Invent. math. 65, 209-219 (1981)

Inventiones mathematicae

Distinguishing Prymians from Jacobians

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1. Main Result, Terminology, and Notation

1.1. Throughout this paper we fix an algebraically closed field k of characteristic $\neq 2$; all varieties considered are defined over k.

We denote by $\tilde{C}$ a connected curve with an involution i: $\tilde{C} \to \tilde{C}$ ($i^2 = \mathrm{Id}$). Throughout this paper we also suppose that the pair $(\tilde{C}, i)$ satisfies the following conditions:

(i) $\tilde{C}$ has only ordinary double points;

(ii) The fixed points of i are exactly the singular points, and at a singular point the two branches are not exchanged under i.

In this situation, due to Mumford [M] in the non-singular case and to Beauville [B] in the general case, we have the principally polarized *Prym variety*, or *Prymian* for short, (P, Ξ), i.e. an Abelian variety P with a principal polarization Ξ. In this paper we are interested in establishing when (P, Ξ) is not a Jacobian of some smooth curve or a sum of them. Of course, we consider Jacobians with their principal polarizations.

Note that (i), (ii) imply

(iii) For any decomposition $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ we have $\#(\tilde{C}_1 \cap \tilde{C}_2) \equiv 0 \pmod 2$.

If $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ with $\tilde{C}_1 \cap \tilde{C}_2 = \{p, q\}$, and $\tilde{C}_i'$ $(i = 1, 2)$ is the curve obtained from $\tilde{C}_i$ by identifying p, q, then by Lemma (4.11) [B] $P \cong P_1 \times P_2$, where P_i is the Prym variety associated to $\tilde{C}_i'$ with the involution induced by i. So in view of (iii) we may restrict our interest to curves which satisfy the following condition:

(iv) For any non-trivial decomposition $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$, $\# C_1 \cap C_2 \geqq 4$.

The main result is

1.2. Theorem. *Let $(\tilde{C}, i)$ be a pair consisting of a curve $\tilde{C}$ of genus $2p-1$ and an involution i of $\tilde{C}$ satisfying* (i), (ii) *and* (iv); *let $C = \tilde{C}/(i)$ be the quotient curve, (P, Ξ) the associated Prym variety. Recall that $p_a(C) = p$ and* $\dim P = p-1$. *Assume that $p \geqq 8$. Then (P, Ξ) is a Jacobian or a sum of them iff one of the following holds:*

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(a) *C is hyperelliptic;*
(b) *C is obtained from a hyperelliptic curve by identifying two points;*
(c) *C is trigonal.*

By a *hyperelliptic* (*trigonal*) curve we mean a curve that possesses a *finite* morphism on $\mathbb{P}^1$ of degree 2 (of degree 3) in a general point.

From Theorem (4.10) [B] we know all suspected cases (see Sect. 3) where (P, Ξ) may be a Jacobian. To prove our theorem we need to eliminate all irrelevant cases. In case (f) we shall find a curve on the Ξ-divisor which goes into a point under the Gauss map, and so (P, Ξ) in this case is not a Jacobian. The main idea of the proof in cases (d), (e), (g) is as follows: assuming $P=J(S)$, the construction of a subvariety $Z' \subset \mathrm{Mult}_3(\Theta)$ of dimension $p-7$ implies that S is superelliptic, which gives a new component in $\mathrm{Sing}(\Xi)$, which leads to a contradiction. In Sect. 2 we investigate superelliptic curves and their Jacobians.

1.3. A more detailed treatment with a complete distinguishing theorem will be given in Russian [S]. This paper will also contain results, obtained by Beauville, Mumford, Tyurin, relating to the theory of Prym varieties and its application to conic bundles and intersections of three quadrics.

The author expresses hearty thanks to his father V.N. Shokurov for his help in preparation of this English extract from [S].

1.4. We use mainly the terminology and notation of [B]. We shall also need the following concepts. For a curve $S \subset \mathbb{P}^n$ and an effective divisor D on S we denote by $\langle D \rangle$ the linear span of D in $\mathbb{P}^n$, i.e. the intersection of all hyperplanes H in $\mathbb{P}^n$ for which $H \cdot S \geqq D$. By $\mathrm{Mult}_3 X$ we denote the set of all singular points of multiplicity $\geqq 3$ on X; Ω_C denotes the canonical sheaf of a curve C.

2. Superelliptic Curves

2.1. In this section S denotes a smooth connected curve of genus $g=g(S) \geqq 7$. To begin with, we recall one general fact

2.2. Proposition. *Let* $Jv(S) = \mathrm{Pic}^{g-1}(S)$ *and* $\Theta = \{L \in Jv(S) \mid h^0(S, L) \geqq 1\}$ *be the canonical principal polarization of* $Jv(S)$. *Then*
(2.2.1) $g-4 \leqq \dim \mathrm{Sing}\, \Theta \leqq g-3$,
(2.2.2) *If S is non-hyperelliptic, then* $\dim \mathrm{Sing}\, \Theta = g-4$.

2.3. Now we describe curves with "lots" of singularities of multiplicity 3 on Θ. These will be hyperelliptic or superelliptic curves. The former do not interest us, so we suppose in what follows that S is non-hyperelliptic.

A curve S is said to be *superelliptic* if there exists a morphism

$$\varepsilon\colon S \to E$$

of degree 2 on E, a smooth elliptic curve.

It turns out that the structure ε is unique if $g \geqq 6$. Indeed, consider a general divisor $e_1 + e_2$ of degree 2 on E. Then $|\varepsilon^*(e_1+e_2)|$ is a linear system on S of degree 4 and dimension $\geqq 1$. Identify our curve S with its canonical model $S \subset \mathbb{P}^{g-1}$.

By geometrical interpretation of the Riemann-Roch formula the points of $\varepsilon^*(e_1+e_2)$ span a plane $\langle\varepsilon^*(e_1+e_2)\rangle$. This implies that any two lines $\langle\varepsilon^*(e_1)\rangle$, $\langle\varepsilon^*(e_2)\rangle$ meet in a common point. So all of them meet in a common point $O\notin S$. We say that O is a *center of a superelliptic projection*. Now we may identify ε with a projection of S from O, and E with the image $E\subset\mathbb{P}^{g-2}$ of this projection. E is a projectively normal curve in $\mathbb{P}^{g-2}$ of degree $g-1$. It is a well-known fact that

$(*)$ Any divisor F of degree $<g-1$ on such a curve spans the projective subspace $\langle F\rangle$ of dimension $\deg F-1$, i.e. the "points" of F are in a general position in $\langle F\rangle$.

If we have another superelliptic structure $\varepsilon'\colon S\to E'$, then we have another center O'. So for general points $e_1, e_2\in E'$ a divisor $\varepsilon_*\varepsilon'^*(e_1+e_2)$ has degree 4 and lies in a plane, i.e. $\dim\langle\varepsilon_*\varepsilon'^*(e_1+e_2)\rangle=2$. But by $(*)$ this is impossible, since $\deg\varepsilon_*\varepsilon'^*(e_1+e_2)=4<g-1$ for $g\geqq 6$. Therefore when $g\geqq 6$ there exists a unique superelliptic structure on S, i.e. this structure is inherent. In particular, this structure allows us to separate in $\operatorname{Sing}\Theta$ for such a curve a component

$$\Lambda=\{\varepsilon^*(M)(F)\,|\,M\in\operatorname{Pic}^2(E)\ \text{and}\ F\in S^{(g-5)}\},$$

where $S^{(k)}$ denotes the k-th symmetric product of S. Moreover, by the Riemann-Kempf singularity theorem we can separate in $\operatorname{Mult}_3\Theta$ a component

$$\Lambda'=\{\varepsilon^*(M)(F)\,|\,M\in\operatorname{Pic}^3(E)\ \text{and}\ F\in S^{(g-7)}\}$$

of dimension $g-6$ for $g\geqq 7$. We may now state and prove two results which we shall need later on.

2.4. Proposition. *Let S be a non-hyperelliptic curve of genus $g\geqq 7$ with* $\dim\operatorname{Mult}_3\Theta\geqq g-6$. *Then S is a superelliptic curve and* $\operatorname{Mult}_3\Theta=\Lambda'$.

Proof. Let M be a general sheaf of Λ' (for a component of dimension $\geqq g-6$). Subtracting the fixed component of M we obtain $\Lambda''\subset G_d^2=\{L\in\operatorname{Pic}^d(S)\,|\,h^0(S,L)\geqq 3\}$ with $d\leqq g-1$ and $\dim\Lambda''\geqq d-5$. Now we may suppose that for a general $L\in\Lambda''$ the system $|L|$ has no base points. From Marten's theorem [Ma] we see that $\dim|L|=2$. Therefore $\bigcup_{L\in\Lambda''}|L|\subset S^{(d)}$ has dimension $\geqq d-3$; a general divisor D from this set spans the space $\langle D\rangle$ of dimension $d-3$ and $d-1$ points of D also span $\langle D\rangle$ since D has no base points. So if we take $d-1$ points of such a general divisor D we obtain a variety of irregular divisors of degree $d-1$, and dimension 1 of their general linear systems. This variety has dimension $\geqq d-3$. So there exists a subvariety $Z\subset G_{d-1}^1$ of dimension $\geqq d-4$. Then by the theorem in Appendix [M] we have one of the following cases:

(2.4.1) S is trigonal, and for a general point $x\in S$ and a general $M\in\operatorname{Mult}_3\Theta$, any divisor of $|M(-x)|$ may be written as g_3^1+F, where g_3^1 is a divisor of the trigonal series and F is an effective divisor on S;

(2.4.2) S is a superelliptic curve, and for a general point $x\in S$ and a general $M\in\operatorname{Mult}_3\Theta$, a divisor of $|M(-x)|$ may be written as $\varepsilon^*(e_1+e_2)+F$, where F is an effective divisor on S.

We establish that (2.4.1) is impossible. Indeed, in this case every $D\in|M|$ has the form g_3^1+F. We may assume that g_3^1 and F have no points in common. Then if x is a point of g_3^1, the divisor g_3^1+F-x also has the previous form and so $F=(g_3^1)'+F'$, i.e. every M may be written as $\mathcal{O}_S(2g_3^1+F'')$, where F'' is an effective divisor of degree $g-7$. Therefore the dimension of such singularities of Θ is at most $g-7$, which is a contradiction.

Similarly, we obtain in case (2.4.2) that every M may be represented as $\varepsilon^*(\mathcal{O}_E(e_1+e_2+e_3))(F)$, i.e. $M\in\Lambda'$. More precisely, we prove that for a superelliptic S, Λ' is the only component of dimension $g-6$ of $\mathrm{Mult}_3(\Theta)$. The full statement is implied by the following lemma (which is essentially proved below): Let D be an effective divisor of degree $g-1$ on S, and let A be the greatest effective divisor on E such that $|D-\varepsilon^*A|\neq\emptyset$. Then either $A=0$ and $h^0(S,\mathcal{O}_S(D))\leqq 2$, or $h^0(S,\mathcal{O}_S(D))=\deg(A)$, that is $|D|=\varepsilon^*|A|+$fixed part. Here we omit the details since in what follows we shall need only what has been proved. Q.E.D.

2.5. Proposition. *Let S be a superelliptic curve of genus $g\geqq 7$. Then*
(2.5.1) *Λ, Λ' are irreducible subvarieties in $Jv(S)$ of dimensions $g-4$ and $g-6$, respectively;*
(2.5.2) *$\mathrm{Sing}\,\Theta$ contains one more component of dimension $g-4$ besides Λ;*
(2.5.3) *If $t\in J(S)$ is an element of the Jacobian $J(S)$ such that $t\cdot\Lambda'\subset\mathrm{Sing}\,\Theta$* ($\cdot$ denotes the natural action of $J(S)$ on $Jv(S)=\mathrm{Pic}^{g-1}(S)$), *then $t\cdot\Lambda'\subset\Lambda$.*

Proof. (2.5.1) is obvious.

To prove (2.5.2) consider a projection $\pi\colon S\to\mathbb{P}^2$ from $g-3$ general points $x_1,\dots,x_{g-3}\in S$. Then $\pi(S)\subset\mathbb{P}^2$ is a curve of degree $2g-2-(g-3)=g+1$ with the ordinary singular point $\pi(O)$ of multiplicity $g-3$, and $p_a(\pi(S))-\frac{(g-3)(g-4)}{2}=\frac{g(g-1)}{2}-\frac{(g-3)(g-4)}{2}=g+2(g-3)>g$ for $g\geqq 7$. So we have a singular point $y\in\pi(S)$ other than $\pi(O)$. Hence there exists an irregular divisor of the form $D=x_1+\dots+x_{g-3}+x_{g-2}+x_{g-1}$ on S. It is obvious that the dimension of the variety of such D is $g-3$ and these divisors determine a component of $\mathrm{Sing}\,\Theta$ of dimension $g-4$ distinct from Λ. This proves (2.5.2).

Now we show that if $F\in S^{(g-1)}$ and F contains a divisor $\varepsilon^*(e)$ then $\mathcal{O}_S(F)\in\mathrm{Sing}\,\Theta$ iff $\mathcal{O}_S(F)\in\Lambda$. Sufficiency is clear. We prove necessity. So $F=\varepsilon^*(e)+F'$ where $F'\in S^{(g-3)}$. If F' contains $\varepsilon^*(e')$ we get what we need. Otherwise, for some point x of $\varepsilon^*(e)$, $F'+x$ does not contain $\varepsilon^*(e')$ and $\varepsilon\langle F'+x\rangle=\langle\varepsilon_*(F'+x)\rangle$, and hence by $(*)$ $\dim\langle\varepsilon_*(F'+x)\rangle=g-3$. It follows that $\dim\langle F'\rangle=g-3$ and $O\notin\langle F'\rangle$. Therefore $\dim\langle F\rangle=g-2$ since $O\in\langle\varepsilon^*(e)\rangle\subset\langle F\rangle$, which by Riemann-Roch contradicts our assumption $\mathcal{O}_S(F)\in\mathrm{Sing}\,\Theta$, i.e. $\dim|F|\geqq 1$.

Now note that if $L\in t\cdot\Lambda'\subset\mathrm{Sing}\,\Theta$ then, for any $e_1,e_2\in E$, $L(\varepsilon^*(e_1)-\varepsilon^*(e_2))\in\mathrm{Sing}\,\Theta$ since $\Lambda'(\varepsilon^*(e_1)-\varepsilon^*(e_2))=\Lambda'$. If $\dim|L|=2$, then $L\in\mathrm{Mult}_3\,\Theta=\Lambda'\subset\Lambda$ by Proposition 2.4. Now consider the case $\dim|L|=1$. If $|L|$ is base point free, then a general $F\in|L|$ consists of $g-1$ different points $x_1,\dots,x_{g-1}$ and $\left\langle\sum_{i=1}^{g-1}x_i\right\rangle=g-3$. Moreover $\dim\left\langle\sum_{i=1}^{g-2}x_i\right\rangle=g-3$ and again by $(*)$ $O\notin\langle F\rangle$ or $F=\varepsilon^*(e)+$(an effective divisor). In the last case by the previous discussion $L\in\Lambda$. So let $O\notin\langle F\rangle$. Then $\varepsilon(\langle F\rangle)$ is a hyperplane in $\mathbb{P}^{g-2}$ and, for a general $e_1\in E$, $\dim\langle F+\varepsilon^*(e_1)\rangle=g-1$. It follows that $h^0(S,\Omega_S\otimes L(\varepsilon^*(e_1))^{-1})=0$, and by Riemann-Roch $h^0(S,L(\varepsilon^*(e_1))$

$=g+1-g+1=2=h^{\circ}(S, L)$. Hence $|L(\varepsilon^*(e_1))|=|L|+\varepsilon^*(e_1)$. But $|L(\varepsilon^*(e_1)-\varepsilon^*(e_2))| \neq \emptyset$ for a general $e_2 \in E$, so $|L(-\varepsilon^*(e_2))| \neq \emptyset$ for such e_2. Therefore $\varepsilon^*(e_2)+$(an effective divisor)$\in|L|$, i.e. again by the previous consideration $L \in \Lambda$. The case where L has base points is much easier to investigate using (*) to obtain $L \in \Lambda$.

Q.E.D.

3. The Proof of the Main Result

3.1. First note that in cases (a), (b), (c) of Theorem 1.2 a Prymian is a Jacobian or a sum of Jacobians, by Mumford [M], Dalalyan, Recillas [R]. So we must prove that if this takes place, then C has one of the prescribed forms. By (iv) and Theorem (4.10) [B] this is obvious when $\dim \operatorname{Sing} \Xi \geqq p-4$. Therefore by Proposition 2.2 we may suppose that $\dim \operatorname{Sing} \Xi = p-5$, *$(P, \Xi)$ is isomorphic to $(J(S), \Theta)$ for some smooth connected non-hyperelliptic curve S of genus $g=p-1=7$*. Then by the same Theorem (4.10) [B] one of the following cases, besides (a), (b), (c) above, occurs:

(d) C is a double cover of an irreducible curve of genus one;

(e) C is obtained from a hyperelliptic curve by identifying two pairs of points;

(f) $C=C_1 \cup C_2$ with $\# C_1 \cap C_2 = 4$, and neither C_1 nor C_2 is a smooth rational curve;

(g) $C=C_1 \cup C_2$ with $\# C_1 \cap C_2=4$, where C_1 is a smooth rational curve and C_2 is a hyperelliptic curve of genus $\geqq 5$. Recall that a curve with ordinary double points is hyperelliptic if it can be realized as a two-sheeted covering of $\mathbb{P}^1$.

We prove that none of the last cases (d), (e), (f), (g) is possible when (P, Ξ) is a Jacobian.

It is convenient to look at the Prymian in $\operatorname{Pic}(\tilde{C})$, after translation by $\pi^* L_0$, where $\pi: \tilde{C} \to C$ is the natural projection and L_0 is a theta-characteristic as in Proposition (3.10) [B]. Then the Prymian becomes the variety

$$Pv = Pv(\tilde{C}, i) = \{L \in \operatorname{Pic}^{2p-2}(\tilde{C}) \mid Nm(L) = \Omega_C \text{ and } h^0(\tilde{C}, L) \equiv 0 \pmod 2\},$$

and Ξ becomes the canonical polarization

$$\Xi = \Xi(\tilde{C}, i) = \{L \in Pv(\tilde{C}, i) \mid h^0(\tilde{C}, L) > 0\}.$$

So under our assumption

$$(Pv(\tilde{C}, i), \Xi) = (Jv(S), \Theta).$$

3.2. We first consider case (f).

The concept of a Prym variety may be extended (without the canonical principal polarization) to include the case where $\pi: \tilde{C} \to C$ has non-singular branch points, i.e. in (ii) we may admit non-singular fixed points of i. In this situation we can construct

$$Pv(\tilde{C}, i) = \{L \in \operatorname{Pic}^{2p-2}(\tilde{C}) \mid Nm(L) = \Omega_C\}.$$

It will be a complete Abelian variety (a connected one when there are branch points) and of dimension $p+(b/2)-1$, where b is the number of non-singular fixed points for i. For example, take components C_1, C_2 corresponding to our $C_1, C_2 \subset C$ with an involution induced by i and also denoted by i. Let $\tilde{f}\colon C_1, C_2 \to \tilde{C}$ be the desingularization in 4 intersection points $\{u_i\}_{i=1}^4 = \tilde{C}_1 \cap \tilde{C}_2$. Then we have an isogeny

$$\tilde{f}^0\colon\ Pv(\tilde{C}, i) \to Pv(\tilde{C}_1, i) \times Pv(\tilde{C}_2, i)$$

$$L \mapsto \left(L\Big|_{C_1}\left(-\sum_{i=1}^{4} u_i\right), L\Big|_{C_2}\left(-\sum_{i=1}^{4} u_i\right)\right)$$

of degree 4. The reader who has trouble at this point is referred to a detailed treatment of this subject in [S]. Let $Z \subset Pv(\tilde{C}, i)$ be the inverse image under $\tilde{f}^0$ of the subvariety $Pv(\tilde{C}_1, i) \times \{M \in Pv(\tilde{C}_2, i) \mid h^0(\tilde{C}_2, M) > 0\} \subset Pv(\tilde{C}_1, i) \times Pv(\tilde{C}_2, i)$. It is easy to see that $h^0(\tilde{C}, L) > 0$ for all $L \in Z$, so $Z \subset \Xi$; L has a section vanishing in component $\tilde{C}_1$. Also one can prove that $\dim Z = \dim Pv(\tilde{C}_1, i) + \dim \{M \in Pv(\tilde{C}_2, i) \mid h^0(\tilde{C}_2, M) > 0\} = p_a(C_1) + (4/2) - 1 + \dim |\Omega_{C_2}| = p_a(C_1) + 1 + p_a(C_2) - 1 = p_a(C_1) + p_a(C_2) = p_a(C) - 3 = p - 3$. So at a general point of Z we have the well-defined Gauss map $\Gamma\colon Z \to (\mathbb{P}^{p-2})^*$ which maps a point $z \in Z$ into the projectivization of the tangent space to Ξ at z; henceforth we identify the projectivizations of all tangent spaces to $Pv(\tilde{C}, i)$ with one fixed $\mathbb{P}^{p-2}$. Obviously, all hyperplanes in $\mathbb{P}^{p-2}$ corresponding to general points $z \in Z$ under Γ contain a subspace $P \subset \mathbb{P}^{p-2}$ which is the projectivization of the tangent space to an Abelian subvariety $(\tilde{f}^0)^{-1}(Pv(\tilde{C}_1, i) \times \{x\})$. P has the dimension $\dim Pv(\tilde{C}_1, i) - 1 = p_a(C_1) + (4/2) - 1 - 1 = p_a(C_1)$. So the image of Z under Γ has the dimension $p - 2 - p_a(C_1) - 1 = p - 3 - p_a(C_1) < p - 3$ since under our assumption in (f) C_1 is not a smooth rational curve. Therefore on $\Xi = \Theta$ there exists a one-dimensional family of non-singular points which Γ maps into one point. This is impossible for Jacobians by the geometrical interpretation of the Riemann-Roch formula and the Riemann-Kempf singularity theorem. Indeed, if $L \in \Theta \subset Jv(S)$ is a non-singular point, then $\dim |L| = 0$, $\deg L = g - 1$ and a unique divisor $F \in |L|$ spans a hyperplane H in the space $\mathbb{P}^{g-1} = \mathbb{P}^{p-2}$ of the canonical imbedding for S. H is the image of L under Γ, and it determines L with $\Gamma(L) = H$ up to a finite choice. This contradiction shows that case (f) is impossible.

3.3. Now we consider case (g). So let $C = C_1 \cup C_2$, $\# C_1 \cap C_2 = 4$, where C_1 is a smooth rational curve and C_2 is a hyperelliptic curve, i.e. we have a finite morphism

$$\gamma\colon\ C_2 \to \mathbb{P}^1$$

of degree 2 at a general point. Let

$$\tilde{f}^0\colon\ Pv(\tilde{C}, i) \to Pv(\tilde{C}_1, i) \times Pv(\tilde{C}_2, i)$$

be the isogeny as above. Now we denote by Z the inverse image under $\tilde{f}^0$ of a subvariety $Pv(\tilde{C}_1, i) \times \{(\gamma \circ \pi)^*(M)(F) \in Pv(\tilde{C}_2, i) \mid M \in \operatorname{Pic}^1(\mathbb{P}^1)$ and F is a non-singular effective divisor$\}$, where $\pi\colon \tilde{C}_2 \to C_2$ is the canonical projection. It is a well-known fact from [B] that $Z \subset \operatorname{Sing} \Xi$ and $\dim Z = p - 5$. Also we introduce a variety Z' which is the inverse image under $\tilde{f}^0$ of $Pv(\tilde{C}_1, i) \times$

$\{(\gamma\circ\pi)^*(M)(F)\in Pv(\tilde{C}_2, i) | M\in \mathrm{Pic}^2(\mathbb{P}^1)$ and F is a non-singular effective divisor$\}$. More precisely, we consider the projective closure of these varieties. We shall establish that $Z'\subset \mathrm{Mult}_3\,\Xi$ and $\dim Z'=p-7$. To prove it we need the following

3.4. Lemma. *Let D be a connected curve with ordinary double points such that $\# D_1\cap D_2\geqq 2$ for any non-trivial decomposition $D=D_1\cup D_2$, and let L be a non-singular* (see [B]) *sheaf of* $\mathrm{Pic}(D)$ *with $h^0(D,L)\geqq \deg L/2+1$. Then $|\Omega_D\otimes L^{-1}|$ contains a non-singular divisor iff $\deg((\Omega_D\otimes L^{-1})|_{D'})\geqq 0$ for any component $D'\subset D$.*

Proof. Necessity is obvious, so we must prove sufficiency. Denote by D_1 the maximal component of D on which not all sections of $H^0(D, \Omega_D\otimes L^{-1})$ vanish simultaneously, and by U the maximal set of points of $\mathrm{Sing}\,D\cap D_1$ in which all sections of $H^0(D, \Omega_D\otimes L^{-1})$ vanish. Let $f\colon D_1'\to D_1$ be the desingularization of D_1 in points of $U\cap \mathrm{Sing}\,D_1$, and $L_1=f^*((\Omega_D\otimes L^{-1})|_{D_1})(-\sum_{f(x)\in U} x)$; the summation is made over different points. Then sheafs $f^*(L|_{D_1})$ and $L_1=\Omega_{D_1'}\otimes f^*(L|_{D_1})^{-1}$ are non-singular in the sense of Sect. 4 in [B]. Hence by Riemann-Roch and our hypotheses $h^0(D,\Omega_D\otimes L^{-1})=h^0(D,L)+\deg(\Omega_D\otimes L^{-1})-\deg\Omega_D/2\geqq \deg L/2+1+\deg\Omega_D/2-\deg L=\deg(\Omega_D\otimes L^{-1})+1$. By Lemma 4.7 [B] $1+\deg(\Omega_D\otimes L^{-1})/2\leqq h^0(D,\Omega_D\otimes L^{-1})\leqq h^0(D_1', L_1)\leqq \deg L_1/2+$(the number of components of D_1'). But under our assumption all components of the multidegree of $\Omega_D\otimes L^{-1}$ are positive, so $\deg L_1\leqq \deg\Omega_D\otimes L^{-1}-\#\{x\in D_1' | f(x)\in U\}$. This implies the inequality $\#\{x\in D_1' | f(x)\in U\}/2\leqq$(the number of components of $D_1'\}-1$. So there exists a component of D_1' which contains at most one point x with $f(x)\in U$. But any component of D meets other components in at least two points, so this component of D_1' coincides with D_1' and $D_1=C$. Therefore $\#\{x\in D_1' | f(x)\in U\}\leqq 0$, i.e. $U=\emptyset$, what we need. Q.E.D.

3.5. To prove the assertion before Lemma 3.4 we show that
(3.5.1) A general divisor of $|\Omega_{C_2}\otimes\gamma^*(M)^{-2}|$, where $M\in\mathrm{Pic}^2(\mathbb{P}^1)$, is non-singular;
(3.5.2) $\dim|\Omega_{C_2}\otimes\gamma^*(M)^{-2}|=p_a(C)-8$.
The former follows from Lemma 3.4, since $h^0(C_2,\gamma^*(M)^2)=h^0(\mathbb{P}^1, M^2)=5=\deg\gamma^*(M)^2/2+1$ and $p_a(C_2)\geqq 5$. Lemma 4.7 [B] and Riemann-Roch imply (3.5.2). Indeed, $h^0(C_2,\gamma^*(M)^2)=5$ and

$$\begin{aligned} h^0(C_2,\Omega_{C_2}\otimes\gamma^*(M)^{-2}) &= \deg\Omega_{C_2}\otimes\gamma^*(M)^{-2}+h^0(C_2,\gamma^*(M)^2)\\ &\quad -\deg\Omega_{C_2}\otimes\gamma^*(M)^{-2}/2+1=p_a(C_2)-1-4+1\\ &= p_a(C_2)-4=p_a(C)-7.\end{aligned}$$

So there exists a $(p-8)$-dimensional family of effective non-singular divisors F on $\tilde{C}_2$ with $\pi_*F\in|\Omega_{C_2}\otimes\gamma^*(M)^{-2}|$; $\dim|F|=0$ for such a general F, since if $\dim|F|>0$ then $|F+x-i(x)|<\dim|F|$ for a general $x\in\tilde{C}_2$ and $|F+x-i(x)|$ also contains a nonsingular divisor F' with $\pi_*F'\in|\Omega_{C_2}\otimes\gamma^*(M)^{-2}|$. Therefore $\dim\{(\gamma\circ\pi)^*(M)(F) | M\in\mathrm{Pic}^2(\mathbb{P}^1)$ and F is such a non-singular effective divisor on $\tilde{C}_2\}=p-8$. On the other hand, $\dim Pv(\tilde{C}_1, i)=0+(4/2)-1=1$; and so $\dim Z'=p-7$.

If L is a general sheaf on Z', then $h^0(\tilde{C},L)=6$ or $h^0(\tilde{C},L)=4$ and $H^0(\tilde{C},L)$ contains three sections s_1,s_2,s_3 for which we have $s_l\otimes i^*s_j=s_j\otimes i^*s_l$, $1\leqq l$, $j\leqq 3$. In the former situation it is obvious that $L\in\mathrm{Mult}_3\,\Xi$ by the Riemann-Kempf singularity theorem, since $2\Xi=\tilde{\Theta}|_P$ for the polarization $\tilde{\Theta}$ of $Jv(\tilde{C})$. In the other case we add s_4 to s_1, s_2, s_3 to form a basis of $H^0(\tilde{C}, L)$ and consider $Pf(\omega_{\bar{l}\bar{j}})$ as in Sect. 6 [M]. This Pfaffian vanishes because $\omega_{\bar{l}\bar{j}}=(s_l\otimes i^*s_j-s_j\otimes i^*s_l)/2=0$ for $1\leq l, j\leq 3$. So the first terms of the Taylor expansion for a function h giving $\tilde{\Theta}$ in a neighbourhood of L has order 4, and they vanish on P. From this we see that in this case $L\in \mathrm{Mult}_3\,\Xi$ also. Therefore $Z'\subset \mathrm{Mult}_3\,\Xi$.

Now we prove the existence of a surface $T\subset P=P(\tilde{C}, i)$ such that $T\cdot Z'=Z$.

Note that just as we extend the notion of a Prym variety Pv we do the same with a Prymian P, ignoring polarization. Namely, let

$$P_l=P(\tilde{C}_l, i)=\{L\in \mathrm{Pic}(\tilde{C}_l)\,|\,Nm\,L=0\},\qquad l=1,2.$$

The desingularization $\tilde{f}$ gives an isogeny

$$\begin{aligned}\tilde{f}^*\colon\ P&\to P_1\times P_2\\ L&\mapsto \tilde{f}^*(L)\end{aligned}$$

and the action of P on $Pv(\tilde{C}, i)$ and that of $P_1\times P_2$ on $Pv(\tilde{C}_1, i)\times Pv(\tilde{C}_2, i)$ are concordant with $\tilde{f}$, i.e. $\tilde{f}^0(p\cdot x)=\tilde{f}^*(p)\cdot \tilde{f}^0(x)$ for any $p\in P$ and $x\in Pv(\tilde{C}, i)$. So it is enough to construct a surface $T'\subset P_2$ such that $T'\cdot\{(\gamma\circ\pi)^*(M_2)(F)\,|\,M_2\in \mathrm{Pic}^2(\mathbb{P}^1)$ and F as earlier$\}=\{(\gamma\circ\pi)^*(M_1)(F)\,|\,M_1\in\mathrm{Pic}^1(\mathbb{P}^1)$ and F as earlier$\}$.

For a general sheaf $(\gamma\circ\pi)^*(M_1)(F)\in Pv(\tilde{C}_2, i)$, the divisor F is non-singular and $\pi_*F\in|\Omega_{C_2}\otimes\gamma^*(M_1)^{-2}|$. As before it is easy to establish that $\dim|\pi_*F|=\deg\Omega_{C_2}\otimes\gamma^*(M_1)^{-2}/2=p_a(C_2)-3$ and $\dim|\Omega_{C_2}\otimes\gamma^*(M_1)^{-3}|=p_a(C_2)-5$. So $|\pi_*F|=|\gamma^*(M_2)|+|\Omega_{C_2}\otimes\gamma^*(M_1)^{-4}|$ and $\pi_*F=\gamma^*(x+y)+($an effective non-singular divisor on $C_2)$ for some points $x, y\in\mathbb{P}^1$. Let D_1 and D_2 be components of F over $\gamma^*(x)$ and $\gamma^*(y)$, i.e. $\pi_*D_1=\gamma^*(x)$ and $\pi_*D_2=\gamma^*(y)$. Then $F=D_1+D_2+($an effective non-singular divisor on $\tilde{C}_2)$. It follows that $(\gamma\circ\pi)^*(M_1)(F)(-D_1+i^*D_2)=(\gamma\circ\pi)^*(M_2)$ (an effective non-singular divisor on C_2).

Now we may define the desired surface T' which is the closure of $\{\mathcal{O}_{\tilde{C}_2}(D_1-D_2)\in P_2\,|\,D_1, D_2$ are non-singular divisors of degree 2 on $\tilde{C}_2$ such that $\pi_*D_1, \pi_*D_2\in|\gamma^*(M_1)|$. We have proved that $T'\cdot\{(\gamma\circ\pi)^*(M_2)(F)\}\supseteq\{(\gamma\circ\pi)^*(M_1)(F)\}$. The inverse inclusion is obvious.

Thus we have $Z'\subset\mathrm{Mult}_3\,\Xi=\mathrm{Mult}_3\,\Theta$, $\dim Z'=p-7$, and so by Proposition 2.4 S is a superelliptic curve and $Z'=\Lambda'$. Moreover, there exists a surface $T\subset P=J(S)$ such that $T\cdot\Lambda'=T\cdot Z'=Z$. Then by Proposition 2.5 $Z=\Lambda$ and there exists another component of dimension $p-5$ in $\mathrm{Sing}\,\Xi=\mathrm{Sing}\,\Theta$. This component according to the proof of Theorem (4.10) [B] corresponds to cases (d) and (e). In all of these cases one pair of the points in $C_1\cap C_2$ is compatible with γ, i.e. there exist $u_1, u_2\in C_1\cap C_2$ such that $\gamma(u_1)=\gamma(u_2)$. Let $\tilde{f}_1\colon\tilde{D}_1\to\tilde{C}$ be the desingularization relating to the other two points $u_3, u_4\in C_1\cap C_2$, and $\gamma_1\colon D_1=\tilde{D}_1/(i)\to\mathbb{P}^1$ be the hyperelliptic structure induced by γ; $\gamma_1|_{C_2\subset D_1}=\gamma$, $\gamma_1(C_1)=\gamma(u_1)=\gamma(u_2)$. Consider the isogeny

$$\begin{aligned}\tilde{f}_1^0\colon\ Pv(\tilde{C}, i)&\to Pv(\tilde{D}_1, i)\\ L&\mapsto \tilde{f}_1^*(L)\left(-\sum_{i=1}^{4}x_i\right)\end{aligned}$$

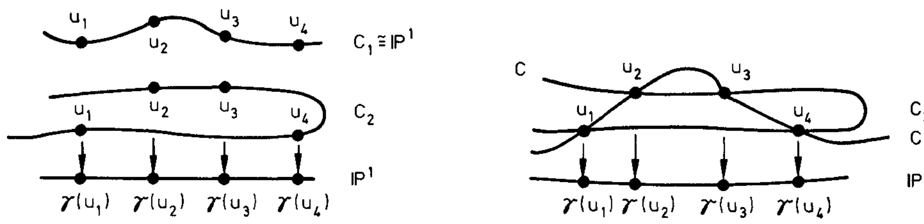

Fig. 1

of degree 2 where x_i are points resulting from the desingularization. The corresponding component Z_1 of $\operatorname{Sing}\Xi$ is the inverse image under $\tilde{f}_1^0$ of $\{(\gamma_1 \circ \pi_1)^*(M)(F) \in Pv(\tilde{D}_1, i) | M \in \operatorname{Pic}^1(\mathbb{P}^1)$ and F is an effective non-singular divisor on $\tilde{D}_1\}$, where $\pi_1: \tilde{D}_1 \to D_1$ is the natural projection. More precisely, we take the closure of this variety in $Pv(\tilde{C}, i)$. As before we may also define Z'_1 for $M \in \operatorname{Pic}^2(P^1)$ and prove that $Z'_1 \subset \operatorname{Mult}_3 \Xi$, $\dim Z'_1 = p-7$. In this situation too there exists a surface $T_1 \subset P$ such that $T_1 \cdot Z'_1 = Z_1$. So as earlier $Z_1 = \Lambda$, and $Z = Z_1$. But this is impossible, since in fact $Z \neq Z_1$. Indeed for a general $L \in Z_1$, $\tilde{f}_1^0(L) = (\gamma_1 \circ \pi_1)^*(M)(F)$ with an effective non-singular divisor F on $\tilde{D}_1$ and $M \in \operatorname{Pic}^1(\mathbb{P}^1)$. So $\tilde{f}_1^0(L)|_{\tilde{C}_1} = 0$ and $\tilde{f}^0(L)|_{\tilde{C}_1} = \tilde{f}_1^0(L)|_{\tilde{C}_1}(-\tilde{u}_1 - \tilde{u}_2) = \mathcal{O}_{\tilde{C}_1}(-\tilde{u}_1 - \tilde{u}_2) = \text{const} \in Pv(\tilde{C}_1, i)$ where $\tilde{u}_1$, $\tilde{u}_2$ are points of $\tilde{C}_1$ over $u_1, u_2 \in C_1$. On the other hand, $\dim Pv(\tilde{C}_1, i) = 0 + (4/2) - 1 = 1$ and there is a general $L \in Z$ such that $\tilde{f}^0(L)|_{\tilde{C}_1} \neq \mathcal{O}_{\tilde{C}_1}(-u_1 - u_2)$. This contradiction means that case (g) does not hold when (P, Ξ) is a Jacobian.

We have slightly misled the reader here. In fact there may occur one more case, the trigonal case (c), when the harmonic relation of the quadruple $\gamma(u_1)$, $\gamma(u_2)$, $\gamma(u_3)$, $\gamma(u_4) \in \mathbb{P}^1$ is equal to that of the quadruple $u_1, u_2, u_3, u_4 \in C_1 \cong \mathbb{P}^1$. (See Fig. 1.) But then by degeneration of the Recillas theorem [R] (P, Ξ) is a Jacobian. To avoid such trivial cases we suppose in what follows that (a), (b), (c) of Theorem 1.2 do not hold.

3.6. Case (e) can be rejected in much the same way as (g). Namely, let $\gamma: D \to \mathbb{P}^1$ be a hyperelliptic curve, and after identifying points u_1, u_2 and u_3, u_4 we obtain C. We outline the main steps. First denote by $\tilde{f}: \tilde{D} \to \tilde{C}$ the desingularization over two identified points. Let

$$\tilde{f}^0: Pv(\tilde{C}, i) \to Pv(\tilde{D}, i)$$
$$L \mapsto \tilde{f}^*(L)\left(-\sum_{i=1}^{4} \tilde{u}_i\right)$$

be the corresponding isogeny, where $\tilde{u}_i$ is a branch point of $\tilde{D}$ over $u_i \in D$. In this situation we consider closures Z, Z' of the inverse image under $\tilde{f}^0$ of $\{(\gamma \circ \pi)^*(M)(F) \in Pv(\tilde{D}, i) | M \in \operatorname{Pic}^1(\mathbb{P}^1)$ and F as before$\}$ and that of $(\gamma \circ \pi)^*(M)(F) \in Pv(\tilde{D}, i) | M \in \operatorname{Pic}^2(\mathbb{P}^1)$ and F as before$\}$, where $\pi: \tilde{D} \to D$ is the natural projection. Then one can prove that $\dim Z = p-5$, $\dim Z' = p-7$, $Z \subset \operatorname{Sing}\Xi$, $Z' \subset \operatorname{Mult}_3 \Xi$ and there exists a surface $T \subset P$ with $T \cdot Z' = Z$. This implies that S is a superelliptic curve and $Z' = \Lambda'$, $Z = \Lambda$. So $\operatorname{Sing}\Xi = \operatorname{Sing}\Theta$ must contain another component, besides Λ, of dimension $p-5$. This is possible only when C is a double cover of an irreducible curve of genus one. But we shall soon see that the corresponding singularity component will be again Λ. Therefore case (e) does not hold.

3.7. Let E be an irreducible curve of genus one and let $\varepsilon\colon C \to E$ be a finite morphism of degree 2 over a general point of E. Consider the closure Z of $\{(\varepsilon \circ \pi)^*(M)(F) \in Pv(\tilde{C}, i) | M \in \mathrm{Pic}^2(E)$ and F is an effective non-singular divisor on $C\}$, where $\pi\colon \tilde{C} \to C$ is the natural projection. Similarly we define Z' for $M \in \mathrm{Pic}^3(E)$. As earlier $Z' \subset \mathrm{Mult}_3 \Xi$. Suppose $\dim Z = p-5$. Then we establish that *S is a superelliptic curve and* $Z = \Lambda$. This will complete our proof because we have no other component of dimension $p-5$ in $\mathrm{Sing}(\Xi)$ besides $Z = \Lambda$.

Indeed, $\pi_*(F) \in |\Omega_C \otimes \varepsilon^*(e+e')^{-2}|$ is an effective non-singular divisor on C for a general $L = (\varepsilon \circ \pi)^*(\mathcal{O}_E(e+e'))(F) \in Z$. Just as for a non-singular superelliptic curve C, $\pi_*(F) = \varepsilon^*\left(\sum_{i=1}^{p-5} e_i\right) = \varepsilon^*(e_1+e_2+e_3) +$ (an effective non-singular divisor on C), where e_1, e_2, e_3 are general non-singular points of E. Recall that $p \geqq 8$ under our assumption. Denote by h_{lj} a half of $e_l + e_j$, $1 \leqq l, j \leqq 3$, i.e. $2h_{lj} \sim e_l + e_j$; h_{lj} are regarded as non-singular points of E for general e_l, e_j ($\sim$ denotes linear equivalence). So $F = D_1 + D_2 + D_3 +$ (an effective non-singular divisor on $\tilde{C}$) and $\pi_* D_i = \varepsilon^*(e_i)$. Let

$$t_{lj} = (\varepsilon \circ \pi)^*(h_{lj}) - D_l - D_j$$

be a divisor on $\tilde{C}$. Then $\pi_* t_{lj} = 2\varepsilon^*(h_{lj}) - \varepsilon^*(e_l + e_j) \sim 0$. So $\mathcal{O}_{\tilde{C}}(t_{lj}) \in \mathrm{Ker}\, \pi_* = P(\tilde{C}, i) \oplus \mathbb{Z}/2\mathbb{Z}$. On the other hand, we may choose h_{lj} such that $\sum_{1 \leqq l < j \leqq 3} h_{lj} \sim \sum_{l=1}^{3} e_l$. Then $\sum_{1 \leqq l < j \leqq 3} (\varepsilon \circ \pi)^*(h_{lj}) \sim \sum_{l=1}^{3} (\varepsilon \circ \pi)^*(e_l)$ and

$$\begin{aligned} \sum_{1 \leqq l < j \leqq 3} t_{lj} &= (\varepsilon \circ \pi)^* \ (h_{12} + h_{13} + h_{23}) - 2D_1 - 2D_2 - 2D_3 \\ &\sim (\varepsilon \circ \pi)^* \ (e_1 + e_2 + e_3) - 2D_1 - 2D_2 - 2D_3 \\ &= \sum_{l=1}^{3} i^* D_l - D_l \end{aligned}$$

and the sheaf corresponding to this divisor lies in $P(\tilde{C}, i)$. So one of $\mathcal{O}_{\tilde{C}}(t_{lj}) \in P(\tilde{C}, i)$ and $(\varepsilon \circ \pi)^*(\mathcal{O}_E(e+e'))(F + t_{lj}) = (\varepsilon \circ \pi)^*(\mathcal{O}_E(e+e'+h_{lj}))$ (an effective non-singular divisor on $\tilde{C}$), i.e. $L(t_{lj}) = L' \in Z'$. Now we define a maximal subvariety $T \subset P$ the general points of which have the form

$$\mathcal{O}_{\tilde{C}}(D_1 + D_2 - (\varepsilon \circ \pi)^*(h_{12}))$$

where D_1, D_2 are effective non-singular divisors on C of degree 2 with $\pi_* D_1 = \varepsilon^*(e_1)$, $\pi_* D_2 = \varepsilon^*(e_2)$, and h_{12} is a half of $e_1 + e_2$. Thus we have proved that $T \cdot Z' \supseteq Z$. The inverse inclusion is clear. So $T \cdot Z' = Z$. Since $\dim Z = p-5$ under our assumption, $\dim T \leqq 2$, $\dim Z' \leqq p-7$ by simple check, we have $\dim Z' = p-7$, $\dim T = 2$. Then as before S is a superelliptic curve and $Z = \Lambda$. Therefore case (d) is impossible.

Q.E.D.

3.8. *Note.* When both $\tilde{C}$ and C are non-singular, only one suspected case, (d), may occur, i.e. when C is a superelliptic curve. In particular, we have proved

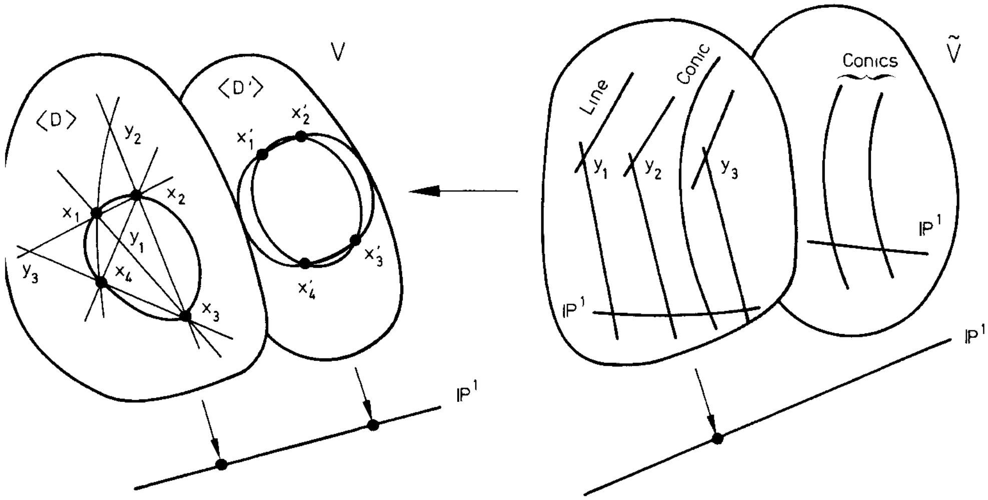

Fig. 2

that in this situation (P, Ξ) *is not a Jacobian*. This is the answer to Mumford's question [M] for $p \geqq 8$. This is also true, according to Ph.D. thesis of Dalalyan, for $p=6$.

3.9. In conclusion we give a geometrical interpretation of the Recillas theorem: *if C is trigonal, then* (P, Ξ) *is a Jacobian* (J, Θ) *of a curve S*. Moreover this curve S is tetragonal, i.e. there exists a g_4^1 series (a one-dimensional linear system of divisors of degree 4) on S. Consider the canonical imbedding $S \subset \mathbb{P}^{p-2}$. Let $V \to \mathbb{P}^1$ be a $\mathbb{P}^2$-bundle, or in other words a three-dimensional rational scroll, whose fibres are planes $\langle D \rangle$ for $D \in g_4^1$. So we have the inclusion $S \subset V$ and the induced map $S \to \mathbb{P}^1$ corresponds to g_4^1. Denote by $\tilde{V}$ the blowing up of V in S. It is a well-known fact that the intermediate Jacobian of $\tilde{V}$ is (J, Θ). On the other hand, $\tilde{V}$ possesses the structure of a conic bundle over F_n: $q: \tilde{V} \to F_n$, where F_n is a rational scroll surface (see Figure 2). Conics on $\langle D \rangle$ through x_1, x_2, x_3, x_4 of D after blowing up become conics of $\tilde{V}$ over F_n. By Recillas' construction C is the curve of degeneration of q: $\tilde{V} \to F_n$ and the lines of $\tilde{V}$ over C form a curve $\tilde{C}$. So the intermediate Jacobian of V is also the Prymian (P, Ξ) and $(P, \Xi) = (J, \Theta)$.

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Received October 22, 1980