamata's moderate degenerations ([6], 1.3), again by [8]. Another application is related to the original Kulikov theorem ([12], Theorem I). We consider a little more general case including semistable degenerations of Enriques surfaces. Let $f: X \to \Delta$ be a projective degeneration of surfaces with numerically trivial canonical divisors (e.g., K3 surfaces) whose degenerate scheme fiber $\sum dD_i$ is d-multiple of a divisor with normal crossings and nonsingular D_i. Then by the classification of surfaces, mK is linearly equivalent to $X_0 = \sum d_i D_i$; more exactly, $m = 1$, except for degenerations of Enriques surfaces where $m = 2$, and degenerations of hyperelliptic surfaces where $m = 2, 3, 4$, or 6 (a proper divisor of 12). We denote by I the product md, which can be referred to as the index of this degeneration. Note that the index is invariant under extremal modifications and $\mathbb{Q}$-factorializations, as are m and d. Note also that f is numerically semistable for $D = \sum D_i$. Thus, Corollary 5.2 implies

5.3. Corollary. *There exists a bimeromorphic modification $g: Y \to \Delta$ of the degeneration f such that K_Y is numerically trivial on Y (IK_Y is linearly trivial) and (only numerically for $d \geq 2$) semistable for modified D, having only semi-stable singularities of types $V_2(r, a)$ and $V_1(r, a; n)$ with indices $r|m$. We have only singularities $V_1(r, a; n)$ if D has no triple points.*

Proof. We take a Kulikov model of f as a required modification g. By our assumption, K_Y is numerically equivalent to $\sum d_i D_i$ with integer d_i, where the D_i are components of the modified degenerate fiber $Y_0 = \sum dD_i$. Moreover, since $D = \sum D_i$ is numerically trivial, we can assume that all $d_i \leq 0$ and at least one $d_i = 0$. Then all $d_i = 0$, and K_Y is numerically trivial because K_Y is nef with respect to g. Hence mK_Y and IK_Y are linearly equivalent to a multiple of D, and 0, respectively. So, by Lemma 1.4, mK_Y is Cartier, and the indices of the singularities of Y divide m.

For $m = d = I = 1$ (e.g., semistable degeneration of K3 surfaces), this includes Kulikov's theorem ([12], Theorem I) when Gorenstein singularities are nonsingular (cf. [20]). For $m = I = 2$ and $d = 1$, we have degenerations of Enriques or hyperelliptic surfaces, and then a Kulikov model has only singularities of type $V_2(2, 1)$ and $V_1(2, 1; n)$ (see (1.2.4)). Moreover, if the initial degeneration has no triple points, we get only singularities $V_1(2, 1; n)$ with a "flower pot" resolution, which gives Persson's result ([19], 3.3.1). But the same singularities appear even for any $d \geq 1$.

Acknowledgment

Partial financial support was provided by the Institute for Advanced Study in Princeton, NJ, and by the NSF grant DMS-9200933.

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Original published in English

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