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ABSTRACT. In this paper the author determines when the principally polarized Prymian $P(\tilde{C}, I)$ of a Beauville pair $(\tilde{C}, I)$ satisfying a certain stability type condition is isomorphic to the Jacobian of a nonsingular curve. As an application, he points out new components in the Andreotti-Mayer variety N_{g-4} of principally polarized abelian varieties of dimension g whose theta-divisors have singular locus of dimension $\geqslant g-4$; he also proves a rationality criterion for conic bundles over a minimal rational surface in terms of the intermediate Jacobian. The first part of the paper contains the necessary preliminary material introducing the reader to the modern theory of Prym varieties.

Figures: 10. Bibliography: 32 titles.

Statement of the main theorem. Let $(\tilde{C}, I)$ be a pair consisting of a connected curve $\tilde{C}$ with only ordinary double points as singularities over an algebraically closed ground field k of characteristic $\neq 2$ and an involution $I: \tilde{C} \to \tilde{C}$ satisfying the following condition:

(B) *The set of fixed points of the involution I coincides with the set of singular points of the curve $\tilde{C}$, and the involution I preserves the branches at all the singular points.*

Such pairs $(\tilde{C}, I)$ will be called *Beauville pairs*. Generalizing a result from [17], in [3] Beauville associated to such a pair a principally polarized abelian variety $P(\tilde{C}, I)$, the so-called *Prymian* of the pair $(\tilde{C}, I)$. Details of this construction will be discussed in §3. In addition, we assume that the curve $\tilde{C}$ satisfies the following stability type condition:

(S) *For each decomposition* $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, $\#\tilde{C}_1 \cap \tilde{C}_2 \geqslant 4$.

MAIN THEOREM. *The variety $P(\tilde{C}, I)$ is isomorphic, as principally polarized abelian variety, to a sum of Jacobians of nonsingular curves if and only if the quotient curve $C = \tilde{C}/I$ is of one of the following types*:

(a) *C is a hyperelliptic curve*;

(b) *C is a trigonal curve*;

(c) *C is a quasi-trigonal curve*;

(d) *C is a plane quintic, and the pair $(\tilde{C}, I)$ is odd.*

A curve C is called *hyperelliptic* if there exists a finite morphism $C \to \mathbf{P}^1$ of degree 2, *trigonal* if there exists a finite morphism $C \to \mathbf{P}^1$ of degree 3, and *quasi-trigonal* if it is obtained from a hyperelliptic curve by gluing two nonsingular points. This last name is

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explained by the fact that the intersection of quadrics passing through the canonical image of a quasi-trigonal curve in $\mathbf{P}^{g-1}$ is the cone over a nonsingular normal rational curve of degree $g-2$ in $\mathbf{P}^{g-2}$ with vertex at the image of the glued points (here $g = p_a(C)$). If $C \subset \mathbf{P}^2$ is a plane quintic, i.e. a curve of degree 5 in the projective plane $\mathbf{P}^2$, then we say that the pair $(\tilde{C}, I)$ is *even* (*odd*) if the number $h^0(\tilde{C}, \pi^* M)$ is even (odd), where $M = \mathcal{O}_{\mathbf{P}^2}(1)|_C$ and $\pi\colon \tilde{C} \to C = \tilde{C}/I$ is the natural projection. If $\tilde{C}$ is a smooth plane quintic, this means that the theta-characteristic $M \otimes \eta$ is odd (even), where η is the point of order two in the Jacobian $J(C)$ corresponding to the unramified covering $\pi\colon \tilde{C} \to C$.

The problem of reducing the general case to the case when condition (S) required in the proof of our main theorem is satisfied is considered in Corollary 3.16 and Remark 3.17(a).

The proof of sufficiency in the main theorem is obtained by passage to the limit from the results of Dalalyan [9] and Mumford [17] in case (a), Recillas [30] in case (b) (a geometric interpretation of this case can be found in [27]), Dalalyan [10] in case (c), and Tyurin [23] and Masiewicki [19] in case (d). The necessity is proved in §§7–9. In §10 we give some direct applications of the main theorem, and §§1–6 contain the necessary preliminary material. The main theorem for $p_a(C) \geqslant 8$ was announced in [26], and a sketch of proof for that case was given in [27]. The problem of distinguishing Prymians from Jacobians for smooth hyperelliptic curves of genus 6 was considered in Dalalyan's thesis.

The ground field k is algebraically closed, and in Theorem 1.6 and §§3, 5 7–10 we assume that char $k \neq 2$.

§1. Orthogonal sheaves

1.1. Let $X = \bigcup_1^c X_i$ be the decomposition of a curve X into irreducible components. For each locally free sheaf L on X we denote by $\deg L$ the vector $(d_1, \ldots, d_c)$, where $d_i = \deg L|_{X_i}$ and deg denotes the ordinary degree. The vector $\deg L$ is called the *multidegree* of the sheaf L. It is clear that $\deg L = \sum_1^c d_i$.

1.2. Let X be a variety. We denote by $\mathcal{R}_X$ the sheaf of rings of rational functions on X, i.e. the sheaf whose ring of sections $\mathcal{R}_X(U)$ on a Zariski open set U is the ring of rational functions on the variety U, i.e. a product of the fields of rational functions on the components of this variety. It is clear that on $\mathcal{R}_X$ there is a natural structure of $\mathcal{O}_X$-algebra. A sheaf L of $\mathcal{O}_X$-modules is called *torsion-free* if the natural map

$$L \overset{\mathrm{id} \otimes 1}{\to} L \otimes_{\mathcal{O}_X} \mathcal{R}_X$$

is an inclusion.

1.3. We recall that X is called a *Gorenstein variety* if its dualizing sheaf ω_X is invertible.

1.4. LEMMA. *Let $f\colon N \to X$ be the desingularization of a Gorenstein curve X, and let L_0 and L_1 be invertible sheaves on X and N respectively such that $2 \deg L_0 = \deg \omega_X$ and $2 \deg L_1 = \deg \omega_N$. Then there exist a nonsingular irreducible variety S and a coherent sheaf $\mathcal{L}$ of $\mathcal{O}_{X \times S}$-modules which is flat over S and has the following properties*:

(1.4.1) *For each point $s \in S$, $\mathcal{L}_s = \mathcal{L}|_{X \times \{s\}}$ is a sheaf without torsion having rank* 1 *at all generic points.*

(1.4.2) *$L_0 \approx \mathcal{L}_{s_0}$ for some point $s_0 \in S$.*

(1.4.3) *$f_* L_1 \approx \mathcal{L}_{s_1}$ for some point $s_1 \in S$.*

1.5. *A well-known construction.* Let $f\colon N \to X$ be the desingularization of a curve X. The exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_N \to \delta \to 0$$

defines a skyscraper sheaf δ on X. Let L be an arbitrary locally free sheaf of rank r on X; we assume that the curve X is connected. Fixing an isomorphism $L|_U \approx (\mathcal{O}_X|_U)^r$ on a neighborhood U of the singular subset, we obtain the following exact sequence:

$$0 \to L \to f_*f^*L \to \delta^r \to 0.$$

Thus each locally free sheaf L of rank r on X is the kernel of an epimorphism h: $f_*L_1 \to \delta^r$, where $L_1 = f^*L$ is a locally free sheaf on N (the descent data).

Consider a nonsingular irreducible variety T and a locally free sheaf $\mathcal{M}$ of rank r on $N \times T$.

Let p and q be the natural projections in the following diagram:

$$\begin{array}{c} N \times T \\ \downarrow f_T \\ X \xleftarrow{p} X \times T \xrightarrow{q} T \end{array}$$

Then the sheaf $H = q_* \operatorname{Hom}_{\mathcal{O}_{X\times T}}(f_{T_*}\mathcal{M}, p^*\delta^r)$ on T is locally free. In fact, since δ is zero outside the singular points, over a neighborhood of a point $t \in T$ the sheaf $\mathcal{M}$ may be replaced by $\mathcal{O}^r_{N\times T}$. Hence H is locally isomorphic to $q_* p^*G$, where $G = \operatorname{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}^r_N, \delta^r)$. But the sheaf $q_* p^*G \approx \mathcal{O}_T \otimes_k H^0(X, G)$ is locally free.

Consider the vector bundle S_1 associated to H. We denote by κ the natural projection $X \times S_1 \to X \times T$. On $X \times S_1$ there is a canonical homomorphism of coherent sheaves

$$h\colon\ \kappa^*f_{T_*}\mathcal{M} \to \kappa^*p^*\delta^r$$

such that the points $s \in S_1$ are in a one-to-one correspondence with the pairs consisting of a point $t = q \circ \kappa(s) \in T$ and a homomorphism $h|_{X\times\{s\}}\colon f_*(\mathcal{M}|_{N\times\{t\}}) \to \delta^r$. Now we can take S to be the open subset in S_1 consisting of those points $s \in S_1$ at which $h|_{X\times\{s\}}$ is surjective and has a locally free kernel. It is clear that S is a nonsingular irreducible variety and the sheaf $\mathcal{L} = \ker h|_{X\times S}$ is locally free (cf. [15], Chapter II, §5) and has the required properties.

PROOF OF LEMMA 1.4. In the preceding construction, let T be the variety parametrizing the invertible sheaves on N of multidegree $\deg L_0$ (modulo isomorphism), and let $\mathcal{M}$ be the corresponding Poincaré sheaf on $N \times T$, i.e. the sheaf $\mathcal{M}$ for which $\mathcal{M}|_{N\times\{M\}} \approx M$ for each $M \in T$. This time we denote by S the open subset in S_1 consisting of those points $s \in S_1$ for which $h|_{X\times\{s\}}$ is surjective. Then S is a nonsingular irreducible variety and its points parametrize the pairs (L, h_s), where L is an invertible sheaf on N of multidegree $\deg L_0$ and $h_s\colon f_*L \to \delta$ is an epimorphism. We claim that the sheaf $\mathcal{L} = \ker(h|_S)$ has the required properties. In fact, a local verification shows that the sheaves $\kappa^*p^*\delta$ and $\kappa^*f_{T_*}\mathcal{M}$ are flat over S_1 and therefore also over S. Hence from the surjectivity of $h|_S$ it follows that $\mathcal{L}$ is flat over S (cf. [7], §0.6). Condition (1.4.1) holds since $\mathcal{L}_s$ is a subsheaf of the torsion-free sheaf f_*L. The argument presented in the beginning of §1.5 shows that there exists a point $s_0 \in S$ satisfying (1.4.2).

Thus it remains to verify (1.4.3). In analogy with the conductor of the integral closure of a domain, by the *desingularization conductor* of a local ring $\mathcal{O}_{X,x}$ we mean the largest ideal c_x in $\mathcal{O}_{X,x}$ which is also an ideal in $f_*(\mathcal{O}_N)_x$. This ideal is given by an effective divisor $\Sigma_P\, n_{P,x}P$ on N; namely, c_x is identified with the set of functions g from the semilocal ring $f_*(\mathcal{O}_N)_x = \bigcap_{f(P)=x} \mathcal{O}_{N,P}$ for which $g \equiv 0 \bmod \Sigma\, n_{P,x}P$; here $n_{P,x} > 0$ only for $P \in f^{-1}(x)$.

In what follows we write n_P instead of $n_{P,f(P)}$. Since $f^*(\omega_X) = \omega_N(\Sigma_{P\in N} n_P P)$ (cf. [21], Chapter 4), we have

$$d_i = \deg(\omega_X|_{X_i}) = \deg(\omega_{N_i}) + \sum_{P\in N_i} n_P,$$

where X_i denotes an irreducible component of the curve X and N_i denotes its normalization. In view of the Gorenstein property,

$$2\delta_2 \overset{\mathrm{df}}{=} 2\dim_k f_*(\mathcal{O}_N)_x/\mathcal{O}_{X,x} = \sum_P n_{P,x}.$$

Next we consider an invertible sheaf M on N and an effective divisor $\Sigma_{P\in N} k_P P$ with $\Sigma_{f(P)=x} k_P = \delta_x$ for all points $x \in X$. Then, trivializing M on a sufficiently small neighborhood over the singular points of the curve X, we obtain an exact sequence

$$0 \to f_*\big(M\big(-\textstyle\sum k_P P\big)\big) \to f_*(M) \to \delta \to 0$$

on X. Hence there exists an epimorphism h: $f_*(M) \to \delta$ with $\ker h \approx f_*(M(-\Sigma k_P P))$. Thus to verify (1.4.3) *it suffices to find in the family* T *a sheaf* M *of the form* $L_1(\Sigma k_P P)$, i.e. to find a sheaf M of this form whose multidegree is equal to deg L_0. That means that the following equalities hold:

$$\frac{1}{2}\deg(\omega_X|_{X_i}) = \deg(M|_{N_i}) = \deg(L_1|_{N_i}) + \sum_{P\in N_i} k_P = \frac{1}{2}\deg(\omega_{N_i}) + \sum_{P\in N_i} k_P.$$

Together with the preceding equalities, this yields the following system of equations with respect to the nonnegative integers k_P:

$$\sum_{P\in N_i} k_P = \Big(\sum_{P\in N_i} n_P\Big)\Big/2 \qquad (i = 1,\dots,c),$$

$$\sum_{f(P)=x} k_P = \delta_x = \Big(\sum_{f(P)=x} n_P\Big)\Big/2 \qquad (x \in X).$$

We shall seek a solution of this system in the form $k_P = [n_P/2] + \varepsilon_P$ (as usual, [ ] denotes the integer part), where $\varepsilon_P = 0$ if n_P is even and $\varepsilon_P = 1$ or 0 if n_P is odd. Thenit remains to solve the following system:

$$\sum_{\substack{P\in N_i\\ n_P\equiv 1 \bmod 2}} \varepsilon_P = \Big(\sum_{P\in N_i} n_P\Big)\Big/2 - \sum_{P\in N_i}[n_P/2],$$

$$\sum_{\substack{f(P)=x\\ n_P\equiv 1 \bmod 2}} \varepsilon_P = \Big(\sum_{f(P)=x} n_P\Big)\Big/2 - \sum_{f(P)=x}[n_P/2].$$

For each singular point $x \in X$ we partition the set of points $P \in f^{-1}(x) \subset N$ for which n_P is odd into ordered pairs (P, P') such that $\varepsilon_P = 0$ and $\varepsilon_{P'} = 1$. It is clear that thus we obtain all solutions of the second part of our system. Now we construct an oriented graph Γ whose vertices correspond to the irreducible components $N_1,\dots,N_c$ and whose edges correspond to the pairs (P, P'); more precisely, to a pair (P, P') we associate an edge joining the component N_i of the point P with the component N_j of the point P' in the direction from N_i to N_j. We observe that in our case the sum $\Sigma_{P\in N_i} n_P$ is even since each component of the multidegree deg ω_X is even. Therefore the number of edges passing through (coming in and going out) an arbitrary vertex of this graph is even. It is clear that the equations in the second part of our system for the corresponding ε_P still hold if we change orientation of the edges of the graph Γ. On the other hand, by Euler's theorem (see, for example, [20], II.3) there exists an orientation on Γ for which the number of incoming edges is equal to the number of outgoing ones. For such orientation the ε_P also satisfy the first part of our system:

$$\sum_{\substack{P\in N_i \\ n_P\equiv 1 \bmod 2}} \varepsilon_P = \frac{1}{2}\left(\#\{P\in N_i \mid n_P\equiv 1 \bmod 2\}\right)$$

$$= \Big(\sum_{P\in N_i} n_P\Big)\Big/2 - \sum_{P\in N_i}[n_P/2]. \quad \blacksquare$$

1.6. THEOREM. *Suppose that the following conditions hold:*

(1.6.1) $\pi\colon \mathscr{X}\to S$ *is a proper flat family of curves.*

(1.6.2) $\mathscr{L}$ *is a coherent sheaf of* $\mathscr{O}_X$*-modules which is flat over* S, *and the sheaves* $L_s = \mathscr{L}|_{\pi^{-1}(s)}$ *on the curves* $X_s=\pi^{-1}(s)$ *are torsion-free sheaves whose rank at a generic point of* X_s *is locally constant on* $\mathscr{X}$.

(1.6.3) $Q\colon \mathscr{L}\to \omega_{\mathscr{X}/S}$ *is a nonsingular quadratic form.*

Then the function $s\mapsto h^0(X_s, L_s) \bmod 2$ *is constant on each connected component of the base* S.

1.7. REMARKS. (a) In the statement of the theorem we denote by $\omega_{\mathscr{X}/S}$ the relative dualizing sheaf $\pi^!\mathscr{O}_S$ [25], i.e. the sheaf whose restriction to each curve X_s of our family coincides with the dualizing sheaf ω_{X_s} on X_s.

(b) We say that the above quadratic form Q is *nonsingular* if the corresponding bilinear form induces an isomorphism $L_s \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_{X_s}}(L_s, \omega_{X_s})$ for each $s\in S$.

In the proof of this theorem we follow Mumford [16]. His main idea is to interpret the space $H^0(X_s, L_s)$ for each point s in some neighborhood of a point s_0 from S as the intersection of two maximal isotropic subspaces $W_{1,s}$ and $W_{2,s}$ in some larger even-dimensional vector space V_s with a nonsingular quadratic form q_s. The index s indicates that all these objects $W_{1,s}$, $W_{2,s}$, V_s and q_s vary along with s. Then from the known fact that $\dim(W_{1,s}\cap W_{2,s}) \bmod 2$ is invariant with respect to continuous deformations (cf. [5], IX.6, Exercise 18.d) it follows that the function $h^0(X_s, L_s) \bmod 2$ is locally constant, hence constant on each connected component of the base S.

1.8. We begin by interpreting the space $H^0(X_s, L_s)$. So, let X be a curve, and let L be a torsion-free sheaf on X. Consider the divisor $D=\sum_1^n P_i$ on X defined by distinct nonsingular points P_i of X. We have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L(-D) & \to & L & \to & L/L(-D) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & L(-D) & \to & L(D) & \to & L(D)/L(-D) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L(D)/L & = & L(D)/L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Suppose that n is so large that $h^0(X, L(-D)) = 0$. Then the above diagram gives rise to a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & H^0(X,L) & \to & H^0(X,L/L(-D)) & \to & H^1(X,L(-D)) \\ & & \downarrow & & \downarrow\beta & & \| \\ 0 & \to & H^0(X,L(D)) & \xrightarrow{\alpha} & H^0(X,L(D)/L(-D)) & \to & H^1(X,L(-D)) \\ & & \downarrow & & \downarrow & & \\ & & H^0(X,L(D)/L) & = & H^0(X,L(D)/L) & & \end{array}$$

with exact rows and columns. Identifying the space $H^0(X, L)$ with its image in $H^0(X, L(D)/L(-D))$ we obtain a representation $H^0(X, L) = W_1 \cap W_2$, where $W_1 = \operatorname{im}\alpha$ and $W_2 = \operatorname{im}\beta$.

Suppose now that $L \approx \operatorname{Hom}_{\mathcal{O}_X}(L, \omega_X)$. We remark that in view of 1.7(b) this holds for each L_s. They by the Serre-Grothendieck duality

$$h^1(X, L) = h^1\big(X, \mathbf{Hom}_{\mathcal{O}_X}(L, \omega_X)\big) = \dim \operatorname{Ext}^1_X(L, \omega_X) = h^0(X, L),$$

since $\mathbf{Ext}^1_X(L, \omega_X) = 0$ in view of the local duality (cf. the isomorphism on p. 213 in [25]). Hence $\chi(L) = 0$ and $\chi(L(D)) = \Sigma_1^n r_i$, where r_i is the rank at the point P_i. But

$$h^1(X, L(D))\text{-}h^1\big(X, \mathbf{Hom}_{\mathcal{O}_X}(L(-D), \omega_X)\big) = h^0(X, L(-D)) = 0$$

in view of the choice of D. Hence

$$\dim W_1 = h^0(X, L(D)) = \sum_{i=1}^n r_i.$$

Simple computations with the skyscraper sheaves $L/L(-D)$ and $L(D)/L(-D)$ show that $\dim W_2 = \Sigma_1^n r_i$ and $\dim V = 2\ \Sigma_1^n r_i$, where $V = H^0(X, L(D)/L(-D))$. Thus $\dim V = 2\dim W_1 = 2\dim W_2$.

Suppose in addition that $Q\colon L \to \omega_X$ is a nonsingular quadratic form on L. This form induces a canonical isomorphism $L \approx \operatorname{Hom}_{\mathcal{O}_X}(L, \omega_X)$. Furthermore, it defines a nonsingular quadratic form $Q_D\colon L(D) \to \omega_X(2D)$. Let $q\colon V \to k$ be the quadratic form on the space V such that

$$q(v) = \sum_{i=1}^n \operatorname{Res}_{P_i} Q_D(l),$$

where $l \in \Gamma(U, L(D))$ is the representative of the element $v \in V = H^0(X, L(D)/L(-D))$ on a neighborhood U of the set $\{p_i\}_1^n$. Easy local computations show that q is a well-defined nonsingular quadratic form and the subspace $W_2 \subset V$ is isotropic, i.e. $q(W_2) = 0$. The subspace W_1 is also isotropic with respect to q. In fact, for each element $w \in W_1$ there exists a global representative $l \in \Gamma(X, L(D))$, and so

$$q(w) = \sum_{i=1}^n \operatorname{Res}_{P_i} Q_D(l) = 0$$

by the sum of residues theorem [1].

Thus we have constructed a vector space V with nonsingular quadratic form q and two maximal isotropic subspaces W_1 and W_2 with $W_1 \cap W_2 = H^0(X, L)$. It is known that for each orthogonal (with respect to q) automorphism φ of the space V which transforms W_1 to W_2 (it is easy to verify that such automorphisms always exist) we have the following equality:

$$\det \varphi = (-1)^{(\dim V)/2 + h^0(X, L)}$$

(cf. the exercise from [5] indicated above).

To prove Theorem 1.6 we globalize the above construction. For each point $s_0 \in S$ there exist a neighborhood U and a relative Cartier divisor $D \subset \mathscr{X}$ whose restriction to each fiber over $s \in U$ satisfies the conditions indicated in 1.8. Over U one can globalize the above construction; namely, there exist locally free sheaves $\mathscr{W}_1 = \pi_* \mathscr{L}(D)$, $\mathscr{W}_2 = \pi_*(\mathscr{L}/\mathscr{L}(-D))$ and $\mathscr{V} = \pi_*(\mathscr{L}(D)/\mathscr{L}(-D))$ (cf. [16], Corollary 2) and canonical inclusions $\mathscr{W}_1 \overset{\alpha}{\hookrightarrow} \mathscr{V}$ and $\mathscr{W}_2 \overset{\beta}{\hookrightarrow} \mathscr{V}$ such that for each point $s \in U \subset S$ the space $H^0(X_s, L_s)$ is identified with $(\mathscr{W}_1 \cap \mathscr{W}_2)_{\mathscr{O}_S} \otimes k(s)$. Using residues, it is easy to construct a nonsingular quadratic form $\mathfrak{q}\colon \mathscr{V} \to \mathscr{O}_S$. It is easy to verify that for each point $s \in U$ the quadruple $\mathscr{W}_1$, $\mathscr{W}_2$, $\mathscr{V}$, $\mathfrak{q}$ restricts to the quadruple $W_{1,s}$, $W_{2,s}$, V_s, q_s constructed in 1.8 for $X = X_s$, $L = L_s$ and $D = \mathscr{D}_{X_s}$.

We recall that the ground field has characteristic $\neq 2$. Globalization of standard constructions from linear algebra allows to construct (possibly, after replacing U by a smaller neighborhood of the point s_0) decompositions $\mathscr{V} \approx \mathscr{W}_1 \oplus \mathscr{W}_1^*$ and $\mathscr{V} \approx \mathscr{W}_2 \oplus \mathscr{W}_2^*$, where $\mathscr{W}_1^*$ and $\mathscr{W}_2^*$ are isotropic subsheaves and the pairing corresponding to $\mathfrak{q}$ is induced by the duality $*$. It is clear that locally there exists an automorphism $\Phi\colon \mathscr{V} \to \mathscr{V}$ which is orthogonal with respect to $\mathfrak{q}$ and transforms $\mathscr{W}_1$ to $\mathscr{W}_2$. Replacing, if necessary, U by a smaller neighborhood we see that $\det \Phi \in \Gamma(U, \mathscr{O}_S^*)$ and $(\det \Phi)^2 = 1$. Hence either $\det \Phi = 1$ or $\det \Phi = -1$. By 1.8,

$$\det \varphi_s = (-1)^{(\dim V)/2 + h^0(X_s, L_s)},$$

where φ_s is the restriction of Φ to the fiber over $s \in U$. Hence the map $s \mapsto h^0(X_s, L_s) \bmod 2$ is locally constant, which proves the theorem.

§2. Polarization

Let X be an arbitrary curve. We denote by $J(X)$ its Jacobian. Recall that $J(X)$ is a smooth connected commutative algebraic group whose closed points are in a natural one-to-one correspondence with the classes of isomorphic invertible sheaves on X of multidegree $(0, \dots, 0)$; the group structure corresponds to tensor product of invertible sheaves.

If $k = \mathbf{C}$, then to $J(X)$ there corresponds the analytical Jacobian $J^{\mathrm{an}}(X)$ which is isomorphic to the quotient variety $H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z})$; this immediately follows from the cohomology sequence corresponding to the exact triple

$$0 \to \mathbf{Z} \to \mathscr{O}_X^{\mathrm{an}} \xrightarrow{\exp} \mathscr{O}_X^{*,\mathrm{an}} \to 0.$$

In the case when X is an irreducible nonsingular curve over $\mathbf{C}$ this yields the well-known representation of $J^{\mathrm{an}}(X)$ in the form of complex torus $\mathbf{C}^g\{$*lattice of periods*$\}$ (see, for example, [8]), where g is the genus of the curve X. Algebraic structure on $J^{\mathrm{an}}(X)$ can be recovered if we use a remarkable additional structure which is called *polarization*.

If the ground field k is arbitrary, the introduction of polarization also has deep geometric meaning. Usually polarization is given by the algebraic equivalence class of a divisor which is called the *polarization divisor*; if our variety is complete, this divisor is assumed to be ample. For abelian varieties there also exist other equivalent approaches (see, for example, [8] or [17]).

Let $f\colon N \to X$ be the desingularization of the curve X. Then $f^*\colon J(X) \to J(N)$ is an epimorphism onto an abelian variety of maximum dimension. Hence it is natural to define polarization on $J(X)$ as the lifting of polarization on $J(N)$. For those curves X which have the property

(RK) *X is a Gorenstein curve (i.e. the dualizing sheaf ω_X is invertible) and each component of the multidegree* $\mathbf{deg}\,\omega_X$ *of the sheaf ω_X is even*

there exists a unique (modulo translation) natural polarization divisor on $J(X)$ satisfying the Riemann-Kempf theorem (the abbreviation RK refers to this theorem). It is clear that this class of curves contains smooth curves and irreducible Gorenstein curves.

So, let X be a curve with property (RK). Then the variety

$$\mathrm{Jv}(X) \overset{\mathrm{df}}{=} \mathrm{Pic}^{(\mathbf{deg}\,\omega_X)/2}(X),$$

whose closed points are in a natural one-to-one correspondence with the isomorphism classes of invertible sheaves on X of multidegree $(\mathbf{deg}\,\omega_X)/2$, is nontrivial. Moreover, this variety is a principal homogeneous space with respect to the action of the Jacobian $J(X)$ defined by the following formulas:

$$J(X) \times \mathrm{Jv}(X) \to \mathrm{Jv}(X), \qquad [M], [L] \mapsto [M \otimes L]$$

(here $[L]$ denotes the isomorphism class of a sheaf L). The variety $\mathrm{Jv}(X)$ will be called the *Jacobi variety* of the curve X. Fixing a class $[L] \in \mathrm{Jv}(X)$, we obtain a noncanonical isomorphism

$$j_{[L]}\colon J(X) \xrightarrow{\approx} \mathrm{Jv}(X), \qquad [M] \mapsto [M \otimes L].$$

If D is a divisor in $\mathrm{Jv}(X)$, then we denote by $D_{[L]}$ the image of D on $J(X)$ under the isomorphism $j_{[L]}$.

2.1. PROPOSITION. *Let X be a curve satisfying* (RK). *Then*

(2.1.1) *the subvariety*

$$\{[L] \in \mathrm{Jv}(X) \mid h^0(X, L) \geqslant 1\} \subset \mathrm{Jv}(X)$$

defines an effective reduced divisor Θ on $\mathrm{Jv}(X)$.

If X' is another curve satisfying (RK), *Θ' is the divisor on* $\mathrm{Jv}(X')$ *defined by the preceding formula, and $g\colon X' \to X$ is a birational morphism, then, for arbitrary classes $[L_0] \in$* $\mathrm{Jv}(X)$ *and $[L_0'] \in$* $\mathrm{Jv}(X')$,

(2.1.2) *the divisor $\Theta_{[L_0]}$ on $J(X)$ is algebraically equivalent to the divisor $(g^*)^{-1}(\Theta'_{[L_0]})$, where $g^*\colon J(X) \to J(X')$ is the morphism corresponding to g.*

PROOF. First of all, we observe that the proposition easily reduces to the situation when X is a connected curve and $g = f\colon X' = N \to X$ is a desingularization of X. It is well known that for a smooth curve X the divisor Θ corresponds to the standard polarization and each effective divisor which is algebraically equivalent to $\Theta_{[L_0]}$ is obtained from this divisor by a translation, so that set-theoretically it has the form

$$\Theta_{[L]} = \{[M] \in J(X) \mid h^0(X, M \otimes L) \geqslant 1\}, \tag{2.1.3}$$

where L is an invertible sheaf of multidegree $(\mathbf{deg}\,\omega_X)/2$. The last assertion follows from the fact that our polarization is principal (cf. (2.2)). For a singular curve this is not quite so, and, although by the proposition the divisor $(f^*)^{-1}(\Theta'_{[L'_0]})$ is algebraically equivalent to $\Theta_{[L_0]}$, there need not exist representation (2.1.3) with an invertible sheaf L. Thus the required algebraic equivalence is not simply a translation on $J(X)$, but something like this. Namely, $(f^*)^{-1}(\Theta'_{[L'_0]})$ is represented in the form (2.1.3) with $L = f_* L'_0$, and by Lemma 1.4 the last sheaf can be deformed to an invertible sheaf of a suitable multidegree.

More previsely, let $\mathscr{L}$, S, s_1 and s_2 be as in Lemma 1.4, where L_1 is replaced by L'_0. We construct a divisor Z on $J(X) \times S$ such that Z is flat over an open subset in S containing s_0 and s_1 and for each point $s \in S$

$$\operatorname{supp}(Z|_{J(X)\times(s)}) = \{[M] \in J(X) | h^0(X, M \otimes L_s) \geqslant 1\}.$$

Here we use the Kempf construction (cf. [29]). Let $\mathscr{P}$ be the Poincaré sheaf on $X \times J(X)$, and let p, q, r and m be the projections of the product $X \times J(X) \times S$ onto $X \times S$, $J(X) \times S$, $X \times J(X)$ and X respectively. Consider the sheaf $\mathscr{F} = r^*\mathscr{P} \otimes p^*\mathscr{L}$ on $X \times J(X) \times S$. For closed points $s \in S$ and $[M] \in J(X)$

$$\mathscr{F}|_{X\times\{[M]\}\times\{s\}} \approx M \otimes L_s.$$

We fix g nonsingular points $x_1, \ldots, x_g$ on X in such a way that the divisor $D = \sum_1^g x_i$ has multidegree $\mathbf{deg}\, D \geqslant \frac{1}{2} \deg \omega_X$ ($\geqslant$ with respect to all components), where g is the genus of the curve X. Since the sheaf $\mathscr{L}$ is invertible in a neighborhood of the subvariety $\{x_i\}_1^g \times S$, there is an exact sequence

$$0 \to \mathscr{F} \otimes m^*\mathscr{O}_X(-D) \to \mathscr{F} \to \mathscr{F} \otimes m^*\mathscr{O}_D \to 0.$$

Consider the corresponding sequence of direct images with respect to q. We claim that $q_*(\mathscr{F} \otimes m^*\mathscr{O}_D)$ and $R^1q_*(\mathscr{F} \otimes m^*\mathscr{O}_X(-D))$ are locally free sheaves of rank g, and

$$R^1q_*(\mathscr{F} \otimes m^*\mathscr{O}_D) = 0.$$

By Corollary 2 from [15], Chapter II, §5, it suffices to verify that

$$h^0(X, M \otimes L_s \otimes \mathscr{O}_D) = g,$$
$$h^1(X, M \otimes L_s \otimes \mathscr{O}_D) = 0,$$
$$h^1(X, M \otimes L_s \otimes \mathscr{O}_X(-D)) = g$$

for all closed points $s \in S$, $[M] \in J(X)$. The first two equalities are obvious, and the third follows from the Riemann-Roch theorem and the construction of D. Thus we obtain an exact sequence

$$E_1 \xrightarrow{u} E_2 \to R^1q_*\mathscr{F} \to 0, \tag{2.1.4}$$

where $E_1 = q_*(\mathscr{F} \otimes m^*\mathscr{O}_D)$ and $E_2 = R^1q_*(\mathscr{F} \otimes m^*\mathscr{O}_X(-D))$ are locally free sheaves of rank g on $J(X) \times S$.

Since the functors R^iq_* commute with base change, for each pair $p = (M, s) \in J(X) \times S$ (2.1.4) induces an exact sequence

$$E_1|_p \xrightarrow{u|_p} E_2|_p \to H^1(X, M \otimes L_s) \to 0.$$

By the Riemann-Roch theorem,

$$h^1(X, M \otimes L_s) = h^0(X, M \otimes L_s).$$

Therefore $h^0(X, M \otimes L_s) \geqslant 1$ if and only if $\det(u)|_P = 0$. This proves (2.1.1), since from Lemma 1.4 it easily follows that $h^0(X, L) = 0$ for a generic $[L] \in \mathrm{Jv}(X)$ (compare with Lemma 2.1 in [3]).

Let Z be the divisor on $J(X) \times S$ locally defined by the equation $\det(u) = 0$. Let $U \subset S$ be the open subset consisting of those points $s \in S$ for which $Z_s = Z|_{J(X)\times\{s\}} \neq J(X)$ and is reduced. To prove (2.1.2) it remains to verify that $s_1 \in U$, since then the set of points s corresponding to invertible sheaves L_s is open and dense in U. To do this we observe that

$$f_*(L_0') \otimes M \simeq f_*(L_0' \otimes f^*M)$$

for each $[M] \in J(X)$. Hence set-theoretically $Z_{s_1} = (f^*)^{-1}(\Theta'_{[L_0']})$. Since $\Theta'_{[L_0']}$ is reduced, our divisor is also reduced because for a sheaf f^*M corresponding to a generic point of $\Theta'_{[L_0']}$ we have

$$h^1(X, f_*(L_0') \otimes M) = h^0(X, f_*(L_0') \otimes M) = h^0(N, L_0' \otimes f^*M] = 1. \quad \blacksquare$$

The above result justifies the following:

2.2. DEFINITION. Let $f\colon N \to X$ be the desingularization of the curve X, and let Θ be the polarization divisor of the Jacobian $J(N)$. A divisor which is algebraically equivalent to $(f^*)^{-1}(\Theta)$ is called a *polarization divisor* of the Jacobian $J(X)$ or, in more traditional terminology, a *theta-divisor*. The complete algebraic equivalence class of such divisors is called the *polarization* of the Jacobian $J(X)$. In what follows all Jacobians will be provided with this additional structure. From a category viewpoint we thus obtain a contravariant *Jacobi functor* from the category of algebraic curves to the category of polarized commutative algebraic groups; under this correspondence birational morphisms of curves are transformed to morphisms of polarized groups.

From a geometric point of view, effective polarization divisors present special interest. To such divisors there correspond various tangent maps. If there exists a canonical choice of effective polarization divisor, at least modulo translation, then these maps define geometric invariants of the Jacobian, such as, for example, the ramification divisor of the Gauss map. Such a choice automatically exists for a *principal* polarization of an abelian variety A, i.e. when the algebraic group A is complete and its polarization has degree 1. We recall that the *degree of polarization* of an abelian variety A is by definition the quotient $\Theta^{\dim A}/(\dim A)!$, where Θ is a polarization divisor. For some special algebraic groups, among the polarization divisors there are similar distinguished classes of effective divisors modulo translation. By Proposition 2.1, examples are given by the theta-divisors Θ_L on $J(X)$, where $J(X)$ is a curve satisfying condition (RK). However it is possible that these are not all polarization divisors. The divisor Θ on the Jacobian variety $\mathrm{Jv}(X)$ introduced in Proposition 2.1 will be called the *canonical polarization divisor* or *canonical theta-divisor* on $\mathrm{Jv}(X)$. The corresponding divisors $\Theta_{[L]} \subset J(X)$ will also be called *canonical.*

Let r and d be two nonnegative integers. We denote by $G_d^r(X)$ the subvariety

$$\{[L] \in \mathrm{Pic}(X) | \deg L = d.\ h^0(X, L) \geqslant r + 1\}$$

in $\mathrm{Pic}(X)$ (if there is no danger of confusion, we write simply G_d^r). Similarly, for an integral vector l with c components (where c is the number of components of the curve X)

we obtain a subvariety $G_l^r \subset \mathrm{Pic}^l(X)$. For example, for a curve X satisfying condition (RK) we have $G^0_{(\deg \omega_X)/2} = \Theta$.

Our next goal is to study infinitesimal properties of G_d^r. To do this, we first of all give a local description of the subvariety G_d^r. Take an arbitrary point $[L_0] \in G_d^r \cap \mathrm{Pic}^{l_0}(X)$, where $l_0 = \deg L_0$. We fix nonsingular points $x_1, \dots, x_m$ on X such that $m = h^0(X, L_0) \geqslant r + 1$ and $h^0(X, L_0(-D)) = 0$ for $D = \Sigma_1^m x_i$. The vector l_0 gives rise to a Poincaré divisor $\mathcal{P}$ on $X \times \mathrm{Pic}^{l_0}(X)$. We denote by p and q the projections $X \times \mathrm{Pic}^{l_0}(X) \to X$ and $X \times \mathrm{Pic}^{l_0}(X) \to \mathrm{Pic}^{l_0}(X)$. Now we apply the construction from the proof of Proposition 2.1 in the case when $S = \{L\}$. This construction yields an exact sequence of sheaves

$$E_1 \xrightarrow{u} E_2 \to R^1 q_* \mathcal{P} \to 0$$

on $\mathrm{Pic}^{l_0}(X)$, where in a neighborhood of the point $[L_0]$ the sheaves $E_1 = q_*(\mathcal{P} \otimes p^*\mathcal{O}_D)$ and $E_2 = R^1 q_*(\mathcal{P} \otimes p^*\mathcal{O}_X(-D))$ are locally free and have ranks m and $n = m - d - \chi(\mathcal{O}_X)$ respectively. For $[L] \in \mathrm{Pic}^d(X)$ the Riemann-Roch theorem shows that

$$h^0(X, L) \geqslant r + 1 \Leftrightarrow h^1(X, L) \geqslant r + 1 - d - \chi(\mathcal{O}_X).$$

Hence a point $[L]$ in a neighborhood of $[L_0] \in \mathrm{Pic}^{l_0}(X)$ lies in G_d^r if and only if $\mathrm{rk}(u|_{[L]}) \leqslant m - r - 1$. Defining the map u in a neighborhood of $[L_0]$ by an $m \times n$ matrix (u_{ij}) whose entries are functions on $\mathrm{Pic}^{l_0}(X)$ defined in a neighborhood of $[L_0]$, we see that G_d^r is locally defined by the vanishing of all $(m - r)$ minors of this matrix. By our construction, $u|_{[L_0]} = 0$, i.e. the exact sequence

$$0 \to H^0(X, L_0) \to E_1|_{[L_0]} \xrightarrow{u_{[L_0]}} E_2|_{[L_0]} \to H^1(X, L_0) \to 0$$

reduces to a pair of isomorphisms

$$H^0(X, L_0) \approx E_1|_{[L_0]}, \qquad H^1(X, L_0) \approx E_2|_{[L_0]}.$$

The linear parts of the germs of sections of the sheaf $\mathcal{O}_{\mathrm{Pic}^{l_0}(X)}$ at $[L_0]$ can be identified with the cotangent space

$$T^*_{\mathrm{Pic}^{l_0}(X), [L_0]} = H^0(X, \omega_X)$$

at the point $[L_0]$. Therefore the space of linear parts of the germs of sections of the sheaf E_2 can be identified with

$$H^0(X, \omega_X) \otimes E_2|_{[L_0]} \approx H^0(X, \omega_X) \otimes H^1(X, L_0).$$

After performing all these identifications we have the following:

2.3. LEMMA. *The linear part of the map u at the point $[L_0]$ coincides with the linear homomorphism*

$$H^0(X, L_0) \to H^0(X, \omega_X) \otimes H^1(X, L_0),$$

corresponding to the $\cup$*-product*

$$H^0(X, \omega_X)^* \otimes H^0(X, L_0) = H^1(X, \mathcal{O}_X) \otimes H^0(X, L_0) \xrightarrow{\cup} H^1(X, L_0).$$

Before proving this lemma, we give several corollaries. We begin by restating the lemma in terms of coordinates. To do this, we fix a basis $(s_i)_1^m$ in $H^0(X, L_0)$ and a basis $(t_j)_1^n$ in $H^1(X, L_0)$. Then for E_1 and E_2 there exist local trivializations which under the restriction

$|_{[L_0]}$ induce the trivializations of the corresponding vector spaces given by the bases (s_i) and (t_j) respectively, i.e. the bases of these trivializations correspond to the fixed bases at $[L_0]$ modulo terms of higher order. Under these trivializations, the map u is given by an $m \times n$ matrix (u_{ij}) composed of functions on $\mathrm{Pic}^{l_0}(X)$ which are defined in a neighborhood of $[L_0]$. Since $(u_{ij})|_{[L_0]} = 0$, the linear part of the matrix (u_{ij}) at $[L_0]$ does not depend on the choice of such trivializations. As above, the linear parts of the functions u_{ij} are identified with the elements of the cotangent space $T^*_{\mathrm{Pic}^{l_0}(X),[L_0]} = H^0(X, \omega_X)$.

2.4. COROLLARY. *The linear part of the matrix* (u_{ij}) *is equal to*

$$\left((s_i \otimes t_j^*)_{[L_0]}\right), \qquad 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n,$$

where

$$(\ )_{[L_0]}\colon H^0(X, L_0) \otimes H^1(X, L_0)^* = H^0(X, L_0) \otimes H^0\left(X, \omega_X \otimes L_0^{-1}\right) \to H^0(X, \omega_X)$$

is the natural pairing and (t_j^*) *is the basis in* $H^1(X, L_0)^*$ *dual to* (t_j).

This immediately follows from Lemma 2.3 and some standard facts from linear algebra. ■

One of the most important applications of the coordinate version of Lemma 2.3 is the following:

2.5. COROLLARY (RIEMANN - KEMPF THEOREM). *Let X be a curve satisfying condition* (RK), *and let Θ be the canonical polarization divisor on the Jacobian variety* $\mathrm{Jv}(X)$. *Then for each pont* $[L] \in \mathrm{Jv}(X)$ *the following assertions hold*:

(2.5.1) *If* $\det((s_j \otimes t_j^*)_{[L]}) \not\equiv 0$, *then*

$$\mathrm{Multi}_{[L]}\Theta = h^0(X, L)$$

($\mathrm{Multi}_x X$ *denotes the multiplicity of the point x on the divisor $X \subset Y$*), *and, moreover, the form* $\det((s_i \otimes t_j^*)_{[L]})$ *gives the equation of the tangent cone to Θ at the point $[L]$ in the tangent space* $T_{\mathrm{Jv}(X),[L]}$.

(2.5.2) *If* $\det((s_i \otimes t_j^*)_{[L]}) \equiv 0$, *then*

$$\mathrm{Multi}_{[L]}\Theta > h^0(X, L).$$

(2.5.3) *In particular, for* $[L] \in \mathrm{Jv}(X)$ *we always have*

$$\mathrm{Multi}_{[L]}\Theta \geqslant h^0(X, L).$$

2.6. REMARKS. (a) The matrix $((s_i \otimes t_j^*)_{[L]})$ considered in the preceding corollary is square, i.e. $m = n$, since $\chi(\mathcal{O}_X) = (\det \omega_X)/2$.

(b) It is easy to verify that $\det((s_i \otimes t_j^*)_{[L]}) \equiv 0$ if and only if there exists a nonzero section $s \in H^0(X, L)$ such that $(s \otimes H^0(X, \omega_X \otimes L^{-1})) = 0$. If in $H^0(X, L)$ or in $H^0(X, \omega_X \otimes L^{-1})$ there is a section which does not identically vanish on each component of the curve X, this is impossible. Hence for an irreducible curve X condition (2.5.1) always holds, which yields the equality $\mathrm{Multi}_{[L]}\Theta = h^0(X, L)$ and a convenient description of the tangent cone to Θ at each point $[L] \in \Theta$ similar to the classical version of the Riemann theorem on singularities (see, for example, [8]).

(c) Below we shall consider a similar theorem for the divisor Ξ on a Prym variety.

PROOF OF COROLLARY 2.5. By our main construction, in some neighborhood U of the point $[L] \in \mathrm{Jv}(X)$ there is an exact sequence

$$\mathcal{O}_U^m \xrightarrow{(u_{ij})} \mathcal{O}_U^m \to R^1 q_* \mathcal{P} \to 0$$

such that Θ is locally defined by the equation $\det(u_{ij}) = 0$ (here $m = h^0(X, L)$). As at the end of the proof of 2.1, we observe that the equation $\det(u_{ij}) = 0$ is reduced at the generic points of Θ. All other assertions follow immediately from Corollary 2.4 and the fact that $(u_{ij})|_{[L]} = 0$. ■

2.7. COROLLARY. *Let $Z \subset G_d^r$ be a subvariety, and let $[L] \in Z$ be a point such that $h^0(X, L) = r + 1$. Then the tangent subspace $T_{Z,[L]} \subset T_{\mathrm{Pic}(X),[L]}$ lies in the zero subset of the forms from* $\mathrm{im}(\ )_{[L]} \subset H^0(X, \omega_X) = T^*_{\mathrm{Pic}(X),[L]}$, *where* $(\ )_{[L]}$ *denotes the natural pairing*

$$H^0(X, L) \otimes H^0(X, \omega_X \otimes L^{-1}) \to H^0(X, \omega_X).$$

Hence, if Z is irreducible, then

$$\dim Z \leqslant g - \dim \mathrm{im}(\ )_{[L]}.$$

PROOF. By the main construction of this section, in some neighborhood U of the point $[L]$ there exists an $m \times n$ matrix (u_{ij}) such that $u_{ij} \in \mathcal{O}_U$ and $Z \cap U \subset \{[M] \in \mathrm{Pic}(X) | M \in U \text{ and } (u_{ij}(M)) = (0)\}$. By Corollary 2.4, the linear parts of the functions u_{ij} at $[L]$ generate the space $\mathrm{im}(\ )_{[L]}$. ■

PROOF OF LEMMA 2.3. We need to show that for each tangent vector $t \in H^1(X, \mathcal{O}_X) = H^0(X, \omega_X)^*$ at the point $[L_0]$ the composition

$$\begin{array}{ccccc} E_1|_{[L_0]} & \to & H^0(X, \omega_X) \otimes E_2|_{[L_0]} & \to & E_2|_{[L_0]} \\ \wr\wr & & \wr\wr & & \wr\wr \\ H^0(X, L_0) & \to & H^0(X, \omega_X) \otimes H^1(X, L_0) & \to & H^1(X, L_0) \\ & & \omega \otimes s \mapsto t(\omega) \cdot s & & \end{array}$$

is the $\cup$-multiplication by t. The tangent vector t defines a morphism $\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2) \to \mathrm{Pic}(X)$ which is equivalent to defining an invertible sheaf L_ε on $x_\varepsilon = X \times \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$ which restricts to L_0. The sheaf L_ε as sheaf of $\mathcal{O}_X$-modules is represented by the extension

$$0 \to L_0 \to L_\varepsilon \to L_0 \to 0,$$

corresponding to our tangent vector

$$t \in T_{\mathrm{Pic}(X),[L_0]} = H^1(X, \mathcal{O}_X) = \mathrm{Ext}^1_{\mathcal{O}_X}(L_0, L_0).$$

Consider the following commutative diagram of sheaves of $\mathcal{O}_X$-modules:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_0(-D) & \to & L_0 & \to & \mathcal{O}_D & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_\varepsilon(-D) & \to & L_\varepsilon & \to & \mathcal{O}_D \otimes k[\varepsilon]/(\varepsilon^2) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_0(-D) & \to & L_0 & \to & \mathcal{O}_D & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The rows and columns in this diagram are exact. This diagram induces the commutative diagram of vector spaces over k

$$\begin{array}{ccccccc}
 & & & & E_2|_{[L_0]} \quad 0 & & \\
 & & & & \| \quad \downarrow & & \\
 & & H^0(D, \mathcal{O}_D) & \rightarrow & H^1(X, L_0(-D)) & \overset{\approx}{\rightarrow} & H^1(X, L_0) \\
 & & \downarrow & & \downarrow \alpha & & \\
k[\varepsilon]/(\varepsilon^2) & \otimes & H^0(D, \mathcal{O}_D) & \overset{t^*u}{\rightarrow} & H^1(X, L_\varepsilon(-D)) & & \\
 & & \downarrow \beta & & \downarrow & & \\
H^0(X, L_0) & \overset{\approx}{\rightarrow} & H^0(D, \mathcal{O}_D) & \rightarrow & H^1(X, L_0(-D)) & & \\
\downarrow \cup t & & \downarrow \quad \| & & & & \\
H^1(X, L_0) & & 0 \quad E_1|_{[L_0]} & & & &
\end{array}$$

whose rows and columns are also exact. The map

$$\alpha^{-1} \circ t^*u \circ \beta^{-1}: H^0(X, L_0) \approx E_1|_{[L_0]} \to H^1(X, L_0) \approx E_2|_{[L_0]}$$

is well defined and coincides with the composition considered in the beginning of the proof. Using the definition of the boundary homomorphism and a diagram search, it is easy to show that this map coincides with the lower left-hand vertical arrow in the second diagram, i.e. with the $\cup$-multiplication by t. ■

§3. Prymians and Prym varieties

We begin with some notation and conventions. In what follows $C, C_1, \tilde{C}, \ldots$ denote curves whose only singularities are *ordinary double points*. We recall that a singular point $s \in C$ is called an *ordinary double point* if the computation $\hat{\mathcal{O}}_{C,s}$ of its local ring $\mathcal{O}_{C,s}$ is isomorphic to $k[[u, v]]/(u \cdot v)$.

A morphism I: $\tilde{C} \to \tilde{C}$ with $I^2 = \mathrm{id}$ is called an *involution*. In what follows we shall assume that the involutions under consideration are not the identity on any irreducible components of our curve. The symbol denoting involution will have the same indices as the symbol denoting curve, e.g. I_1: $\tilde{C}_1 \to \tilde{C}_1$, $I''_{2,3}$: $\tilde{C}''_{2,3} \to \tilde{C}''_{2,3}$. Similarly, as a rule, each object corresponding to a pair of the above type will have the same indices, e.g. the canonical projection will be denoted by $\pi''_{2,3}$: $\tilde{C}''_{2,3} \to C''_{2,3}$.

So, let $(\tilde{C}, I)$ be a pair consisting of a curve $\tilde{C}$ and an involution I on $\tilde{C}$. We denote by C the quotient curve $\tilde{C}/I$ and by π: $\tilde{C} \to C$ the corresponding projection. This notation is justified by the fact that, as is easy to see, the only singularities of C are ordinary double points. We observe that the morphism π is finite, but is not necessarily flat.

Let $\tilde{K}$ (respectively K) denote the ring of rational functions $\Gamma(\tilde{C}, \mathcal{R}_{\tilde{C}})$ on $\tilde{C}$ (on C), i.e. the product of the fields of rational functions of the irreducible components of this curve. As usual, $\mathrm{Div}(X)$ denotes the group of Cartier divisors of the curve X. Clearly I induces an involution I^*: $\tilde{K} \to \tilde{K}$. The norm homomorphism from $\tilde{K}$ to K is by definition the multiplicative homomorphism

$$N_{\tilde{K}/K}: \tilde{K} \to \tilde{K}^{I^*} = K, \qquad h \mapsto h \cdot I^*(h),$$

where $\tilde{K}^{I^*}$ is the subring of I^*-invariant rational functions on $\tilde{C}$. The norm induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\tilde{K}^* & \to & \mathrm{Div}(\tilde{C}) & \to & \mathrm{Pic}(\tilde{C}) & \to & 0 \\
\downarrow N_{\tilde{K}/K} & & \downarrow \pi_* & & \downarrow \mathrm{Nm} & & \\
K^* & \to & \mathrm{Div}(C) & \to & \mathrm{Pic}(C) & \to & 0,
\end{array}$$

where $\mathrm{Nm} = \mathrm{Nm}_{\tilde{C}/C}$: $\mathrm{Pic}(\tilde{C}) \to \mathrm{Pic}(C)$ is the usual norm of invertible sheaves (cf. [7], II.6.5).

The involution I gives rise to an involution I^*: $\mathrm{Pic}(\tilde{C}) \to \mathrm{Pic}(C)$. It is clear that

$$\pi^* \circ \mathrm{Nm} = \mathrm{id} + I^*, \tag{3.1.1}$$

$$\mathrm{Nm} \circ \pi^* = 2, \tag{3.1.2}$$

where 2 denotes the isogeny corresponding to raising to the second tensor power and the map π^*: $\mathrm{Pic}(C) \to \mathrm{Pic}(\tilde{C})$ is induced by the lifting of sheaves.

3.1. Lemma-Definition. *The connected commutative algebraic group*

$$\mathrm{Pr}(\tilde{C}, I) \underset{\mathrm{def}}{=} \ker(\mathrm{Nm})^0 = \ker(\mathrm{id} + I^*)^0 = (\mathrm{id} - I^*) J(\tilde{C})$$

with polarization induced from the Jacobian $J(\tilde{C})$ under the natural inclusion $\mathrm{Pr}(\tilde{C}, I) \subset J(\tilde{C})$ is called the Prymian of the pair $(\tilde{C}, I)$;

$$\dim \mathrm{Pr}(\tilde{C}, I) = g(\tilde{C}) - g(C),$$

where $g(\tilde{C}) = h^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$ and $g(C) = h^1(C, \mathcal{O}_C)$ are the genera of the curves $\tilde{C}$ and C.

Proof. In view of (3.1.2), the kernel of the map π^* is finite and Nm: $J(\tilde{C}) \to J(C)$ is an epimorphism. Hence by (3.1.1) $\ker(\mathrm{Nm})^0 = \ker(\mathrm{id} + I^*)^0$ and has the required dimension. Moreover, we clearly have the inclusions $(\mathrm{id} - I^*)J(\tilde{C}) \subset \ker(\mathrm{id} + I^*)^0$ and $\ker(\mathrm{id} - I^*)^0 \supset \pi^*J(C)$. Now we observe that the homomorphism $\pi^* \circ \mathrm{Nm}$ on $\ker(\mathrm{id} - I^*)$ coincides with the isogeny defined by multiplication by 2. Therefore $\ker(\mathrm{id} - I^*) = \pi^*J(C)$ and $\dim(\mathrm{id} - I^*)J(\tilde{C}) = g(\tilde{C}) - g(C) = \dim \ker(\mathrm{id} + I^*)^0$. Hence

$$(\mathrm{id} - I^*)J(\tilde{C}) = \ker(\mathrm{id} + I^*)^0. \qquad \blacksquare$$

In this section we study the main properties of Prymians. Here we consider only curves with ordinary double points. However it should be noted that in certain cases it is necessary to consider curves with cuspidal double points; this problem will be dealt with in one of the following parts. Nowe we are going to determine when $\mathrm{Pr}(\tilde{C}, I)$ is a *complete* algebraic group, i.e. an *abelian* variety. After that we shall study the question of when the polarization on such an abelian variety *corresponds to a principal polarization*. We say that two polarizations correspond to each other if they are proportional. It turns out that the polarization of a Prymian $\mathrm{Pr}(\tilde{C}, I)$ can never be principal. We shall see that the best thing one may hope for is that our polarization is equal to a principal polarization multiplied by two. The most important cases when this is so were first described by Mumford in the nonsingular case and by Beauville in the singular case (cf. [17] and [3]).

3.2. DEFINITION. A birational morphism $X' \to X$ will be called a *partial desingularization* (or *resolution of singularities*) of the curve X. Similarly, a birational morphism $\tilde{f}$: $\tilde{C}' \to \tilde{C}$ fitting into a commutative diagram

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow{\tilde{f}} & \tilde{C} \\ \downarrow I' & & \downarrow I \\ \tilde{C}' & \xrightarrow{\tilde{f}} & \tilde{C} \end{array}$$

will be called a *partial desingularization* (or *resolution of singularities*) of the pair $(\tilde{C}, I)$ and will be denoted by $\tilde{f}$: $(\tilde{C}', I') \to (\tilde{C}, I)$. Occasionally we shall specify the set of singular points resolved by a given morphism.

A partial desingularization of a pair gives rise to a commutative diagram:

$$\begin{array}{ccc} J(\tilde{C}) & \xrightarrow{\tilde{f}^*} & J(\tilde{C}') \\ I^* \downarrow & & \downarrow I'^* \\ J(\tilde{C}) & \xrightarrow{\tilde{f}^*} & J(\tilde{C}') \end{array}$$

It is clear that $\tilde{f}^* \operatorname{Pr}(\tilde{C}, I) = \tilde{f}^* \circ (\mathrm{id} - I^*)J(\tilde{C}) = (\mathrm{id} - I'^*) \circ \tilde{f}^* J(\tilde{C}) \subseteq \operatorname{Pr}(\tilde{C}', I')$. Therefore a partial desingularization $\tilde{f}$: $(\tilde{C}', I') \to (\tilde{C}, I)$ induces a morphism of the corresponding Prymians

$$\tilde{f}^*\colon \operatorname{Pr}(\tilde{C}, I) \to \operatorname{Pr}(\tilde{C}', I') \qquad [L] \mapsto [\tilde{f}^* L].$$

By definition of polarization of a Prymian, this is a morphism of polarized algebraic groups, i.e. the polarization on $\operatorname{Pr}(\tilde{C}, I)$ is the preimage of the polarization on $\operatorname{Pr}(\tilde{C}', I')$ with respect to $\tilde{f}^*$.

We observe that a partial desingularization $\tilde{f}$: $\tilde{C}' \to \tilde{C}$ gives rise to a partial desingularization of the pair if and only if the singular points x and $I(x)$ are resolved simultaneously and the involution I' is induced by the involution I. It is easy to verify that $\tilde{f}$ defines a partial desingularization of the quotient

$$f\colon C' = \tilde{C}/I' \to C = \tilde{C}/I.$$

The following obvious result often helps to simplify situation.

3.3. LEMMA. *Suppose that $(\tilde{C}, I)$ is a disjoint union of two pairs $(\tilde{C}_1, I_1)$ and $(\tilde{C}_2, I_2)$, i.e.*

$$\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2, \quad \tilde{C}_1 \cap \tilde{C}_2 = \varnothing, \quad I(\tilde{C}_1) = \tilde{C}_1, \quad I(\tilde{C}_2) = \tilde{C}_2,$$
$$I|_{\tilde{C}_1} = I_1, \qquad I|_{\tilde{C}_2} = I_2.$$

Then the polarized Prymian splits into a direct sum:

$$\operatorname{Pr}(\tilde{C}, I) = \operatorname{Pr}(\tilde{C}_1, I_1) \oplus \operatorname{Pr}(\tilde{C}_2, I_2). \quad \blacksquare$$

3.4. REMARKS. (a) Let A_1 and A_2 be two commutative polarized algebraic groups. By a *direct sum* $A_1 \oplus A_2$ we understand the commutative algebraic group $A_1 \times A_2$ with polarization defined by polarization divisors of the form $p_1^* D_1 + p_2^* D_2$, where D_1 and D_2 are polarization divisors on A_1 and A_2. It is easy to verify that this is a direct sum in the category of commutative algebraic groups.

(b) The above lemma reduces all questions involving Prymians to the case of a connected curve $\tilde{C}$, with the exception of one trivial possibility when $\tilde{C}$ consists of two connected components C_1 and C_2 transposed by the involution I. It is easy to verify that in the last case $\operatorname{Pr}(\tilde{C}, I) = J(C_1) = J(C_2)$ with polarization equal to the standard polarization on each of the Jacobians multiplied by two; hence in this case *the Prymian is complete if and only if the irreducible components of the curve C_1 (and C_2) are nonsingular and form a tree*. In view of this, in what follows we shall often assume that $\tilde{C}$ is a connected curve; this will allow us to simplify the exposition. However it should be noted that in general it is not advisable to disregard the disconnected case, since it naturally arises in some auxiliary constructions.

We introduce the following invariants:

$$2n_e = \#\{x \in \operatorname{Sing}\tilde{C} \mid I(x) \neq x\},$$

$$n'_f = \#\{x \in \operatorname{Sing}\tilde{C} \mid I(x) = x, \pi(x) \in \operatorname{Reg} C\},$$

$$2c_e = \#\{C' \subset \tilde{C} \mid C' \text{ is an irreducible component with } I(C') \neq C'\},$$

$$r = \#\{x \in \operatorname{Reg}\tilde{C} \mid I(x) = x\}.$$

Here, as usual, $\operatorname{Sing} X$ denotes the subvariety of all singular points from X and $\operatorname{Reg} X$ denotes the open subvariety of nonsingular points from X.

3.5. THEOREM. *Let $\tilde{C}$ be a connected curve. Then the Prymian $\operatorname{Pr}(\tilde{C}, I)$ is an abelian variety only in the following cases*:

(3.5.1) *$n'_f = 1$, $n_e = c_e - 2$, the set $\{x \mid I(x) = x\}$ consists of a single point, and the resolution of $\tilde{C}$ at this point consists of two connected components C_1 and C_2 transposed by the involution; the irreducible components of each of the curves C_1 and C_2 are nonsingular and form a tree (Figure 1). In this case $\operatorname{Pr}(\tilde{C}, I) = J(C_1) = J(C_2)$ and the polarization is equal to two times the polarization on each of the Jacobians.*

(3.5.2) *$n'_f = 0$ and $n_3 = c_e$. In this case the polarization on $\operatorname{Pr}(\tilde{C}, I)$ is divisible by two if and only if $r \leqslant 2$. More precisely, here there are two possibilities*:

(3.5.3) *$n'_f = 0$ and $2n_e = 2c_e = \tilde{c}$, where $\tilde{c}$ is the number of irreducible components of the curve $\tilde{C}$; the resolution of $\tilde{C}$ at a pair of points x, $I(x) \in \{x \in \operatorname{Sing}\tilde{C} \mid I(x) \neq x\}$ consists of two connected components C_1 and C_2 transposed by the involution, and the irreducible components of each of the curves C_1 and C_2 are nonsingular and form a tree (cf. Figure 2). In this case $\operatorname{Pr}(\tilde{C}, I) = J(C_1) = J(C_2)$, and the polarization is equal to two times the polarization on each of the Jacobians.*

(3.5.4) *$n'_f = 0$ and $2n_e = 2c_e < \tilde{c}$; denote by $\tilde{C}_0$ a union of all irreducible components $C' \subset \tilde{C}$ with $I(C') = C'$; $\tilde{C}_0$ is a connected curve; the resolution of $\tilde{C}$ at the points of the set $\{x \in \operatorname{Sing}\tilde{C} \mid I(x) \neq x,\ x \in \tilde{C}_0\}$ consists of the connected component $\tilde{C}_0$ and $\#\{x \in \operatorname{Sing}\tilde{C} \mid I(x) \neq x, x \in \tilde{C}_0\}$ connected components C_1, $I(C_1)$, C_2, $I(C_2), \dots$, where the curves of each pair are transposed by the involution; and the irreducible components of each of the curves C_1, $I(C_1)$, C_2, $I(C_2), \dots$ are nonsingular and form a tree. In this case*

$$\operatorname{Pr}(\tilde{C}, I) = \operatorname{Pr}(\tilde{C}_0, I_0) \oplus J(C_1) \oplus J(C_2) \oplus \cdots,$$

where $I_0 = I|_{\tilde{C}_0}$, and the polarization is equal to the polarization on the Jacobians multiplied by two (*cf. Figure* 3).

3.6. DEFINITION. Let $(\tilde{C}, I)$ be a pair such that $\Pr(\tilde{C}, I)$ is an abelian variety whose polarization is equal to a principal polarization multiplied by two. In this case we shall say that $(\tilde{C}, I)$ is a *principal* pair, and $\Pr(\tilde{C}, I)$ with the above principal polarization equal to one-half of the standard polarization will be called a *principal* Prymian. This principal Prymian will be denoted by $P(\tilde{C}, I)$. Thus, as a group scheme $P(\tilde{C}, I)$ coincides with $\Pr(\tilde{C}, I)$; the difference is only in the choice of polarization divisor.

3.7. REMARKS. (a) The case (3.5.3), like (3.5.1), is trivial. However, it plays an important role in global questions, since such pairs naturally arise as degenerations of the principal case, where $\tilde{C}$ is a nonsingular curve and I is an involution without fixed points. Moreover, such pairs and the pairs described in (3.5.4) allow us to make the Prym map $P(\ ,\ )$ proper (cf. [3]). The pairs described in (3.5.3) are sometimes called *Wirtinger* pairs. Wirtinger [6] was the first to introduce them in the case when $2n_e = 2c_e = 2$ and $C_1 \approx_I C_2$ is a nonsingular irreducible curve.

(b) The pairs $(\tilde{C}, I)$ with $r = 0$ from (3.5.2) are usually called *admissible*. It is these pairs that make the Prym map proper.

(c) As we can see from the theorem, the question of distinguishing principal Prymians from the Jacobians of nonsingular curves naturally reduces to the case when $\tilde{C}$ is a connected curve and $n'_f = n_e = c_e = 0$. Below we shall see that we may also assume that $r = 0$. Thus it is natural to consider the pairs $(\tilde{C}, I)$ satisfying the following condition:

(B) $\operatorname{Sing} \tilde{C} = \{x \in \tilde{C} | I(x) = x\}$ *and* $\pi(\operatorname{Sing} \tilde{C}) = \operatorname{Sing} C$, *i.e. the fixed points of the involution I are precisely the singular points of the curve* $\tilde{C}$, *and the involution I preserves the branches at these points.*

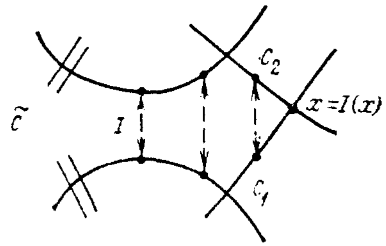

FIGURE 1

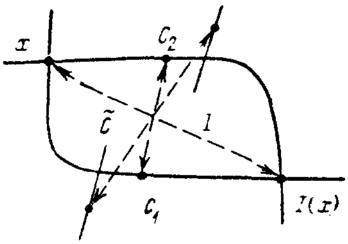

FIGURE 2

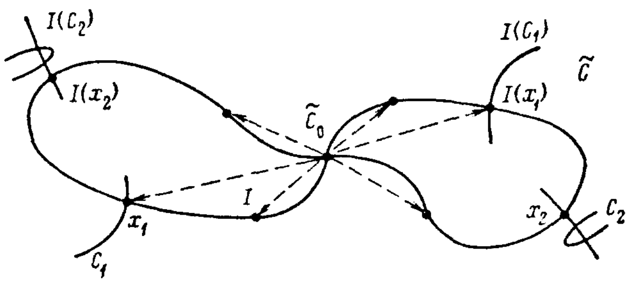

FIGURE 3

We shall call such pairs $(\tilde{C}, I)$ *Beauville* pairs, since Beauville was the first to study the main properties of their Prymians in the singular case.

3.8. Below we shall need an explicit description of the Cartier divisors on $\tilde{C}$ and C and an explicit description of the action of the homomorphisms I^* and π_* on these divisors. The group of Cartier divisors on $\tilde{C}$ is as follows:

$$\operatorname{Div}(\tilde{C}) = \left(\bigoplus_{x \in \operatorname{Reg} \tilde{C}} \mathbf{Z}x\right) \oplus \left(\bigoplus_{s \in \operatorname{Sing} \tilde{C}} \mathscr{R}_{\tilde{C},s}/\mathscr{O}^*_{\tilde{C},s}\right).$$

Let $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ be a partial desingularization at a point $s \in \operatorname{Sing} \tilde{C}$, and let s_1 and s_2 be the preimages of s under this desingularization. Then there is an exact sequence of groups

$$1 \to k^* \to \mathscr{R}^*_{\tilde{C},s}/\mathscr{O}^*_{\tilde{C},s} \xrightarrow{(v_1, v_2)} \mathbf{Z} \oplus \mathbf{Z} \to 0,$$

where v_1 and v_2 are the valuations at the points s_1, $s_2 \in \tilde{C}'$ respectively; the map $h \mapsto \tilde{f}^*(h)(s_1)/\tilde{f}^*(h)(s_2)$ identifies the kernel of the arrow (v_1, v_2) with k^*. A choice of local parameters t_1 and t_2 at the points s_1 and s_2 splits this triple and defines an isomorphism

$$(\ ,\ ,\ )_s\colon k^* \times \mathbf{Z} \times \mathbf{Z} \overset{\approx}{\leftarrow} \mathscr{R}^*_{\tilde{C}^*,s}/\mathscr{O}^*_{\tilde{C},s},$$

$$((\tilde{f}^*(h)/t_1^{v_1})(s_1)/(\tilde{f}^*(h)/t_2^{v_2})(s_2), \qquad v_1 = v_1(\tilde{f}^*(h)),$$
$$v_2 = v_2(\tilde{f}^*(h))) \leftarrow \textit{the class of } h.$$

The description of the above homomorphisms falls into three cases:

(3.8.1) $I(s) \neq s$. Then $\pi(s) \in \operatorname{Sing} C$ and for a suitable choice of local parameters

$$I^*(z, m, n)_s = (z, m, n)_{I(s)},$$
$$\pi_*(z, m, n)_s = (z, m, n)_{\pi(s)},$$
$$\pi_*(z, m, n)_{I(s)} = (z, m, n)_{\pi(s)}.$$

(3.8.2) $I(s) = s$ and $\pi(s) \in \operatorname{Reg} C$. In this case for a suitable choice of local parameters

$$I^*(z, m, n)_s = (z^{-1}, n, m)_s,$$
$$\pi_*(z, m, n)_s = (m + n)\pi(s).$$

(3.8.3) $I(s) = s$ and $\pi(s) \in \operatorname{Sing} C$. In this case for a suitable choice of local parameters

$$I^*(z, m, n)_s = ((-1)^{m+n}z, m, n)_s,$$
$$\pi_*(z, m, n)_s = ((-1)^{m+n}z^2, m, n)_{\pi(s)}.$$

PROOF OF THEOREM 3.5. Suppose that $\Pr(\tilde{C}, I)$ is a complete variety. First we consider the case when $n'_f \neq 0$, i.e. there exists a point $s \in \operatorname{Sing} \tilde{C}$ at which the branches are permuted by the involution. We denote by $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ the desingularization at s. If $\tilde{C}'$ is a connected curve, then there is an exact sequence

$$1 \to (k^*, 0, 0)_s \to J(\tilde{C}) \xrightarrow{\tilde{f}^*} J(\tilde{C}') \to 0 \tag{3.8.4}$$

and $\pi_*(k^*, 0, 0)_s = \{0\}$. Hence in this case there exists an inclusion $k^* \hookrightarrow \Pr(\tilde{C}, I)$ and the Prymian is not complete. Therefore the curve $\tilde{C}'$ is not connected. But then it is easy to see that $\tilde{C}'$ is a disjoint union of two connected components C_1 and C_2 transposed by the involution I', and $n'_f = 1$. Besides that, in this case we have an obvious isomorphism $f^*\colon J(\tilde{C}) \xrightarrow{\approx} J(\tilde{C}')$ which is compatible with the action of the involutions. Therefore

$$\Pr(\tilde{C}, I) = \Pr(\tilde{C}', I') = J(C_1) = J(C_2)$$

and the polarization is equal to the polarization on the Jacobians multiplied by two. In order that these Jacobians be complete it is necessary and sufficient that the irreducible components of each of the curves C_1, and C_2 be nonsingular and form a tree.

Suppose now that $n'_f = 0$. First we reduce everything to the case when $n_e = 0$. If $n_e \neq 0$, then there exists a point $s \in \operatorname{Sing} \tilde{C}$ with $I(s) \neq s$. We denote by $\tilde{f}\colon (\tilde{C}', I') \to (\tilde{C}, I)$ the partial resolution of our pair at the points s and $I(s)$. If the curve $\tilde{C}'$ is connected, then there is an exact sequence

$$1 \to (k^*,0,0)_s \oplus (k^*,0,0)_{I(s)} \to J(\tilde{C}) \xrightarrow{\tilde{f}^*} J(\tilde{C}') \to 0,$$

and $\pi_*((z,0,0)_s + (z^{-1},0,0)_{I(s)}) = (1,0,0)_{\pi(s)}$ in view of (3.8.1). Therefore in this case there exists an inclusion $k^* \hookrightarrow \operatorname{Pr}(\tilde{C}, I)$, and the Prymian is not complete. Hence the curve $\tilde{C}'$ is not connected. If $\tilde{C}'$ has two connected components C_1 and C_2 and the involution I' preserves these components (cf. Figure 4), then there is an exact sequence

$$1 \to \Big\{\underset{k^*/\pm 1}{\underset{\shortparallel}{(z,0,0)_s + (z^{-1},0,0)_{I(s)}}}\Big\}_{z\in k^*} \pmod{\sim} \to J(\tilde{C}) \xrightarrow{\tilde{f}^*} J(\tilde{C}') \to 0$$

and

$$\pi_*\big((z,0,0)_s + (z^{-1},0,0)_{I(s)}\big) = (1,0,0)_{\pi(s)}$$

(here we consider the isomorphism $(\, , \, , \,)_{I(s)}$ induced by $(\, , \, , \,)_s$ with respect to I). From this it follows that there is an inclusion $k^*/\pm 1 \hookrightarrow \operatorname{Pr}(\tilde{C}, I)$, and so in this case $\operatorname{Pr}(\tilde{C}, I)$ is also noncomplete. Hence, if $\tilde{C}'$ has two connected components C_1 and C_2, then these components are transposed by the involution I'. We shall show that in this case

$$\operatorname{Pr}(\tilde{C}, I) \underset{\tilde{f}^*}{\overset{\approx}{\to}} \operatorname{Pr}(\tilde{C}', I'),$$

which, combined with the argument given at the end of the preceding paragraph, leads us to the case of Wirtinger pairs (3.5.3). First of all, we observe that there is an exact sequence

$$\begin{aligned} 1 &\to \{(z,0,0)_s + (z^{-1},0,0)_{I(s)}\}_{z\in k^*} \to (k^*,0,0)_s \oplus (k^*,0,0)_{I(s)} \\ &\to J(\tilde{C}) \to J(\tilde{C}') \to 0 \end{aligned}$$

and

$$\pi_*\big((z,0,0)_s + (z',0,0)_{I(s)}\big) = (z \cdot z',0,0)_{\pi(s)};$$

here we consider the isomorphism $(\, , \, , \,)_{I(s)}$ induced by $(\, , \, , \,)_s$ with respect to I. Therefore $\tilde{f}^*\colon \operatorname{Pr}(\tilde{C}, I) \to \operatorname{Pr}(\tilde{C}', I')$ is a monomorphism. However by 3.1

$$\dim \operatorname{Pr}(\tilde{C}, I) = g(\tilde{C}) - g(C) = g(\tilde{C}') + 1 - g(C') - 1 = \dim \operatorname{Pr}(\tilde{C}', I'),$$

and therefore $\operatorname{Pr}(\tilde{C}, I) \approx \operatorname{Pr}(\tilde{C}', I')$. It remains to consider the case when the curve consists of three connected components. One of these components, which will be denoted by $\tilde{C}_0$, is stable with respect to I', and the two other components C_1 and C_2 are transposed by I'. Moreover, since $\tilde{f}^*\colon J(C) \overset{\approx}{\to} J(\tilde{C}')$, 3.3 shows that

$$\operatorname{Pr}(\tilde{C}, I) = \operatorname{Pr}(\tilde{C}', I') = \operatorname{Pr}(\tilde{C}_0, I_0) \oplus \operatorname{Pr}(C_1 \cup C_2, I''),$$

where $I_0 = I'|_{C_0}$ and $I'' = I'|_{C_1 \cup C_2}$. We have already verified that the last summand is complete if the irreducible components of each of the curves C_1 and C_2 are nonsingular and form a tree; the polarization of this summand is always divisible by two. From this it follows that $n_e(\tilde{C}) - n_e(\tilde{C}_0) = c_e(\tilde{C}) - c_e(\tilde{C}_0) > 0$ and $r(\tilde{C}_0) = r(\tilde{C})$. This lets us reduce everything to the case $n_e = 0$.

Thus we may assume that $n'_f = 0$ and $n_e = 0$. Then from the connectedness it follows that $c_e = 0$. In particular, in the above reduction we have $n_e = c_e$, which proves the necessity. To prove the sufficiency, it remains to show that if $n'_f = n_e = c_e = 0$, then $\Pr(\tilde{C}, I)$ is an abelian variety whose polarization is divisible by two if and only if $r \leqslant 2$. This forms an essential part of the following proposition. ■

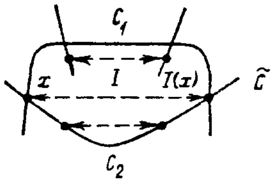

FIGURE 4

3.9. PROPOSITION. *Let $\tilde{C}$ be a connected curve, and suppose that the pair $(\tilde{C}, I)$ satisfies the following condition*:

(F) $\operatorname{Sing}\tilde{C} \subset \{x \in \tilde{C} \mid I(x) = x\}$ *and* $\pi(\operatorname{Sing}\tilde{C}) = \operatorname{Sing} C$, *i.e. each singular point of $\tilde{C}$ and the branches at these singular points are stable under the action of the involution I.*

Then the following assertions are true:

(3.9.1) *If $r > 0$, then* $\Pr(\tilde{C}, I) \approx \ker(\mathrm{Nm})$, *i.e.* $\ker(\mathrm{Nm}) = \ker(\mathrm{Nm})^0$.

(3.9.2) *If $r = 0$, i.e. the pair $(\tilde{C}, I)$ satisfies condition* (B), *then there is an exact sequence of groups*

$$0 \to \Pr(\tilde{C}, I) \to \ker(\mathrm{Nm}) \to \mathbf{Z}/2\mathbf{Z} \to 0.$$

In other words, $\ker(\mathrm{Nm})$ *has two connected components.*

(3.9.3) $\Pr(\tilde{C}, I)$ *is an abelian variety of dimension $g(C) - 1 + r/2$ with polarization of degree $2^{\dim \Pr(\tilde{C}, I)}$ in the case when condition* (B) *holds and polarization of degree $2^{g(C)}$ otherwise.*

(3.9.4) *The polarization of* $\Pr(\tilde{C}, I)$ *is divisible by two if and only if $r \leqslant 2$.*

3.10. REMARK. It is easy to show that if a pair $(\tilde{C}, I)$ satisfies condition (F), then the number r is even. This follows from the fact that the number of fixed points of an involution on a smooth curve is even.

3.11. LEMMA. *In the conditions of Propositions* 3.9, *let L be an invertible sheaf on $\tilde{C}$ with* $\mathrm{Nm}(L) \approx \mathcal{O}_C$. *Then $L \approx M \otimes I^*(M^{-1})$ for some invertible sheaf M on $\tilde{C}$. Moreover, M can be chosen so that its multidegree is equal to* $(0, 0, \ldots, 0)$ *or* $(1, 0, \ldots, 0)$.

PROOF. We choose a Cartier divisor $D'' \in \operatorname{Div}(\tilde{C})$ in such a way that $L \approx \mathcal{O}_{\tilde{C}}(D'')$. Then $\pi_* D'' = (h)$ is a principal divisor, where $h \in K^*$ is a rational function. Since $\tilde{K}$ is a product of fields of type C^1 (cf. [14]), by Tsen's theorem there exists a function $\tilde{h} \in \tilde{K}^*$ with $\mathrm{Nm}_{\tilde{K}/K}(\tilde{h}) = h$. Hence $L \approx \mathcal{O}_{\tilde{C}}(D')$, where $D' = D'' - (\tilde{h})$ and $\pi_* D' = 0$. That means that

$$D' = \sum_{x \in \operatorname{Reg}\tilde{C}} n_x(x - I(x)) + \sum_{s \in \operatorname{Sing}\tilde{C}} D'_s$$

and $\pi_* D'_s = 0$ for all $s \in \operatorname{Sing} \tilde{C}$. But then, in view of (3.8.3), $D'_s = (\pm 1, 0, 0)_s$, where $(-1, 0, 0)_s = (1, 0, 1)_s - I^*(1, 0, 1)_s$. Hence $D' = D - I^*D$ for some divisor $D \in \operatorname{Div}(\tilde{C})$, from which it follows that $L \approx M \otimes I^*(M^{-1})$ for $M = \mathcal{O}_{\tilde{C}}(D)$. Now we observe that $(\mathrm{id} - I^*)(1, 1, -1)_s = 0$. Therefore we may replace M by $M(1, 1, -1)_s$. Twisting by $(1, 1, -1)_s$, we can transfer the number one in the multidegree from one component to an arbitrary component intersecting the chosen one. Since the curve $\tilde{C}$ is connected, after a suitable number of such transformations the multidegree of M becomes equal to $(d, 0, \ldots, 0)$. Multiplying M by $\pi^* M'$, where M' is an invertible sheaf on C of multidegree $(-[d/2], 0, \ldots, 0)$ (we remark that, since $c_e = 0$, the irreducible components of the curves $\tilde{C}$ and C are in a one-to-one correspondence), we obtain the required sheaf. ■

PROOF OF PROPOSITION 3.9. By the above lemma, $\ker(\mathrm{Nm})$ has at most two connected components. First we consider the case $r > 0$. Then we may assume that on the curve corresponding to the first component of the multidegree for a suitable ordering of the irreducible components there exists a nonsingular point x which is a fixed point of the involution I, so that $x - I^*(x) = 0$. Hence in this case $\ker(\mathrm{Nm}) = \ker(\mathrm{Nm})^0$, which proves (3.9.1).

Suppose now that $r = 0$, i.e. condition (B) holds. We set

$$\operatorname{Pr}'(\tilde{C}, I) = \{[L] \in \operatorname{Pic}(\tilde{C}) \mid L = M \otimes I^*(M^{-1}), \text{ and } [M] \in \operatorname{Pic}^{(1,0,\ldots,0)}(\tilde{C})\}.$$

It is easy to verify that the variety $\operatorname{Pr}'(\tilde{C}, I) \subset \operatorname{Pic}(\tilde{C})$ does not depend on the choice of irreducible components of the curve $\tilde{C}$ on which the sheaves have degree 1.

3.12. REMARKS. (a) Let $(\tilde{C}, I)$ be a pair satisfying condition (B). Then $\tilde{C}$ has the following property:

(E) *For each decomposition $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, $\dim(\tilde{C}_1 \cap \tilde{C}_2) \leqslant 0$, of the curve $\tilde{C}$ the number of intersection points of the components $\tilde{C}_1$ and $\tilde{C}_2$ is even, i.e.*

$$\#(\tilde{C}_1 \cap \tilde{C}_2) \equiv 0 \mod 2.$$

This follows from the fact that the number of fixed points of an involution on a nonsingular curve is even. Since $c_e = 0$ and in this case π defines an isomorphism between $\operatorname{Sing} \tilde{C}$ and $\operatorname{Sing} C$, condition (E) holds also for the curve C.

(b) Let $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ be the partial desingularization of a curve $\tilde{C}$ with ordinary double points at a subset S. Then from the description of the dualizing sheaves on curves (see, for example, [21])

$$\tilde{f}^* \omega_{\tilde{C}} = \omega_{\tilde{C}'}\Big(\sum_{\tilde{f}(x) \in S} x\Big). \tag{3.12.1}$$

In particular, if $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ is a decomposition of $\tilde{C}$, then we have the following adjunction-type formula:

$$\omega_{\tilde{C}}|_{\tilde{C}_i} = \omega_{\tilde{C}_i}\Big(\sum_{s \in \tilde{C}_1 \cap \tilde{C}_2} s\Big), \qquad i = 1, 2. \tag{3.12.2}$$

(c) From the above two remarks it follows that if $(\tilde{C}, I)$ is a Beauville pair, then $\tilde{C}$ and the quotient curve C satisfy condition (RK).

(d) We also indicate the following formula which holds for the Beauville pairs:

$$\pi^* \omega_C = \omega_{\tilde{C}}.$$

This formula can be easily derived from the description of dualizing sheaves (cf. Lemma (3.2) in [3]).

We go on with the proof of Proposition 3.9. We consider a Beauville pair. In this case in view of 3.12(c) there exists an invertible sheaf L_0 on C with $L_0^2 \approx \omega_C$. Consider the function

$$\ker(\mathrm{Nm}) \to \mathbf{Z} \mod 2, \qquad [L] \mapsto h^0(\tilde{C}, L \otimes \pi^*L_0) \mod 2.$$

We claim that this function is constant on each connected component of ker(Nm) and assumes distinct values on $\mathrm{Pr}(\tilde{C}, I)$ and $\mathrm{Pr}'(\tilde{C}, I)$, which proves (3.9.2). Since

$$h^0(\tilde{C}, L \otimes \pi^*L_0) = h^0(C, \pi_*L \otimes L_0),$$

in view of Theorem 1.6, the first assertion follows from the existence of a nondegenerate quadratic form $Q_0\colon \pi_*L \otimes L_0 \to \omega_C$ or, which is equivalent, a form $Q\colon \pi_*L \to \mathrm{Nm}(L) \approx \mathcal{O}_C$ varying with L. This form is induced by the norm map

$$Q = \mathrm{Nm}\colon \pi_*\mathcal{O}_{\tilde{C}}|_U \to \mathcal{O}_C|_U, \qquad h \mapsto \mathrm{Nm}(h)$$

on sufficiently small open subsets $U \subset C$. The local verification of the fact that Q is nondegenerate is rather simple and is left to the reader. To prove the second assertion we need to find $[L_1], [L_2] \in \ker(\mathrm{Nm})$ such that

$$h^0(\tilde{C}, L_i \otimes \pi^*L_0) \equiv i \mod 2, \qquad i = 1,2.$$

By (E), the curve C does not have nonsingular irreducible rational components intersecting the other components at a single point. From this it is easy to deduce that in the linear system $|\omega_C|$ there exists a divisor D consisting of distinct nonsingular points (cf. 3.13). For each point belonging to the divisor D we pick a point on $\tilde{C}$ lying in the fiber over this point; thus we obtain a divisor $\tilde{D} \in \mathrm{Div}(\tilde{C})$ with $\pi_*\tilde{D} = D$. Let $M = \mathcal{O}_{\tilde{C}}(\tilde{D})$. Then $h^0(\tilde{C}, M) > 0$ and $\mathrm{Nm}\, M = \mathcal{O}_C(D) \approx \omega_C$. Next we show that $h^0(\tilde{C}, M') = h^0(\tilde{C}, M) - 1$, where $M' = M(I(x) - x)$ and x is a generic point of $\tilde{C}$. In what follows we shall need a similar result in a more general situation than the one considered in the case (B). Therefore we now prove this result in a more general setting.

3.13. The following concepts enable one to generalize some facts which are well known in the smooth case to the case of a singular curve X. Let L be an invertible sheaf on X. The *linear system* of the sheaf L is by definition the set $|L|$ of effective Cartier divisors D with $\mathcal{O}_X(D) \approx L$. For a connected curve X it is more convenient to consider the system $|L|$ as an open subvariety in $\mathbf{P}(H^0(X, L))$ which does not necessarily coincide with $\mathbf{P}(H^0(X, L))$ (and may be even empty) if the curve X is reducible. We shall say that an effective divisor is *nonsingular* if its support lies in the set of nonsingular points. A sheaf L, or its linear system $|L|$, is called *nonsingular* if $|L|$ contains a nonsingular divisor. This is equivalent to the existence for each singular point $x \in X$ of a global section $s \in H^0(X, L)$ with $s(x) \neq 0$.

3.14. LEMMA. *Let X be a Gorenstein curve with involution I which is nontrivial on all irreducible components of the curve X, and let L be a nonsingular invertible sheaf for which the sheaf $\omega_X \otimes L^{-1} \otimes I^*(L^{-1})$ is nonsingular. Then there exists a generic point $x \in X$ such that*

$$h^0(X, L(I(x) - x)) = h^0(X, L) - 1.$$

Furthermore, if $\dim|L| > 0$, *then we may assume that the sheaf $L(I(x) - x)$ is also nonsingular; the sheaf*

$$\omega_X \otimes L(I(x) - x^{-1}) \otimes I^*\big(L(I(x) - x)^{-1}\big) \approx \omega_X \otimes L^{-1} \otimes I^*(L^{-1})$$

is automatically nonsingular.

PROOF. It is clear that all fixed points of the systems $|L|$ and $|\omega_X \otimes L^{-1}|$ lie in Reg X. Let $x \in \operatorname{Reg} X$ be a sufficiently general point which, in particular, is distinct from these fixed points; if $\dim|L| > 0$, then we also assume that x belongs to a sufficiently general divisor from $|L|$. Then

$$h^0\left(X, \omega_X \otimes L(I(x))^{-1}\right) = h^0\left(X, \omega_X \otimes L^{-1}\right) - 1,$$

from which, by the Riemann-Roch theorem, it follows that

$$h^0(X, L(I(x))) = h^0(X, L).$$

Hence $h^0(X, L(I(x) - x)) = h^0(X, L) - 1$ and the sheaf $L(I(x) - x)$ is nonsingular provided that $\dim|L| > 0$. ■

We return to the proof of the proposition. Since $\omega_{\tilde{C}} \approx M \otimes I^*(M)$ (cf. 3.12(d)), by the preceding lemma for M there exists a point $x \in \tilde{C}$ with the required properties. As a result, we obtain two invertible sheaves M and M' on $\tilde{C}$ such that $\operatorname{Nm} M \approx \operatorname{Nm} M' \approx \omega_C$ and $h^0(\tilde{C}, M') = h^0(\tilde{C}, M) - 1$. After a suitable change of notation, the sheaves $M \otimes \pi^*L_0^{-1}$ and $M' \otimes \pi^*L_0^{-1}$ yield the required sheaves L_1 and L_2. This completes the proof of (3.9.2).

The following lemma shows that by performing a resolution of singularities we can reduce (3.9.3) to the case of a nonsingular curve $\tilde{C}$.

3.15. LEMMA. *In the conditions of Proposition* 3.9, *let* $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ *be the desingularization at a singular point of* $\tilde{C}$. *Then the morphism of Prymians*

$$\tilde{f}^*\colon \operatorname{Pr}(\tilde{C}, I) \to \operatorname{Pr}(\tilde{C}', I')$$

is an isogeny of degree 2 *or* 1. *More precisely, this isogeny has degree* 1 *if and only if* $(\tilde{C}, I)$ *satisfies condition* (B) *or* $\tilde{C}'$ *is not connected.*

3.16. COROLLARY. *Let* $(\tilde{C}, I)$ *be a Beauville pair. Suppose that there exists a decomposition* $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ *with* $\tilde{C}_1 \cap \tilde{C}_2 = \{p, q\}$. *Denote by* $\tilde{C}_1'$ *and* $\tilde{C}_2'$ *the curves obtained by identifying the points p and q by means of the involutions* I_1' *and* I_2' *induced by* I (*cf. Figure* 5). *Then*

$$\operatorname{Pr}(\tilde{C}, I) = \operatorname{Pr}(\tilde{C}_1', I_1') \oplus \operatorname{Pr}(\tilde{C}_2', I_2'). \quad ■$$

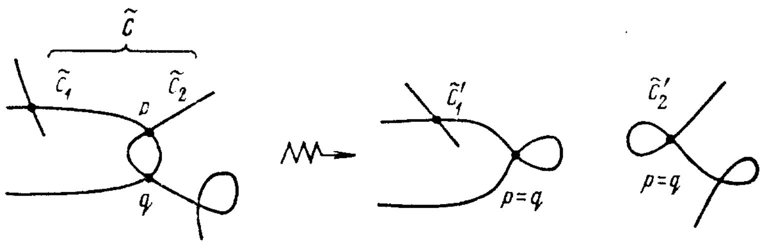

FIGURE 5

3.17. REMARKS. (a) The pairs $(\tilde{C}_1', I_1')$ and $(\tilde{C}_2', I_2')$ are again Beauville pairs, and, as we shall see below, the polarization of their Prymians is divisible by two. Hence we have the following decomposition of principal Prymians:

$$P(\tilde{C}, I) = P(\tilde{C}_1', I_1') \oplus P(\tilde{C}_2', I_2').$$

This reduces the problem of distinguishing Prymians from Jacobians to the case when the curve $\tilde{C}$ and therefore also the curve C in the case (B) satisfy the following condition:

(S) *For each decomposition* $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$

$$\#(\tilde{C}_1 \cap \tilde{C}_2) \geqslant 4.$$

(b) If $r = 2$, then, gluing the two fixed points of the involution I on $\tilde{C}$, we obtain a Beauville pair $(\tilde{C}, I')$. By Lemma 3.15, we have an isomorphism

$$\Pr(\tilde{C}, I') \tilde{\to} \Pr(\tilde{C}, I).$$

This sometimes allows us to consider only the Beauville case with $r = 0$ (cf. also 3.7(c)).

PROOF OF LEMMA 3.15. The case when the curve $\tilde{C}'$ is not connected is obvious, since then there exists an isomorphism $\tilde{f}^*\colon J(\tilde{C}) \to J(\tilde{C}')$ which is compatible with the involutions. If $\tilde{C}'$ is a connected curve, then by (3.8.4)

$$\ker(\Pr(\tilde{C}, I) \to \Pr(\tilde{C}', I')) \subseteq \ker\left(k^* \xrightarrow{a^2} k^*\right) = \{\pm 1\}.$$

By (3.9.1), if $(\tilde{C}, I)$ is not a Beauville pair, then the above inclusion is an equality; in case (B) the kernel is trivial. The dimension of the Prymians $\Pr(\tilde{C}, I)$ and $\Pr(\tilde{C}', I')$ coincide in view of 3.1. ■

Now we turn to the proof of (3.9.3). Using Lemmas 3.3 and 3.15 and examining the variation of dimension and degree under the resolution of singularities of $\tilde{C}$, we reduce the problem to the case of a smooth curve $\tilde{C}$; as above, we assume that this curve is connected. In that case it is clear that $\Pr(\tilde{C}, I)$ is an abelian variety. From 3.1 and the Hurwitz formula, it is easy to derive a formula for the dimension of $\Pr(\tilde{C}, I)$. It is harder to compute the degree of polarization. Now we turn to this problem. We use Mumford's argument from [17].

So, let C be a smooth curve, and let Θ be an effective polarization divisor on the Jacobian $J(C)$. Since the polarization is principal, there is an isomorphism

$$\lambda_\Theta\colon J(C) \to \operatorname{Pic}^0(J(C)), \qquad x \mapsto \left[\mathcal{O}_{J(C)}(T_x^*\Theta - \Theta)\right],$$

where $T_x\colon J(C) \to J(C)$ is the translation by x (cf. 2.1). A similar isomorphism is defined by $\tilde{\Theta} \subset J(\tilde{C})$. We fix points $\tilde{x}_0 \in \tilde{C}$ and $x_0 = \pi(\tilde{x}_0) \in C$. Consider the Abel-Albanese mappings

$$\begin{aligned} t\colon C \to J(C), &\qquad x \mapsto [\mathcal{O}_C(x - x_0)], \\ \tilde{t}\colon \tilde{C} \to J(\tilde{C}), &\qquad \tilde{x} \mapsto [\mathcal{O}_{\tilde{C}}(\tilde{x} - \tilde{x}_0)] \end{aligned}$$

and the commutative diagram:

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\tilde{t}} & J(\tilde{C}) \\ \pi\downarrow & & \downarrow \mathrm{Nm} \\ C & \xrightarrow{t} & J(C) \end{array}$$

Applying $\operatorname{Pic}^0(\ )$ to this diagram, we obtain the following commutative diagram:

$$\begin{array}{ccc} J(\tilde{C}) & \xleftarrow{\tilde{t}^*} & \operatorname{Pic}^0(J(\tilde{C})) \\ \pi^*\uparrow & & \uparrow \mathrm{Nm}^* \\ J(C) & \xleftarrow{t^*} & \operatorname{Pic}^0(J(C)) \end{array}$$

By the Riemann theorem (cf. [8], 2.7),

$$(t^*)^{-1} = -\lambda_\Theta, \qquad (\tilde{t}^*)^{-1} = -\lambda_{\tilde{\Theta}}.$$

Hence the diagram

$$\begin{array}{ccc} J(\tilde{C}) & \xrightarrow{\lambda_{\tilde{\Theta}}} & \mathrm{Pic}^0(J(\tilde{C})) \\ \pi^*\uparrow & & \uparrow \mathrm{Nm}^* \\ J(C) & \xrightarrow{\lambda_\Theta} & \mathrm{Pic}^0(J(C)) \end{array}$$

is also commutative, i.e. $\pi^* = \lambda_{\tilde{\Theta}}^{-1} \circ \mathrm{Nm}^* \circ \lambda_\Theta$. Since $\pi^{**} \circ \lambda_{\tilde{\Theta}} \circ \pi^* = (\mathrm{Nm} \circ \pi^*)^* \circ \lambda_\Theta = 2 \cdot \lambda_\Theta$, the diagram

$$\begin{array}{ccc} J(\tilde{C}) & \xrightarrow{\lambda_{\tilde{\Theta}}} & \mathrm{Pic}^0(J(\tilde{C})) \\ \pi^*\uparrow & & \downarrow \pi^{**} \\ J(C) & \xrightarrow{2\lambda_\Theta} & \mathrm{Pic}(J(C)) \end{array}$$

is commutative. Moreover, in view of the duality and one of the preceding equalities, $\pi^{**} \circ \tilde{\lambda} = \lambda_\Theta \circ \mathrm{Nm}$. Therefore $\mathrm{Pr}(\tilde{C}, I) = \lambda_{\tilde{\Theta}}^{-1}(\ker \pi^{**})^0$. The morphism π^* and the inclusion $\mathrm{Pr}(\tilde{C}, I) \subset J(\tilde{C})$ define an isogeny

$$\sigma\colon J(C) \times \mathrm{Pr}(\tilde{C}, I) \to J(\tilde{C}), \qquad (x, y) \mapsto \pi^*x + y.$$

Set $H = \ker \sigma$. It is clear that

$$(x, y) \in H \Rightarrow \pi^*x + y = 0 \Rightarrow \pi^{**}(\lambda_{\tilde{\Theta}}(\pi^*x)) = 0 \Rightarrow 2x = 0 \Rightarrow 2y = 0.$$

Therefore $H \subset J_2(C) \times \mathrm{Pr}_2(\tilde{C}, I)$ (the index 2 indices that we consider the subgroup of points of order 2 on the corresponding abelian variety). Since $H \cap \{0\} \times \mathrm{Pr}_2(\tilde{C}, I) = \{(0,0)\}$, there exist a subgroup $H_1 \subset J_2(C)$ and a homomorphism $\psi\colon H_1 \to \mathrm{Pr}_2(\tilde{C}, I)$ such that

$$H = \{(\alpha, \psi\alpha) | \alpha \in H_1\}.$$

Let $H_0 = \ker \pi^*$. Then $H_0 \subset H_1$ and ψ defines an inclusion $H_1/H_0 \hookrightarrow \mathrm{Pr}_2(\tilde{C}, I)$. The lifting $\sigma^*\tilde{\Theta}$ of the polarization divisor defines a morphism $\lambda_{\sigma^*\tilde{\Theta}}$ which coincides with the composition

$$J(C) \times \mathrm{Pr}(\tilde{C}, I) \xrightarrow{\sigma} J(\tilde{C}) \xrightarrow{\lambda_{\tilde{\Theta}}} \mathrm{Pic}^0(J(\tilde{C})) \xrightarrow{\sigma^*} \mathrm{Pic}^0(J(C)) \times \mathrm{Pic}^0(\mathrm{Pr}(\tilde{C}, I)).$$

This morphism can be represented by a matrix $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, where

$$\alpha\colon J(C) \to \mathrm{Pic}^0(J(C)), \qquad \beta\colon \mathrm{Pr}(\tilde{C}, I) \to \mathrm{Pic}^0(J(C)),$$
$$\gamma\colon J(C) \to \mathrm{Pic}^0(\mathrm{Pr}(\tilde{C}, I)), \qquad \delta\colon \mathrm{Pr}(\tilde{C}, I) \to \mathrm{Pic}^0(\mathrm{Pr}(\tilde{C}, I)).$$

In view of the self-duality of $\lambda_{\sigma^*\tilde{\Theta}}$, we have the equality $\gamma = \beta^*$. But by the definition of $\mathrm{Pr}(\tilde{C}, I)$ the morphism β is equal to 0, and $\delta = \lambda_\Xi$ corresponds to the polarization divisor Ξ on $\mathrm{Pr}(\tilde{C}, I)$. Hence $\gamma = 0$ and the lifting $\sigma^*\tilde{\Theta}$ of our polarization splits. From one of the preceding diagrams it follows that $\alpha = 2\lambda_\Theta$. Now we observe that since the morphism $(2\lambda_\Theta, \delta)$ of the polarization on $J(C) \times \mathrm{Pr}(\tilde{C}, I)$ corresponds to the lifting of the principal polarization with respect to σ, $\ker\sigma$ is a maximal isotropic subgroup in $\ker(2\lambda_\Theta, \delta)$ with respect to the skew-symmetric form defined by the polarization (cf. [15], §23). Thus

$$H \subset J_2(C) \times \ker\delta$$

and

$$e_{2,J(C)}(\alpha, \beta) \cdot e_\delta(\psi(\alpha), \psi(\beta)) = 1,$$

if $(\alpha, \psi(\alpha)), (\beta, \psi(\beta)) \in H$. Therefore $\psi(H_1/H_0) \subset \ker\delta$ and

$$e_\delta(\psi(\alpha), \psi(\beta)) = e_{2,J(C)}(\alpha, \beta)$$

for all $\alpha, \beta \in H_1$. In view of the maximality of H,

$$(\#H_1)^2 = (\#H)^2 = \#(J_2(C) \times \ker\delta),$$

from which it follows that

$$\#(H_1^\perp) = \frac{\#J_2(C)}{\#H_1} = \frac{\#H_1}{\ker\delta} \leqslant \frac{\#H_1}{\#\operatorname{im}\psi} = \#H_0,$$

where $H_1^\perp = \{\alpha \in J_2(C) | e_{2,J(C)}(\alpha, \beta) = 1 \text{ for all } \beta \in H_1\}$. But $H_0 \subset H_1^\perp$, since for all $\alpha \in H_0$ and $\beta \in H_1$

$$e_{2,J(C)}(\alpha, \beta) = e_\delta(\psi(\alpha), \psi(\beta)) = 1.$$

Hence $H_0 = H_1^\perp$, $\operatorname{im}\psi = \ker\delta$ and $H_1 = H_0^\perp$. Therefore the polarization degree of $\Pr(\tilde{C}, I)$ is equal to

$$\sqrt[2]{\#\ker\lambda_\Xi} = \sqrt[2]{\#\ker\delta} = \sqrt[2]{\#(H_1/H_0)} = 2^c,$$

where c is the integer defined by the equality $\#H_0 = 2^{a-c}$ and $a = \dim J(C)$. Now we recall that $H_0 = \ker\pi^*$. To complete the proof of (3.9.3) it remains to recall the following result (cf. [17]).

3.18. LEMMA. *Let $\tilde{C}$ and C be two connected smooth curves, and let π: $\tilde{C} \to C$ be a morphism of degree* 2. *Then*:

(3.18.1) $\ker\pi^* = \{0\}$ *when π has branch points*; *and*

(3.18.2) $\ker\pi^* = \{0, U\}$ *where π is the unramified covering defined according to the Kummer theory by a point $U \in J_2(C)$ of order two.*

Now we turn to the proof of (3.9.4). By (3.9.3), the polarization of the Prymian $\Pr(\tilde{C}, I)$ is divisible by two only for $r = 0$ and 2. In view of Remark 3.17(b), to prove the sufficiency it suffices to consider only the case $r = 0$. So, let $(\tilde{C}, I)$ be a Beauville pair. Consider the subvariety

$$\operatorname{Pv}(\tilde{C}, I) = \{[L] \in \operatorname{Pic}(\tilde{C}) | \operatorname{Nm} L \approx \omega_C, h^0(\tilde{C}, I) \equiv 0 \bmod 2\}$$

in $\operatorname{Pic}^{(\deg \Omega_{\tilde{C}})/2}(\tilde{C})$ (compare with 3.12(d)). From the preceding proof it follows that this subvariety is nonempty. Moreover, this is a principal homogeneous space with respect to $\Pr(\tilde{C}, I)$. The variety $\operatorname{Pv}(\tilde{C}, I)$ is called the *Prym variety*. We claim that the divisor

$$\Xi = \{[L] \in \operatorname{Pv}(\tilde{C}, I) | h^0(\tilde{C}, L) > 0\},$$

which from now on will be called *canonical*, corresponds to the polarization of $\Pr(\tilde{C}, I)$, i.e. for each $[L_0] \in \operatorname{Pv}(\tilde{C}, I)$, under the isomorphism

$$\Pr(\tilde{C}, I) \xrightarrow{\approx} \operatorname{Pv}(\tilde{C}, I) \subset \operatorname{Jv}(\tilde{C}), \qquad [L] \mapsto [L \otimes L_0]$$

the divisor Ξ is transformed to one-half of the polarization divisor on $\Pr(\tilde{C}, I)$. In fact, let $\Theta = \{[M] \in \mathrm{Jv}(\tilde{C}) | h^0(\tilde{C}, M) > 0\}$ be the canonical polarization divisor on $\mathrm{Jv}(\tilde{C})$. From Lemma 3.14 it is easy to deduce that there exists an invertible sheaf L on $\tilde{C}$ with $[L] \in \mathrm{Pv}(\tilde{C}, I)$ and $h^0(\tilde{C}, L) = 0$. Hence the divisor Θ intersects $\mathrm{Pv}(\tilde{C}, I)$ along an effective divisor $F = \sum \alpha_i F_i$, where the F_i are irreducible reduced components. Since $h^0(\tilde{C}, M) > 0$ and $h^0(\tilde{C}, M) \equiv 0 \bmod 2$ for $[M] \in \Theta \cap \mathrm{Pv}(\tilde{C}, I)$, so that $h^0(\tilde{C}, M) \geqslant 2$ for such $[M]$, the Riemann-Kempf theorem shows that the divisor Θ is singular along $\Theta \cap \mathrm{Pv}(\tilde{C}, I)$. Therefore $\alpha_i \geqslant 2$ for all i. On the other hand, the degree of polarization of the Prymian $\Pr(\tilde{C}, I)$ is equal to $2^{\dim(\Pr(\tilde{C}, I))}$, and so

$$(F)^{\dim(\Pr(\tilde{C}, I))} = 2^{\dim(\Pr(\tilde{C}, I))} \times ((\dim(\Pr(\tilde{C}, I)))!).$$

Now it is easy to show that all α_i are equal to 2, our polarization divisor is divisible by two, and the quotient $\Xi = \sum F_i$ defines a principal polarization. This complete the proof of Proposition 3.9. ■

We emphasize that *the canonical polarization divisor Ξ on a Prym variety* $\mathrm{Pv}(\tilde{C}, I)$ *is defined only in the case of Beauville pairs.*

3.19. *Prym varieties and their canonical subvarieties.* Let $(\tilde{C}, I)$ be a pair with $c_e = 0$ satisfying condition (F). If $\tilde{C}$ is a connected curve and $r = 0$, then $\mathrm{Pv}(\tilde{C}, I)$ and $\Xi \subset \mathrm{Pv}(\tilde{C}, I)$ are defined as above. If $\tilde{C}$ is a connected curve and $r > 0$, then we set

$$\mathrm{Pv}(\tilde{C}, I) = \{[L] \in \mathrm{Pic}(\tilde{C}) | \mathrm{Nm}\, L \approx \omega_C\}$$

and

$$\Xi(\tilde{C}, I) = \{[M] \in \mathrm{Pv}(\tilde{C}, I) | h^0(\tilde{C}, M) > 0\}.$$

It is clear that $\mathrm{Pv}(\tilde{C}, I)$ is a principal homogeneous space with respect to the natural action of the group $\Pr(\tilde{C}, I)$. Therefore in this case $\mathrm{Pv}(\tilde{C}, I)$ also is a complete connected nonsingular variety. If $\tilde{C}$ is not connected, then $\mathrm{Pv}(\tilde{C}, I)$ is defined to be a product of the Pv corresponding to all connected components, and $\Xi(\tilde{C}, I)$ is defined as above. We remark that if we are not in the Beauville case, then the subvariety $\Xi(\tilde{C}, I)$ is not necessarily a divisor. This is clear from the following result.

3.20. LEMMA. *Let $\tilde{C}$ be a connected curve, and let $(\tilde{C}, I)$ be a pair satisfying condition* (F). *Let Z be an irreducible closed subvariety in Ξ whose generic point $[L]$ corresponds to a nonsingular sheaf L. Then*

$$\dim Z \leqslant g(C) - 1 = \dim \mathrm{Pv}(\tilde{C}, I) - r/2.$$

PROOF. At a generic point of the variety $\mathscr{D} = \bigcup_{[L] \in Z} |L|$ there is a well-defined quasifinite morphism

$$\mathscr{D} \to |\omega_C|, \qquad D \mapsto \pi_* D.$$

Therefore it is clear that $\dim Z \leqslant \dim \mathscr{D} \leqslant \dim|\omega_C| = g(C) - 1$. ■

Varieties $\mathrm{Pv}(\tilde{C}, I)$ will be called *Prym varieties*, and their subvarieties $\Xi(\tilde{C}, I)$ will be called *canonical* subvarieties. For the sake of brevity, we often write simply Pv, Ξ, Pr and P with the indices of the curve $\tilde{C}$. For example, $\mathrm{Pv}_1 = \mathrm{Pv}(\tilde{C}_1, I_1)$, $\Xi' = \Xi(\tilde{C}', I')$, $\Pr^* = \Pr(\tilde{C}^*, I^*)$, and $P_1' = P(\tilde{C}_1', I_1')$.

3.21. *The isogeny $\tilde{f}^0$.* Let $(\tilde{C}, I)$ be a pair with $c_e = 0$ satisfying condition (F), and let $\tilde{f}$: $\tilde{C}' \to \tilde{C}$ be its partial desingularization at a subset $S \subset \mathrm{Sing}\, \tilde{C}$. Then the map

$$\mathrm{Pic}(\tilde{C}) \to \mathrm{Pic}(\tilde{C}'), \qquad [L] \mapsto \left[\tilde{f}^* L\left(-\sum_{\tilde{f}(x) \in S} x\right)\right],$$

induces a mapping

$$\tilde{f}^0\colon \operatorname{Pv}(\tilde{C}, I) \to \operatorname{Pv}(\tilde{C}', I'),$$

since

$$\operatorname{Nm}\circ \tilde{f}^* L\Big(-\sum_{\tilde{f}(x)\in S} x\Big) = f^*\circ \operatorname{Nm}(L)\Big(-\sum_{f(x)\in\pi(S)} x\Big) = f^*\omega_C\Big(\sum_{f(x)\in\pi(S)} x\Big) = \omega_{C'}$$

(cf. (3.12.1)).

(3.21.1) We observe that for the pair $(\tilde{C}', I')$ we have $c_e = 0$, and this pair again satisfies condition (F). Let $\tilde{g}\colon \tilde{C}'' \to \tilde{C}'$ be its partial desingularization. Then

$$(\tilde{f}\circ\tilde{g})^0 = \tilde{g}^0\circ\tilde{f}^0.$$

(3.21.2) $\tilde{f}^0$ is an *equivariant* morphism of homogeneous spaces, i.e. for each $[L]\in \operatorname{Pv}(\tilde{C}, I)$ and each $[M]\in\operatorname{Pr}(\tilde{C}, I)$

$$\tilde{f}^0(M\otimes L)\approx \tilde{f}^*(M)\otimes\tilde{f}^0(L).$$

From these facts and Lemma 3.15 it is clear that $\tilde{f}^0$ is a finite epimorphism which, like the corresponding mapping of abelian varieties

$$\tilde{f}^*\colon \operatorname{Pr}(\tilde{C}, I)\to\operatorname{Pr}(\tilde{C}', I')$$

will be called an *isogeny*. We observe that

$$\deg\tilde{f}^0 = \deg\tilde{f}^*. \tag{3.21.3}$$

§4. Special curves

In §2 we defined a subvariety $G_d^r\subset\operatorname{Pic}(X)$ whose points correspond to classes of isomorphic invertible sheaves L on X with $\deg L = d$ and $h^0(X, L)\geqslant r+1$. The geometric properties of the variety G_d^r reflect essential information about the structure of the curve X. For example, if X is a nonsingular irreducible curve of genus g and $0<d<g-1$, then by Martens' theorem [18] $\dim G_d^r\leqslant d-2r$ and the equality is possible only if $r=0$ or X is a hyperelliptic curve. A further refinement of the estimate for the dimension of G_d^r was suggested by Mumford in the Appendix to [17]. In this section, following Beauville [3], we extend the theorems of Martens and Mumford to the case of curves with singularities.

4.1. LEMMA. *Let L and M be two nonsingular invertible sheaves on a curve X, and let*

$$\varphi\colon H^0(X, L)\otimes H^0(X, M)\to H^0(X, L\otimes M)$$

be the natural pairing. Then

$$\dim\operatorname{im}\varphi\geqslant h^0(X, L)+h^0(X, M)-h^0(X, \mathcal{O}_X),$$

and therefore

$$h^0(X, L\otimes M)\geqslant h^0(X, L)+h^0(X, M)-h^0(X, \mathcal{O}_X).$$

PROOF. By our assumption, generic divisors from the linear systems $|L|$, $|M|$ and $|L\otimes M|$ are nonsingular, and the canonical morphism

$$|\varphi|\colon |L|\times|M|\to|L\otimes M|,\qquad (D_1, D_2)\mapsto D_1+D_2$$

always has finite fiber over a nonsingular divisor. On the other hand, on $\operatorname{im}\varphi$ we have a rational map

$$(\ )_0\colon H^0(X, L \otimes M) \to |L \otimes M|$$

which transforms a section $s \in H^0(X, L \otimes M)$ to its divisor of zeros $(s)_0$. The dimension of the fibers of this map is equal to the number of components of the curve X, i.e. to $h^0(X, \mathcal{O}_X)$. Therefore

$$\begin{aligned} \dim \operatorname{im} \varphi &= h^0(X, \mathcal{O}_X) + \dim \operatorname{im} |\varphi| \\ &\geqslant h^0(X, L) + h^0(X, M) - h^0(X, \mathcal{O}_X). \quad \blacksquare \end{aligned}$$

4.2. LEMMA (CLIFFORD'S THEOREM). *Let X be a Gorenstein curve, and let L be a nonsingular invertible sheaf on X such that the sheaf $\omega_X \otimes L^{-1}$ is also nonsingular. Then*

$$h^0(X, L) \leqslant (\deg L)/2 + h^0(X, \mathcal{O}_X).$$

Moreover, if $\omega_X \otimes L^{-1} \not\approx \mathcal{O}_X$ and the curve X is irreducible or the sheaf $\omega_X \otimes L^{-2}$ is nonsingular, then the equality holds if and only if L is isomorphic to a tensor product $\otimes_i M_i$ of free sheaves $M_i \in G_2^1$; in a natural sense, this decomposition is unique.

4.3. REMARKS. (a) Let L be an invertible sheaf. We recall that this sheaf is called *free* if for each point $x \in X$ there exists a section $s \in H^0(X, L)$ with $s(x) \neq 0$. The linear system $|L|$ is called *free* if L is free. That means that the linear system $|L|$ does not have fixed points. A free invertible sheaf L on a connected curve X defines a morphism

$$\varphi_L\colon X \to \mathbf{P}(H^0(X, L)^*) \approx \mathbf{P}^{\dim |L|}.$$

(b) From the preceding lemma it is easy to deduce that if the dualizing sheaf is nonsingular, thenit is free; hence for a connected curve X this sheaf defines a morphism

$$\kappa = \varphi_{\omega_X}\colon X \to \mathbf{P}(H^0(X, \omega_X)^*) \approx \mathbf{P}^{g-1},$$

which is called *canonical* (here g is the genus of the curve X). The image of this morphism is called a *canonical curve*.

(c) In accordance with the classical terminology, we say that an invertible sheaf L on a curve X is *special* if the sheaves L and $\omega_X \otimes L^{-1}$ are nonsingular; in particular, $h^1(X, L) > 0$. The linear systems of such sheaves admit a good description on the canonical curve.

An invertible sheaf L with $h^0(X, L) = r + 1$ and $\deg L = d$ is called a *sheaf of type G_d^r* (compare with §2). It is possible that for fixed r and d a generic curve X does not have invertible sheaves of type G_d^r although they exist on certain special curves. Such curves are called *special* (in the sense of moduli). A typical example of such curves is given by the curves from 4.2 with $M_i \in G_2^1$. These sheaves are special and free, and $h^0(X, M_i) = 2$. To these sheaves there corresponds a morphism

$$\gamma_i = \varphi_{M_i}\colon X \to \mathbf{P}^1,$$

which has degree 2 over the generic point of $\mathbf{P}^1$. Such a morphism will be called a *hyperelliptic structure* on X. If γ_i is a finite morphism, then the curve X is called *hyperelliptic*. Equivalently, a curve X is hyperelliptic if it is connected and Gorenstein, its dualizing sheaf ω_X is free, and the corresponding canonical curve $\kappa(X) \subset \mathbf{P}^{g-1}$ coincides with the normally embedded rational curve of degree $g - 1$ in $\mathbf{P}^{g-1}$. From this it follows that $\gamma_i \approx \kappa$ and the hyperelliptic structure γ_i on such X is *unique*.

In what follows we shall encounter some other types of special structures and the corresponding special curves.

PROOF OF LEMMA 4.2. By the Riemann-Roch theorem,

$$h^0(X, L) - h^0(X, \omega_X \otimes L^{-1}) = \deg L + h^0(X, \mathcal{O}_X) - h^0(X, \omega_X).$$

But by Lemma 4.1,

$$h^0(X, L) + h^0\big(X, \omega_X \otimes L^{-1}\big) \leqslant h^0(X, \mathcal{O}_X) + h^0(X, \omega_X).$$

Summing up, we obtain

$$2h^0(X, L) \leqslant \deg L + 2h^0(X, \mathcal{O}_X).$$

This yields the required inequality.

Suppose now that we have equality, i.e. $\dim|L| = (\deg L)/2$. It clearly suffices to consider the case when the curve X is connected. Then we have a canonical morphism

$$\kappa = \varphi_{\omega_X}\colon X \to \mathbf{P}^{g-1}.$$

The above relations yield the equality

$$\dim|L| + \dim|\omega_X \otimes L^{-1}| = \dim|\omega_X|.$$

Therefore a generic hyperplane section H intersects $\kappa(X)$ along two nonintersecting divisors D_1 and D_2 such that $\kappa^{-1}(D_1) \in |L|$ and $\kappa^{-1}(D_2) \in |\omega_X \otimes L^{-1}|$. In view of the geometric interpretation of the Riemann-Roch theorem, $\dim\langle D_1\rangle = (\deg L)/2 - 1$(1) and $\dim\langle D_2\rangle = \deg(\omega_X \otimes L^{-1})/2 - 1$. For a generic hyperplane section, $\langle D_1 + D_2\rangle = H$. Hence $\langle D_1\rangle \cap \langle D_2\rangle \neq \varnothing$. We also assume that the divisor $D_1 = \sum_i x_i$ is nonsingular and consists of distinct points. It is clear that $\deg D_1 \geqslant \dim\langle D_1\rangle + 1 = (\deg L)/2$. Using the above inequality, it is easy to verify that the degree of the mapping κ over an arbitrary generic point $x \in X$ does not exceed 2, and if this degree is equal to 2, then $\mathcal{O}_X(\kappa^{-1}(x))$ is a free sheaf of type G_2^1. In view of this, the last assertion of the lemma holds for X provided that $\deg D_1 = (\deg L)/2$, and in that case $M_i = -X(\kappa^{-1}(x_i))$.

It remains to show that under our assumptions $\deg D_1 \leqslant (\deg L)/2$. In fact, suppose that $\deg D_1 > (\deg L)/2$, and let D be a nonsingular divisor on $\kappa(X)$ consisting of $(\deg L)/2 - 1$ points $x_i \in \operatorname{Supp} D_1$ in general position among which there are all those points x_i of the divisor D_1 for which $\#(\kappa^{-1}(x_i)) = 2$. Set $D' = D_1 - D$; this is also a nonsingular divisor. Since

$$\dim\langle D\rangle + \dim\langle D_2\rangle = (\deg L)/2 - 2 + \deg\big(\omega_X \otimes L^{-1}\big)/2 - 1 = (\deg \omega_X)/2 - 3,$$

the linear system $|\kappa^{-1}(D')|$ has dimension 1. It is easy to see that this linear system has at least three nonfixed points. Therefore a generic divisor $\kappa^{-1}(D'') \in |\kappa^{-1}(D')|$ consists of $\deg D'$ distinct points, and $\#(\operatorname{Supp} D'' - \operatorname{Supp} D_1) \geqslant 3$. By construction, the divisors $\kappa^{-1}(D + D')$ and $\kappa^{-1}(D + D'')$ are linearly equivalent. Hence

$$\dim\langle D + D''\rangle = (\deg L)/2 - 1.$$

But $\dim\langle D + D'\rangle \cap \langle D + D''\rangle = \dim\langle D\rangle = (\deg L)/2 - 2$. Hence

$$\dim\langle D_3\rangle = (\deg L)/2,$$

where D_3 is the divisor on $\kappa(X)$ formed by the distinct points of the set $\operatorname{Supp}(D_1 + D'')$. By the above,

$$\deg \kappa^{-1}(D_3) \geqslant \deg L + 3.$$

The sheaf $\mathcal{O}_X(\kappa^{-1}(D_3))$ is nonsingular. On the other hand, if the sheaf $\omega_X \otimes L^{-2}$ is nonsingular, so is the sheaf $\omega_X(-\kappa^{-1}(D_3))$. So in this case we obtain a contradiction with

(1) $\langle\ \rangle$ denotes the linear span (author's remark).

the above inequality. If the curve X is irreducible, then so is $\kappa(X)$, and in this case it can be shown that the sheaf $\omega_X(-\kappa^{-1}(D_3))$ is nonsingular, which again leads to a contradiction. ∎

The following lemma extends the main result from [18] to the case of curves with singularities.

4.4. LEMMA (MARTENS' THEOREM). *Let X be a connected Gorenstein curve, and let Z be an irreducible subvariety in G_d^r such that for a generic point $[L] \in Z$ the sheaves L and $\omega_X \otimes L^{-1}$ are nonsingular. Then $\dim Z \leqslant d - 2r$. Moreover, if $\dim Z = d - 2r$ and for a generic point $[L] \in Z$ the sheaf $\omega_X \otimes L^{-2}$ is nonsingular, then on X there are r free sheaves M_i of type G_2^1 such that for a generic point $[L] \in Z$*

$$L \approx \left(\bigtimes_{i=1}^{r} M_i \right) \otimes \mathcal{O}_X(D),$$

where the fundamental subset D of the linear system $|L|$ is a nonsingular divisor of degree $d - 2r$ on X.

PROOF. By Corollary 2.7 and Lemma 4.1, for a generic point $[L]$ of the component of G_d^r containing Z we have

$$\dim Z \leqslant g - \dim \operatorname{im}(\ )_{[L]} \leqslant g + 1 - h^0(X, L) - h^0(X, \omega_X \otimes L^{-1}).$$

From the Riemann-Roch theorem it follows that

$$\dim Z \leqslant \deg L - 2h^0(X, L) + 2 \leqslant d - 2r.$$

Suppose now that $\dim Z = d - 2r$ and that for a generic point $[L] \in Z$ the sheaves L, $\omega_X \otimes L^{-1}$ and $\omega_X \otimes L^{-2}$ are nonsingular. By the above, $h^0(X, L) = r + 1$. Subtracting the fixed divisors of general linear systems $|L|$ for $[L] \in Z$, we reduce the problem to the case when the system $|L|$ for $[L] \in Z$, we reduce the problem to the case when the system $|L|$ is free. We fix two general divisors D_1, $D_2 \in |L|$ such that $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2 = \varnothing$. From the exact sequence

$$0 \to \omega_X(-D_1 - D_2) \xrightarrow{(s,-s)} \omega_X(-D_1) \oplus \omega_X(-D_2) \xrightarrow{(s+t)} \omega_X \to 0$$

we infer that

$$\dim \operatorname{im}(\ )_{[L]} \geqslant 2h^0(X, \omega_X \otimes L^{-1}) - h^0(X, \omega_X \otimes L^{-2}).$$

Applying again Corollary 2.7, we see that

$$d - 2r \leqslant g - 2(g + r - d) + h^0(X, \omega_X \otimes L^{-2}),$$

from which it follows that

$$h^0(X, \omega_X \otimes L^{-2}) \geqslant g - d = (\deg \omega_X \otimes L^{-2})/2 + 1.$$

We have already proved that the variety $\{[\omega_X \otimes L^{-2}] | [L] \in Z\}$ has dimension $\leqslant 0$. Therefore $\dim Z \leqslant 0$ and $d = 2r$. The last assertion of Martens' theorem now follows from Clifford's theorem. ∎

4.5. THEOREM. *Let X be a connected Gorenstein curve, and let Z be an irreducible subvariety in G_d^1 such that for a generic point $[L] \in Z$ the sheaves L, $\omega_X \otimes L^{-1}$ and $\omega_X \otimes L^{-2}$ are nonsingular and the sheaf L is free. Suppose also that $(X_1, X_2) \geqslant 2$ for each decomposition $X_1 \cup X_2 = X$. Then*

$$\dim Z \leqslant d - 4,$$

with the exception of the following cases:

(4.5.1) $d = 2$, $\dim Z = 0$; $Z = \{[L]\}$, *where* L *is a free sheaf of type* G_2^1;

(4.5.2) $d = 3$, $\dim Z = 0$; $Z = \{[L]\}$, *where* L *is a free sheaf of type* G_3^1;

(4.5.3) $d = 4$; $\dim Z = 1$; *a generic point* $[L]$ *from* Z *has the form* $[L] = \varepsilon^*([M])$, *where* $\varepsilon\colon X \to E$ *is a morphism onto a connected curve* E *consisting of at most two irreducible components*; *the* (*arithmetic*) *genus of* E *is equal to* 1; *the morphism* ε *has degree* 2 *over the generic points of* E, *and* M *is a nonsingular ample sheaf of degree* 2 *on* E; *the morphism* ε *is determined uniquely by* Z;

(4.5.4) $d = 4$, $\dim Z = 1$; *a general point from* Z *has the form* $[L] = q^*([M(x-x)])$, *where*

$$q\colon X \rangle\!\rangle\!\rangle\!\to Q \subset \mathbf{P}^2$$

is a morphism onto a plane curve of degree 5 (*a quintic*), M *is the sheaf on* Q *induced by a hyperplane section, and* x *is a nonsingular point of an irreducible component* $Q_1 \subset Q$; *for* $g(X) \geqslant 7$, Q_1 *is a line*.

4.6. Remarks. (a) In accordance with the definitions given above, we shall call a morphism $\tau\colon X \to \mathbf{P}^1$ which has degree 3 over the generic point a *trigonal structure* on X. The sheaf L from (4.5.2) defines such a structure on X. If the above morphism is finite, then we say that the curve X is *trigonal*.

(b) A morphism $\varepsilon\colon X \to E$ satisfying (4.5.3) will be called a *superelliptic structure*. A curve X will be called *superelliptic* if ε is finite. The *ampleness* of M means that all components of its multidegree are positive; from this it follows that we may assume that E is a plane curve of degree 3.

(c) In order to get a better understanding of the case (4.5.4), we consider the case when the curve X is *stable* in the sense of Mumford and Deligne, i.e. X is a connected curve whose only singularities are ordinary double points and there are no nonsingular rational components $X_i \approx \mathbf{P}^1$ intersecting the other components of X along a set consisting of at most two points. In this case the canonical morphism $\kappa = \varepsilon_{\omega_X}\colon X \to \mathbf{P}^{g-1}$, where g is the genus of X, is well defined and finite. We observe that in view of the one-dimensionality of Z, $\deg(\omega_X \otimes L^{-2}) > 0$. Therefore $g \geqslant 6$, and for $g = 6$ q is an isomorphic embedding, i.e. X is a *plane quintic*. It is easy to verify that in a natural sense the structure of plane quintic is unique. If $g \geqslant 7$, then the morphism

$$q\colon X \to Q \subset \mathbf{P}^2$$

is not birational, i.e. there exists a component $Q' \subset Q$ over which the degree of q is greater than 1. But $\deg q^*(M(-x)) = 4$. Hence q is not birational only over the line $Q' = Q_1$ formed by the points x. We denote by Q_2 the complement of Q_1 in Q. The curve Q_2 has degree 4 and genus 3. If in addition the curve satisfies condition (S), i.e. $(X_1 \cap X_2) \geqslant 4$ for each decomposition $X_1 \cup X_2$ of X, then the fibers of q over the points from $Q_2 - Q_1$ are finite, since a plane curve of degree 4 may have a singular poinit of degree 4 only if this curve is a union of four lines passing through this point. This is also impossible under our assumptions. Hence the proper preimage Y of the curve Q_2 with respect to q is a connected curve of genus $\leqslant 3$ having at most four intersection points with the other components of X. From the nonsingularity of $\omega_X \otimes L^{-2}$ it follows that

$$\deg\left(\omega_X|_Y\right) \geqslant \deg\left(L^{\otimes 2}|_Y\right) = 8.$$

Therefore, in view of (3.12.2), $Y \approx_q Q_2$ has genus 3 and intersects the other components at 4 points; moreover, $L|_Y \approx \omega_Y \approx \mathcal{O}_Y(Y, X')$, where X' is the complement of Y in X which is clearly equal to $q^{-1}(Q_1)$ (compare with [3], 4.9).

(d) In particular, if X is an irreducible curve, then in Theorem 4.5 $\dim Z \leqslant d - 4$ with the exception of the cases when X is a hyperelliptic, trigonal or superelliptic curve or a plane quintic (for $g = 6$) (compare with the Appendix in [17]).

(e) If the curve X is nonsingular and irreducible, then from the condition $d \leqslant g - 2$ it follows that the sheaf $\omega_X \otimes L^{-1}$ is nonsingular, and from the condition $\dim Z \geqslant d - 3$ it follows that the sheaf $\omega_X \otimes L^{-2}$ is nonsingular (compare with the inequality $h^1(X, L^{\otimes 2}) \geqslant g - d - 1$ in the subsequent proof). Therefore in the case of a nonsingular irreducible curve X, replacing the condition of nonsingularity of the sheaves $\omega_X \otimes L^{-1}$ and $\omega_X \otimes L^{-2}$ by the conditions $0 < \deg L < g - 1$ and $\dim Z \geqslant d - 3$, we obtain the same exceptions as in Mumford's theorem on special divisors in the Appendix to [17].

PROOF OF THEOREM 4.5. By 4.4, $\dim Z \leqslant g - 3$ with the exception of the case (4.5.1). Therefore in what follows we assume that $\dim Z = d - 3$ and find out when this is possible. Let $[L]$ be a general point from Z. As in the proof of Lemma 4.4, we prove the inequality

$$\dim \operatorname{im}(\ )_{[L]} \geqslant 2h^0(X, \omega_X \otimes L^{-1}) - h^0(X, \omega_X \otimes L^{-2}).$$

On the other hand, by Corollary 2.7

$$\dim \operatorname{im}(\ )_{[L]} \leqslant g - d + 3,$$

where g is the genus of X. By duality,

$$2h^1(X, L) - h^1(X, L^{\otimes 2}) \leqslant g - d + 1.$$

We observe that, by Lemma 4.4, $h^0(X, L) = 2$. Therefore $h^1(X, L) = g - d + 1$ and

$$h^1(X, L^{\otimes 2}) \geqslant g - d - 1,$$

which by the Riemann-Roch theorem yields

$$h^0(X, L^{\otimes 2}) \geqslant d.$$

Thus a general point $[L] \in Z$ is transformed to $[L^{\otimes 2}] \in G_{2d}^{d-1}$; we denote by $Z^{\otimes 2}$ the corresponding image $\{[L^{\otimes 2}] \in \operatorname{Pic}(X) | [L] \in Z\}$. It is clear that $\dim Z^{\otimes 2} = \dim Z$. Applying this time Lemma 4.4 to $Z^{\otimes 2}$, we come to the inequality

$$d - 3 = \dim Z^{\otimes 2} \leqslant 2d - 2(d - 1) = 2.$$

Hence $3 \leqslant d \leqslant 5$. If $d = 3$, we obtain (4.5.2).

Next we show that the case $d = 5$, $\dim Z = 2$ is impossible. In fact, otherwise $h^0(X, L) = 2$, $h^0(X, L^{\otimes 2}) = 5$ and we have equality in the above inequalities. In particular, $\dim \operatorname{im}(\ )_{[L]} = g - 2$ and, at a general point $[L] \in Z$, $\operatorname{im}(\ )_{[L]}$ defines a complete system of linear equations for the tangent subspace $T_{Z,[L]} \subset T_{\operatorname{Pic}(X),[L]}$ to Z at $[L]$. It is clear that the map $[L] \mapsto [L^{\otimes 2}]$ preserves the tangent spaces. Therefore $\operatorname{im}(\ )_{[L^{\otimes 2}]} \subseteq \operatorname{im}(\ )_{[L]}$. We fix two general divisors $D_1 \in |L|$ and $D_2 \in |\omega_X \otimes L^{-2}|$ with $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2 = \varnothing$. Then we obtain the following exact sequence:

$$0 \to \omega_X(-D_1 - D_2) \xrightarrow{(s,-s)} \omega_X(-D_1) \oplus \omega_X(-D_2) \xrightarrow{(s+t)} \omega_X \to 0.$$

Now we observe that

$$\operatorname{im}(s + t) \subseteq \operatorname{im}(\ )_{[L]} + \operatorname{im}(\ )_{[L^{\otimes 2}]} = \operatorname{im}(\ )_{[L]}.$$

Hence

$$g - 2 = \dim \operatorname{im}(\ )_{[L]} \geqslant h^0\left(X, \omega_X \otimes L^{-1}\right) + h^0\left(X, L^{\otimes 2}\right) - h^0(X, L),$$

and the Riemann-Roch theorem yields a contradiction, namely

$$2 = h^0(X, L) \geqslant 3.$$

Thus it remains to consider the case when $d = 4$ and $\dim Z = 1$. Let L' be another general sheaf from $[L'] \in Z$. Since $|L|$ and $|L'|$ are one-dimensional free linear systems and $\dim Z \geqslant 1$, $|L \otimes L'|$ is a free system of dimension > 2, from which it follows that $h^0(X, L \otimes L') \geqslant 4$. Applying 4.4, we see that $h^0(X, L^{\otimes 2}) \leqslant 4$. In view of the semicontinuity of h^0, $h^0(X, L \otimes L') = 4$ (for general $[L], [L'] \in Z$). From this it follows that for such L and L' the sheaf $\omega_X \otimes L^{-1} \otimes L'^{-1}$ is nonsingular. By the Riemann-Roch theorem,

$$h^0\left(X, \omega_X \otimes L^{-1}\right) = g - 3,$$

and

$$h^0\left(X, \omega_X \otimes L^{-1} \otimes L'^{-1}\right) = g - 5.$$

We fix a set of $g - 6$ general points $P_1, \dots, P_{g-6}$ on X such that for general $[L], [L'] \in Z$

$$H^0\left(X, \omega_X\left(-\sum_{i=1}^{g-6} P_i\right) \otimes L^{-1} \otimes L'^{-1}\right) \neq \{0\}$$

and the sheaf

$$\omega_X\left(-\sum_{i=1}^{g-6} P_i\right) \otimes L^{-1} \otimes L'^{-1}$$

is nonsingular. Then each global section $s_{[L']}$ of this sheaf defines an inclusion

$$H^0(X, L') \subset \xrightarrow{\otimes s_{[L']}} H^0(X, M),$$

where $M \overset{\text{def}}{=} \omega_X(-\Sigma_{i=1}^{g-6} P_i) \otimes L^{-1}$. The free part of the linear system $|M|$ defines a morphism

$$q = \varphi_M \colon X \to \mathbf{P}^2,$$

and the morphism $\varphi_{L'} \colon X \to \mathbf{P}^1$ is a composition of φ_M and the projection from a point in $\mathbf{P}^2$. In view of the irreducibility of Z and the fact that $|L'|$ is free, there exists a unique irreducible component Q_1 of the image $Q = \varphi_M(X) \subset \mathbf{P}^2$ containing the centers of these projections. We denote by Q_i the other irreducible components of Q. Let

$$d_i = \deg Q_i, \qquad r_i = \deg\left(L|_{\varphi_M^{-1}(Q_i)}\right).$$

If $\deg Q = 5$, we obtain (4.5.4) (compare with (4.6c)). Moreover, it is clear that $\deg Q \leqslant 5$. Therefore in what follows it suffices to consider the case

$$\deg Q = \sum_{i \geqslant 1} d_i \leqslant 4.$$

An easy computation of degrees yields the following equality:

$$r_1(d_1 - 1) + \sum_{i \geqslant 2} r_i d_i = 4.$$

Next we show that $r_i d_i \geqslant 2$ for $i \geqslant 2$. Otherwise X would have an irreducible component $X_i \approx \mathbf{P}^1$ which is isomorphically mapped onto a line $Q_i \subset \mathbf{P}^2$ and for which $\deg(L|_{X_i}) = 1$. Since the sheaf $\omega_X \otimes L^{-2}$ is nonsingular, $\deg(\omega_X|_{X_i}) \geqslant 2$. Therefore the other components of X intersect X_i along a divisor of degree $\geqslant 4$ (compare with (3.12.2)), which contradicts the assumption that $\deg Q \leqslant 4$. Here we also use the condition on the decompositions of the curve X.

On the other hand, $\deg Q \geqslant 3$ since $\dim Z = 1$. If $\deg Q = 3$, then we obtain (4.5.3) with $\varepsilon = q$ and $E = Q$, possibly after blowing down the component Q_1 when $d_1 = 1$. Using the above inequalities, it is easy to verify that q has degree $\neq 2$ over one of the components of E only in the case when $d_1 = 2$, $r_1 = 1$, $d_2 = 1$ and $r_2 = 3$ (cf. Figure 6). Proceeding as in the above proof of the inequality $r_i d_i \geqslant 2$, one can show that this contradicts the nonsingularity of $\omega_X \otimes L^{-2}$.

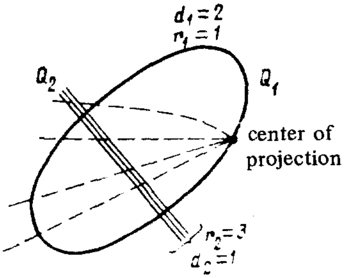

FIGURE 6

Thus it remains to show that the case $\deg Q = \Sigma_{i \geqslant 1} d_i = 4$ is impossible. In view of the above relations, there exists an irreducible component $Q_i \subset Q$ with $r_i = 1$ and $d_i \geqslant 2$ if $i = 1$. Since $r_i d_i \geqslant 2$ for $i \geqslant 2$, we conclude that $d_i \geqslant 2$. If $d_i = 2$, then $Q_i \approx \mathbf{P}^1$ is a plane conic. Reasoning as above for $r_i = d_i = 1$ and $i \geqslant 2$, we see that $i = 1$. Hence

$$\sum_{i \geqslant 2} d_i = 2, \qquad r_i d_i \geqslant 2, \qquad \sum_{i \geqslant 2} r_i d_i = 3.$$

This system of relations does not have solutions. Therefore $d_i \geqslant 3$. But it is clear that $d_i \leqslant 3$. Hence $d_i = 3$. From the relations for the general case it follows that $i = 1$. Let X_1 be the proper preimage of the curve Q_1. By the condition on the decompositions of the curve X the other components of X intersect X_1 along a divisor of degree $\leqslant 3$ if $X_1 \approx Q_1$ and of degree $\leqslant 5$ if $X_1 \approx \mathbf{P}^1$. This contradicts the nonsingularity of the divisor

$$\omega_X \otimes L^{-2}|_{X_1}.$$

The uniqueness in (4.5.3) immediately follows from the ampleness of M, since then $M^{\otimes 2}$ is very ample. ∎

4.7. REMARK. From the proof it is easy to see that if we discard the condition $(X_1, X_2) \geqslant 2$ for the decompositions $X_1 \cup X_2 = X$, then we obtain a few new possibilities in the case when $d = 4$, $\dim Z = 1$ which can easily be described quite explicitly. This condition on the decompositions holds for each connected curve satisfying (RK).

§5. Sing Ξ

Before formulating an analogue of the Riemann-Kempf theorem on singularities in the case of Prymians, we give a definition of the skew-symmetric pairing $[\ ]_\varphi$.

Let $(\tilde{C}, I)$ be a pair consisting of a curve $\tilde{C}$ with ordinary double points and an involution I on this curve. As in 3.19, we assume that $c_e = 0$ and that condition (F) is satisfied. Consider a point $[L] \in \mathrm{Pv}(\tilde{C}, I)$. By definition $\mathrm{Nm}\, L \approx \omega_C$, from which by (3.1.1) it follows that

$$L \otimes I^*L \approx \pi^*\mathrm{Nm}\, L \approx \pi^*\omega_C.$$

We fix such an isomorphism $\varphi\colon L \otimes I^*L \xrightarrow{\approx} \pi^*\omega_C$. Then we obtain a pairing

$$(\ )_\varphi\colon H^0(\tilde{C}, I) \otimes H^0(\tilde{C}, I) \to H^0(\tilde{C}, \pi^*\omega_C),$$
$$s \otimes t \mapsto (s \otimes t)_\varphi = \varphi(s \otimes I^*t).$$

The involution I^* acts on the space $\mathrm{H}^0(\tilde{C}, \pi^*\omega_C)$. With respect to this involution, $H^0(\tilde{C}, \pi^*\omega_C)$ splits into a direct sum

$$H^0(\tilde{C}, \pi^*\omega_C)^+ \oplus H^0(\tilde{C}, \pi^*\omega_C)^-,$$

where $H^0(\tilde{C}, \pi^*\omega_C)^+ = \pi^*(H^0(C, \omega_C))$ is the space of invariant differentials and $H^0(\tilde{C}, \pi^*\omega_C)^-$ is a subspace in the space $H^0(\tilde{C}, \omega_{\tilde{C}})^-$ of anti-invariant differentials which are also called *Prym differentials*. It is easy to verify that

$$I^*(s \otimes t)_\varphi = (t \otimes s)_\varphi.$$

Therefore the above pairing gives rise to the following two pairings:

$$\mathrm{Symm}^2\, H^0(\tilde{C}, I) \to H^0(\tilde{C}, \pi^*\omega_C)^+ = H^0(C, \omega_C)$$

and

$$[\ ]_\varphi\colon \Lambda^2 H^0(\tilde{C}, I) \to H^0(\tilde{C}, \pi^*\omega_C)^-,$$
$$s \wedge t \mapsto [s \wedge t]_\varphi = \varphi(s \otimes I^*t - t \otimes I^*s).$$

In what follows we shall often write simply $[s \wedge t]$, assuming that φ is fixed.

Unless otherwise explicitly specified, in this section we shall asssume that $(\tilde{C}, I)$ is a Beauville pair. Then by the analogue of the Hurwitz formula (cf. (3.12d)) $\pi^*\omega_C = \omega_{\tilde{C}}$. Since the Prymian is given by the anti-invariant part of the corresponding Jacobian, for each point $[L] \in \mathrm{Pv} = \mathrm{Pv}(\tilde{C}, I)$ there are canonical identifications

$$T^*_{\mathrm{Pv},[L]} = H^0(\tilde{C}, \omega_{\tilde{C}})^- = H^0(\tilde{C}, \pi^*\omega_C)^-,$$

where $T^*_{\mathrm{Pv},[L]}$ is the tangent space to the Prym variety Pv at the point $[L]$. The above skew-symmetric pairing can be written in the form

$$[\ ]_\varphi\colon \Lambda^2 H^0(\tilde{C}, I) \to T^*_{\mathrm{Pv},[L]}, \qquad s \wedge t \mapsto \varphi(s \otimes I^*t - t \otimes I^*s).$$

Let $s_1, \ldots, s_m$ be a basis of the space $H^0(\tilde{C}, L)$, and let $m = h^0(\tilde{C}, L)$. Set $\omega^-_{ij} = [s_i \wedge s_j] \in T^*$. Then we obtain a skew-symmetric matrix $\omega^- = (\omega^-_{ij})_{1 \leqslant i,j \leqslant m}$. We denote by $\mathrm{Pf}(\omega^-)$ the Pfaffian of this matrix; modulo sign, this is equivalent to the equality $\mathrm{Pf}(\omega^-)^2 \overset{\mathrm{def}}{=} \det(\omega^-)$. The Pfaffian $\mathrm{Pf}(\omega^-)$ is a polynomial homogeneous form of degree $m/2$ on the tangent space T; for $m = 0$ we assume that this form is identically equal to 1.

5.1. THEOREM. *Let $(\tilde{C}, I)$ be a Beauville pair, and let Ξ be the canonical polarization divisor on the Prym variety $\mathrm{Pv}(\tilde{C}, I)$. Then for each point $[L] \in \mathrm{Pv}(\tilde{C}, I)$ the following assertions hold*:

(5.1.1) *If* $\mathrm{Pf}(\omega^-) \not\equiv 0$, *then*

$$\mathrm{Mult}_{[L]}\Xi = h^0(\tilde{C}, L)/2$$

and the form $\mathrm{Pf}(\omega^-)$ *gives an equation of the tangent cone to* Ξ *at the point* $[L]$ *in the tangent space* $T_{\mathrm{Pv},[L]}$.

(5.1.2) *If* $\mathrm{Pf}(\omega^-) \equiv 0$, *then*

$$\mathrm{Mult}_{[L]}\Xi > h^0(\tilde{C}, L)/2.$$

(5.1.3) *In particular, for each point* $[L] \in \mathrm{Pv}(\tilde{C}, I)$

$$\mathrm{Mult}_{[L]}\Xi \geqslant h^0(\tilde{C}, L)/2.$$

We remark that although $\mathrm{Pf}(\omega^-) \not\equiv 0$ on a generic $\mathrm{Pv}(\tilde{C}, I)$ for all $[L]$, even for some nonsingular irreducible curves $\tilde{C}$ there may exist points $[L] \in \mathrm{Pv}(\tilde{C}, I)$ with $\mathrm{Pf}(\omega^-) \equiv 0$. They define singular points $[L] \in \Xi$ which are often called *Mumford singular points*. Below we shall describe the most typical of these points.

PROOF OF THEOREM 5.1. The divisor 2Ξ is defined by the intersection $\Theta \cdot \mathrm{Pv}(\tilde{C}, I)$ on $\mathrm{Jv}(\tilde{C})$. By the Riemann-Kempf theorem, the first term of the formal expansion of the function defining Θ at the point $[L]$ is equal to $\det(s_i \otimes t_j^*)$. Therefore the first term in the expansion of 2Ξ at $[L]$ is defined by the restriction of $\det(s_i \otimes t_j^*)$ to the tangent space $T_{\mathrm{Pv},[L]}$. If we consider the basis $\{t_j^*\}_1^m$ of the space $H^0(\tilde{C}, \omega_{\tilde{C}} \otimes L^{-1}) \approx H^0(\tilde{C}, I^*L)$ formed by the sections $\{i^*s_j\}_1^m$, then

$$\left(s_i \otimes I^*s_j\right)_{[L]}\big|_{T_{\mathrm{Pv},[L]}} = \omega_{ij}^-/2.$$

Therefore the first term of the expansion is equal to

$$\left(\tfrac{1}{2}\right)^{m/2}\det\left(\omega_{ij}^-\right)^{1/2} = \left(\tfrac{1}{2}\right)^{m/2}\mathrm{Pf}(\omega^-).$$

All assertions of the theorem easily follow from this fact. ■

The second useful tool in the study of $\mathrm{Sing}\,\Xi$ is the following:

5.2. LEMMA. *Let* Pv^r *be the subvariety of those points* $[L] \in \mathrm{Pv}(\tilde{C}, I)$ *for which* $h^0(\tilde{C}, L) \geqslant r + 1$, *let* $Z \subset \mathrm{Pv}^r$ *be a subvariety, and let* $[L] \in Z$ *be a point for which* $h^0(\tilde{C}, L) = r + 1$. *Then the tangent subspace* $T_{Z,[L]} \subset T_{\mathrm{Pv},[L]}$ *lies in the set of zeros of the forms from* $\mathrm{im}[\ ]_\varphi \subset T^*_{\mathrm{Pv},L}$, *where* $[\ ]_\varphi$ *denotes the skew-symmetric pairing* $[\ ]_\varphi\colon \Lambda^2 H^0(\tilde{C}, L) \to T^*_{\mathrm{Pv},[L]}$ *introduced above. Hence if* Z *is irreducible, then*

$$\dim Z \leqslant p - 1 - \dim \mathrm{im}[\ ]_\varphi,$$

where $p = g(C)$ *is the genus of the curve* $C = \tilde{C}/I$.

PROOF. In view of Corollary 2.7, the subspace $T_{Z[L]} \subset T_{\mathrm{Pic}(\tilde{C}),[L]}$ lies in the set of zeros of the forms from $\mathrm{im}(\ )_{[L]} \subset T^*_{\mathrm{Pic}(\tilde{C}),[L]}$. It is easy to verify that the restrictions of these forms to the tangent space $T_{\mathrm{Pv},[L]}$ sweep out $\mathrm{im}[\ ]_\varphi$, which yields the required result; we would also like to point out the equality $\dim T_{\mathrm{Pv},[L]} = \dim \mathrm{Pv} = p - 1$. ■

5.3. DEFINITION. A curve C whose partial desingularization at an ordinary double point $c \in C$ is a hyperelliptic curve will be called *quasitrigonal*. As in the trigonal case, the canonical morphism is birational, and its image is not defined by an intersection of quadrics if the curve is quasitrigonal but not hyperelliptic.

The proof of the following assertion will serve a model for the proof of Theorem 5.12, which is the main result of this section.

5.4. PROPOSITION. *Let $(\tilde{C}, I)$ be a Beauville pair satisfying condition* (S). *Then*

$$\dim \operatorname{Sing} \Xi \leqslant p - 5,$$

with the exception of the following cases:

(5.4.1) *C is a hyperelliptic curve*;

(5.4.2) *C is a quasitrigonal curve*;

(5.4.3) *$p \leqslant 3$, where p denotes the genus of the curve $C = \tilde{C}/I$.*

In the last three cases the principal Prymian $P(\tilde{C}, I)$ is isomorphic to a direct sum of Jacobians of hyperelliptic curves.

Here $\dim \operatorname{Sing} \Xi$ denotes the maximal dimension of irreducible components of the subvariety of singular points $\operatorname{Sing} \Xi$ of the canonical polarization divisor.

The proof is based on the following assertion.

5.5. LEMMA. *Let $Z \subset \operatorname{Sing} \Xi$ be an irreducible component of dimension $\geqslant p - 5$. Then for a generic point $[L] \in Z$*

(P) *there exist two linearly independent sections $s_1, s_2 \in H^0(\tilde{C}, L)$ such that $s_1 \otimes I^* s_2 = s_2 \otimes I^* s_1$ or equivalently $[s_1 \wedge s_2] = 0$.*

Conversely, if $[L] \in \mathrm{Pv}(\tilde{C}, I)$ satisfies condition (P), *then $[L] \in \operatorname{Sing} \Xi$.*

5.6. REMARK. It will be clear from the proof that we may assume that s_1 and s_2 depend "continuously" on $[L]$, i.e. there exist a set $U \subset Z$ which is open in the etale topology and two sections ξ_1, ξ_2 of the sheaf $p_* \mathscr{L}$ over U whose restrictions on $[L] \in U$ coincide with sections s_1 and s_2 satisfying (P); here p denotes the projection $\tilde{C} \times Z \to Z$ and $\mathscr{L}$ is the universal Poincaré sheaf on $\tilde{C} \times Z$.

PROOF. Let $[L]$ be a general point from Z. If $h^0(\tilde{C}, L) = 2$, then, in view of 5.1, from the fact that $[L]$ is a singular point of Ξ it follows that $\mathrm{Pf}(\omega^-) \equiv 0$. In this case the matrix ω^- has the form

$$\begin{pmatrix} 0 & [s_1 \wedge s_2] \\ [s_2 \wedge s_1] & 0 \end{pmatrix},$$

where s_1, s_2 is an arbitrary basis in $H^0(\tilde{C}, L)$. Therefore $[s_1 \wedge s_2] = 0$.

Since $h^0(\tilde{C}, L)$ is even, it remains to consider the case when $h^0(\tilde{C}, L) \geqslant 4$. But in view of Lemma 5.2 $\dim \operatorname{im}[\ ]_\varphi \leqslant 4$ and the affine subvariety of decomposable forms $s_1 \wedge s_2$ in $\wedge^2 H^0(\tilde{C}, L)$ has dimension $\geqslant 5$ since $h^0(\tilde{C}, L) \geqslant 4$. Therefore $\ker[\ ]_\varphi$ contains a decomposable form $s_1 \wedge s_2$ for which $[s_1 \wedge s_2] = 0$, from which it follows that $s_1 \otimes I^* s_2 = s_2 \otimes I^* s_1$.

The converse immediately follows from 5.1 (compare with the proof of Lemma 5.7). ■

5.7. LEMMA. *Let $[L]$ be a point of $\mathrm{Pv}(\tilde{C}, I)$ for which there exist three linearly independent sections $s_1, s_2, s_3 \in H^0(\tilde{C}, L)$ such that $s_i \otimes I^* s_j = s_j \otimes I^* s_i$ for all $1 \leqslant i, j \leqslant 3$. Then*

$$\mathrm{Mult}_{[L]} \Xi \geqslant 3.$$

PROOF. In view of 5.1 it suffices to consider the case when $h^0(\tilde{C}, L) = 4$. Considering the basis formed by the family $\{s_i\}_1^3$ and a section $s_4 \in H^0(\tilde{C}, L)$ we see that the 4×4 matrix ω^- has the form

$$\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & * \end{pmatrix}.$$

Therefore $\mathrm{Pf}(\omega^-) \equiv 0$, and applying 5.1 once more we conclude that $\mathrm{Mult}_{[L]} \Xi > 2$. ■

Property (P) has a convenient geometric interpretation.

5.8. LEMMA. *Let $\tilde{C}$ be an arbitrary connected curve with an involution I, and let L be an invertible sheaf on $\tilde{C}$ satisfying condition* (P) *with sections s_1 and s_2 which do not simultaneously vanish at each singular point of the curve C. Then $L \approx \pi^* M(D)$, where*

(5.8.1) *M is a free invertible sheaf on $C = \tilde{C}/I$ and $h^0(C, M) \geqslant 2$; and*

(5.8.2) *D is a nonsingular divisor on $\tilde{C}$ for which $\pi_* D \in |\mathrm{Nm}\, L \otimes M^{-2}|$; in particular, the sheaves* $\mathrm{Nm}\, L \otimes M^{-1}$ *and* $\mathrm{Nm}\, L \otimes M^{-2}$ *are nonsingular.*

Conversely, if M is a free invertible sheaf on C with $h^0(C, M) = m$ and D is a nonsingular sheaf on $\tilde{C}$, then the sheaf $L = \pi^* M(D)$ has m sections $s_1, \ldots, s_m$ which pairwise satisfy condition (P).

PROOF. Replacing s_1 and s_2 by their linear combinations, we may assume that they do not simultaneously vanish at each singular point of $\tilde{C}$. Consider the rational function $\varphi = s_1/s_2$ on $\tilde{C}$. Since $I^*\varphi = \varphi$ by (P), we have $\varphi = \pi^*\psi$ for some rational function ψ on C. Let D be the divisor of common zeros of the sections s_1 and s_2, and let $(s_1)_0$, $(\varphi)_0$ and $(\psi)_0$ be the divisors of zeros of the section s_1, the function φ and the function ψ. All these divisors are nonsingular, and

$$(s_1)_0 = (\varphi)_0 + D, \qquad (\varphi)_0 = \pi^*(\psi)_0.$$

Therefore $L \approx \pi^* M(D)$, where $M = \mathcal{O}_C((\psi)_0)$. By our assumption, there are not components on which the involution acts identically. Hence $\pi_*(s_1)_0 = \pi_*(\varphi_0) + \pi_* D = 2(\psi)_0 + \pi_* D$, from which it follows that $\pi_* D \in |\mathrm{Nm}\, L \otimes M^{-2}|$.

Conversely, let $s_1', \ldots, s_m' \in H^0(\tilde{C}, \pi^* M)$ be the lifting of a basis of the space $H^0(C, M)$. Performing the natural identification $\pi^* M = I^*\pi^* M$, we obtain the equalities $I^* s_i' = s_i'$, $1 \leqslant i \leqslant m$. From this it follows that condition (P) holds for the pairs of sections $s_i = s_i' \otimes s$, where s is a section from $H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(D))$. ■

5.9. REMARK. Thus the triple (L, s_1, s_2) defines M and D uniquely. Furthermore, for such a "continuously varying" triple at a generic point there exists a morphism to the pairs (M, D).

The condition that the sections s_1 and s_2 in (P) do not simultaneously vanish at all singular points of $\tilde{C}$ is not always satisfied. To work also in these cases, we shall need partial desingularizations. So, we return to the beginning of this section and suppose that the pair $(\tilde{C}, I)$ satisfies condition (F) and $c_e = 0$.

Let $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ be a partial desingularization of the curve $\tilde{C}$ or, which is equivalent, of the pair $(\tilde{C}, I)$. Let $[L] \in \mathrm{Pv}(\tilde{C}, I)$. The isomorphism $\varphi\colon L \otimes I^* L \xrightarrow{\approx} \pi^*\omega_C$ induces an isomorphism $\tilde{f}^* L \otimes I'^* \tilde{f}^* L \approx \pi'^* f^* \omega_C$ and hence, in view of (3.12.1), an isomorphism

$$\varphi'\colon \tilde{f}^0 L \otimes I'^* \tilde{f}^0 L \xrightarrow{\approx} \pi'^* \omega_{C'}.$$

The last isomorphism induces a pairing

$$[\ ]_{\varphi'}\colon \Lambda^2 H^0(\tilde{C}', \tilde{f}^0 L) \to H^0(\tilde{C}', \pi'^* \omega_{C'}).$$

We denote by S the set of those points $s \in \tilde{C}$ which are resolved by the desingularization $\tilde{f}$ and by $H^0(\tilde{C}, L)_S$ the subspace of sections from $H^0(\tilde{C}, L)$ vanishing at all points of S.

5.10. LEMMA. *There is an isomorphism*

$$H^0(\tilde{C}', \tilde{f}^0 L) \underset{s' \leftarrow s}{\xleftarrow{\approx}} H^0(\tilde{C}, L)_S,$$

compatible with the skew-symmetric pairings; in particular, for all $s_1, s_2 \in H^0(\tilde{C}, L)_S$

$$[s_1, s_2]_\varphi = 0 \Leftrightarrow [s_1', s_2']_{\varphi'} = 0.$$

PROOF. Let $\mathscr{I}_S$ be the sheaf of ideals of the desingularized singularities. In our case

$$\mathscr{I}_S = \tilde{f}_* \mathscr{O}_{\tilde{C}'}\Big(-\sum_{\tilde{f}(x) \in S} x\Big),$$

and, using the local construction from 1.5, it is easy to construct an isomorphism $L \otimes \mathscr{I}_S \xrightarrow{\approx} \tilde{f}_* \tilde{f}^0 L$. Hence

$$H^0(\tilde{C}, L)_S = H^0(\tilde{C}, L \otimes \mathscr{I}_S) \xrightarrow{\approx} H^0(\tilde{C}, \tilde{f}_* \tilde{f}^0 L) = H^0(\tilde{C}', \tilde{f}^0 L).$$

The compatibility is obvious. ∎

PROOF OF PROPOSITION 5.4. Let Z be an irreducible component of Sing Ξ of dimension $\geqslant p - 4$. For $p \leqslant 3$ everything is obvious. So we assume that $p \geqslant 4$. Then $Z \neq \varnothing$ and condition (P) holds for a generic point $[L] \in Z$ and $s_1, s_2 \in H^0(\tilde{C}, L)$. Let $\tilde{C}_0$ be the maximal component of the curve $\tilde{C}$ on which the sections s_1 and s_2 simultaneously vanish, and let $\{z_i\}_1^n$ be the set of all singular points of $\tilde{C}$ at which both s_1 and s_2 vanish. In view of Remark 5.6, we may assume that $\tilde{C}_0$ and $\{z_i\}_1^n$ do not depend on the choice of generic point $[L] \in Z$ for suitable s_1 and s_2. Consider the partial singularization $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ of $\tilde{C}$ at the points of the set $\{z_i\}_1^n$. The isogeny $\tilde{f}^0\colon \mathrm{Pv}(\tilde{C}, I) \to \mathrm{Pv}(\tilde{C}', I')$ maps Z onto an irreducible variety $Z' \subset \mathrm{Pv}(\tilde{C}', I')$ of the same dimension. By the preceding remark and Lemma 5.10, condition (P) holds for a generic point $[L'] \in Z'$ and sections s_1 and s_2 which do not simultaneously vanish at each singular point of $\tilde{C}'$. Moreover, we may assume that s_1 and s_2 do not simultaneously vanish at each generic point of $\tilde{C}_1'$, where $\tilde{C}_1'$ is the preimage of the complementary curve $\tilde{C}_1$ to $\tilde{C}_0$ in $\tilde{C}$. Set $\tilde{C}_0' = \tilde{f}^{-1}(\tilde{C}_0)$. For the Beauville curves, the preimage of a singular point under the desingularization consists of two nonsingular points which are stable with respect to the involution. Hence the curve $\tilde{C}_1'$ is connected, since otherwise

$$\dim Z = \dim Z' \leqslant \dim \mathrm{Pv}(\tilde{C}_0', I_0') + \sum_i \dim \Xi(\tilde{C}_{1,i}', I_{1,i}'),$$

where $\tilde{C}_{1,i}'$ are the connected components of $\tilde{C}_1'$, and by Lemma 3.20 and assumption (S) this expression is

$$\leqslant \dim \mathrm{Pv}(\tilde{C}_0', I_0') + \dim \mathrm{Pv}(\tilde{C}_1', I_1') - 4 = \dim \mathrm{Pv}(\tilde{C}', I') - 4$$

and also

$$= \dim \mathrm{Pv}(\tilde{C}, I) - 4 = p - 5$$

by Lemma 3.15 and Proposition 3.9, which contradicts the hypothesis on the dimension of Z. Thus $\tilde{C}_1'$ is a connected curve. By Lemma 5.8, $L'|_{\tilde{C}_1'}$ is representable in the form $\pi_1'^* M(D)$, where M is a free invertible sheaf on $C_1' = \tilde{C}_1'/I_1'$ for which $h^0(C', M) \geqslant 2$ and D is a nonsingular divisor on $\tilde{C}_0'$. We denote by $Z_1' \subset \mathrm{Pic}(C_1')$ the irreducible variety of such sheaves $[M]$ (cf. 5.9). For a generic point $[M] \in Z_1'$, the sheaves M, $\omega_{C_1'} \otimes M^{-1}$ and $\omega_{C_1'} \otimes M^{-2}$ are nonsingular and the divisor D is finitely mapped onto $\pi_{1*}' D \in |\omega_{C_1'} \otimes M^{-2}|$. Therefore

$$\begin{aligned} \dim \mathrm{Pv}(\tilde{C}, I) - 3 \leqslant \dim Z &= \dim Z' \\ &\leqslant \dim \mathrm{Pv}(\tilde{C}_0', I_0') + \dim Z_1' + \dim \big\|\omega_{C_1'} \otimes M^{-2}\big\|, \end{aligned}$$

from which it follows that

$$\dim \mathrm{Pv}(\tilde{C}_1', I_1') - 3 \leqslant \dim Z_1' + \dim \left|\omega_{C_1'} \otimes M^{-2}\right|.$$

By Lemmas 4.4 and 4.2,

$$\dim Z_1' \leqslant d - 2, \qquad \dim \left|\omega_{C_1'} \otimes M^{-2}\right| \leqslant g(C_1') - 1 - d,$$

where $d = \deg M$. Hence $\dim \mathrm{Pv}(\tilde{C}_1', I_1') \leqslant g(C_1')$ and the equality is possible only if M is a nonsingular sheaf of type G_2^1. Furthermore, by (3.9.3) $-1 + r/2 \leqslant 0$, where $r = r(\tilde{C}_1', I_1')$, from which it follows that $r \leqslant 2$. Applying (S) once again, we see that $\tilde{C}_0 = \varnothing$ and $\#\{z_i\}_1^n = n = r/2 \leqslant 1$ and $\tilde{C}_1'$ is a hyperelliptic curve.

The last assertion for a nonsingular hyperelliptic curve and a quasitrigonal curve with a unique singular point was proved by Dalalyan [9], [10]; in the quasitrigonal case one should take into consideration Remark 3.17(b). The singular case is obtained from the nonsingular one by passing to the limit. ■

The main result of this section is the description of Prym varieties with $\dim \mathrm{Sing}\,\Xi \geqslant p - 5$. In view of a well-known property of the singularities of the polarization divisor of the Jacobian of a nonsingular curve, all Prymians which are isomorphic to Jacobians satisfy the above condition. In the following sections we shall find out which of them are really isomorphic to Jacobians. Before formulating Theorem 5.12, we describe the types of singularities that we shall need.

5.11. *Varieties of special singularities.* Each of the varieties $Z(\cdots)$ described below is contained in $\mathrm{Sing}\,\Xi$, and the varieties $Z'(\cdots)$ are contained in $\mathrm{Mult}_3\,\Xi$, the subvariety of points of the divisor Ξ of multiplicity $\geqslant 3$. The proof of this fact is based on Lemmas 5.5, 5.7, 5.8, 5.10 and the obvious fact that if $\tilde{C} = \tilde{C}_1 \sqcup \tilde{C}_2$ is a disjoint union, $I(\tilde{C}_i) = \tilde{C}_i$, L is an invertible sheaf on $\tilde{C}$ and $s_1, s_2 \in H^0(\tilde{C}, L)$ are sections with $s_i|_{\tilde{C}_i} \equiv 0$, $i = 1, 2$, then $s_1 \otimes I^*s_2 \equiv 0 \equiv s_2 \otimes I^*s_1$ and therefore $[L]$ satisfies condition (P) for s_1 and s_2 provided that both these sections $\not\equiv 0$. In what follows we assume that $(\tilde{C}, I)$ is a Beauville pair satisfying condition (S) for which the quotient curve C is *neither hyperelliptic nor quasitrigonal.* Under these conditions, all subvarieties $Z(\cdots)$ constructed below have dimension $\leqslant p - 5$, and $\dim Z'(\cdots) \leqslant p - 7$. As a matter of fact, it will be clear that some of these subvarieties satisfy the equality $\dim Z(\cdots) = p - 5$; and if this is so, we shall point out this fact. Describing the subvarieties $Z(\cdots)$ and $Z'(\cdots)$ in $\mathrm{Pv}(\tilde{C}, I)$, we determine the structure of the sheaf L or its lifting to some partial desingularization, where $[L] \in Z(\cdots)$, $Z'(\cdots)$ are generic points. For this construction we need some special structures on C which exist on only a few of the curves C corresponding to the pair $(\tilde{C}, I)$. The introduction of these structures allows to select the Prymians whose polarization divisor has a large singular subset ($\dim \mathrm{Sing}\,\Xi \geqslant p - 5$) and the more refined analysis carried out in the following sections enables one to pick those Prymians that are isomorphic to Jacobians.

Type I (*trigonal*). Let $\tau\colon C \to \mathbf{P}^1$ be a trigonal structure on C. The subvariety $Z(\tau) \subset \mathrm{Pv}$ has generic points $[L]$ for which $L \approx (\tau \circ \pi)^* M(D)$, where $M = \mathcal{O}_{\mathbf{P}^1}(1)$ and D is a nonsingular divisor on $\tilde{C}$. We observe that by (S) the curve C in this case is trigonal.

Type II (*superelliptic*). Let $\varepsilon\colon C \to E$ be a superelliptic structure, where E is a plane $(\subset \mathbf{P}^2)$ curve of arithmetic genus 1 which has at most two irreducible components. The subvariety $Z(\varepsilon) \subset \mathrm{Pv}$ has generic points $[L]$ for which $L \approx (\varepsilon \circ \pi)^* M(D)$, where M is a free invertible ample (i.e. such that all components of its multidegree are positive) sheaf of degree 2 on E and D is a nonsingular divisor on $\tilde{C}$. Taking a sheaf M of degree 3 and proceeding in a similar way, we define the subvariety $Z'(\varepsilon) \subset \mathrm{Pv}$.

Type III. Consider the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{f} & C \\ \gamma \downarrow & & \\ \mathbf{P}^1 & & \end{array}$$

where γ is a hyperelliptic structure on C_1 and f is the partial desingularization at two singular points c_1 and c_2. We denote by $\tilde{f}\colon \tilde{C}_1 \to \tilde{C}$ the corresponding partial desingularization of the curve $\tilde{C}$ at the points $\tilde{c}_1 = \pi^{-1}(c_1)$ and $\tilde{c}_2 = \pi^{-1}(c_2)$. The subvariety $Z(\gamma;\, c_1, c_2) \subset \mathrm{Pv}$ has generic points $[L]$ for which $\tilde{f}^0 L \approx (\gamma \circ \pi_1)^* M(D)$, where $M = \mathcal{O}_{\mathbf{P}^1}(1)$ and D is a nonsingular divisor on $\tilde{C}_1$. Taking $M = \mathcal{O}_{\mathbf{P}^1}(2)$ and proceeding in a similar manner, we define the subvariety $Z'(\gamma;\, c_1, c_2) \subset \mathrm{Pv}$. For the above types we have not verified the inclusions $Z(\cdots) \subset \operatorname{Sing}\Xi$ and $Z'(\cdots) \subset \operatorname{Mult}_3 \Xi$, and we will not always do this in what follows, but the present case $Z'(\gamma;\, c_1, c_2) \subset \operatorname{Mult}_3 \Xi$ is typical and we consider it in more detail. In view of the converse statement in 5.8, the sheaf $(\gamma \circ \pi_1)^* M(D)$ has three sections which pairwise satisfy condition (P). In view of 5.10, each sheaf L with $\tilde{f}^0 L \approx (\gamma \circ \pi_1)^* M(D)$ has the same property. By 5.7, from this it follows that $\operatorname{Mult}_{[L]} \Xi \geqslant 3$. In the case of $Z(\gamma;\, c_1, c_2)$ one should use the converse statement in 5.5.

Type IV. Let C_0 be a connected component of the curve C whose complementary component C_1 is connected and admits a hyperelliptic structure $\gamma\colon C_1 \to \mathbf{P}^1$, and suppose that $C_0 \cap C_1 = \{c_i\}_1^4$ is a set consisting of four points. Let $\tilde{f}\colon \tilde{C}_0 \sqcup \tilde{C}_1 \to \tilde{C}$ be the partial desingularization at the set $\{\tilde{c}_i\}_1^4 = \{\pi^{-1}(c_i)\}_1^4$. The subvariety $Z(\gamma;\, c_1) \subset \mathrm{Pv}$ has generic points $[L]$ for which

$$\tilde{f}^0[L] = \big([L_0], [(\gamma \circ \pi_1)^* M(D)]\big),$$

where $[L_0] \in \mathrm{Pv}_0 = \mathrm{Pv}(\tilde{C}_0, I_0)$, $M = \mathcal{O}_{\mathbf{P}^1}(1)$ and D is a nonsingular divisor on $\tilde{C}_1$. Taking $M = \mathcal{O}_{\mathbf{P}^1}(2)$, in a similar manner we define the subvariety $Z'(\gamma;\, C_1) \subset \mathrm{Pv}$.

Type V. Suppose that the curve C has a component C_1 which is a plane curve of genus 3 whose complementary component C_2 intersects C_1 along a set consisting of four points $\{c_i\}_1^4$ and $\omega_{C_1} = \mathcal{O}_{C_1}(\Sigma_1^4 c_i)$. In view of (S), the curves C_1 and C_2 are connected. The subvariety $Z(C_1) \subset \mathrm{Pv}$ has generic points $[L]$ for which $L \approx \pi^* M(D)$, where D is a nonsingular divisor on $\tilde{C}$ and M is a nonsingular sheaf of degree 4 on C with $h^0(C, M) \geqslant 2$ such that $M|_{C_1} \approx \omega_{C_1}$ (compare with (4.5.4) + (4.6(c))).

Type VI. Suppose that the curve C admits a decomposition $C = C_1 \cup C_2$ with $C_1 \cap C_2 = \{c_i\}_1^4$. Let $\tilde{f}\colon \tilde{C}_1 \sqcup \tilde{C}_2 \to \tilde{C}$ be the partial desingularization at the set $\{\tilde{c}_i\}_1^4 = \{\pi^{-1}(c_i)\}_1^4$. Set

$$Z(C_1, C_2) = (\tilde{f}^0)^{-1}(\Xi_1 \times \Xi_2),$$

where $\Xi_i = \Xi(\tilde{C}_i, I_i)$ (cf. 3.19). We verify the inclusion $Z(C_1, C_2) \subset \operatorname{Sing}\Xi$. If $[L'] \in \Xi_1 \times \Xi_2$, then there exist two sections $s_1, s_2 \in H^0(\tilde{C}_1 \sqcup \tilde{C}_2, L')$ with $s_i|_{\tilde{C}_i} \equiv 0$, $i = 1, 2$. Hence L' satisfies condition (P). By 5.10, (P) holds for L with $\tilde{f}^0 L \approx L'$, and therefore $\operatorname{Mult}_{[L]} \Xi \geqslant 2$. Using Lemma 3.14, it is easy to verify that $\dim Z(C_1, C_2) = p - 5$ if neither of the curves C_1 and C_2 is isomorphic to $\mathbf{P}^1$.

Type VII. Suppose that the curve C has a decomposition $C = C_0 \cup C_1 \cup C_2$ with $C_i \cap C_j = \{c_{ij}\}$, $0 \leqslant i \leqslant j \leqslant 2$. Let $\tilde{f}\colon \tilde{C}_0 \sqcup \tilde{C}_1 \sqcup \tilde{C}_2 \to \tilde{C}$ be the partial desingularization at the set $\{\tilde{c}_{ij}\} = \{\pi^{-1}(c_{ij})\}$. Set

$$Z(C_0, C_1, C_2) = (\tilde{f}^0)^{-1}(\mathrm{Pv}_0 \times \Xi_1 \times \Xi_2).$$

Type VIII. Let C_0 be a component of C whose complement consists of two connected components C_1 and C_2 such that $\#C_0 \cap C_1 = \#C_0 \cap C_2 = 4$. We denote by $\tilde{f}\colon \tilde{C}_0 \sqcup \tilde{C}_1 \sqcup \tilde{C}_2 \to \tilde{C}$ the corresponding partial desingularization at 8 points. We set

$$Z(C_0, C_1, C_2) = (\tilde{f}^0)^{-1}(\mathrm{Pv}_0 \times \Xi_1 \times \Xi_2).$$

Type IX. Suppose that C has decomposition $C = C_0 \cup C_1 \cup C_2$ such that $C_1 \cap C_2 = \{c_0\}$ and $\#C_0 \cap C_1 = \#C_0 \cap C_2 = 3$. We denote by $\tilde{f}\colon \tilde{C}_0 \sqcup \tilde{C}_1 \sqcup \tilde{C}_2 \to \tilde{C}$ the corresponding partial desingularization at 7 points. We set

$$Z(C_0, C_1, C_2) = (\tilde{f}^0)^{-1}(\mathrm{Pv}_0 \times \Xi_1 \times \Xi_2).$$

For types VII, VIII and IX, $\dim Z(C_0, C_1, C_2) = p - 5$ if $C_1, C_2 \not\approx \mathbf{P}^1$. This is proved in the same way as in VI, with the help of 3.14. More precisely, using this lemma we establish the upper bound in 3.20, i.e. we show that

$$\dim \Xi_i = g(C_i) - 1 = \dim \mathrm{Pv}(\tilde{C}_i, I_i) - r(\tilde{C}_i, I_i)/2,$$

if $\tilde{C}_i$ is a connected curve with involution I_i satisfying (F) and $r(C_i, I_i) > 0$. Using this, it is easy to determine the dimension of $\mathrm{Pv}_0 \times \Xi_1 \times \Xi_2$ and hence to compute the dimension of $Z(C_0, C_1, C_2)$.

Type X. Let C be a plane quintic, so that there exists a very ample invertible sheaf M of degree 5 on C such that $h^0(C, M) = 3$. In particular, $p = g(C) = 6$. The subvariety $Z(C)$ has generic points $[L]$ for which $L \approx \pi^*M(x - I(x))$, where x is a nonsingular point on $\tilde{C}$. In this case $\dim Z(C) = 0$ or 1 depending on the evenness or oddness of the type of the pair $(\tilde{C}, I)$ (compare with [17], p. 347).

The varieties $Z(\cdots)$ and $Z'(\cdots)$ described above will be called *varieties of special singularities*, and their points (as a rule, generic points) will be called *special singularities*.

5.12. THEOREM (compare with Theorem 4.10 from [3]). *Let $(\tilde{C}, I)$ be a Beauville pair satisfying condition* (S) *such that the quotient curve $C = \tilde{C}/I$ has genus $p = g(C) \geqslant 6$, and let $\Xi = \Xi(\tilde{C}, I)$ be the canonical polarization divisor on* $\mathrm{Pv}(\tilde{C}, I)$. *Suppose that the curve C is neither hyperelliptic nor quasitrigonal. Then*

$$\dim \mathrm{Sing}\, \Xi \leqslant p - 5,$$

i.e. the dimension of each irreducible component Z of the variety $\mathrm{Sing}\,\Xi$ *is $\leqslant p - 5$. Moreover, if $Z \subset$* $\mathrm{Sing}\,\Xi$ *is an irreducible component of dimension $p - 5$, then Z lies in one of the varieties of special singularities $Z(\cdots)$ described in* 5.11, *i.e. each point from Z is a special singularity.*

PROOF. The inequality immediately follows from Proposition 5.4. Suppose now that Z is an irreducible component of $\mathrm{Sing}\,\Xi$ of dimension $p - 5$. In view of Lemma 5.5, we may follow the proof of Proposition 5.4. The difference is that the curve $\tilde{C}_1'$ has at most two connected components. First we consider the case when $\tilde{C}_1'$ is not connected; we denote by $\tilde{C}_1$ and $\tilde{C}_2$ the corresponding components of $\tilde{C}$. If $\tilde{C}_0 \neq \varnothing$, then there is a decomposition $\tilde{C} = \tilde{C}_0 \cup \tilde{C}_1 \cup \tilde{C}_2$. In view of (S) and the inequalities from the proof of Proposition 5.4,

$$\#\tilde{C}_1 \cap (\tilde{C}_0 \cup \tilde{C}_2) = \#\tilde{C}_2 \cap (\tilde{C}_0 \cup \tilde{C}_1) = 4.$$

It is easy to verify that in this case $Z \subset Z(C_0, C_1, C_2)$ and is of type VIII if $\tilde{C}_1 \cap \tilde{C}_2 = \varnothing$, type IX if $\#\tilde{C}_1 \cap \tilde{C}_2 = 1$, and type VII if $\#\tilde{C}_1 \cap \tilde{C}_2 = 2$. If $\tilde{C}_0 = \varnothing$, then we obtain an inclusion $Z \subset Z(C_1, C_2)$ in a special variety of type VI. In what follows we assume that $\tilde{C}_1'$ is a connected curve. As in the proof of Proposition 5.4, we come to the inequality

$$\dim \mathrm{Pv}(\tilde{C}_1', I_1') - 4 \leqslant \dim Z_1' + \dim \left|\omega_{C_1'} \otimes M^{-2}\right|,$$

and for a generic point $[L] \in Z$

$$\tilde{f}^0[L] = ([L_0], [\pi_1'^* M(D)]),$$

where $[L_0] \in \mathrm{Pv}_0 = \mathrm{Pv}(C_0, I_0)$, M is a free invertible sheaf on $C_1' = \tilde{C}_1'/I_1'$, $h^0(C_1', M) \geqslant 2$, $[L] \in Z_1'$, and D is a nonsingular divisor on $\tilde{C}_1'$. By Lemmas 4.4 and 4.2,

$$\dim Z_1' \leqslant d - 2, \qquad \dim \left|\omega_{C_1'} \otimes M^{-2}\right| \leqslant g(C_1') - 1 - d,$$

where $d = \deg M$. Hence $\dim \mathrm{Pv}(\tilde{C}_1', I_1') \leqslant g(C_1') + 1$, and equality holds only if M is a nonsingular sheaf of type G_2^1. By (3.9.3), $-1 + r/2 \leqslant 1$, where $r = r(\tilde{C}_1', I_1')$, from which it follows that $r \leqslant 4$. Thus if $\tilde{C}_0 \neq \varnothing$, then $r = 4$ and M is a nonsingular sheaf of type G_2^1. In this case $Z \subseteq Z(\gamma; C_1)$, where γ is the hyperelliptic structure for M and C_1 is the component of C corresponding to $\tilde{C}_1'$. These singularities are of type IV.

Thus it remains to consider the case when $C_0 = \varnothing$ and $\tilde{C}_1'$ is a connected curve. It is clear that $2n = r(\tilde{C}_1', I_1')$. If $r = 4$, then we obtain an inclusion $Z \subseteq Z(\gamma; c_1, c_2)$ in a special variety of type III. Since r is even and $r \leqslant 4$, we may assume that $n \leqslant 1$. By the condition of the theorem, M is not a sheaf of type G_2^1. Then by 4.4 $\dim Z_1' \leqslant d - 3$ and by 4.2 $\dim|M| < d/2$. If $r = 2$, then

$$\dim \mathrm{Pv}(\tilde{C}_1', I_1') = g(C_1'),$$

from which it follows that $\dim Z_1' = d - 3$ and

$$\dim \left|\omega_{C_1'} \otimes M^{-2}\right| = g(C_1') - 1 - d.$$

In view of 4.4 and the Riemann-Roch theorem, this is possible if and only if $d = 3$, $\dim|M^{\otimes 2}| = 3$, and by the above $\dim|M| = 1$. By (S), the curve $\tilde{C}_1'$ is stable and does not have nonsingular sheaves of type G_2^1. Therefore the canonical mapping $\kappa\colon C_1' \to \mathbf{P}^{\cdots}$ is birational. By the geometric interpretation of the Riemann-Roch theorem, the linear spans $\langle D \rangle$ of the divisors from $|M|$ define trisecants of the image $\kappa(C_1')$, and besides that, any two generic trisecants of such type intersect with each other. All these lines must have a common point $O \in \mathbf{P}^{\cdots}$, since otherwise the points of the divisors $D \in |M|$ sweep out a plane curve of degree 3 and $|M^{\otimes 2}|$ is cut out by conics, which is impossible in view of the nonsingularity of $|\omega_{C_1'} \otimes M^{-2}|$. Thus the trisecants $\langle D \rangle$ intersect at a point $O \notin \kappa(C_1')$. Applying Clifford's theorem, we can show that these trisecants sweep out a quadratic cone. We denote by $C_{1,1}'$ the component of the curve C_1' for which $\kappa(C_{1,1}')$ lies on this cone. By our construction, $\kappa(C_{1,1}')$ is a curve of degree 6 and the preimages of its hyperplane sections of with respect to κ make up the system $|M^{\otimes 2}|$ at a generic point. Let $C_{1,i}'$ be a connected component of the complementary curve $C_1' - C_{1,1}'$. By (S), $\#C_{1,1}' \cap C_{1,i}' \geqslant 3$, and, arguing as in the beginning of our treatment of the case of equality, from the proof of Lemma 4.2 we infer that

$$\dim \langle \kappa(C_{1,1}') \rangle \cap \langle (C_{1,i}') \rangle \leqslant 0,$$

since $\langle D_{1,1}' \rangle \cap \langle D_{1,i}' \rangle = \varnothing$ for a generic hyperplane section $D_{1,1}' + D_{1,i}'$ of the curve $\kappa(C_{1,1}' \cup C_{1,i}')$. But since the curve $\kappa(C_{1,1}')$ has degree 6, it cannot have more than one point of multiplicity $\geqslant 3$. Hence $C'_{1,2} = C'_1 - C'_{1,1}$ is a connected curve. The morphism φ_M corresponding to the linear system $|M|$ maps the curve $C'_{1,2}$ to a point. Therefore $\#C'_{1,1} \cap C'_{1,2} = 3$. Now we observe that $\deg(\omega_{C'_1}) \otimes M^{-2}|_{C'_{1,1}} = 0$, from which it follows that $\deg \omega_{C'_{1,1}} = 3$, which contradicts the fact that $\deg \omega_{C'_{1,1}}$ is even. Here we left out the case when $C'_{1,2} = \varnothing$, which is impossible if $p \geqslant 6$.

So it remains to consider the case when $C_0 = \{z_i\}_{i=1}^n = \varnothing$, i.e. $C'_1 = C$. As above, $\dim Z'_1 \leqslant d - 3$. If $\dim Z'_1 = d - 3$, then in view of Theorem 4.5 and Remarks 4.6 we have inclusions $Z \subseteq Z(\cdots)$ of type I, II, V or X. To complete the proof it suffices to show that the case $\dim Z'_1 \leqslant d - 4$ is impossible. Using the inequality

$$\dim \mathrm{Pv} - 4 \leqslant \dim Z'_1 + \dim |\omega_C \otimes M^{-2}|$$

it is easy to verify that this is possible only if $d = 4$, $\dim|M^{\otimes 2}| = 4$ and $\dim|M| = 1$. Now we can argue as in the preceding paragraph. The vertex O is replaced by a line which does not intersect $\kappa(C)$, and the planes intersecting $\kappa(C)$ along four points corresponding to divisors from $|M|$ serve as generatrices. The component of C whose canonical image lies on the quadratic cone will again be denoted by $C'_{1,1}$. Then $C'_{1,2} = C - C'_{1,1}$ is a connected curve, and $\#C'_{1,1} \cap C'_{1,2} = 4$. Since $p \geqslant 6$, $C'_{1,2} \neq \varnothing$. To get a contradiction, we shall use an argument which differs from the argument at that end of the preceding paragraph; we remark that this argument can also be used in the above situation. The divisor

$$D = \sum_{x \in C'_{1,1} \cap C'_{1,2}} x$$

on $C'_{1,2}$ belongs to the linear system $|M|_{C'_{1,1}}|$. Since the hyperplane sections of the curve $\kappa(C'_{1,1})$ correspond to the elements of the linear system $|M^{\otimes 2}|$ on $C'_{1,1}$ and $\kappa(D)$ is a point because $\dim\langle \kappa(C'_{1,1})\rangle \cap \langle \kappa(C'_{1,2})\rangle = 0$,

$$\dim \left|M|_{C'_{1,1}}\right| \geqslant 3.$$

In view of 3.12.2 and the nonsingularity of the systems $|M|$ and $|\omega_C \otimes M^{-2}|$, the systems $|M|_{C'_{1,1}}|$ and $|\omega_{C'_{1,1}} \otimes (M|_{C'_{1,1}})^{-1}|$ are nonsingular. This, as expected, yields a contradiction with Clifford's theorem. ∎

CONCLUDING REMARK. It turns out that the difficulties encountered in the end of the proof of Theorem 5.12 are related to the problem of finding out when there is an equality in Clifford's theorem. Everything would be greatly simplified if we could prove the following result, which is certainly true. Let C be a connected curve with ordinary quadratic singularities which satisfies condition (S) and on which there do not exist free sheaves of type G^1_2. If L and $\omega_C \otimes L^{-1}$ are nonsingular invertible sheaves, then $\dim|L| \leqslant (\deg L)/2$, and equality holds only if $L \approx \mathcal{O}_C$ or $L \approx \omega_C$, as in the classical version of Clifford's theorem (compare with 4.2).

§6. Superelliptic curves and certain curves of small genera

In this section S denotes a *connected smooth* curve of genus $g = g(S)$. First we recall a general fact clarifying the role of the Mumford-Beauville Theorem 5.12 (details can be found e.g. in [2] and [8]).

6.1. PROPOSITION. *Let* $\Theta \subset \mathrm{Jv}(S)$ *be the canonical polarization divisor* (*cf.* §2). *Then*
(6.1.1) $g - 4 \leqslant \dim \operatorname{Sing} \Theta \leqslant g - 3$;
(6.1.2) $\dim \operatorname{Sing} \Theta = g - 3$ *if and only if* S *is a hyperelliptic curve.*

6.2. REMARK. Assertion (6.1.2) is an immediate consequence of the classical Martens theorem (cf. [18]).

The main part of this section is devoted to the study of those curves whose theta-divisor Θ has "many" singularities of multiplicity $\geqslant 3$. After making more precise what we mean by "many", we show that the only such curves are hyperelliptic curves, superelliptic curves with $g \geqslant 7$ (cf. Proposition 6.4) and plane quintics with $g = 6$ (cf. Proposition 6.8). In view of Proposition 5.4, hyperelliptic curves do not present much interest for our purposes. Therefore in what follows we assume that *the curve S is not hyperelliptic.*

In accordance with Remark 4.6(b), a curve S will be called *superelliptic* if there exists a morphism $\varepsilon\colon S \to E$ of degree 2 onto a smooth elliptic curve E (in this case E is always smooth because S is connected, smooth and not hyperelliptic).

It is easy to verify that for $g \geqslant 6$ such structure ε is unique. In fact, consider a general divisor of the form $e_1 + e_2$ on E, where $e_1, e_2 \in E$. Then the linear system $|\varepsilon^*(e_1 + e_2)|$ has degree 4 and dimension 1 (the last assertion holds by Clifford's theorem). In what follows we identify the curve S with its canonical model $S \subset \mathbf{P}^{g-1}$, i.e. with the image of the morphism $\kappa = \varphi_{\omega_S}$ corresponding to the canonical system on S. By the geometric interpretation of the Riemann-Roch theorem, the points of the divisor $\varepsilon^*(e_1 + e_2)$ span a plane $\langle \varepsilon^*(e_1 + e_2) \rangle$. From this it follows that any two lines of the form $\langle \varepsilon^* e_1 \rangle$, $\langle \varepsilon^* e_2 \rangle$ intersect at some point. Since all these lines cannot lie in one plane, all of them pass through a common point $O \notin S$. This point O will be called the *center of the superelliptic projection*. Clearly, the morphism ε is identified with the projection of the curve S from the point O, and the curve E is identified with the image of this projection in $\mathbf{P}^{g-2}$. We observe that under this identification E is a projectively normal curve of degree $g - 1$ in $\mathbf{P}^{g-2}$. The following result is well known.

6.3. Lemma-Exercise. *Each effective divisor D of degree $\langle g - 1 \rangle$ on E spans a projective subspace $\langle D \rangle$ of dimension* $\deg D - 1$, *i.e. the "points" of each such divisor D are in general position.* ■

Let $\varepsilon'\colon S \to E'$ be another superelliptic structure on the curve S. Then we obtain another center of projection $O' \neq O$. But in this situation, for generic points $e_1, e_2 \in E'$ the divisor $\varepsilon_* \varepsilon'^*(e_1 + e_2)$ has degree 4 and lies in a plane, i.e. $\dim\langle \varepsilon_* \varepsilon'^*(e_1 + e_2) \rangle \leqslant 2$. But by 6.3 this is impossible for $g \geqslant 6$ since then $\deg \varepsilon_* \varepsilon'^*(e_1 + e_2) = 4 \leqslant g - 1$. Therefore for $g \geqslant 6$ on the curve S there exists *at most one* superelliptic structure, which means that if such a structure does exist, then it reflects some intrinsic properties of the curve and its Jacobian. In particular, this structure allows us to point out the following very important component in the variety $\operatorname{Sing} \Theta$ of singular points of the theta-divisor:

$$\Lambda = \{ [\varepsilon^* M(D)] \mid [M] \in \operatorname{Pic}^2(E) \text{ and } D \in S^{(g-5)} \};$$

here we implicitly use the Torelli theorem for curves and, as usual, $S^{(k)}$ denotes the kth symmetric power of the curve S or, in other words, the variety of effective divisors of degree k on S. Moreover, by 2.5, for $g \geqslant 7$, $\operatorname{Mult}_3 \Theta$ contains the subvariety

$$\Lambda' = \{ [\varepsilon^* M(D)] \mid [M] \in \operatorname{Pic}^3(E) \text{ and } D \in S^{(g-7)} \}$$

of dimension $g - 6$.

6.4. Proposition. *Let S be a nonhyperelliptic curve of genus $g \geqslant 7$ such that* $\dim \operatorname{Mult}_3 \Theta \geqslant g - 6$. *Then S is a superelliptic curve and* $\operatorname{Mult}_3 \Theta = \Lambda'$.

Proof. Let M be a general sheaf corresponding to a component $\mathscr{M} \subset \operatorname{Multi}_3 \Theta$ of dimension $\geqslant g - 6$. Subtracting the fixed components of such sheaves we obtain an irreducible subvariety $\Lambda'' \subseteq G^2_d = \{[L] \in \mathrm{Pic}^d(S) | h^0(S, L) \geqslant 3\}$, where $d \leqslant g - 1$ and $\dim \Lambda'' \geqslant d - 5$. But now for a generic point $[L] \in \Lambda''$ the sheaf L is free. By Martens' theorem (cf. Lemma 4.4), $\dim|L| = 2$. Therefore the variety $\bigcup_{[L] \in \Lambda''} |L| \subset S^{(d)}$ has dimension $\geqslant d - 3$. Furthermore, a general divisor D from this subvariety spans a space $\langle D \rangle$ of dimension $d - 3$, and any $d - 1$ points of this divisor also span $\langle D \rangle$ because $|D|$ is a linear system without basepoints. Hence, choosing $d - 1$ points in each general divisor D of this type and passing to the isomorphism classes of the corresponding sheaves, we obtain a subvariety $Z \subset G^1_{d-1}$ of dimension $\geqslant d - 4$ because for a generic point $[L] \in Z$ the sheaf L is of type G^1_{d-1}. Subtracting the fixed components of these sheaves and taking into consideration 4.5 and Remark 4.6(e), we obtain the following possibilities:

(6.4.1) S is a trigonal curve, and for a generic point $x \in S$ and a generic point $[M] \in \mathcal{M}$ each divisor from $|M(-x)|$ has the form $g^1_3 + F$, where g^1_3 is a divisor from the unique trigonal linear series $|g^1_3|$ on S, and F is an effective divisor.

(6.4.2) S is a superelliptic curve, and for a generic point $x \in S$ and a generic point $[M] \in \mathcal{M}$ each divisor from $|M(-x)|$ has the form $\varepsilon^*(e_1 + e_2) + F$, where F is an effective divisor and e_1 and e_2 are two points on E.

We claim that the case (6.4.1) is impossible. In fact, in this case each divisor from $|M|$ has the form $g^1_3 + F$. It is clear that for a suitable divisor from $|M|$ the divisors g^1_3 and F do not have common points. Suppose now that x is a point of the divisor g^1_3. By the above, the divisor $g^1_3 + F - x$ again has the form $(g^1_3)' + F'$, where $(g^1_3)'$ is an element of the trigonal series and F' is an effective divisor. It is clear that $\mathrm{Supp}\, g^1_3 \cap \mathrm{Supp}(g^1_3)' = \varnothing$. Therefore $M \approx \mathcal{O}_S(2g^1_3 + F'')$, where F'' is an effective divisor of degree $g - 7$. So $\dim \mathcal{M} \leqslant g - 7$, which yields a contradiction.

Arguing similarly in the case (6.4.2), we obtain an isomorphism

$$M \approx \varepsilon^* \mathcal{O}_E(e_1 + e_2 + e_3)(F)$$

for a generic point $[M] \in \mathcal{M}$, so that $\mathcal{M} = \Lambda'$. Thus S is a superelliptic curve and Λ' is the only component of dimension $g - 6$ in $\mathrm{Mult}_3 \Theta$. The other assertions of the proposition follow from the following lemma whose statement was suggested by Arnaud Beauville. ■

6.5. LEMMA. *Let $\varepsilon \colon S \to E$ be a superelliptic structure, let $D \in S^{(g-1)}$ be an effective divisor on S of degree $g - 1$, and let A be the maximal effective divisor on E for which $\varepsilon^* A \leqslant D$. Then either*

(6.5.1) $A = 0$ *and* $h^0(S, \mathcal{O}_S(D)) \leqslant 2$

or

(6.5.2) $h^0(S, \mathcal{O}_S(D)) = \deg A$ *and* $|D| = \varepsilon^*|A| +$ *the fixed part.*

PROOF. If $A = 0$, then it is clear that $h^0(S, \mathcal{O}_S(D)) \leqslant 2$. Therefore, subtracting the part $\varepsilon^* A$, it suffices to show that an effective divisor D on S of degree $< g - 1$ whose complete linear system $|D|$ does not contain elements of the form $\varepsilon^* e + D'$, where $D' \in S^{(\deg D - 2)}$ and $e \in E$, is necessarily linearly fixed, i.e. $h^0(S, \mathcal{O}_S(D)) = 1$. Moreover, it suffices to verify that in this case the space $\langle D \rangle$ does not contain the center 0 of the superelliptic projection. Then our assertion follows from the geometric interpretation of the Riemann-Roch theorem. So, let D be such a divisor. By assumption, each hyperplane $H \subset \mathbf{P}^{g-1}$ containing the divisor D (i.e. such that $(H \cdot S) \geqslant D$) and passing through the point O is projected onto a hyperplane $\varepsilon(H) \subset \mathbf{P}^{g-2}$ containing the divisor $\varepsilon_* D$ on E. Hence $\dim\langle D \rangle \geqslant \dim\langle \varepsilon_* D \rangle$, and equality is possible only when $\langle D \rangle \not\ni 0$. But by 6.3

$\dim\langle \varepsilon_* D\rangle = \deg D - 1$. Hence $\dim\langle D\rangle \geqslant \deg D - 1$, and in view of the geometric interpretation of the Riemann-Roch theorem the divisor D is linearly fixed and we have equality, from which it follows that $\langle D\rangle \not\ni O$. ∎

6.6. Proposition. *Let S be a superelliptic curve of genus $g \geqslant 7$. Then the following assertions are true*:

(6.6.1) *Λ and Λ' are irreducible subvarieties in* $\mathrm{Jv}(S)$ *of dimensions $g - 4$ and $g - 6$ respectively.*

(6.6.2) *The variety of singularities* $\operatorname{Sing}\Theta$ *has besides Λ at least one other component of dimension $g - 4$.*

(6.6.3) *If $t \in J(S)$ is a point such that $t \cdot \Lambda' \subseteq \operatorname{Sing}\Theta$ ($\cdot$ denotes the natural action of the Jacobian $J(S)$ on the variety $\mathrm{Jv}(S) = \mathrm{Pic}^{g-1}(S)$), then $t \cdot \Lambda' \subseteq \Lambda$.*

6.7. Remark. From the following proof it will be clear that the assertion (6.6.2) concerning the existence of another component holds also for $g = 6$ and 5.

Proof of Proposition 6.6. Assertion (6.6.1) is obvious.

To prove (6.6.2), we consider the projection $\pi\colon S \to \mathbf{P}^2$ from $g - 3$ generic points $x_1, \ldots, x_{g-3} \in S$. Then the image $\pi(S) \subset \mathbf{P}^2$ is a curve of degree $2g - 2 - (g - 3) = g + 1$ with an ordinary singular point $\pi(O)$ of multiplicity $g - 3$, and

$$p_a(\pi(S)) - \frac{(g-3)(g-4)}{2} = \frac{g(g-1)}{2} - \frac{(g-3)(g-4)}{2} = 3g - 6 > g.$$

Hence $\pi(S)$ has at least one other singular point $y \neq \pi(O)$. Therefore on S there exist irregular special divisors of the form $x_1 + \cdots + x_{g-3} + x_{g-2} + x_{g-1}$. It is clear that the dimension of the variety of such divisors is equal to $g - 3$, and they define a component of dimension $g - 4$ in $\operatorname{Sing}\Theta$ which is distinct from Λ. This proves (6.6.2).

As for (6.6.3), we first observe that if $[L] \in t \cdot \Lambda' \subseteq \operatorname{Sing}\Theta$, then for each pair of points $e_1, e_2 \in E$ there is an inclusion $[L(\varepsilon^*(e_1 - e_2))] \in \operatorname{Sing}\Theta$, since $\Lambda'\varepsilon^*(e_1 - e_2) = \Lambda'$. If $\dim|L| \geqslant 2$, then, by 6.4, $[L] \in \mathrm{Mult}_3\Theta = \Lambda' \subsetneq \Lambda$. Now we consider the case when $\dim|L| = 1$. If the linear system $|L|$ is free, then a general divisor $D \in |L|$ consists of $g - 1$ distinct points, and $\dim\langle \sum_{i=1}^{g-1} x_i\rangle = g - 3$. Moreover, $\dim\langle \sum_{i=1}^{g-2} x_i\rangle = g - 3$ and, by 6.3, $O \notin \langle D\rangle$ or $D = \varepsilon^* e +$ an effective divisor. In the last case, by Lemma 6.5, $[L] \in \Lambda$. If $O \notin \langle D\rangle$, then $\varepsilon(\langle D\rangle)$ is a hyperplane in $\mathbf{P}^{g-2}$ and a generic point $e_1 \in E$ satisfies the equality $\dim\langle D + \varepsilon^* e_1\rangle = g - 1$. From this it follows that $h^0(S, \omega_S \otimes L(\varepsilon^* e_1)^{-1}) = 0$, and by the Riemann-Roch theorem

$$h^0(S, L(\varepsilon^* e_1)) = g + 1 - g + 1 = 2 = h^0(S, L).$$

Therefore $|L(\varepsilon^* e_1)| = |L| + \varepsilon^* e_1$, but $|L(\varepsilon^* e_1 - \varepsilon^* e_2)| \neq \varnothing$ for a generic point $e_2 \in E$. Hence for such points e_2 we have $|L(-\varepsilon^* e_2)| \neq \varnothing$. Therefore $\varepsilon^* e_2 +$ an effective divisor $\in |L|$ and, again by Lemma 6.5, $[L] \in \Lambda$. If the system $|L|$ has basepoints, then it contains a divisor of the form $\varepsilon^* a +$ an effective divisor. Therefore, again by Lemma 6.5, $[L] \in \Lambda$. ∎

6.8. Proposition. *Let S be a nonhyperelliptic curve of genus 6. Then the theta-divisor Θ has at most one singular point of multiplicity $\geqslant 3$, and if such a point $[M]$ does exist, then S is a plane quintic embedded in $\mathbf{P}^2$ via the morphism φ_M.*

Proof. Let $[M] \in \Theta$ be a singular point of multiplicity $\geqslant 3$. Then, in view of 2.5 and Remark 2.6(b), $h^0(S, M) \geqslant 3$. But $\deg M = g(S) - 1 = 5$. In this case by Clifford's theorem the sheaf M is free and $h^0(S, M) = 3$. Thus we obtain a morphism $\varphi_M: S \to \mathbf{P}^2$ of degree 1 onto a curve $\varphi_M(S)$ of degree 5. Applying the Plücker formula for the genus of a plane curve and using the fact that S is the normalization of the curve $\varphi_M(S)$, we see that φ_M is an embedding and S is a plane quintic. By the Noether-Enriques theorem [28], the structure of plane quintic is defined intrinsically, namely by the intersection of quadrics in the canonical embedding of a given curve, which implies the uniqueness of the point $[M]$. ∎

6.9. PROPOSITION. *Let S be a nonhyperelliptic curve of genus 6. Then for a generic point $[M]$ of one of the two-dimensional components of the variety* $\operatorname{Sing}\Theta$ *there exists only one nonsingular point on* $\operatorname{Sing}\Theta$ *the tangent plane at which to* $\operatorname{Sing}\Theta$ *is "parallel" to the tangent plane at $[M]$, namely the point $[\omega_S \otimes M^{-1}]$.*

6.10. REMARK. It is easy to generalize the last result to the case of an even $g \geqslant 8$, but the assertion fails for $g = 5$. When we speak about "parallel" planes, we have in mind the natural connection on an abelian variety.

PROOF OF PROPOSITION 6.9. First we consider the case when S is a plane quintic. By the Noether-Enriques theorem, in this case the intersection of quadrics through the curve $S \subset \mathbf{P}^5$ defines the image of the plane $\mathbf{P}^2$ under the Veronese map defined by the linear series of conics. We shall show that on S there are no complete free linear systems $|D|$ of degree 5 and dimension 1. In view of Theorem 4.5 and Remark 4.6(e), from this it is easy to conclude that

$$\operatorname{Sing}\Theta = \{[M(s_1 - s_2)] \mid s_1, s_2 \in S\}$$

(at least modulo components of dimension $\leqslant 1$), where M is the sheaf of type G_5^2 defining the structure of plane quintic on S. But the tangent plane to $\operatorname{Sing}\Theta$ at the point $[M(s_2 - s_1)]$ is "parallel" only to the tangent plane at the point $[\omega_S \otimes (M(s_1 - s_2))^{-1}]$ since it is easy to verify that, in a natural sense, the projectivization of this plane coincides with the line in $\mathbf{P}^5$ passing through the points $s_1, s_2 \in S \subset \mathbf{P}^5$. So, let $D = \sum_1^5 x_i$ be a divisor consisting of 5 distinct points on S and varying in the linear system $|D|$ of type G_5^1. By the Riemann-Roch theorem, the linear span $\langle D \rangle \subset \mathbf{P}^5$ has dimension 3. It intersects the Veronese image of the plane $\mathbf{P}^2$ at at least 5 points. This is possible only if the intersection is not proper and contains a conic in $\mathbf{P}^5$ corresponding to a line in $\mathbf{P}^2$. Since $|D| \neq |M|$, this line does not contain all five points of the divisor D. If this line contains four points from $\operatorname{Supp} D$, then $[\mathcal{O}_S(D)]$ belongs to the variety $\operatorname{Sing}\Theta$ described above, i.e. $|M|$ has a fixed point. Finally, if this line contains no more than three points of the divisor D, then the intersection $\langle D \rangle \cap S$ contains at least 7 points, from which it follows that the curve S is trigonal, which is impossible by the Noether-Enriques theorem [28]. Thus our proposition is true in the case of a plane quintic, and so in what follows we may assume that S is not a quintic.

Next we observe that $\operatorname{Sing}\Theta$ has a two-dimensional component $\mathcal{M}$ with generic point $[M]$, where M is a free invertible sheaf. In fact, by Theorem 4.5 and Remark 4.6(e) this holds for arbitrary two-dimensional components provided that the curve S is not trigonal or superelliptic (and is not a plane quintic). If S is a trigonal curve, then it is easy to verify that the component which is symmetric to the component $\{[M(D)] \mid D \in S^{(2)}\}$ with respect to the point $[\omega_S] \in \operatorname{Pic} S$, where M is the invertible sheaf of type G_3^1 defining the trigonal structure on S, satisfies our condition. In the superelliptic case it suffices to take a component which is complementary to Λ; such a component exists in view of Remark 6.7.

So, let $\mathscr{M}$ be a two-dimensional component in Sing Θ for whose generic point $[M]$ the sheaf M is free. Clearly we may assume that the point $[M]$ is not equal to one-half of the canonical class $[\omega_S]$ and is nonsingular on Sing Θ. Then by [2] to the sheaf M there corresponds a quadric $Q \subset \mathbf{P}^5$ of rank 4 containing S, and in a natural sense the vertex of this quadric (which is a line) is the projectivization of the tangent plane to Sing Θ at the point $[M]$ (cf. 2.7). Hence it suffices to verify that S is contained in a unique quadric of rank 4 with this vertex. This can be easily shown using the birationality of the projection of the curve S with center at this vertex and the fact that the degree of the image of S is not less than 8 (the degree can be equal to 8 only in the trigonal case when the vertex contains two points of S; in the remaining cases the degree is equal to 10). ∎

Suppose now that S is a nonhyperelliptic and nontrigonal curve of genus $g = 5$. By Masiewicki's thesis [19], the pair Sing Θ and the natural involution -1 induced by the symmetry with respect to the class $[\omega_S]$ is a Beauville pair for which the quotient curve Sing $\Theta/\pm 1$ is a *plane quintic* and the Prymian is the *Jacobian of* S.

6.11. PROPOSITION. *Let S be a canonical curve of genus 5 which is a complete intersection of three quadrics. Then for a generic point $[M]$ of an arbitrary irreducible component of the variety* Sing Θ *which is not an elliptic curve there exists only one other nonsingular point of* Sing Θ *the tangent space at which to* Sing Θ *is "parallel" to the tangent space at $[M]$, namely the point $[\omega_S \otimes M^{-1}]$. The elliptic components in* Sing Θ *correspond to lines in $\mathbf{P}^2$ under the mapping of the quotient curve* Sing $\Theta/\pm 1$ *onto a plane quintic. In particular, in view of* (B), *two such components intersect at exactly one point.*

PROOF. Since the curve S is neither hyperelliptic nor trigonal, for a generic point $[M]$ of each of the components of Sing Θ the sheaves $[M]$ and $[\omega_S \otimes M^{-1}]$ are free and the projectivization of the tangent line to Sing Θ at $[M]$ is the vertex of the corresponding quadric of rank 4. Therefore it suffices to verify that the curve S is contained in a unique quadric with this vertex, with the exception of the case when $[M]$ lies on a component which is an elliptic curve. This easily follows from the fact that the projection of S from this vertex has degree 2 or 1. In the first case the image of our projection is an elliptic curve.

The remaining assertions are obvious consequences of the Beauville condition and the fact that each component of a plane quintic which is not a line either intersects the other components at at least 6 points or coincides with the quintic. ∎

We conclude this section by proving several simple results about curves of genus 4.

6.12. PROPOSITION. *Let S be a nonhyperelliptic curve of genus* 4.

(6.12.1) $\#\operatorname{Sing}\Theta \leqslant 2$.

(6.12.2) *A point $x \in \operatorname{Sing}\Theta$ is a symmetry point for the divisor Θ if and only if* $\#\operatorname{Sing}\Theta = 1$.

(6.12.3) *If x_1 and x_2 are two distinct points from* Sing Θ, *then $(x_1 - x_2) \notin J_2(S)$, where $J_2(S)$ is the set of points of order* 2 *on the Jacobian $J(S)$.*

PROOF. By the Noether-Enriques theorem [28], the curve S is the intersection of a quadric and a cubic in $\mathbf{P}^3$. Therefore by the Riemann-Kempf theorem and the geometric interpretation of the Riemann-Roch theorem the points from Sing Θ correspond to the sheaves of type G_3^1 cut by the lines on the quadric $Q \supset S$. Since there are at most two such lines modulo rational equivalence, $\#\operatorname{Sing}\Theta \leqslant 2$.

By the above, if $\#\operatorname{Sing}\Theta = 1$, then the quadric $Q \subset \mathbf{P}^3$ has rank 3, i.e. is a quadratic cone. Hence two lines on this cone are cut out by a hyperplane section, so that $2M \approx \omega_S$, where $[M]$ is the sole point of $\operatorname{Sing}\Theta$. Conversely, since Θ is a principal polarization divisor, all its symmetry points belong to one-half of the canonical class or, in other words, are theta-characteristics. The necessity assertion in (6.12.2) easily follows from this fact.

The last assertion follows from the fact that if S lies on a quadric Q of rank 4, then the linear systems $|2l_1|$, $|2l_2|$, where l_1 and l_2 are the line generatrices of the quadric Q, cut nonequivalent complete linear series on S. ■

§7. Proof of the main theorem: case $p \geqslant 8$

In this section we turn to the proof of our main theorem on distinguishing Prymians from Jacobians. More precisely, as we have already remarked in the Introduction, we point out some properties of Beauville pairs which are necessary in order that their Prymians be Jacobians. The sufficiency is clear in view of the results mentioned in the Introduction, and we shall discuss it in more detail in one of the following parts.

7.1. So, let $(\tilde{C}, I)$ be a Beauville pair such that $P(\tilde{C}, I) \simeq J(S)$, where S is a nonsingular curve of genus g. As above, as we assume that *the corresponding quotient curve C is neither hyperelliptic nor quasitrigonal* (compare with §5). Besides that, in this section, unless specified otherwise, we assume that $p = p(C) \geqslant 8$ (compare with [26] and [27]). Then, by Propositiions 5.4 and 6.1, S is a nonsingular connected nonhyperelliptic curve of genus $g = p - 1 \geqslant 7$. The goal of the present section is to show that C is a trigonal curve (7.12).

The above isomorphism between the Prymian and the Jacobian can be rewritten in the form of a canonical isomorphism $\operatorname{Pv}(\tilde{C}, I) = \operatorname{Jv}(S)$ under which the canonical polarizations are identified.

We begin by proving the following assertion, which greatly simplifies our situation.

7.2. LEMMA. *Suppose that the curve C has two connected components C_1 and C_2 such that*

(7.2.1) $C_1 \cup C_2 = C$

and

(7.2.2) $\#C_1 \cap C_2 = 4$.

Then one of these components C_1 or C_2 is a nonsingular rational curve.

PROOF. Suppose the converse. Then, by (S), $p(C_1), p(C_2) \geqslant 1$. Let $f\colon C_1 \sqcup C_2 \to C$ be the partial desingularization at the points of the set $C_1 \cap C_2$. To this desingularization there corresponds a partial desingularization of the pair $(\tilde{C}, I)$, namely $\tilde{f}\colon \tilde{C}_1 \sqcup \tilde{C}_2 \to \tilde{C}$, where $\tilde{C}_1$ and $\tilde{C}_2$ are the proper preimages of the curves C_1 and C_2 with respect to the projection $\pi\colon \tilde{C} \to C$. In view of (3.18), this last desingularization induces an isogeny

$$\tilde{f}^0\colon \operatorname{Pv}(\tilde{C}, I) \to \operatorname{Pv}(\tilde{C}_1, I_1) \times \operatorname{Pv}(\tilde{C}_2, I_2).$$

We denote by $Z \subset \operatorname{Pv}$ the preimage of the subvariety $\operatorname{Pv}_1 \times \Xi_2 \subseteq \operatorname{Pv}_1 \times \operatorname{Pv}_2$. By Lemma 5.10, $h^0(\tilde{C}, L) > 0$ for all points $[L] \in Z$. Therefore $Z \subset \Xi$. Now we observe that $\dim Z = p - 3$, since from 3.14 it easily follows that the codimension of Ξ_2 in Pv_2 is equal to two and $\Xi_2 \neq \varnothing$ if $p(C_2) \geqslant 1$. Thus the Gauss map $\Gamma\colon Z \dashrightarrow (\mathbf{P}^{g-1})^*$ associating to a point $z \in Z$ the projectivization of the tangent space to Ξ at z is well-defined at generic points of Z (here the projectivizations of all tangent spaces to Pv are naturally identified with one fixed $\mathbf{P}^{g-1}$, namely the space of the canonical embedding of the curve S). By construction, all hyperplanes from $\mathbf{P}^{g-1}$ belonging to the image of the Gauss map pass through a subspace $P \subset \mathbf{P}^{g-1}$ coinciding with the projectivization of the tangent space to the abelian variety $(\tilde{f}^0)^{-1}\mathrm{Pv}_1 \times \{x\}$ (more precisely, to a principal homogeneous space with respect to the action of $P(\tilde{C}_1, I_1)$). The dimension of the subspace P is equal to $\dim \mathrm{Pv}_1 - 1 = p(C_1) + \frac{4}{2} - 1 - 1 = p(C_1)$. Hence the image of Γ has dimension $g - 1 - p(C_1) - 1 = p - 3 - p(C_1) < p - 3$, since by our assumption $p(C_1) \geqslant 1$. Hence there exists a one-dimensional family of smooth points on $\Xi = \Theta$ which is mapped to a single point by the Gauss map Γ. But this is impossible for the Jacobians. In fact, for each smooth point $[L] \in \Theta \subset \mathrm{Jv}(S) = \mathrm{Pv}$ we have $\dim|L| = 0$, $\deg L = g - 1$ and the unique divisor $D \in |L|$ spans the hyperplane $\langle D \rangle \subset \mathbf{P}^{g-1}$. Using 2.7, it is easy to verify that $\Gamma([L]) = \langle D \rangle$. But the hyperplane $\langle D \rangle$ intersects the curve $S \subset \mathbf{P}^{g-1}$ along finitely many points and so defines the point $[L]$ modulo a finite number of possibilities (under the assumption that $[L]$ is a smooth point of Θ). This contradiction completes the proof. ■

7.3. REMARK. From the proof it is easy to see that Lemma 7.2 holds for *all* p.

7.4. By Proposition 6.1, the variety $\mathrm{Sing}\,\Xi = \mathrm{Sing}\,\Theta$ has an irreducible component Z of dimension $\geqslant g - 4 = p - 5$. Moreover, in view of our assumptions $\dim Z = p - 5$. We fix such an irreducible component Z. By Theorem 5.12, Z is a subvariety of one of the varieties of special singularities $Z(\cdots)$ described in 5.11. We say that Z *has type corresponding to the type of a special variety* $Z(\cdots)$ if $Z \subseteq Z(\cdots)$, and *does not have* it otherwise. Here and in what follows in the study of special singularities we use the notation from 5.11.

7.5. First we verify that the component Z *does not have type* V, VI, VII, VIII, IX or X. In fact, case X does not occur already for $p \geqslant 7$. For the types VI, VII, VIII and IX we have $p(C_1), p(C_2) \geqslant 1$, since otherwise $Z(\cdots) = \varnothing$. After a suitable modification of the components C_1 and C_2 this leads to a contradiction with Lemma 7.2. For example, in the case of type IX it suffices to replace C_1 by the curve $C_1 \cup C_0$ and to leave C_2 intact. The same argument shows that Z cannot have type V, because in this case $p(C_1) = 3$ and in view of (S) the complementary curve C_2 in C is connected and has genus $\geqslant 1$ for $p(C) \geqslant 7$.

7.6. REMARK. As a matter of fact, we have proved a somewhat stronger assertion:

(7.6.1) *For* $p \geqslant 7$ *the component* Z *does not have any of the types* V $-$ X.

(7.6.2) *For* $p \geqslant 6$ *the component* Z *does not have any of the types* VI $-$ IX.

7.7. Suppose now that Z has type IV, i.e. $Z \subseteq Z(\gamma, C_1)$. We claim that then S is a *superelliptic curve* and $Z = \Lambda$. We note that by 7.2 the curve C_0 is a nonsingular rational curve. On the other hand, applying (S) and 7.2 again, we see that the structure γ is finite, i.e. C_1 is a hyperelliptic curve of genus $p(C_1) = p - 3 \geqslant 5$.

7.8. The following assertions are immediate consequences of the fact that the canonical image of the curve C_1 is a projectively normal rational curve of degree $p(C_1) - 1$ in $\mathbf{P}^{p(C_1)-1}$:

(7.8.1) *The linear system* $|\omega_{C_1} \otimes \gamma^*(M_1)^{-2}|$ *is nonsingular and has dimension* $g(C_1) - 3$, *where* $[M_1] \in \mathrm{Pic}^1(\mathbf{P}^1)$.

(7.8.2) *The linear system* $|\omega_{C_1} \otimes \gamma^*(M_2)^{-2}|$ *is nonsingular and has dimension* $g(C_1) - 5 \geqslant 0$, *where* $[M_2] \in \mathrm{Pic}^2(\mathbf{P}^1)$.

But by 3.14 a general nonsingular divisor D on $\tilde{C}_1$ such that $\pi_{1*}D \in |\omega_{C_1} \otimes \gamma^*(M_1)^{-2}|$ is linearly fixed. Hence

$$\dim Z(\gamma; C_1) = \dim P(\tilde{C}_0, I_0) + \dim\left|\omega_{C_1} \otimes \gamma^*(M_1)^{-2}\right| = p - 5,$$

and $\dim Z'(\gamma; C_1) = p - 7$. From this it follows that S is a superelliptic curve (cf. Proposition 6.4) and $Z'(\gamma; C_1) = \Lambda'$.

Now we construct a surface $T \subseteq P(\tilde{C}, I)$ for which

$$T \cdot Z'(\gamma; C_1) = Z(\gamma; C_1).$$

Since the isogeny $\tilde{f}^0$: $\mathrm{Pv} \to \mathrm{Pv}_0 \times \mathrm{Pv}_1$ is compatible with the actions of the corresponding Prymians (cf. 3.21.2), it suffices to construct a surface $T_1 \subset P_1 = P(\tilde{C}_1, I_1)$ for which the following condition holds:

$$T_1 \cdot \{[(\gamma\pi_1)^*(M_2)(D)] \in \mathrm{Pv}_1 | [M_2] \in \mathrm{Pic}^2(\mathbf{P}^1) \text{ and } D \text{ is a nonsingular divisor on } \tilde{C}_1\}$$
$$= \{[(\gamma\pi_1)^*(M_1)(D)] \in \mathrm{Pv}_1 | [M_1] \in \mathrm{Pic}^1(\mathbf{P}^1) \text{ and } D \text{ is a nonsingular divisor on } \tilde{C}_1\}.$$

Here we understand equality in the sense of coincidence of generic points. In fact, by (5.8.2) for a generic point $[(\gamma \circ \pi_1)(M_1)(D)] \in \tilde{f}^0 Z(\gamma; C_1)$ we have

$$\pi_{1*} D \in \left|\omega_{C_1} \otimes \gamma^*(M_1)^{-2}\right|.$$

Now we observe that, modulo a nonsingular fixed divisor, the linear system $|\pi_{1*}D|$ can be viewed as a subsystem of the complete hyperelliptic system $|\omega_{C_1}|$ and

$$\dim |\pi_{1*}D| = \deg \omega_{C_1} \otimes \gamma^*(M_1)^{-2}/2 = p(C_1) - 3.$$

On the other hand,

$$\dim \left|\omega_{C_1} \otimes \gamma^*(M_1)^{-4}\right| = p(C_1) - 5 \geqslant 0.$$

Using the fact that C_1 is a hyperelliptic curve, it is easy to show that $\pi_{1*}D = \gamma^*(x + y) + a$ *nonsingular divisor on* C_1, where x and y are generic points on $\mathbf{P}^1$. Let D_1 and D_2 be the components of the divisor D on $\tilde{C}_1$ lying over $\gamma^*(x)$ and $\gamma^*(y)$, so that $\pi_{1*}D_1 = \gamma^*(x)$ and $\pi_{1*}D_2 = \gamma^*(y)$. Then $D = D_1 + D_2 +$ *a nonsingular divisor on* $\tilde{C}_1$. Therefore

$$(\gamma \circ \pi_1)^*(M_1)(D - D_1 + I^*D_2) = (\gamma \circ \pi_1)^*(M_2)(\text{a nonsingular divisor on } \tilde{C}_1).$$

Now we define a surface T_1 to be the closure of the following subset of classes of invertible sheaves:

$$\Big\{[\mathcal{O}_{\tilde{C}_1}(D_1 - D_2)] \in P_1 | D_1 \text{ and } D_2 \text{ are nonsingular divisors of degree 2 on } \tilde{C}_1$$
$$\text{such that } \pi_{1*}D_1, \pi_{1*}D_2 \in |\gamma^* M_1|\Big\}.$$

By the above,

$$T_1\{(\gamma \circ \pi_1)^*(M_2)(D)\} \supseteq \{(\gamma \circ \pi_1)^*(M_1)(D)\}.$$

The opposite inclusion is obvious.

Thus we have constructed a surface $T \subseteq P(\tilde{C}, I) = J(S)$ such that

$$T \cdot Z'(\gamma; C_1) = Z(\gamma; C_1).$$

Hence by (6.6.3) there is an inclusion $Z(\gamma; C_1) \subseteq \Lambda$, and in view of the irreducibility of Z and Λ we have $Z = \Lambda$, as required.

7.9. Suppose now that Z has type III, i.e. $Z \subset Z(\gamma; c_1, c_2)$. We shall show that in this case S *is also a superelliptic curve and* $Z = \Lambda$. In this situation the hyperelliptic structure γ: $C_1 \to \mathbf{P}^1$ is not necessarily finite, i.e. there may exist curves $C_0 \subset C_1$ whose image with respect to γ is a point. But by 7.2 and (S), if such a curve does exist, thenit is unique and is a nonsingular rational curve intersecting the residual component C_1' at two distinct points.

This component C_1' is a hyperelliptic curve of genus $p(C_1') \geqslant 5$. We observe that in the case under consideration $\omega_{C_1}|_{C_0} = 0$. Now a word-for-word repetition of the argument from 7.8 for the curve C_1 yields the desired result.

7.10. REMARK. Using the construction of the preceding paragraph, we can show that for $p = 7$ the variety $Z'(\gamma; c_1, c_2)$ consists of two distinct points, since the degree of the corresponding isogeny $\tilde{f}^0$ is equal to two (one should twice apply 3.15).

7.11. Suppose now that Z has type II, i.e. $Z \subseteq Z(\varepsilon)$. We shall verify that in this case *S is also a superelliptic curve and $Z = \Lambda$*. So, let ε: $C \to E$ be a superelliptic structure. If the morphism ε is not finite, then the finiteness may fail only over the ordinary double points $e \in E$. Moreover, by (S) and 7.2 this is possible only if over the point e there lies a nonsingular rational curve C_0. Hence in this situation the morphism extends to a finite superelliptic structure. Since the curve C has only ordinary double singularities, E has the same property. Thus the elliptic curve E is a "wheel" consisting of n rational curves E_i, $i = 1,\ldots,n$. By Lemma 7.2, $n \leqslant 3$. Moreover, for $n = 3$ all the curves $\varepsilon^{-1}(E_1)$, $\varepsilon^{-1}(E_2)$ and $\varepsilon^{-1}(E_3)$ are nonsingular rational curves. In this case it is easy to verify that $p(C) = 4$. Therefore *this is impossible for $p \geqslant 5$*. Thus $n = 1$ or 2.

Now we show that *the case $n = 2$ is also impossible*. By 7.2, one of the components $\varepsilon^{-1}(E_1)$ or $\varepsilon^{-1}(E_2)$ of C is a nonsingular rational curve. We denote this curve (say, $\varepsilon^{-1}(E_2)$) by C_0 and the residual component by C_1. On E there are exactly two singular points, e_1 and e_2. Let c_1 and c_2 be the points of C over e_1, and let c_3 and c_4 be the points of C over e_2. By 7.7 and 7.9, S is a superelliptic curve and $\Lambda = Z(\gamma; c_1, c_2) = Z(\gamma'; C_1)$, where γ is the hyperelliptic structure on the desingularization C_2 of C at the points c_1 and c_2 and γ' is the hyperelliptic structure on C_1 which, as well as γ, is induced by the morphism ε. Consider the partial desingularization $\tilde{f}$: $\tilde{C}_0 \sqcup \tilde{C}_1 \to \tilde{C}$ at the four points $\tilde{c}_1 = \pi^{-1}(c_1)$, $\tilde{c}_2 = \pi^{-1}(c_2)$, $\tilde{c}_3 = \pi^{-1}(c_3)$, $\tilde{c}_4 = \pi^{-1}(c_4)$ as a composition of the desingularization $\tilde{f}_1$: $\tilde{C}_2 \to \tilde{C}$ at the points $\tilde{c}_1$ and $\tilde{c}_2$ and a subsequent desingularization at the points $\tilde{c}_3$ and $\tilde{c}_4$. Then for a generic point $[L] \in Z$

$$\tilde{f}^0([L])|_{\tilde{C}_0} = \left(\tilde{f}_1^0([L])|_{\tilde{C}_0}\right)(-\tilde{c}_3 - \tilde{c}_4) = \left[\mathcal{O}_{\tilde{C}_0}(-\tilde{c}_3 - \tilde{c}_4)\right] = \text{const} \in \mathrm{Pv}(\tilde{C}_0, I_0).$$

To prove this, we use the fact that $\omega_{C_2}|_{C_0} = 0$ (compare with 7.9). But this contradicts the equality $Z = Z(\gamma'; C_1)$. Hence the case $n = 2$ is actually impossible.

Therefore $n = 1$, so that if Z has type II, then $Z \subseteq Z(\varepsilon)$, where ε: $C \to E$ is a finite morphism of degree 2 onto an *irreducible* elliptic curve E having at most one ordinary double point. Consider a generic point $[L] \in Z(\varepsilon)$. Then $L \approx (\varepsilon \circ \pi)^*(\mathcal{O}_E(e + e'))(D)$, where e, e' is a pair of generic points on E and $\pi_* D \in |\omega_C \varepsilon^*(e + e')^{-2}|$ is a nonsingular divisor on C. Using the canonical model of the curve, it is easy to verify that, as in the nonsingular case, there is a representation $\pi_* D = \varepsilon^*(\Sigma_1^{p-5} e_i) = \varepsilon^*(e_1 + e_2 + e_3) +$ *a nonsingular divisor* on the superelliptic curve C, where e_1, e_2 and e_3 are generic nonsingular points on the elliptic curve E. We recall that by our assumption $p \geqslant 8$. We denote by h_{lj} a nonsingular point on E such that $2h_{lj} \sim e_l + e_j$, $1 \leqslant l, j \leqslant 3$ (here $\sim$ denotes the linear equivalence). By the above, there is a representation $D = D_1 + D_2 + D_3 +$ *a nonsingular divisor* on $\tilde{C}$, where $\pi_* D_i = \varepsilon^*(e_i)$, $1 \leqslant i \leqslant 3$. On the curve $\tilde{C}$ we consider the divisor

$$t_{ij} = (\varepsilon \circ \pi)^*(h_{ij}) - D_l - D_j.$$

Then

$$\pi_* t_{ij} = 2\varepsilon^*(h_{ij}) - \varepsilon^*(e_l + e_j) \sim 0.$$

Hence $[\mathcal{O}_{\tilde{C}}(t_{ij})] \in \ker \pi_* = P(\tilde{C}, I) \oplus \mathbf{Z}/2\mathbf{Z}$.

On the other hand, the points h_{lj} can be chosen in such a way that $\Sigma_{1\leqslant l<j\leqslant 3} h_{lj} \sim \Sigma_1^3 e_i$. For such a choice

$$\sum_{1\leqslant l\leqslant j\leqslant 3} (\varepsilon\circ\pi)^* h_{lj} \sim \sum_{l=1}^{3} (\varepsilon\circ\pi)^* e_l,$$

from which it follows that

$$\sum_{1\leqslant l<j\leqslant 3} t_{lj} = (\varepsilon\circ\pi)^*(h_{12} + h_{13} + h_{23}) - 2D_1 - 2D_2 - 2D_3$$

$$\sim (\varepsilon\circ\pi)^*(e_1 + e_2 + e_3) - 2D_1 - 2D_2 - 2D_3 = \sum_{l=1}^{3} I^*D_l - D_l$$

and by definition the class of the sheaf corresponding to this divisor lies in $P(\tilde{C}, I)$. Therefore one of the sheaves $\mathcal{O}_{\tilde{C}}(t_{lj})$ or, more precisely, its isomorphism class lies in $P(\tilde{C}, I)$. Moreover,

$$(\varepsilon\circ\pi)^*(\mathcal{O}_E(e+e'))(D + t_{lj}) = (\varepsilon\circ\pi)^*\mathcal{O}_E(e + e' + h_{lj})$$

(*a nonsingular divisor on* $\tilde{C}$), i.e. $[L(t_{lj})] = [L'] \in Z'(\varepsilon)$. Now we consider the variety $T \subset P = P(\tilde{C}, I)$ whose generic points have the form

$$[\mathcal{O}_{\tilde{C}}(D_1 + D_2 - (\varepsilon\circ\pi)^*(h_{12}))],$$

where D_1 and D_2 are nonsingular divisors of degree 2 on $\tilde{C}$ such that $\pi_* D_1 = \varepsilon^* e_1$ and $\pi_* D_2 = \varepsilon^* e_2$, and h_{12} is one-half of the divisor $e_1 + e_2$. We have already verified that $T \cdot Z'(\varepsilon) \geqslant Z(\varepsilon)$. The reverse inclusion is obvious. Therefore $T \cdot Z'(\varepsilon) = Z(\varepsilon)$. By our assumption $\dim Z(\varepsilon) \geqslant p - 5$ since $Z(\varepsilon) \supseteq Z$, and an easy dimension count shows that $\dim T \leqslant 2$. Hence $\dim Z'(\varepsilon) \geqslant p - 7$. As above, applying Propositions 6.4 and 6.6 we see that S is a superelliptic curve and $Z = \Lambda$.

7.12. In view of 7.5, 7.7, 7.9 and 7.11, *Z does not have type* V, VI, VII, VIII, IX *or* X, *and if Z has type* II, III *or* IV, *then S is a superelliptic curve and* $Z = \Lambda$. By the Torelli theorem, this is an intrinsic result. On the other hand, by (6.6.2) $\operatorname{Sing}\Theta = \operatorname{Sing}\Xi$ has at least one other irreducible component Z of dimension $p - 5$. Therefore by Theorem 5.12 the curve C admits a trigonal structure $\tau\colon C \to \mathbf{P}^1$, and then by (S) the curve C is trigonal, which completes the proof of necessity in the case $p \geqslant 8$.

7.13. REMARK. Type IV can occur when the Prymian is a Jacobian, namely when the glued points c_1, c_2, c_3 and c_4 of the nonsingular rational curve $C_0 \approx \mathbf{P}^1$ are compatible with the hyperelliptic structure $\gamma\colon C_1 \to \mathbf{P}^1$, which defines a trigonal structure on C (cf. Figure 7). Type II and the corresponding quasitrigonal structure can also occur (cf. Figure 8).

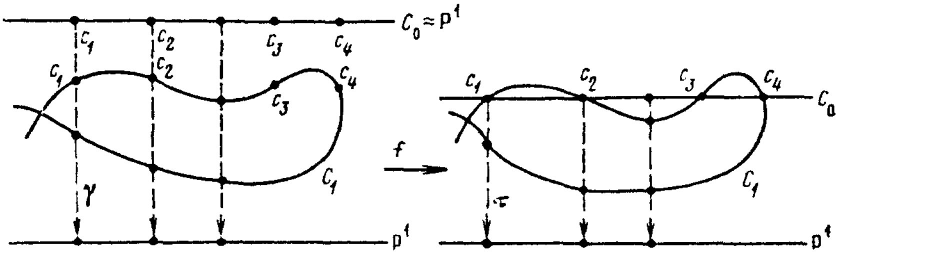

FIGURE 7

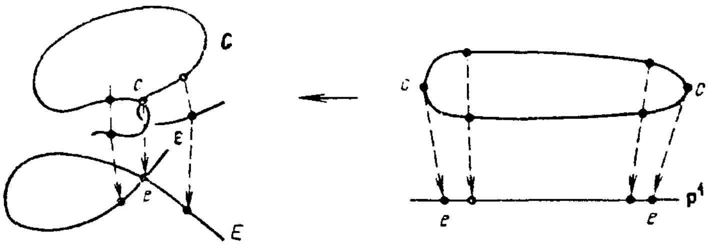

FIGURE 8

§8. Proof of the main theorem: cases $p = 7, 6$

8.1. As in the preceding section, we assume that $P(\tilde{C}, I) \approx J(S)$ or equivalently $\mathrm{Pv}(\tilde{C}, I) = \mathrm{Jv}(S)$, where S *is a connected nonsingular nonhyperelliptic curve whose genus* (this time) *is equal to* $p - 1 = 6$. The case $p - 1 = 5$ will be briefly discussed in the end of this section (cf. 8.6–8.10). As above, we assume that C *is neither hyperelliptic nor quasitrigonal*. Our goal is to show that this curve C is trigonal (cf. 8.5).

8.2. By Proposition 6.9, the variety $\mathrm{Sing}\,\Theta = \mathrm{Sing}\,\Xi$ has a two-dimensional component (a surface) Z such that for a generic smooth point $[M] \in Z$ there exists exactly one other nonsingular point in $\mathrm{Sing}\,\Theta = \mathrm{Sing}\,\Xi$ with a "parallel" tangent plane, namely the point $[\omega_S \otimes M^{-1}]$. We fix such an irreducible component Z. In view of Remark (7.6.1), Z does not have type V–X.

8.3. Now we verify that Z does not have type III or IV. In fact, suppose that $Z \subseteq Z(\cdots)$, where $Z(\cdots)$ is a variety of special singularities of one of these types. By definition, in these cases

$$Z(\cdots) = \left(\tilde{f}^{0}\right)^{-1} \quad (a\ surface \subset \mathrm{Pv}'),$$

where $\tilde{f}^0\colon \mathrm{Pv} \to \mathrm{Pv}'$ is the isogeny corresponding to a partial desingularization at two or four points. A repeated application of Lemma 3.15 shows that the degree of this isogeny is equal to two or four. Since at a generic point from Z the transformation $[M] \mapsto [\omega_S \otimes M^{-1}]$ does not correspond to a translation, we arrive at a contradiction.

8.4. To complete the proof in the case $p = 7$, it remains to consider the situation when Z has type II. So, let $\varepsilon\colon C \to E$ be a superelliptic structure and let $Z \subseteq Z(\varepsilon)$. As in 7.11, we can extend this structure to a *finite* one. Moreover, we may assume that E is a "wheel" consisting of $n = 1$ or 2 rational curves. Since C is not quasitrigonal, for $n = 2$ over each of the two singular points $e_1, e_2 \in E$ there are two ordinary double points c_1, c_2 and c_3, c_4 respectively (compare with 7.13). But then by Lemma 7.2 one of the components $C_i = \varepsilon^{-1}(E_i)$, $i = 1, 2$, of C is a nonsingular rational curve. Furthermore, by Remark 7.10 on the variety $\mathrm{Sing}\,\Xi$ there are two points of multiplicity $\geqslant 3$ with respect to Ξ which lie in $Z'(\gamma; C_1, C_2)$, where γ is the hyperelliptic structure on the resolution of C at two points c_1 and c_2 induced by ε and the "contraction" of a nonsingular rational component to a point. This contradicts Proposition 6.8.

Thus $n = 1$; that is, E is an *irreducible* elliptic curve, possibly with one ordinary double point. Since the curve C is not quasitrigonal, over such a singular point of E there are two singular points c_1, c_2 of C. Hence if E is singular, then, arguing as in the end of the preceding paragraph and applying Remark 7.10 and Proposition 6.8, we arrive at a contradiction. Therefore E is a *nonsingular* elliptic curve.

We denote by $J_2(E)$ the subgroup of points of order 2 in the Jacobian $J(E)$ of E. Consider the homomorphism:

$$(\varepsilon \circ \pi)^* \ : \underset{\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}}{\underset{\parallel}{J_2(E)}} \ \to \ker(\mathrm{Nm}) \approx P(\tilde{C}, I) \oplus \mathbf{Z}/2\mathbf{Z}$$

If it is injective, then there exists an element $[\alpha] \in J_2(E)$ such that $(\varepsilon \circ \pi)^*\alpha \cup 0 \in P(\tilde{C}, I)$. If M is a general invertible sheaf of degree 2 on E, then

$$(\varepsilon \circ \pi)^*(\alpha) \otimes (\varepsilon \circ \pi)^*(M) = (\varepsilon \circ \pi)^*(\alpha \otimes M),$$

where $\alpha \otimes M$ is again a general invertible sheaf of degree 2 on E. Furthermore, $(\alpha \otimes M)^{\otimes 2} = M^{\otimes 2}$. Using this, it is easy to show that $[(\varepsilon \circ \pi)^*\alpha] + Z(\varepsilon) = Z(\varepsilon)$. But this contradicts the choice of $Z \subset Z(\varepsilon) \subset \operatorname{Sing} \Theta$. Hence *the morphism* $(\varepsilon \circ \pi)^*$ *has kernel on* $J_2(E)$.

We remark that, applying the argument from the beginning of §6 to the singular curve C instead of S, we see that $\omega_C \approx \varepsilon^*M$ for some invertible sheaf M of degree 6 on E. Let $[M_1]$, $[M_2]$, $[M_3]$, $[M_4] \in \operatorname{Pic}^3 E$ be four half-divisors for $[M] \in \operatorname{Pic}^6 E$. The group $J_2(E)$ acts transitively on the set of these points. If $[\alpha] \neq 0 \in J_2(E)$ and $(\varepsilon \circ \pi)^*[\alpha] = 0 \in P(\tilde{C}, I)$, then the sheaves $(\varepsilon \circ \pi)^*M_1$ and $(\varepsilon \circ \pi)^*(\alpha \otimes M_1)$ on $\tilde{C}$ are isomorphic to each other. From this it clearly follows that $h^0(\tilde{C}, (\varepsilon \circ \pi)^*M_1) \geqslant 6$. But by the Clifford theorem (4.2)

$$h^0(\tilde{C}, (\varepsilon \circ \pi)^*M_1) \leqslant (\deg(\varepsilon \circ \pi)^*M_1)/2 + 1 = 7.$$

The case $h^0(\tilde{C}, (\varepsilon \circ \pi)^*M_1) = 7$ is impossible, since then by 3.14 and (5.1.3) there exists a one-dimensional family of singular points $[(\varepsilon \circ \pi)^*M_1(c - I(c))]_{c \in \tilde{C}}$ of multiplicity $\geqslant 3$ on Ξ. But this is impossible by Proposition 6.8. Hence

$$h^0(\tilde{C}, (\varepsilon \circ \pi)^*M_1) = 6.$$

But $\mathrm{Nm}(\varepsilon \circ \pi)^*M_1 \approx 2\varepsilon^*(M_1) \approx \varepsilon^*(2M_1) \approx \varepsilon^*(M) = \omega_C$. Using this and Theorem 5.1, we obtain an inclusion $[(\varepsilon \circ \pi)^*M_1] \in \mathrm{Pv}$, and so $\operatorname{Mult}_{[(\varepsilon \circ \pi)^*M_1]}\Xi \geqslant 3$. On the other hand, by Proposition 6.8, $\Xi = \Theta$ has at most one such point. Now we observe that the above argument can be applied to each of the four sheaves M_i, $i = 1, 2, 3, 4$. Therefore *all elements of the group* $J_2(E)$ *lie in the kernel of the homomorphism* $(\varepsilon \circ \pi)^*$. But since the group $J_2(E)$ has rank 2 over $\mathbf{Z}/2\mathbf{Z}$, Lemma 3.18 (applied to the normalizations of the curves of the "tower" $\tilde{C} \to C \to E$) shows that this is impossible. Thus the surface Z does not have type II.

8.5. In view of 8.2–8.4, we see that our surface $Z \subset \operatorname{Sing} \Xi$ has type I, from which, as in the preceding section, it follows that the curve C is trigonal. This completes the proof of necessity for $p = 7$.

8.6. Now we consider the case $p = 6$, $g(S) = p - 1 = 5$. We do not go into details, but give a sketch of the proof of necessity. This time, in addition to the trigonal case one must consider the case of a *plane quintic* C *with an odd pair* $(\tilde{C}, I)$ a special case of which is type V. Therefore we assume that among the singular curves making up $\operatorname{Sing} \Xi$ there are no curves of types I, V and X and show that this yields a contradiction. By Remark (7.6.2), *types* VI–IX *also do not occur.*

8.7. First we verify that S *is not a trigonal curve*. In fact, otherwise Sing Ξ = Sing Θ has exactly two (nonsingular) components with distinct tangents (in the sense of the Gauss map) which are isomorphic to the curve S. One of these components coincides with the curve $g_3^1 + S \subset \mathrm{Jv}(S)$, where g_3^1 is the class of the sheaf of type G_3^1 corresponding to the trigonal structure on S, and the other component is symmetric to the first one with respect to $[\omega_S]$. Arguing as in the case $p = 7$, we obtain a finite superelliptic structure $\varepsilon: C \to E$, where E is a nonsingular curve. Since for type II the divisor D can be chosen in at most two ways and Sing Ξ = Sing Θ has two components, each of these components is birationally isomorphic to the curve $J(E)$, which yields a contradiction.

8.8. *The variety* Sing Ξ *does not have components of type* IV. Suppose to the contrary that on C there is a structure of type IV. Then, by (S) and 7.2, C_0 is a nonsingular rational curve and the hyperelliptic structure γ is finite (compare with 7.7). By 8.7 and the Noether-Enriques theorem [28], the canonical embedding of S is an intersection of three quadrics, and therefore we can apply Proposition 6.11. It follows that the variety $Z(\gamma; C_1)$ is a nonsingular elliptic curve. Hence, again by this proposition, in Sing Ξ there exists at least one other irreducible component $Z' \neq Z$. If this component again has type IV, then $C_1 = C_0' \cup C_1'$, where C_0' and C_1' are nonsingular rational curves such that $C_0 \cap C_0' = \varnothing$ and $\#C_1' \cap C_0' = \#C_1' \cap C_0 = 4$. More precisely, the other structure of type IV is the pair γ', $C_1' \cup C_0$, and the structures γ and γ' yield a single *trigonal* structure (cf. Figure 9), which is impossible by our assumption. If Z' has type III, then by (S) and 7.2 the corresponding structures are compatible with type IV. By what we mean that the points c_1 and c_2 of the structure of type III lie in the set $C_0 \cap C_1$ of intersection points for type IV and the structure γ for type IV is induced by the structure γ for type III. Applying Proposition 6.11 once more, we see that $Z' = Z(\gamma; c_1, c_2)$ is a nonsingular elliptic curve and the intersection of the components Z and Z' consists of exactly one point, which contradicts the definition of the curves Z and Z' as the preimages of isogenies for some partial desingularizations. In fact, since the degree of the isogeny $\tilde{f}^0$ for type II is equal to two, these curves intersect along an *even* number of points. Hence the component Z' must have type II, which, in its turn, again leads to type III and yields a contradiction.

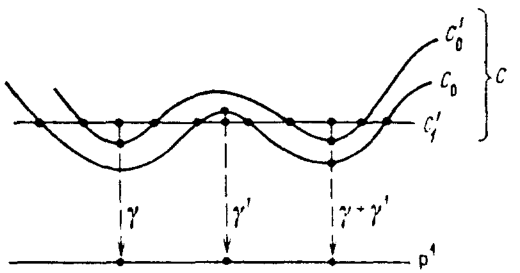

FIGURE 9

8.9. Next we show that *the curve C is irreducible*. By Theorem 5.12, C has a structure of type III or II. First we consider the case of type III. Then the hyperelliptic structure γ is finite. If the curve C_1 is reducible, thenit consists of two nonsingular rational components intersecting along five points. In view of condition (E), C has the form sketched in Figure 10 (after a suitable ordering of the points c_1 and c_2).

On the other hand, using Proposition 6.11 instead of Proposition 6.9 in the argument from 8.3, we see that the curve $Z(\gamma; c_1, c_2) \subset \operatorname{Sing} \Xi$ is an irreducible nonsingular elliptic curve. Hence $\operatorname{Sing} \Xi$ contains other irreducible curves and has other structures of type II or III. But in each of these cases C must have a quasitrigonal structure since after resolving the point c_1 we obtain a hyperelliptic structure on the resolution. Since by our assumption C does not admit quasitrigonal structures, the curve C in this case is irreducible.

The case of a structure of type II is dealt with in a similar way unless the superelliptic structure $\varepsilon\colon C \to E$ is a finite morphism onto a smooth elliptic curve E. But then C is irreducible in view of condition (E).

8.10. Since C is irreducible, so is $\tilde{C}$. Moreover, all curves C for nearby admissible pairs (in the sense of the Zariski topology) are irreducible. But for general pairs $(\tilde{C}, I)$ over Jacobians this is possible only if C is a trigonal (nonsingular) curve or a (nonsingular) plane quintic [12]. But in that case the intersection of quadrics through the canonical model of C is a surface. Hence the same holds for the specialization to our curve C. But this is impossible since if the irreducible curve C is not hyperelliptic, trigonal or quasitrigonal, then the intersection of quadrics through C coincides with itself. This is proved in practically the same way as the Noether-Enriques theorem for $g = 6$. This completes the proof of the necessity for $p = 6$. It remains only to add that the fact that the pair $(\tilde{C}, I)$ is odd when C is a plane quintic follows from Theorem 5.12 (cf. the argument on p. 347 of [17]).

§9. Proof of the main theorem: case $p = 5$

9.1. So, let $\operatorname{Pv}(\tilde{C}, I) = \operatorname{Jv}(S)$, where S is a nonsingular connected nonhyperelliptic curve of genus 4. We need to show that C *is a trigonal curve*. As above, we assume that *the curve C is neither hyperelliptic nor quasitrigonal.*

9.2. First we consider the case when S is a general curve, i.e. its theta-divisor Θ has two distinct singular points. By assumption, $\operatorname{Sing} \Theta = \operatorname{Sing} \Xi$. Therefore the set $\operatorname{Sing} \Xi = \{[L_1], [L_2]\}$ consists of two distinct points $[L_1]$, $[L_2]$. By 5.5, there exist two linearly independent sections $s_1, s_2 \in H^0(\tilde{C}, L_1)$ such that $s_1 \otimes I^*s_2 = s_2 \otimes I^*s_1$. In ivew of 3.15, (S) and 5.10, the sections s_1 and s_2 do not simultaneously vanish on any of the components of the curve $\tilde{C}$, since otherwise we would have $\#\operatorname{Sing} \Xi \geqslant 4$. Similarly, the sections s_1 and s_2 do not simultaneously vanish at every point of any set consisting of at least three points. If the sections s_1 and s_2 simultaneously vanish at two singular points $\tilde{c}, \tilde{c}_2 \in \tilde{C}$, then the same argument shows that $\tilde{f}^0 L_1 \approx \tilde{f}^0 L_2$, where $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ is the partial desingularization at $\tilde{c}_1$ and $\tilde{c}_2$. Hence in this case $[L_1 \otimes L_2^{-1}]$ is a point of order two on $P(\tilde{C}, I) \approx J(S)$, which is impossible by (6.12.3). If the sections s_1 and s_2 simultaneously vanish only at one singular point $\tilde{c} \in \tilde{C}$, then by Lemma 5.8 and the fact that C is not quasitrigonal there is a representation $\tilde{f}^0 L_1 \approx \pi'^* M$, where $\tilde{f}\colon \tilde{C}' \to \tilde{C}$ is the partial desingularization at the point

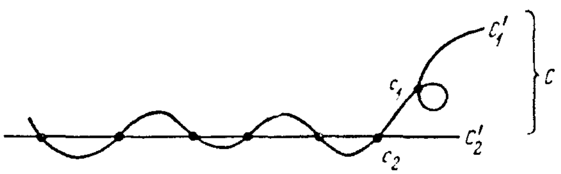

FIGURE 10

$\tilde{c}$ and M is a free invertible sheaf of type G_3^1 on C'. Hence $I'^*\tilde{f}^0L_0 \approx \tilde{f}^0L_1$ and therefore $I^*L_1 \approx L_1$, since in this case the isogeny $\tilde{f}^0$ is an isomorphism. From this it is clear that $[L_1]$ is a symmetry point of the divisor Ξ. In fact, in this case a point $[L'] \in \Xi$ is symmetric to the point corresponding to the sheaf $L'' = L_1^{\otimes 2} \otimes L_1'^{-1} \approx L_1 \otimes I^*L_1 \otimes L'^{-1} \approx \omega_{\tilde{C}} \otimes L'^{-1}$ and $h^0(\tilde{C}, L'') = h^0(\tilde{C}, L')$ (the Serre duality + the Riemann-Roch formula). But in view of (6.12.2) and our assumption this is impossible. Hence the sections s_1 and s_2 do not simultaneously vanish at any of the singular points of $\tilde{C}$. Then, by 5.8, $L_1 \approx \pi^*(M)(D)$, where M is a free invertible sheaf on C such that $h^0(C, M) \geqslant 2$ and D is a nonsingular divisor on $\tilde{C}$. It is clear that $\deg M \leqslant 4$. Moreover, if $\deg M = 4$, then $L_1 \approx \pi^*(M)$ and $I^*L_1 \approx L_1$, which again yields a contradiction with (6.12.2). Hence $\deg M = 3$ and C is a *trigonal* curve.

9.3. Using the result of the preceding paragraph for a general Jacobian and the fact that the Prym map P is an epimorphism for $p = 5$, we see that C is a degeneration of an irreducible trigonal curve. In particular, the intersection of quadrics through the canonical model of our curve $C \subset \mathbf{P}^4$ contains an irreducible surface F. If F linearly spans the space $\mathbf{P}^4$, then, as in the case of a nonsingular curve C, it is easy to verify that the curve is trigonal. So we consider the case when F spans a subspace $\mathbf{P}^3 \subset \mathbf{P}^4$. Since the curve C is contained in a three-dimensional family of quadrics and condition (S) holds, this is possible only if F is a quadric and the intersection of quadrics through C also contains a plane $\mathbf{P}^2 \subset \mathbf{P}^4$. But this is impossible by (S). Hence F is a plane in $\mathbf{P}^4$. In particular, the curve C is reducible. Then by 7.2 one of its components C_0 is a nonsingular rational curve. If C_0 intersects the remaining component C_1 in 6 points, then $C_0 \subset C \subset \mathbf{P}^4$ is a rational normal curve of degree 4. But this easily yields a contradiction since the curve C is contained in three linearly independent quadrics and the curve C_1 does not lie on F. Hence $\#C_0 \cap C_1 = 4$ and $C_0 \subset C \subset \mathbf{P}^4$ is a plane conic. Using this, (S), and Lemma 7.2, one can show that C_1 is irreducible. Then by (S) the curve C_1 linearly spans the space of the canonical embedding $\mathbf{P}^4$. From this it follows that $F = \langle C_0 \rangle$, which yields a contradictions since the curve $C_1 \subset C \subset \mathbf{P}^4$ has degree 6. This completes the proof of necessity for $p = 6$, and by the above, also for all values of p. ■

9.4. To obtain specializations of $P(\tilde{C}, I)$ by general Jacobians in 8.10 and 9.3, one needs to use the results of §7 of [3] for $p = 5$ and an easy count of parameters for $p = 6$.

§10. Some applications

10.1. Rationality Criterion for Conic Bundles over a Minimal Rational Surface. *Let V be a nonsingular threefold over the field $\mathbf{C}$ of complex numbers for which there exists a flat map $\pi\colon V \to S$ onto a nonsingular minimal rational surface S whose general fiber is a nonsingular rational curve* 1 *defining an extremal ray $[l] \in N(V)$ (cf. [31]). The variety V is rational if and only if its intermediate Jacobian $J(V)$ is isomorphic as principally polarized abelian variety to a sum of Jacobians of nonsingular curves, i.e. its Griffiths component $J_G(V)$ is trivial.*

The necessity is well known (cf. [22]). Hence we only prove the sufficiency. So, suppose that $J_G(V)$ is trivial. We shall consider the case when $S = \mathbf{F}_n$ is a rational ruled surface. The case when $S \approx \mathbf{P}^2$ easily follows from [4] and [32]. We denote by b_n the unique irreducible curve on $\mathbf{F}_n$ such that $b_n^2 = -n$, and by s_n a fiber of the ruled structure. Let $C \subset \mathbf{F}_n \approx S$ be the degeneration curve of our conic bundle. We shall show that

$$(C \cdot s_n) \leqslant 3$$

or

$$(C \cdot b_0) \leqslant 3 \quad \text{for } n = 0$$

or

$$(C \cdot b_1) = 0 \quad \text{for } n = 1 \text{ and } b_1 \not\subset C$$

or

$$(C \cdot b_1) = 1 \quad \text{for } n = 1 \text{ and } b_1 \subset C.$$

But in these cases the rationality of V is known (cf. [13] and [32]); moreover, the last two cases are reduced to the case when $S \approx \mathbf{P}^2$.

We denote by $\tilde{C}$ the base of the family of "lines" over the degeneration curve C of our conic bundle. Then the pair $(\tilde{C}, I)$ consisting of $\tilde{C}$ and the natural involution I permuting the two lines in a degenerate fiber is a Beauville pair with $\tilde{C}/I = C$. In view of the computations performed in [4] and [22],

$$J(V)P(\tilde{C}, I).$$

Hence the Prymian $P(\tilde{C}, I)$ is isomorphic to a sum of Jacobians of nonsingular curves.

Consider the linear system $|\omega_S(C)|$ on the surface S. From the exact sequence

$$0 \to \omega_S \to \omega_S(C) \to \omega_C \to 0$$

and the vanishingof $h^0(S, \omega_S) = h^1(S, \omega_S)$ it follows that the linear system $|\omega_S(C)|$ on S restricts isomorphically to the canonical linear system $|\omega_C|$ on C. In view of (E), (B), and the nonexistence of double coverings of $\mathbf{P}^1$ with fewer than two branch points, the linear system $|\omega_C|$ is free and defines the canonical morphism κ: $C \to \mathbf{P}^N$. We remark that the curve C is connected. By the above, the morphism κ is induced by the map φ: $S \to \mathbf{P}^N$ defined by the linear system $|\omega_S(C)|$, possibly after subtracting the fixed component. In what follows we use the following properties of linear systems on $\mathbf{F}_n$: they define a map to a point, or a map onto a rational normal curve of degree N in $\mathbf{P}^N$, or a birational map which possibly blows down the curve b_n for $n \geqslant 1$ and is an isomorphism off b_n. In all these cases the image $\varphi(S)$ coincides with the intersection of quadrics passing through it.

Case $N = 0$. In this case C is an elliptic curve and $|\omega_S(C)| = |\beta b_n|$, where β is a nonnegative and $n \geqslant 1$ for $\beta > 0$. Therefore $C \sim (\beta + 2)b_n + (n + 2)s_n$. We need to show that $\beta \leqslant 1$. In fact, if $b_n \not\subset C$, then

$$\begin{aligned}((\beta + 2)b_n + (n + 2)s_n, b_n) &\geqslant 0\\ \Rightarrow (n + 2) - (\beta + 2)n &\geqslant 0\\ \Rightarrow 2 \geqslant (\beta + 1)n &\Rightarrow 1 \geqslant \beta.\end{aligned}$$

If $b_n \subset C$, then

$$\begin{aligned}((\beta + 1)b_n + (n + 2)s_n \cdot b_n) &\geqslant 2\\ \Rightarrow (n + 2) - (\beta + 1)n \geqslant 2 &\Rightarrow 0 \geqslant \beta \cdot n \Rightarrow 1 \geqslant \beta.\end{aligned}$$

Next we consider the case when $N \geqslant 1$ and the image $\varphi(S)$ is a rational normal curve of degree N in $\mathbf{P}^N$. Then the map κ: $C \to \kappa(C)$ has degree 2 at a generic point and defines a hyperelliptic structure on C. Hence in this case $(C \cdot s_n) = 2$.

Suppose now that $N \geqslant 2$ and the map φ is birational. Then κ either is an embedding or blows down to a point the curve $b_n \subset C$ for $n \geqslant 1$ and is an isomorphism outside this curve. In the last case the curve $b_n \subset C$ intersects the other components of C at exactly two points. If the curve $\kappa(C)$ satisfies condition (S), then by the main theorem the curve C either is trigonal or quasitrigonal or is a plane quintic. In all these cases the quadrics passing through $\kappa(C)$ intersect along an irreducible surface of degree $N-1$ in $\mathbf{P}^N$. By the above, this surface coincides with $\varphi(S)$. Using the classification of such surfaces, it is easy to show that C satisfies one of the conditions formulated in the first paragraph of our proof.

So it remains to show that there does not exist a decomposition $C = C_1 \cup C_2$, where C_1 and C_2 are connected curves, $C_1, C_2 \neq b_n$ and $C_1 \cap C_2 = 2$. More precisely, we shall show that in this case we again come to the alternative described in the first paragraph of our proof. If C_1 or C_2 (say C_1 is linearly equivalent to s_n, then $(C \cdot s_n) = (C_2 \cdot s_n) = 2$. In what follows we shall assume that $b_n \not\subset C_1$ and $C_1 \nsim s_n$. Then $C_1 \sim \beta b_n + \zeta s_n$, where $\beta \geqslant 1$ and $\zeta \geqslant n\beta$. If C_2 contains a curve X which is distinct from s_n and b_n, then $C_2' = C_2 \setminus b_n \sim \beta' b_n + \zeta' s_n$, where $\beta' \geqslant 1$ and $\zeta' \geqslant n\beta'$. But then

$$\beta \cdot \beta' \cdot n \leqslant (C_1 \cdot C_2') \leqslant (C_1 C_2) = 2,$$

from which it easily follows that $(C \cdot s_n) \leqslant 3$ for $n \geqslant 2$. If $n = 1$ and $\beta \cdot \beta' = 2$, then $C_1 \sim \beta(b_1 + s_1)$ and $C_2' \approx \beta'(b_1 + s_1)$, which contradicts the connectedness of C_2 when $b_1 \subset C_2$. Hence $C_2 = C_2'$. Therefore $(C \cdot s_n) = \beta + \beta' = 3$. The case $n = 0$ can be easily treated directly. So, the only unclear case is the case when $C_2 \sim b_n + \zeta' s_n$, where $\zeta' \geqslant 1$. But then it is clear that $\beta \leqslant 2$, from which it again follows that $(C \cdot s_n) \leqslant 3$. ■

Essentially, we have proved the following result.

10.2. THEOREM. *In the conditions of Criterion* 10.1, *if the variety V is rational, then* $|2K_S + C| = \varnothing$, *where K_S is the canonical divisor.* ■

The following conjecture generalizes this result.

10.3. CONJECTURE. *Let V be a nonsingular algebraic threefold over an algebraically closed field k of characteristic $\neq 2$ for which there exists a flat map π: $V \to S$ onto a nonsingular surface S whose general fiber is a nonsingular rational curve* 1 *defining an extremal ray* $[l] \in N(V)$. *If V is a rational variety, then $|2K_S + C| = \varnothing$, where C is the degeneration curve of our conic bundle.*

10.4. REMARK. The necessity rationality condition $|2K_S + C| = \varnothing$ formulated in 10.2 is also sufficient, except in the case when the degeneration curve C is a plane quintic. In the last case we must require that the "theta-characteristic" be even. Using this, it is easy to verify that *in the class of conic bundles over a minimal rational surface there does not exist a smooth family V_b, $b \in B$, such that V_{b_0} is a rational variety and V_b is irrational for* $b \in B - b_0$ (compare with Problem 1 from [24]).

We denote by $\mathscr{B}_p$ the variety of Beauville pairs $(\tilde{C}, I)$ such that $C = \tilde{C}/I$ is a connected curve and $p_a(C) = p$. Let N_{g-4} be the variety of principally polarized abelian varieties of dimension g such that $\dim \operatorname{Sing} \Theta \geqslant g - 4$, where Θ is the effective divisor of principal polarization. This is the so-called *Andreotti-Mayer variety* [2], and according to [2] the variety $J_g = J(\mathscr{M}_g)$ (the closure of the image of the Jacobians of nonsingular curves in the space of principally polarized abelian varieties) is a component of N_{g-4}.

The following theorem describes the intersection of N_{g-4} with the closure $\overline{P(\mathscr{B}_{g+1})}$ of the image of the set of Beauville pairs under the Prym map (compare with Theorem 5.1 in [11]).

10.5. THEOREM. *For $g \geqslant 6$ the variety $N_{g-4} \cap \overline{P(\mathscr{B}_{g+1})}$ has the following components:*

(a) *J_g, the variety of dimension $3g-3$ whose generic points correspond to Jacobians of nonsingular curves of genus g;*

(b) *$\mathscr{H}^{(2)}_{g-1}$, the variety of dimension $2g+1$ whose generic points correspond to the Prymians $P(\tilde{C}, I)$ of Beauville pairs $(\tilde{C}, I)$ such that $C = \tilde{C}/I$ is the curve obtained by identifying two general pairs of points of a nonsingular irreducible hyperelliptic curve of genus $g-1$;*

(c) *$\mathscr{H}^{(4)}_{g-1}$, the variety of dimension $2g$ whose generic points correspond to the Prymians $P(\tilde{C}, I)$ of Beauville pairs $(\tilde{C}, I)$ such that $C = \tilde{C}/I$ is obtained by gluing a nonsingular irreducible hyperelliptic curve of genus $g-2$ and a nonsingular rational curve along four points in general position;*

(d) *$\mathscr{E}_{g+1}$, the variety of dimension $2g-1$ whose generic points correspond to the Prymians $P(\tilde{C}, I)$ of Beauville pairs $(\tilde{C}, I)$ such that $C = \tilde{C}/I$ is a nonsingular irreducible superelliptic curve of genus $g+1$ (this variety consists of two irreducible components); and*

(e) *$\mathscr{P}_d$, $1 \leqslant d \leqslant [(g-2)/2]$, the variety of dimension $3g-4$ whose generic points correspond to the Prymians $P(\tilde{C}, I)$ of Beauville pairs $(\tilde{C}, I)$ such that $C = \tilde{C}/I$ is a union of two nonsingular irreducible curves of genus d and $g-d-2$ respectively intersecting along four points in general position.*

The proof is an immediate consequence of Theorem 5.12, the description of $\overline{P(\mathscr{B}_{g+1})}$ and the main theorem. To compute the dimensions, one needs the local Torelli theorems except in case (d) when the fibers of the Prym map are one-dimensional in view of the tetragonal construction [11]. ■

These components will be discussed in more detail in one of the subsequent parts.

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