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Prelimiting Flips[1]

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You can't always get exactly what you want.

The paper discusses an inductive approach to constructing log flips. In addition to special termination and thresholds, we introduce two new ingredients: the saturation of linear systems, and families of divisors with confined singularities. We state conjectures concerning these notions in any dimension and prove them in general in dimension ≤ 2. This allows us to construct prelimiting flips (pl flips) and all log flips in dimension 4 and to prove the stabilization of an asymptotically saturated system of birationally free (b-free) divisors under certain conditions in dimension 3. In dimension 3, this stabilization upgrades pl flips to directed quasiflips. It also gives for the first time a proof of the existence of log flips that is algebraic in nature, that is, via f.g. algebras, as opposed to geometric flips. It accounts for all the currently known flips and flops, except possibly for flips arising from geometric invariant theory.

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1. INTRODUCTION

1.1. Prelimiting contractions. We write $f\colon X \to X_\vee$ to denote a *prelimiting contraction*, or *pl contraction*, with respect to a divisor S. By this we mean that f is a birational contraction such that

(1) $S = \sum S_i$ is a sum of $s \geq 1$ prime Weil divisors $S_1, \dots, S_s$ on X that are $\mathbb{Q}$-Cartier and *proportional*, that is, $S_i \sim_{\mathbb{Q}} r_{i,j} S_j$ for some rational $r_{i,j} > 0$;

(2) there is a boundary B such that $\lfloor B \rfloor = 0$ and $K + S + B$ is divisorially log terminal, and purely log terminal if $s = 1$; and

(3) $K + S + B$ is numerically negative/$X_\vee$ (that is, it is minus an ample $\mathbb{R}$-Cartier divisor/$X_\vee$).

By conditions (1), (2) and Shokurov [41, следствие 3.8] (see also Kollár and others [27, Ch. 17]), the divisors S_i are normal varieties, with normal varieties in their intersections and normal crossing at the generic points of these intersections. In particular, $Y = \bigcap S_i$ is a normal variety, the *core* of f.

Conditions (2) and (3) together mean that $(X/X_\vee, S+B)$ is a log Fano contraction with only *log terminal* singularities. Thus, by the Connectedness of LCS [27, Theorem 17.4], Y is irreducible of dimension $d = n - s'$ near each fibre of f, where s' is the number of S_i that intersect that fibre and $n = \dim X$. Moreover, f induces a contraction $Y \to f(Y) \subset X_\vee$, and (Y, B_Y) is Klt (Kawamata log terminal) [27, Definition 2.13; 41, c. 110] for the adjoint log divisor

$$K_Y + B_Y = (K + S + B)_{|Y}.$$

These properties establish an induction on Y in our construction of pl flips below (see Section 3 for inductive sequences and compare the proof in Example 3.40).

We always consider the *local* situation, where f is a germ over a neighborhood of a given point $P \in X_\vee$. In particular, all the S_i intersect the *central fibre* $X/P = f^{-1}P$. Thus, $\dim Y = d = n - s \leq n - 1$ by condition (1). The *core dimension* d measures the difficulty of constructing pl flips in the sense indicated in Induction Theorem 1.4 (see Corollary 1.10 and also Examples 3.53 and 3.54).

Sometimes, it is also reasonable to consider a contraction f that is not birational. Then S is not numerically negative/$X_\vee$, in contrast to the case of elementary pl flips. This situation occurs naturally in Section 3 in certain applications of Main Lemma 3.43 (see Example 3.38).

However, in Reduction Theorem 1.2 below, we can assume that the pl contractions f we are most interested in are *elementary*, that is, satisfy in addition the following conditions:

(4) S is numerically negative/$X_\vee$;

(5) $K+S+B$ is strictly log terminal/$X_\vee$, that is, X is $\mathbb{Q}$-factorial and projective/$X_\vee$;

(6) f is *extremal* in the sense that it is the contraction of an extremal ray of the Mori cone; that is, $\rho(X/X_\vee)=1$, where ρ denotes the relative Picard number; and

(7) f is small, or equivalently, under the current assumptions, S is not f-exceptional.

We can omit the final condition (7), but then we meet an additional well-known situation when the contraction is divisorial and is its own S-flip (cf. Example 3.45). Note also that, by (6), f is nontrivial, i.e., is not an isomorphism. (Compare a remark after Example 3.53 when f is an isomorphism.)

A *pl flip* is the S-flip of a pl contraction f (see [45, Section 5] and Example 3.15 below). By a *log flip* we mean the D-flip of a birational log contraction $(X/T,B)$ over any base T (see Conventions 1.14), with $D=K+B$, where we assume that

- $K+B$ is Klt, and
- $-(K+B)$ is nef/T

(cf. Shokurov [41, c. 105, теорема]). Thus, each pl flip for which $-S/X_\vee$ is nef is also the log flip of

$$(X/T,B)=(X/X_\vee,(a+r)S+B+H)$$

for some real numbers $0<r\ll a<1$ and for a general effective divisor $H\sim_{\mathbb{R}}-(K+aS+B)$ (a *complement*) that is numerically ample/$X_\vee$. This holds because $K+(a+r)S+B+H$ is Klt and $\sim_{\mathbb{R}} rS$ (compare the proof of Theorem 3.33). Note also that a D-flip is uniquely determined up to isomorphism by the class of D up to $\sim_{\mathbb{R}}$ and positive real scalar multiples (see Corollary 3.6). By condition (4), each elementary pl flip is also a log flip.

Interest in pl flips rests on the following result:

Reduction Theorem 1.2. *Log flips exist in dimension n provided that*

$(\mathrm{PLF})_n^{\mathrm{el}}$ elementary *pl flips exist in dimension n; and*

$(\mathrm{ST})_n$ *special termination holds in dimension n.*

We discuss Termination (ST) presently in Section 2, where we sketch a proof of Reduction Theorem 1.2. In Special Termination 2.3, we also prove that $(\mathrm{ST})_n$ follows from the log minimal model program (LMMP; see Conventions 1.14) in dimension $n-1$. By our reduction, this is sufficient for the existence of 3-fold and 4-fold log flips.

Corollary 1.3. *In dimension $n\le 4$, $(\mathrm{PLF})_n^{\mathrm{el}}$ implies the existence of all log flips.*

See the proof at the end of Section 2. The following inductive statement is similar in nature, but its proof is more sophisticated. Note, however, that the extremal property is not preserved on passing to covers (cf. Lemma 3.51), so that we are obliged to drop it in our approach to pl flips, together with the assumption that our contractions are elementary.

Induction Theorem 1.4. *The statement*

$(\mathrm{PLF})_n^{\mathrm{small}}$ small *pl flips exist in dimension n*

follows from

$(\mathrm{FGA})_m$ *the finite generation (f.g.) of certain sheaves of algebras*[3] *over varieties of dimension $m\le n-1$.*

[3] $(\mathrm{FGA})_m$ algebras = lca saturated pbd algebras of Section 4; see below. (RFA) stands for restricted functional algebra, which is usually an (FGA) algebra.

More precisely, for any small pl flip (as in $(\mathrm{PLF})_n^{\mathrm{small}}$*) of core dimension* $d = n - s$*, we need the birational version* $(\mathrm{FGA})_d(\mathrm{bir})$ *in dimension* d*. Moreover, for these pl flips to exist, it is enough to assume the birational restricted version* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *in dimension* d*.* We refer forward to Definition 3.47 and Conjecture 3.48 for (RFA). Sections 3 and 4 explain all the statements (FGA) and (RFA) together with variations on them and relations between them.

Corollary 1.5. *LMMP and* $(\mathrm{FGA})_m$ *in dimension* $m \leq n-1$ *imply the existence of log flips in dimension* n*. More precisely, for these flips to exist, it is enough to assume* $(\mathrm{FGA})_m(\mathrm{bir})$ *or even* $(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *in dimension* $m \leq n-1$*.*

See the proof at the end of Section 2. Since LMMP is known in dimension ≤ 3 [45, Theorem 5.2], for 3-fold and 4-fold log flips, we can focus on (FGA) and (RFA).

Corollary 1.6. $(\mathrm{FGA})_m$ *in dimension* $m \leq 3$ *implies the existence of log flips in dimension* $n \leq 4$*. More precisely, for the existence of these flips, it is enough to assume* $(\mathrm{FGA})_m(\mathrm{bir})$ *or even* $(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *in dimension* $m \leq n-1$*.*

Proof. Immediate by Corollary 1.3 and Induction Theorem 1.4. □

Main Theorem 1.7. (FGA) *and* (RFA) *hold in low dimension*:

$(\mathrm{FGA})_m$ *holds for* $m \leq 2$*; and*

$(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *holds for all* $n \leq 4$ *and* $m \leq n-1$*.*

These are among the main results of the paper, and they imply

Corollary 1.8. *Log flips exist in dimension* ≤ 4*.*

Proof. Immediate by Corollary 1.6 and Main Theorem 1.7. □

Corollary 1.9. Directed Klt *flops exist in dimension* ≤ 4*.*

Proof–Explanation. Let $X \to T$ be a log *flopping* contraction of a Klt pair (X, B), that is, a birational 0-*contraction*: $K + B \equiv 0/T$ (thus, $(X/T, B)$ is a 0-log pair as in Remark 3.30(2)). Then, for any Weil $\mathbb{R}$-divisor D on X, there is a *directed* D-flop $X \dashrightarrow X^+/T$ that is a D-flip. (Indeed, X^+/T is small. Therefore, the modified $(X^+/T, B^+)$ is also a 0-contraction, and the modification is a flop.) Since X/T is birational, we can assume that $D \geq 0$ up to $\sim_{\mathbb{R}}$. By Corollary 3.6 below, the flop is a D-flip, and is a log flip with respect to $K + B + \varepsilon D$ for sufficiently small $\varepsilon > 0$ if X/T is $\mathbb{Q}$-factorial. Otherwise, first we need to replace X/T by its $\mathbb{Q}$-factorialization; the latter exists by the same arguments as in the proof of [45, Theorem 3.1] with empty set of blowing up divisors, and Special Termination 2.3 in dimension 4 (see Corollary 2.5, and cf. Example 3.15). The corollary now follows from Corollary 1.8. Strictly speaking, this works only when $-D$ or $-(K + B + \varepsilon D)$ is nef$/T$. The nef assumption on $-(K+B+\varepsilon D)$ can be replaced by small X/T (cf. [41, теорема 1.10], and see the proof of Reduction Theorem 1.2). For this, first replace X/T by T/T and then take its $\mathbb{Q}$-factorialization. □

In dimension 3, we can generalize pl flips to directed quasiflips associated with $(\mathrm{FGA})_3^{\mathrm{pl}}$ algebras as in Corollary 6.44. However, in higher dimensions, for general algebras, we only get

Corollary 1.10. *Small pl flips of core dimension* $d \leq 2$ *exist in any dimension* n*.*

Proof. Immediate by Induction Theorem 1.4 and Main Theorem 1.7. □

The same arguments work for any d provided that the restricted algebra is trivial (see Corollary 3.52 and Example 3.53, and cf. Example 9.11). This gives all the toric flips and flops (Example 3.54). Thus, our approach embraces all the currently known flips and flops, except those arising from geometric invariant theory (GIT; see Dolgachev and Hu [9] and Thaddeus [49]). Of course, GIT flips also arise from f.g. algebras of invariants (Hilbert), but we admit that this is not geometric; compare Conjecture 3.35 versus the log flip conjecture.

Some variations of existence conditions for flips are possible; for example, in Corollary 1.8, the nef assumption for $-(K + B)/T$ can be replaced by the small property for the contraction $X \to T$

(see the end of the proof–explanation of Corollary 1.9 and [27, 18.12]; cf. also [41, теорема 1.10, следствие 1.11]).

1.11. Structure of the paper. To conclude the introduction, we outline the ideas and key points in our arguments, as well as the structure of the paper as a whole. Section 3 proves the (RFA) statement of Induction Theorem 1.4, and Section 4, the (FGA) statement. These two sections explain the role of pl flips in the LMMP as a link between the geometry and algebra of flips.

The starting point is Shokurov [40, предложение 2.12], which states that an adjoint diagram or flip exists if and only if a pluri-ring associated with an adjoint canonical divisor is finitely generated. The same holds for any flips, in particular, for log and pl flips. For the local case, we replace the pluri-ring by the sheaf of algebras associated with a divisor, and finite generation of algebras by that of sheaves of algebras.

Section 3 introduces and discusses the basic properties of graded divisorial sheaves of algebras, their subalgebras, and the restricted functional algebras appearing in (RFA).

After a restriction, we no longer assume that algebras are of adjoint type. But the ambient variety and its subvariety should be appropriate for induction. For our main application, this means a pl contraction $X/X_\vee$ with its normal core Y. Main Lemma 3.43 states that, under certain obvious assumptions, a divisorial algebra is finitely generated if and only if its restriction to Y is finitely generated. The key difficulty is thus to prove that the restricted sheaf of algebras is finitely generated; the point is that *a priori* we do not even know that it is divisorial. We only know that it is pseudo-b-divisorial (pbd, see Definition 4.10), i.e., is a subalgebra of a divisorial algebra. Typically, such an algebra may not be f.g., but we conjecture that this is so for algebras of type (RFA).

Section 4 introduces more general functional algebras, and we state a conjecture that we need about finitely generated algebras (FGA) of a certain type. It is possible that both types are equal (see Remark 4.40(6)). In any case, both types share a saturation property discussed in Section 4. This section also associates with a pbd algebra a sequence of divisors with a limit, so that the algebra is f.g. if and only if the limit stabilizes.

The main body of the paper proves f.g. of these algebras in low dimension, as stated in Main Theorem 1.7. This needs more preparations; we carry these out for (FGA) in Sections 5 and 6, and Section 6 contains the proof of (FGA). For (RFA), Sections 5–10 contain preparations, and Section 11, the proof. Section 5 explains how saturation can guarantee the stabilization of a limit of divisors on an appropriate model (or *prediction model*). Section 6 suggests a choice of such a model and introduces conjectures on confinement of singularities for a saturated linear system or for its generic member. The conjectures are proved in dimension ≤ 2, which is enough for Main Theorem 1.7. For higher dimensions, in the theorem it is also enough to prove these conjectures in the same dimensions (cf. Theorem 6.45).

We expect that the classes of (FGA) and (RFA) algebras generalize in the form of log canonical algebras, in particular, for rather *big* subalgebras of a pluri-log canonical algebra (see Example 4.49 and Remarks 4.40(3) and 6.15(10)).

1.12. History. While not going back so deep into the past, the history is still impressive. Here we do not pretend to give a complete or very accurate account.[4] The history of flips can be divided into four periods.

Flips first arise as flops in birational modifications of Fano threefolds; geometrically, these are very classical, and appear as certain projections (see Iskovskikh and Prokhorov [16] for a modern

[4] A better one may be given in Reid's lecture "Flips 1980–2001" at the Algebraic Geometry Conference in memory of Paolo Francia in Genova 2001. (See the volume of proceedings of the Newton Institute program Higher dimensional geometry.)

treatment). Flops also appear in a paper of Atiyah of 1958 and, in modern terms, as a variation of geometric invariant theory (VGIT) and the Thaddeus principle [9, 49].

Kulikov used flops systematically in his construction of minimal models for semistable degenerations of K3 surfaces; they appear under the name "standard perestroika" in the Russian original of [30]. Almost at the same time, the appearance of the simplest genuine 3-fold flip (Francia [11]) manifested something weird that happens in higher dimensions.

The real history of flips starts with the Mori theory [31], which rapidly led to the Cone and Contraction Theorems in the LMMP [40] and Kawamata [19]. This also led to the understanding that flips and their termination are required to complete the LMMP. Formal definitions and conjectures on flips and flops were devised that complete the LMMP as a conjectural program and that invite one to establish it. (In positive characteristic, Mori's results remain unsurpassed.)

The search for 3-fold flips started in the mid-1980s in the geometric framework of the 3-fold minimal model program (MMP), that is, with only terminal singularities, and was successfully completed by Mori [32]. This approach uses

(1) the existence of flops; and

(2) a very deep result on the generic member of the relative anticanonical system ("general elephant" in Reid's glossary) of a 3-fold flipping contraction with only terminal singularities.

(1) was proved independently by Kawamata, Kollár, Mori, Shokurov, and Tsunoda with some variations in the methods and level of generality. Essentially, this can be viewed as rooted in the Brieskorn–Tyurina theory of simultaneous resolution of deformations of rational double points. Kawamata [21] also gave a proof in positive and mixed characteristic, and Corti gave an updated version in characteristic 0 [8] (see also Kollár and Mori [29, Ch. 7]). Actually, a weak form of (2) is enough to prove 3-fold flips, which is the main part of [32]. The existence of flips in the LMMP, that is, log flips, was also established for 3-folds [41, 27]. In place of (1) and (2), [41] used, respectively, semistable flips and complements more general than elephants. The most difficult part of [41] is concerned with *special* log flips (that is, pl flips with $B = 0$) having a 2-complement. This is still only understood by few people, even among experts (for an updated treatment, see Takagi [48]). These results on flips and flops are so technical that they cannot be included in textbooks or even monographs, except for the semistable case [8]. Nonetheless, they reduce 3-fold flips to very explicit geometric flips, and finally to Reid's pagodas.

With the aim of generalizing Mori's approach to dimension 4, Kachi [17, 18] obtained some results on special flips, but these are still far from general 4-fold flips, even semistable ones. (2) is quite doubtful in dimension ≥ 4, even in MMP.

This paper treats flips from a more algebraic point of view, based on pl flips and their f.g. properties. It may eventually be possible to generalize our methods to produce general log flips directly (cf. stabilization in Section 9). Such a tendency to algebraization (formalization) in geometry is not new but a rather common phenomenon. Of course, our approach gives rather little geometric information about the flips and flops that are constructed. However, we hope that this can be remedied *a posteriori* [47].

Finally, general log termination remains an open problem in dimension ≥ 4.

Remark 1.13. Minor modifications of our arguments also prove the existence of 3-fold and 4-fold log flips and directed flops in the analytic category; the same applies where relevant to most of our other results.

1.14. Conventions. By LMMP we usually mean that of [45, Section 5]. Note that [45] develops the LMMP in the log canonical category, which is more general than that of Kollár and others [27]; it involves contractions to *log canonical* singularities, possibly contracting a locus of log Kodaira dimension ≥ 0 along its Iitaka fibration. Thus, the LMMP in dimension n involves the classification of log varieties, in particular, the abundance conjecture, in dimension $n-1$. The work [45] settles the case $n=3$.

X/T is used for any contraction. $X/X_\vee$ is used exclusively for a pl contraction. X/Z is used when X is complete or projective$/Z$ but not necessarily birational; writing $X \to Y/Z$ where $X \to Y$ is birational and $Y \to Z$ is a fibre space allows us to treat families of contractions.

Since the construction of flips is local, we fix a point $P \in S_\vee \subset X_\vee$ and view $f\colon X \to X_\vee$ throughout as a germ$/P$. All other varieties and their objects are considered locally$/P$. For example, if D is a divisor on a variety X/Z, $P \in Z$, then $|D|$ denotes its linear system in a neighborhood of the fibre X/P.

$\operatorname{Mov} D$ and $\operatorname{Fix} D$ denote the *mobile* and *fixed* components of an $\mathbb{R}$-Weil divisor D, that is, divisors $M = \operatorname{Mov} D$ and $F = \operatorname{Fix} D$ such that $|D| = |M| + F$ and $|M|$ does not have fixed components. These are well defined if $|D| \neq \varnothing$. Then $D = M + F$ with $F \geq 0$ and M integral (Cartier in codimension 1). If $|D| = \varnothing$, we set $F = +\infty$; then $M = -\infty$ and $\mathcal{O}_X(M) = 0$.

To say that an $\mathbb{R}$-divisor D is nef always means, in particular, that D is $\mathbb{R}$-Cartier (compare b-nef in Lemma 4.23). This last condition holds for every $\mathbb{R}$-divisor if X is $\mathbb{Q}$-factorial.

The base field k is of characteristic 0. For example, $k = \mathbb{C}$.

Much of the paper uses the language of *birational divisors*, or b-divisors; see [45, 14] for the discussion of these ideas. In particular, $\mathcal{A} = \mathcal{A}^X = \mathcal{A}(X,B)$ denotes the *discrepancy*, or *coboundary*, b-divisor of the pair (X,B) for some $\mathbb{R}$-divisor B on X. (This is R in [45, Definition 1.1.4], with $C = 0$ as a b-divisor, and $C_X = B$ in the formula.) $\mathcal{B} = \mathcal{B}^X = \mathcal{B}(X,B) = -\mathcal{A}$ denotes the *codiscrepancy*, or *pseudo-boundary*, b-divisor of (X,B).

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2. SPECIAL TERMINATION

Let $(X/Z,B)$ be a log pair such that

- $f\colon X \to Z$ is a proper morphism; and
- (X,B) is log canonical.

Let $g\colon X \to Y/Z$ be a *birational contraction* of $(X/Z,B)$ that is *log canonical*, that is, g satisfies the following property:

(log canonical) $\quad K + B$ is numerically negative$/Y$.

Equivalently, this is a contraction $\operatorname{cont}_F = g$ of an extremal face F of $\overline{\mathrm{NE}}(X/Z)$ with $(K+B)\cdot F < 0$. We say that the $(K+B)$-flip of g is a *log canonical* flip (see [45, Section 5]).

Caution 2.1. These flips are log flips in the sense of Section 1 only if $K+B$ is Klt.

Special termination in dimension n is the statement

$(\mathrm{ST})_n$ for arbitrary $(X/Z,B)$ with $\dim X = n$, any chain of log canonical flips in extremal faces F with $|F| \cap \lfloor B \rfloor \neq \varnothing$ terminates.

Here the *support* $|F|$ of an extremal face F is the exceptional subvariety of cont_F, that is, the union of curves C/Z with $\operatorname{cont}_F C = \mathrm{pt}$.

Example 2.2. Suppose that we have a chain of log canonical flips of $(X/Z,B)$ in extremal rays R with $S \cdot R < 0$, where S is reduced and $0 < S \leq \lfloor B \rfloor$. Then $|R| \subset \lfloor B \rfloor$ and $|R| \cap \lfloor B \rfloor = |R| \neq \varnothing$. If $(\mathrm{ST})_n$ holds, then any such chain terminates. We meet this situation below in the proof of Reduction Theorem 1.2.

Theorem 2.3 (Special Termination)**.** *LMMP in dimension* $\leq n-1$ *implies* $(\mathrm{ST})_n$.

Remark 2.4. For the proof of Reduction Theorem 1.2, we need the situation where

- $f\colon X \to Z$ is projective;
- (X, B) is log terminal;
- X is $\mathbb{Q}$-factorial; and
- the contractions of $(X/Z, B)$ and their flips are *extremal*, that is, they are contractions g satisfying

(extremal) $\quad \rho(X/Y) = 1.$

Flips preserve this situation. Thus, under these assumptions, in Special Termination 2.3, it would be enough to assume log termination in LMMP (cf. Remark 2.6 below).

Corollary 2.5. $(\mathrm{ST})_n$ *holds for* $n \leq 4$.

Proof. Immediate by Special Termination 2.3 and LMMP in dimension 3 [45, Theorem 5.2]. $\square$

Proof of Special Termination 2.3. Suppose that we have a chain of flips as in $(\mathrm{ST})_n$. We first claim that

(A) after a finite number of flips, the support $|F|$ of any subsequent flip does not contain any log canonical center.

As in the proof of Theorem 4.1 in [41], this holds because the set of log canonical centers is finite, and the flip of F eliminates any log canonical center in $|F|$. We prove the following claim by increasing induction on d:

(B_d) after a finite number of steps, the support $|F|$ of any subsequent flip does not intersect any log canonical center $S \subset \lfloor B \rfloor$ of dimension $\leq d$.

For $d = 0$ this follows from (A), and for $d = n - 1$, it implies $(\mathrm{ST})_n$. For our applications, the only difficult case is $d = 1$ and $n = 4$, that is, a curve $S \subset \lfloor B \rfloor$ in a 4-fold X. Assume (B_{d-1}).

By induction, S is a minimal center and is normal as a subvariety near $|F|$ (cf. Kawamata [22, §1] and [41, лемма 3.6]; alternatively, under the restrictions of Remark 2.4, this follows by [41, следствие 3.8]). By the local divisorial adjunction, $(K + B)_{|S} \sim_{\mathbb{R}} K_S + B_S$ with $B_S = B_{\mathrm{div}}$ (the divisorial part of adjunction) is Klt near a generic point of each prime divisor $P \subset S$ (see [23, 36]), and all possible multiplicities of B_{div} belong to a finite subset of the real interval $[0, 1)$. This subset only depends on the local structure of (X, B) near the generic points of codimension 1 in S and includes the multiplicities of B near P. Moreover, under the restrictions of Remark 2.4, these multiplicities are all of the form

$$\frac{m-1}{m} + \sum \frac{l_i}{m} b_i, \tag{2.5.1}$$

where b_i are the multiplicities of B and m and l_i are natural numbers. This follows from [41, следствие 3.10, лемма 4.2]. A similar formula is proved for $n \leq 4$ and conjectured in general [36].

Introduce the notation $\mu_S = \mathrm{mld}(S, B_S)$ for the minimal log discrepancy of (S, B_S) in its Klt part, for the initial model (X, B). Then there are, in particular, only finitely many numbers of the form (2.5.1) in the interval $[0, 1 - \mu_S]$. Conjecturally, over the whole S, $B_S = B_{\mathrm{div}} + B_{\mathrm{mod}}$, where B_{mod} is b-semiample on S/Z, and (S, B_S) is Klt in the Klt part of (X, B), in particular, near $|F| \cap S$. For example, it is true with $B_{\mathrm{mod}} = 0$ under the restrictions of Remark 2.4. It was also established in a more general situation but still under certain assumptions by Kawamata [24].

Another workable approach is to let $(W, B_W) \to S$ be a projective Iitaka (possibly disconnected) fibration over S, with $\dim W = n - 2$, then replace S by W and use the divisorial adjunction on W for a crepant, strictly log terminal resolution of $S \subset \lfloor B \rfloor$. Then the finiteness of the possible multiplicities for B_S follows from the finiteness of such models for which the vertical boundary multiplicities of the resolution are numbers of the form (2.5.1) in $[0, 1-\mu_W]$, where $\mu_W = \mathrm{mld}(W, B_W)$, except for the horizontal components, for an initial model (W, B_W); the horizontal part of B_W is fixed. Indeed, there are only finitely many possibilities for the vertical boundary multiplicities of (W, B_W) and, thus, for all multiplicities of B_W.

The only interesting case is $d = 1$ and $n = 4$; then W is a log elliptic Iitaka fibration over a curve with a positive lower bound for the mld given by the boundary multiplicities of the above form and fixed horizontal components. The proof in this case and for 3-fold fibrations follows from [45, Second Main Theorem 6.20]. We can use the same arguments together with LMMP for W/Z in higher dimension [45, Remark 6.23.5]; this also includes the finiteness of S/Z when $\dim S \geq 2$. However, this phenomenon should have a better explanation based on complements [36].

Each flip restricts to a birational transformation of $(S/Z, B_S)$ or of the Iitaka fibration $(W/Z, B_W)$. Unfortunately, the restricted transformation may not be a flip (however, it will be a log quasiflip in the terminology of [47]); this happens exactly if the flipped contraction on $(S^+/Z, B_S^+)$ or $(W^+/Z, B_W^+)$ is not small (cf. the end of the proof). In this case, it blows up prime divisors that again have multiplicities of B_S^+ or B_W^+ in our finite set, and $< 1 -$ their log discrepancy on the previous model $(S/Z, B_S)$ or $(W/Z, B_W)$. However, there only exist a finite number of transformations with these properties (cf. the proof of Theorem 4.1 in [41]): we can only decrease B_S^+ or B_W^+ a finite number of times with the above possible multiplicities of the divisorial part of adjunction.

The final monotonicity follows from adjunction and the opposite monotonic property of log discrepancies for flips, strict monotonic in the blown-up prime divisors over $|F|$ under the numerical negativity of $(K+B)/Y$ (see [40, 2.13.3] and [47, Monotonicity]). Only such transformations occur in our main case $d = 1$ and $n = 4$.

Then we use the termination of log flips on $(S/Z, B_S)$ or $(W/Z, B_W)$ by LMMP. For this to be valid, we need to know that they really are log flips. More precisely, for $(S/Z, B_S)$, a birational log contraction $g\colon X \to Y/Z$ and its log flip $g^+\colon X^+ \to Y/Z$ induce the birational log contraction $S \to T = g(S)/Z$ and its log flip $S^+ \to g(S)/Z$ when the latter is small. To prove this, we can complete these modifications locally/T into log flops (of 0-log pairs, see Remark 3.30(2)). As above for S this implies that T and S^+ are normal, and we have contractions. The semiadditivity of the modular part of adjunction [36] (the complement can be non-lc here) implies that the decomposition $B_S = B_{\mathrm{div}} + B_{\mathrm{mod}}$ is preserved under the flips on $(S/Z, B_S)$ because this holds for birational contractions and small modifications of 0-log pairs. For $(W/Z, B_W)$ we can consider the termination of lifted log flips from S/Z to W/Z (actually, it is a composite of log flips up to flops, which do not affect the log termination); the log flips are lifted into log quasiflips of the qlog pair $(W/Z, B_W)$ [2, Section 4]. $\square$

Remark 2.6. The termination at the end of the proof of special termination is that of any chain of successive log flips [45, 5.1.3]. They are *not necessarily extremal*, that is, they may contract extremal faces of dimension ≥ 2. Using LMMP, we can reduce this to a chain of extremal flips. Thus, we need LMMP in dimension $n-1$ in an essential way.

Proof of Reduction Theorem 1.2. We follow the part of the argument in the proof of Reductions 6.4 and 6.5 in [41] that concerns limiting flips, with improvements due to Kollár and others [27, (18.12.1.3–4)]. In this reduction, each small contraction cont_R becomes elementary pl if we discard all the reduced irreducible components S_i in B with $S_i \cdot R \geq 0$. Indeed, by construction, B includes a reduced component S_i with $S_i \cdot R < 0$. Thus, log flips exist by $(\mathrm{PLF})_n^{\mathrm{el}}$ and are the same as before discarding some of the S_i. They terminate by Example 2.2 and $(\mathrm{ST})_n$ (cf. [41, следствие 4.6]). $\square$

Proof of Corollary 1.3. Immediate by Reduction Theorem 1.2 and Corollary 2.5. $\square$

Proof of Corollary 1.5. Immediate by Reduction Theorem 1.2, Induction Theorem 1.4, and Special Termination 2.3. $\square$

3. DIVISORIAL $\mathcal{O}_Z$-ALGEBRAS AND FLIPS

In this section we explain the relation between the geometry and algebra of D-flips and prove Induction Theorem 1.4 in the case of (RFA) algebras. We start with geometry. Definition 3.3, while admittedly somewhat clumsy, aims to unify some standard geometric constructions, in particular, flips and contractions.

Definition 3.1. A *rational contraction* $c\colon X \dashrightarrow Y/Z$ is a dominant rational map with connected fibres. Resolution of indeterminacies expresses c as a composite $c = h \circ g^{-1}$ in a Hironaka hut

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X & & Y/Z \end{tikzcd} \tag{3.1.1}$$

where g is a proper birational morphism and h is a dominant morphism with connected fibres.

We say that $c\colon X \dashrightarrow Y$ is a *rational* 1*-contraction* if it does not blow up any divisor; in other words, in (3.1.1), every exceptional prime divisor E of g is contracted by h, that is,

$$\dim h(E) \le \dim E - 1 = \dim X - 2.$$

For a birational contraction $c\colon X \dashrightarrow Y$, there is no ambiguity about what it means for a divisor E on X to be exceptional (contracted) on Y. But if $c\colon X \dashrightarrow Y$ is a rational fibre space (with positive-dimensional fibres), we need to be more precise.

Definition 3.2. We say that a divisor E of X is

- *exceptional* on Y if it is contracted by c and is *not horizontal*, that is, has $c(E) \ne Y$;
- *very* exceptional on Y if $\operatorname{Supp} E$ does not contain $gh^{-1}\Gamma$ for any prime divisor Γ of Y, that is, E does not contain entire fibres over divisors of Y; all the inclusions and transforms are birational (or as of cycles), that is, $gh^{-1}\Gamma = 0$ or $\operatorname{Supp} gh^{-1}\Gamma = \varnothing$ if it is nondivisorial; in particular, $gh^{-1}\Gamma \ne 0$ if a very exceptional E exists, which is a condition on c (cf. (ZD) below);
- *truly* exceptional if $\dim c(\operatorname{Supp} E) \le \dim Y - 2$, so that $c(E)$ does not contain any prime divisor of Y.

Definition 3.3. We say that a Weil $\mathbb{R}$-divisor D on X is *b-sup-semiample$/Z$* or *bss ample* if one of the following two equivalent conditions holds:

(BSS) there exist a rational 1-contraction $c = \operatorname{cont}_Z D\colon X \dashrightarrow Y/Z$ and a numerically ample $\mathbb{R}$-divisor H on Y/Z such that $D \sim_{\mathbb{R}} c^*H + E/Z$, where E is effective and very exceptional on Y; or

(ZD) "Zariski decomposition": there exist a 1-contraction c with hut (3.1.1) and a numerically ample $\mathbb{R}$-divisor H on Y/Z such that $g^{-1}D + F \sim_{\mathbb{R}} h^*H + E/Z$, where F and E are Weil $\mathbb{R}$-divisors on W that are exceptional on X and Y, respectively, E is effective, and the complement to $\operatorname{Supp} E$ in the fibre over each divisor in Y is not exceptional on X, so that, in particular, $g(E)$ is very exceptional on Y.

In addition, H is the greatest divisor as it is explained below.

The 1-contraction $c = \operatorname{cont}_Z D$ is a generalization of the contraction defined for *semiample* D in [45, Definition 2.5]. The decomposition of D in (BSS) can easily be converted into a Zariski form (see Remark 3.30 and [34, разд. 5]).

For a rational contraction $c\colon X \dashrightarrow Y$ and any $\mathbb{R}$-Cartier divisor H on Y, the Weil $\mathbb{R}$-divisor $c^*H = g(h^*H)$ on X is independent of the choice of the hut (3.1.1). In terms of b-divisors, $c^*H = \mathcal{H}_X$, where $\mathcal{H} = \overline{h^*H}$, see [45, Example 1.1.1]. The fact that the divisor H is the *greatest* means the same for $\mathcal{H}$ up to $\sim_{\mathbb{R}}$, that is, a b-semiample b-divisor $\mathcal{D}^m \sim_{\mathbb{R}} \mathcal{H}$ is the greatest under the property $(\mathcal{D}^m)_X \le D$. In many instances, it is easy to verify this property; thereupon, we do not mention this.

Proof of the equivalence in Definition 3.3. The divisor E in (ZD) gives $E := g(E)$ in the statement (BSS) since F is exceptional on X and $g(E)$ is still very exceptional on Y. Conversely, $D \sim_{\mathbb{R}} c^*H + E/Z$ gives an equivalence $g^{-1}D + F \sim_{\mathbb{R}} h^*H + E/Z$ in which $E := g^{-1}E \ge 0$ and F are still very exceptional on Y and X, respectively. $\square$

Each bss ample divisor is effective up to $\sim_{\mathbb{R}}$:

$$D \sim_{\mathbb{R}} h^*H + E \sim_{\mathbb{R}} h^*H' + E \quad \text{and} \quad \ge 0,$$

where $H' \sim_{\mathbb{R}} H/Z$ is an effective $\mathbb{R}$-divisor. Thus, not every divisor D is bss ample. However, the following uniqueness result holds.

Proposition 3.4. *Suppose that D is bss ample. Then $(X/Z, D)$ uniquely determines the divisor E, the rational contraction $c = \operatorname{cont}_Z D$ (up to isomorphism), and the divisor H (up to $\sim_{\mathbb{R}} /Z$).*

Addendum 3.4.1. *The greatest condition can be replaced by its variation over the generic point of Y:*

if F is a horizontal/Y $\mathbb{R}$-divisor such that $g(F) \le 0$ and F is semiample over the generic point of Y, then $F = 0$. In particular, this is true whenever the same holds with nef instead of semiample.

Addendum 3.4.2. *The horizontal/Y part of the greatest $(\mathcal{D}^m)_W$ given up to $\sim_{\mathbb{R}} /Y$ is uniquely determined by its part that is nonexceptional on X.*

Addendum 3.4.3. *We can omit the greatest condition if $\dim X/Y \le 1$, in particular, if D is big, equivalently, h is birational. We can omit the same condition if $Y = Z$. Thus, always for $\dim X \le 2$.*

Proof. Since $\mathcal{D}^m$ is the greatest, it is unique and $\mathcal{H} \sim_{\mathbb{R}} \mathcal{D}^m/Z$. Hence, $E = D - (\mathcal{D}^m)_X$ and c are also unique.

In Addendum 3.4.1, take F that satisfies the variation. Then there exist $\varepsilon > 0$ and a vertical/Y divisor $E' \ge 0$ such that $\mathcal{D}^m + \varepsilon\overline{F} - \overline{E'}$ is semiample/Z and

$$(\mathcal{D}^m + \varepsilon\overline{F} - \overline{E'})_X \le (\mathcal{D}^m)_X - (\overline{E'})_X \le (\mathcal{D}^m)_X \le D.$$

Thus, $\mathcal{D}^m + \varepsilon\overline{F} - \overline{E'} \le \mathcal{D}^m$ and $F \le 0$; moreover, $F = 0$ by the semiampleness.

Conversely, suppose that $\mathcal{D}^m$ is b-semiample and its contraction c under (ZD) also satisfies the variation of the greatest property. Up to $\sim_{\mathbb{R}} /Z$, we can suppose that $\mathcal{D}^m$ is effective. Let $\mathcal{H}$ be another b-semiample b-divisor which is semiample/Y/Z and with $g(\mathcal{H}_W)_X \le D = (\mathcal{D}^m)_X + E$. If F is the horizontal/Y part of $\mathcal{H}_W$, then it satisfies the variation conditions. Thus, $F = 0$ and $\mathcal{H}_W$ is vertical/Y; moreover, $\mathcal{H}_W = h^*H$ for a semiample/Z divisor H on Y. Over each prime divisor in Y, $(\mathcal{D}^m)_W \ge \mathcal{H}_W$ at least in one nonexceptional divisor on X. Hence, $\mathcal{D}^m \ge \mathcal{H}$ everywhere.

In Addendum 3.4.2, the difference F satisfies the variation of Addendum 3.4.1: $F \sim_{\mathbb{R}} 0/Y$ with $g(F) = 0$. Thus, $F = 0$.

Most of the results in Addendum 3.4.3 are well known. The last one with $Y = Z$ by Negativity 1.1 in [41]. $\square$

Corollary 3.5. *If $D' \sim_{\mathbb{R}} rD$ for some positive $r \in \mathbb{R}$, then D' is also bss ample with the same rational 1-contraction c, $E' = rE$, and $H' \sim_{\mathbb{R}} rH$. Thus, bss ample is a property of the ray $\mathbb{R}_+ \cdot D$ in the space of divisors up to $\sim_{\mathbb{R}}$.*

Proof. Immediate by Proposition 3.4 because $rE > 0$ and $H' = rH$ is ample. Note also that $\sim_{\mathbb{R}}$ and the multiplication preserve $\ge$. $\square$

In the most important applications, $E = 0$; then we say that D is *b-semiample/Z*, that is, semiample on some model W of X, possibly after adding certain divisors that are exceptional on X. Equivalently, $D = \mathcal{D}_X$ for a *b-semiample* b-divisor $\mathcal{D}$ ($\sim_{\mathbb{R}} \mathcal{H}$). For example, if $Y =$ pt., then $H = 0$, $E = 0$, and $D \sim_{\mathbb{R}} \mathcal{H} = 0$.

Corollary 3.6. *Suppose that $f\colon X \to Z = T$ is a birational contraction. Then any D-flip is uniquely determined by D up to $\sim_{\mathbb{R}}$ and multiplication by positive real numbers.*

Lemma 3.7. *The D-flip of a contraction $f\colon X \to T$ exists if and only if*

(BSA) *the divisor $f(D)$ on T/T is bss ample/T,*

and then the flipped contraction $f^+ = c^{-1}\colon Y \to Z = T$ is small, and the flipping modification is $c \circ f$; we have $E = 0$ in (BSS).

The same holds for D itself on X/T if f is small; *that is, if f is small, the D-flip exists if and only if* (BSA) *holds for D on X/T.*

In (BSA) $E = 0$, so $f(D)$ is b-semiample/T.

Proof. If the D-flip exists, then it defines a small contraction $Y = X^+/T$ with $H = D^+ = c(f(D))$ numerically ample/T and a flipping modification $X \dashrightarrow X^+ = Y/T$. Then $f(D)$ is bss ample with the rational 1-contraction $c\colon T \dashrightarrow X^+$ given by H and $E = 0$. Indeed, $f(D) = c^*H = c^*D^+$.

If $f(D)$ is bss ample, then $Y/T = Z$ is small, $E = 0$, and

$$D^+ = c(f(D)) \sim_{\mathbb{R}} c(c^*H) = H.$$

The same arguments apply to D itself for any small f. $\square$

Proof of Corollary 3.6. Immediate by Lemma 3.7, Proposition 3.4, and Corollary 3.5. $\square$

Corollary 3.8. *Suppose that D is nef/Z. Then D is bss ample with $\mathcal{H} \sim_{\mathbb{R}} \overline{D}$ if and only if it is semiample. In addition, $E = 0$. In particular, every semiample divisor D is bss ample.*

Proof. We can replace $(X/Z, D)$ by $(W/Z, g^*D)$ in (ZD). Then $E = 0$ by (BSS) and Negativity 1.1 in [41]. Thus, $D \sim_{\mathbb{R}} h^*H$, $\overline{D} \sim_{\mathbb{R}} \mathcal{H}$, and h contracts all curves C on W/Z and X/Z with $D \cdot C = 0$. $\square$

Thus, if D is bss ample, Proposition 3.4 defines a unique decomposition of D as a sum of two Weil $\mathbb{R}$-divisors:

$$D = D^m + D^e,$$

the exceptional *fixed part* $D^e = E \geq 0$ and the $\mathbb{R}$-*mobile part* $D^m = D - E$. If X/pt. is a surface, this is a *Zariski decomposition* (the converse holds if the positive part of the Zariski decomposition is semiample); then c in (BSS) is a regular contraction. It is also useful to consider the unique b-divisors $\mathcal{D}^m \sim_{\mathbb{R}} \mathcal{H}$ with $\mathcal{D}^m_X = D^m$ and $\mathcal{D}^e = E$ (considered as a b-divisor) with $\mathcal{D}^e_X = D^e$. The divisor

$$D^{\mathrm{sm}} = \sum (\mathrm{mult}_{D_i} D) D_i \qquad \text{for } D_i \text{ not in } \operatorname{Supp} D^e = \operatorname{Supp} E \tag{3.8.1}$$

is also important (here sm = supported in mobile part). We define generalizations of these divisors for any D in Section 4 (see Example 4.30 and Remark 3.30).

Again, if Y is a point, then $D^e = \mathcal{D}^e = 0$, $D^{\mathrm{sm}} = D^m \sim_{\mathbb{R}} 0$, and $\mathcal{D}^m \sim_{\mathbb{R}} 0$.

3.9. The algebra of flips. Here we interpret a flip in terms of a certain sheaf of functional graded algebras. Its *homogeneous* elements (that is, elements of *pure degree*) are sections of *divisorial* $\mathcal{O}_X$-sheaves $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$ with

$$\Gamma(U, \mathcal{O}_X(D)) = \{a \in k(X) \mid (a) + D \geq 0\},$$

where D is a Weil $\mathbb{R}$-divisor and $k(X)$ denotes the function field of X; we write (a) for the divisor of a nonzero element $a \in k(X)$, with the convention that 0 has divisor $(0) = +\infty$. An axiomatic treatment of divisorial sheaves is given in Reid [37, Appendix to §1]. The $\mathcal{O}_X(D)$ are *functional sheaves*, that is, coherent subsheaves of the constant sheaf $k(X)$. They are defined if D is *nonsin-*

gular, that is, every generic point of $\operatorname{Supp} D$ is nonsingular in X. (We assume that D is 0 at any irreducible codimension-1 subvariety D_i along which X is nonnormal; the above sheaf condition means that f is regular at D_i.) Nevertheless, since X is usually assumed to be normal, we can disregard these subtleties. (See, however, Corollary 5.21 and the explanation of the nonnormal case in the proof, p. 148; see also [2].) Any Weil $\mathbb{R}$-divisor on a normal variety X is nonsingular because S is nonsingular in codimension 1.

Definition 3.10. Let $f\colon X \to Z$ be a proper morphism. We define the $\mathbb{N}$-*graded* $\mathcal{O}_Z$-*algebra* of a Weil $\mathbb{R}$-divisor D on X as

$$\mathcal{R}_f D = \mathcal{R}_{X/Z} D \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} f_* \mathcal{O}_X(iD).$$

This is an $\mathbb{N}$-graded *functional* $\mathcal{O}_Z$-subalgebra of the constant sheaf of $\mathcal{O}_Z$-algebras

$$k(X)_\bullet = \bigoplus_{i=0}^{\infty} k(X)$$

(that is, having sections $\bigoplus_{i=0}^{\infty} k(X)$ over every nonempty Zariski open subset of X). The multiplication in $\mathcal{R}_{X/Z} D$ is induced by the multiplication in $k(X)_\bullet$: for homogeneous elements

$$a \in \mathcal{O}_X(iD) \subset k(X) \quad \text{and} \quad b \in \mathcal{O}_X(jD) \subset k(X) \quad \Rightarrow \quad ab \in \mathcal{O}_X((i+j)D) \subset k(X).$$

Indeed, $(ab) + (i+j)D = (a) + iD + (b) + jD \geq 0$.

An $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra is *divisorial* for X/Z (or *relatively divisorial*) if it is isomorphic to $\mathcal{R}_{X/Z} D$ for some D. An $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra is (relatively) *b-divisorial* for X/Z if it is isomorphic to $\mathcal{R}_{Y/Z} D$ for some model Y/Z of X/Z and for some divisor D on Y. Section 4 generalizes these algebras and the notation (see Definition 4.10 and Example 4.12 below).

Remark 3.11. If f is the identity, each homogeneous piece $f_*\mathcal{O}_X(iD) = \mathcal{O}_X(iD)$ is a divisorial sheaf. The algebras $\mathcal{R}_{X/X} D$ have a similar axiomatic description if D is integral (in other words, *locally mobile*: $D = \operatorname{Mov} D$ for X/X). If X/Z is a birational contraction, then $f_*\mathcal{O}_X(iD)$ is a *b-divisorial* sheaf. These also have an axiomatic description, as do b-divisorial algebras when D is mobile on Y/X (cf. Proposition 4.15(4)). All of this is more complicated for $\mathbb{R}$-divisors; in particular, it has a satisfactory description for $\mathbb{R}$-mobile D.

Example 3.12. If D is a Cartier divisor on X, its algebra $\mathcal{R}_{X/X} D$ is the *tensor algebra* of D

$$\mathcal{T}(\mathcal{O}_X(D)) = \bigoplus_{i=0}^{\infty} \mathcal{O}_X(D)^{\otimes i}.$$

This graded algebra is defined for any $\mathbb{R}$-divisor D. However, the above isomorphism does not hold for general $\mathbb{R}$- or $\mathbb{Q}$-Cartier divisors D, even in the local case X/X, because the tensor algebra is generated by finitely many homogeneous elements of degree 1. More generally, any sheaf $\mathcal{M}$ of coherent $\mathcal{O}_X$-modules defines a *symmetric* tensor $\mathcal{O}_X$-algebra

$$\operatorname{Sym} \mathcal{M} = \bigoplus_{i=0}^{\infty} \operatorname{Sym}^i \mathcal{M}.$$

The symmetric tensor algebra of a torsion free sheaf $\mathcal{M}$ of rank r embeds into the symmetric algebra of V, where V is a vector space over $k(X)$ of dimension equal to the rank of $\mathcal{M}$. We only consider rank 1 here. However, such sheaves and their subsheaves appear in the theory of Bogomolov instability of $\mathcal{M}$, sections of the tangent bundle, etc.

For any *integral* divisor D, the divisorial algebra $\mathcal{R}_{X/X} D$ is the *reflexive tensor algebra* of D

$$\mathcal{A}(\mathcal{O}_X(D)) = \mathcal{T}(\mathcal{O}_X(D))^{\vee\vee}.$$

This may, of course, be infinitely generated.

If X/Z is not the identity and/or D is not integral, then divisorial and b-divisorial algebras are more complicated. But they are sometimes f.g., as we see in Section 4.

Definition 3.13. An $\mathcal{O}_Z$-algebra has *finite type* if it is *finitely generated*, or *f.g.* This means that, locally$/Z$, it is generated as an $\mathcal{O}_Z$-algebra by a finite set of sections (*generators*).

In particular, if the algebra is functional (cf. Definition 4.1), each component is a coherent sheaf.

Example 3.14. The divisorial algebra $\mathcal{R}(X/Z,B) = \mathcal{R}_{X/Z}(K+B)$ associated with a log divisor $K+B$ is called *log canonical*. We usually assume that it corresponds to a log pair $(X/Z,B)$ with boundary B and with only log canonical singularities (cf. Example 4.48 below).

In this situation, in dimension ≤ 3, it is known that $K+B$ is *bss ample*$/Z$ when the *relative numerical* log Kodaira dimension [45, Section 2, p. 2673] of $(X/Z,B)$ is big. (See Corollary 3.34 and Theorem 3.33 and [34, предложение 5.6].) The same is conjectured in any dimension. (In general, when the Kodaira dimension is nonnegative, we conjecture that there exists a maximal divisor that is *b-semiample*$/Z$ as in the decomposition of Remark 3.30; see Example 5.8 in [34].) Thus, if B is a $\mathbb{Q}$-divisor, $\mathcal{R}(X/Z,B)$ is *f.g.* by Theorem 3.18 below. LMMP reduces the first conjecture to

(1) the existence of log flips, a very special case of the second conjecture;

(2) the termination of log flips; and

(3) the semiampleness of $K+D$ if it is nef

(cf. the proof of Theorem 3.33 below). For a $\mathbb{Q}$-divisor B, the condition on the Kodaira dimension means that $\mathcal{R}(X/Z,B)$ is nontrivial, but (3) means much more, namely, the global almost generation defined below.

Example 3.15. Let $f\colon X\to T$ be a birational contraction and D be an $\mathbb{R}$-Weil divisor. The *flipping* algebra of $(X/T,D)$ is the divisorial algebra

$$\mathcal{FR}_{X/T}D = \mathcal{R}_{T/T}f(D).$$

For any $\mathbb{Q}$-divisor D, by Corollary 3.32, the D-flip exists if and only if $\mathcal{FR}_{X/T}D$ is f.g. If f is small, then

$$\mathcal{FR}_{X/T}D = \mathcal{R}_{X/T}D$$

by Example 3.21(b). But this fails in general, for example, if f is divisorial and $D=H$ is a hyperplane section of X/T (use Theorem 3.18 below).

Example 3.16 (Stupid Example)**.** Let P be a point on a complete curve C. Take $D=dP$ with nonnegative $d\in\mathbb{R}$. Then $\mathcal{R}_{C/\mathrm{pt.}}D$ has finite type if and only if d is rational (cf. Theorem 3.18 below). Nonetheless, D is semiample for any real $d\geq 0$ and bss ample by Corollary 3.8.

Note that $\mathcal{R}_{C/\mathrm{pt.}}D$ is also f.g. for any real number $d<0$ because then it is *trivial*, that is, it *has no sections of positive degree.*

Thus, f.g. is a somewhat different condition from ample, or bss ample, although the notions are closely related. Here the important phenomena are rationality of multiplicities and nonvanishing in positive degrees. However, to explain f.g. in geometric terms of bss ampleness, we need an additional condition related to generators.

Definition 3.17. A divisorial algebra $\mathcal{R}=\mathcal{R}_{X/Z}D$ is *globally almost generated* (*g.a.g.*) if there is a natural number N and sections of $f_*\mathcal{O}(ND)$ that generate the algebra $\mathcal{R}_{X/X}ND$ in codimension 1 on X, except possibly at a finite set of divisors that are exceptional on $Y=\operatorname{Proj}_Z\mathcal{R}$. (In this form, the definition assumes that $\mathcal{R}$ is finitely generated. But one can generalize it to get rid of this assumption.) In particular, this implies that $\mathcal{R}$ is *nontrivial*, that is, it has a nonzero element of some positive degree $d>0$; this implies that $\mathcal{R}$ has a nonzero element in each degree i *divisible by d*.

Theorem 3.18. *The divisorial algebra* $\mathcal{R} = \mathcal{R}_{X/Z}D$ *is f.g. and g.a.g. if and only if* D *is bss ample/*Z *and* D^{sm} *is a* $\mathbb{Q}$*-divisor* (*see* (3.8.1)); *this* rationality condition *holds, in particular, if* D *is a* $\mathbb{Q}$*-divisor.*

Moreover, if $\mathcal{R}$ *is f.g. and g.a.g., then, in the notation of Definition* 3.3, $Y/Z \cong \mathrm{Proj}_Z \mathcal{R}$, E *is the* stable base divisor *of* $\mathcal{R}$, *and* $\mathrm{Supp}\, E$ *is the* stable divisorial base locus *of* $\mathcal{R}$.

The proof is given on p. 100. The theorem allows us to define divisors D^m, E and b-divisors $\mathcal{D}^m$, $\mathcal{D}^e$, etc., for $\mathcal{R}$. These are well known: E is the *stable base divisor* of D and gives the *stable divisorial base locus* $\mathrm{Supp}\, E$ as the exceptional divisorial subset of Definition 3.17. Either of these divisors can be associated with D or with the *divisorial algebra* $\mathcal{R}$, but as an algebra associated with D, not as an abstract algebra (cf. Truncation Principle 4.6 and the remark in the proof of Proposition 4.15(7) on p. 114).

Note that, if we replace bss ample by semiample in the theorem, we should replace g.a.g. by the usual global generation of the algebra $\mathcal{R}_{X/X}ND$. The next result shows under what circumstances we can add an effective divisor to the mobile part of a multiple linear system.

Lemma 3.19. *Let* D *be a semiample* $\mathbb{Q}$*-divisor and* $c\colon X \to Y/Z$ *be the contraction it defines. Suppose that* E *is an effective Weil* $\mathbb{Q}$*-divisor not contracted by* c, *or is exceptional but not very exceptional on* Y (*see Definition* 3.2). *Then, for some natural numbers* $N \gg 0$ *and* M, *the base locus* $\mathrm{Bs}(ND + ME)$ *is a strict subset of* $\mathrm{Supp}\, E$, *that is,*

$$\mathrm{Supp}\,\mathrm{Fix}(ND + ME) \subsetneq \mathrm{Supp}\, E.$$

Addendum 3.19.1. *Moreover, if* E *is integral, we can take* $M = 1$ *except for the following two cases*:

- D *has numerical dimension* $\nu(X/Z, D) = \dim X - 1$ (*see Kawamata, Matsuda, and Matsuki* [25, Definition 6-1-1]), *the generic fibre of the contraction is a curve of genus* $g \geq 1$, *and every component of* E *dominates* Y; *then we can take* $M = g + 1$; *or*
- E *is contained in multiple fibres of the contraction*; *then we can take* M *to be the least common multiple of the multiplicities.*

Proof. We can assume that all components of $\mathrm{Supp}\, E$ have a single irreducible image under $c\colon X \to Y/Z$. Then we have three cases:

Case 1: E is not contractible and horizontal. Then

$$\nu(X/Z, D) = \dim \mathrm{Supp}\, E = \dim X - 1.$$

The generic fibre of c is a curve of genus g, and $ND+ME$ cuts out a divisor of positive degree $\geq M$. Thus, on the generic fibre we have the required result by RR for curves. This implies the lemma for $N \gg 0$ in this case.

Case 2: E is not contractible and not horizontal. Then $\nu(X/Z, D) = \dim X$ and c is birational. In this case, the lemma follows from the general RR (compare Reid [37, proof of Lemma 1.6]).

Case 3: E is contractible but not very exceptional. The assumption means that $\nu(X/Z, D) < \dim X$, so that c is fibred, and $\mathrm{Supp}\, E$ forms the fibre over the prime divisor $E' = c(\mathrm{Supp}\, E)$ of Y. Let M be the lcm of the multiplicities of c^*E'. Then there exists a natural number m such that $ME - mc^*E'$ is effective and very exceptional. Therefore, we obtain this case from the second one with $E = mE'$. $\square$

On the other hand, in certain cases we certainly cannot add an effective divisor to the mobile part of a multiple of the linear systems.

Proposition 3.20. *Let $c\colon X \dashrightarrow Y/Z$ be a rational 1-contraction, D be an $\mathbb{R}$-Cartier divisor on Y, and E be an effective divisor on X that is very exceptional on Y. Then c induces an isomorphism*

$$c^*\colon i_*\mathcal{O}_Y(D) \xrightarrow{\approx} f_*\mathcal{O}_X(c^*D+E),$$

where $i\colon Y\to Z$.

This statement has the following applications, which can also be proved directly and generalized:

Example 3.21.

(a) If $c\colon X \dashrightarrow X^+/T$ is a D-flip/Z and X/T is small, then c induces an isomorphism

$$c^*\colon f_*^+\mathcal{O}_{X^+}(D^+) \xrightarrow{\approx} f_*\mathcal{O}_X(D),$$

where $D^+=c(D)$, $f\colon X\to Z$, and $f^+\colon X^+\to Z$. Indeed, $c^*D^+=D$ and $E=0$ in this case. The proof is a direct verification; it is enough to do it for $Z=T$.

(b) Moreover, we can replace the D-flip by any small birational transformation, that is, assume that it is a 1-contraction in each direction. Here we can omit the assumption that D is $\mathbb{R}$-Cartier.

Thus, in either case c induces an isomorphism of $\mathcal{O}_Z$-algebras

$$c^*\colon \mathcal{R}_{X^+/Z}D^+ \xrightarrow{\approx} \mathcal{R}_{X/Z}D$$

since c^* is compatible with multiplication.

Lemma 3.22. *In the hut* (3.1.1), *suppose that D is an $\mathbb{R}$-Cartier divisor on W satisfying the following conditions*:

(i) *the negative part of D is very exceptional on Y; and*

(ii) *$-D$ is nef at general curves of $E_i/h(E_i)$ for all prime divisors E_i that occur in D with negative multiplicity.*

Then D is effective.

Note that the statement is only about h and D.

Proof. For birational W/Y, this follows from the negativity of a proper modification [43, Negativity 2.15]; in general, see [34, лемма 1.6]. Indeed, each prime E_i with negative multiplicity has a sufficiently general curve/Z on which D is numerically nonpositive, namely, a general curve in the generic fibre of $E_i/h(E_i)$. By (i), this exists because every prime divisor E_i that occurs in D with negative multiplicity is contractible/Y. $\square$

Proof of Proposition 3.20. First, c induces an inclusion because, for any nonzero rational function $s\in i_*\mathcal{O}_Y(D)\subset k(Y)$, the function $c^*s\in k(X)$ belongs to $f_*\mathcal{O}_X(c^*D+E)$. Indeed, $(s)+D\ge 0$. Hence,

$$(c^*s)+c^*D+E\ge c^*(s)+c^*D=c^*((s)+D)\ge 0.$$

The surjectivity uses Lemma 3.22. For this, we decompose c into $h\circ g^{-1}$ as (3.1.1). We need to show that

$$c^*=(g^{-1})^*\circ h^*\colon i_*\mathcal{O}_Y(D)\to f_*\mathcal{O}_X(g_*(h^*(D))+E)$$

is surjective. Thus, take a nonzero section $s\in f_*\mathcal{O}_X(g_*(h^*(D))+E)$. Then $(s)+g_*(h^*(D))+E\ge 0$. Hence, $(g^*s)+h^*(D)+g^{-1}E\ge 0$ in any prime divisor E_i that is not exceptional on X. We claim that the horizontal part H of the latter divisor is 0. First, $H\ge 0$ because c is a 1-contraction. We can, of course, suppose that W is nonsingular, so that any $\mathbb{R}$-divisor is $\mathbb{R}$-Cartier. Finally, since $g^{-1}E$ is exceptional on Y, $H\equiv (g^*s)\equiv 0$ over the generic point of $Y=h(H)$. Hence, $H=0$ because it is $\equiv 0$ over the generic point of Y.

Thus, g^*s has no horizontal zeros or poles. Therefore, $g^*s = h^*s'$ for some rational function $s' \in k(Y)$. Moreover, $c^*s' = (g^{-1})^* \circ h^*s' = (g^{-1})^*g^*s = s$, and

$$h^*((s') + D) + g^{-1}E = (h^*s') + h^*(D) + g^{-1}E = (g^*s) + h^*(D) + g^{-1}E$$

is ≥ 0 in any prime divisor E_i that is not exceptional on X, in particular, in any prime E_i over a prime divisor $h(E_i)$ that is not exceptional on X and not in $g^{-1}E$. Since $g^{-1}E$ is very exceptional, such E_i exists over each prime divisor in Y. Hence, $h^*((s') + D) \geq 0$ by Lemma 3.22, $(s') + D \geq 0$, and $s' \in i_*\mathcal{O}_Y(D)$. $\square$

Example 3.23. Let $c\colon X \dashrightarrow X^+/T$ be a D-flip$/Z$, where X/T is a D-contraction$/Z$, that is, D is numerically negative$/T$. Then c induces a canonical isomorphism

$$c^*\colon f^+_*\mathcal{O}_Y(D^+) \to f_*\mathcal{O}_X(D),$$

where $D^+ = c(D)$, $f\colon X \to Z$, and $f^+\colon X^+ \to Z$ (see Example 3.21(a)). Indeed, $D = c^*D^+ + E$, where E is effective by Lemma 3.22. We can decompose c into birational contractions g and h as in Lemma 3.22. Then $g^*D = h^*D^+ + E_W$, where E_W is (very) exceptional on X^+ and $-E_W = h^*D^+ - g^*D$ is nef$/X^+$. Thus, the lemma applied to $W \to X^+$ gives $E_W \geq 0$. Hence, $E = g(E_W) \geq 0$, and the required isomorphism follows from Proposition 3.20.

Thus, c induces a canonical isomorphism of $\mathcal{O}_Z$-algebras

$$c^*\colon \mathcal{R}_{X^+/Z}D^+ \to \mathcal{R}_{X/Z}D.$$

In fact, we can replace the D-flip X^+/T by any small contraction (cf. Example 3.21(b)).

By the following result, Proposition 3.20 can also be applied in cases where $c^*D + E$ is replaced by a linearly equivalent divisor.

Lemma 3.24. *Let $D \sim D'$ be linearly equivalent divisors; that is, $D = D' + (s)$, where $s \in k(X)$ is a nonzero rational function. Then the multiplication map $t \mapsto st$ gives an isomorphism*

$$\sim\colon f_*\mathcal{O}_X(D) \to f_*\mathcal{O}_X(D')$$

and an isomorphism of $\mathcal{O}_Z$-algebras

$$\sim\colon \mathcal{R}_{X/Z}D \to \mathcal{R}_{X/Z}D'.$$

These isomorphisms are not unique and depend on the choice of s. But they are compatible with multiplication provided we take s^i for $iD \sim iD'$.

Proof. Immediate from the definitions. $\square$

Corollary 3.25. *Under the assumptions of Proposition* 3.20, *suppose that $D' \sim c^*D + E$. Then c and $\sim$ induce isomorphisms*

$$\sim c^*\colon i_*\mathcal{O}_Y(D) \to f_*\mathcal{O}_X(D') \qquad \text{and} \qquad \sim c^*\colon \mathcal{R}_{Y/Z}D \to \mathcal{R}_{X/Z}D'.$$

Proof. Immediate by Proposition 3.20 since c^* is compatible with multiplication. $\square$

However, we cannot replace $\sim$ by $\sim_{\mathbb{R}}$ in Lemma 3.24 and Corollary 3.25, or even by $\sim_{\mathbb{Q}}$. This last equivalence only gives a quasi-isomorphism of algebras (see Example 4.12). Together with an application of the Truncation Principle 4.6 (see Corollary 3.29), this is enough to prove Theorem 3.18. For this, we first clarify one definition.

Definition 3.26 (compare [45, Definition 2.5])**.** We say that D and D' are $\mathbb{Q}$-*linearly equivalent* if $D - D'$ is a $\mathbb{Q}$-*principal* divisor, that is, a rational linear combination of principal divisors. We write $D \sim_{\mathbb{Q}} D'$. Equivalently, $iD \sim iD'$ for some nonzero integer i.

Note that $\sim_*$ does not need$/Z$ (even if we have a base Z!). However, it does make a difference whether we consider $\sim_*$ on X/Z locally or globally: locally, the $*$-principal divisors are just the $*$-Cartier divisors (in the analytic case, in a neighborhood of a compact subset), whereas, globally, most of these are not $*$-principal.

Since every principal divisor is integral, the $\mathbb{R}$-vector space of $\mathbb{R}$-principal divisors is defined over $\mathbb{Q}$, and all its $\mathbb{Q}$-divisors are $\mathbb{Q}$-principal. Thus, two $\mathbb{Q}$-divisors D and D' are $\sim_{\mathbb{R}}$ if and only if they are $\sim_{\mathbb{Q}}$. Also, if $D \sim_{\mathbb{Q}} D'$ and D is a $\mathbb{Q}$-divisor, then so is D'.

Proposition 3.27. *Suppose that D is bss ample/Z and D^{sm} is a $\mathbb{Q}$-divisor* (*see* (3.8.1))*. Then there exist a $\mathbb{Q}$-divisor H on Y and a natural number I such that*

(i) *H is* (*numerically*) *ample/Z;*

(ii) *$D \sim_{\mathbb{Q}} c^*H + E$; and moreover,*

(iii) *$iD \sim ic^*H + iE$ if and only if $I \mid i$; equivalently,*

(iv) *for all $i \gg 0$, the ith component $\mathcal{R}_i$ of $\mathcal{R} = \mathcal{R}_{X/Z}D$ is nonzero if and only if $I \mid i$.*

The natural number I is unique, and H is unique up to $\sim$.

Addendum 3.27.1. *The divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}D$ is equal to its truncation*

$$\mathcal{R}^{[I]} = \sum_{I|i} \mathcal{R}_i$$

and isomorphic to the truncation

$$(\mathcal{R}_{Y/Z}H)^{[I]},$$

where $[I]$ means that we preserve degrees.

Note that a truncation of a divisorial algebra $\mathcal{R}_{X/Z}D$ is again divisorial: $(\mathcal{R}_{X/Z}D)^{[I]} = \mathcal{R}_{X/Z}(ID)$. The converse does not hold in general, even for subalgebras of a functional algebra; e.g., $(\mathcal{R}_{Y/Z}H)^{[I]}$ is usually not equal to $\mathcal{R}_{Y/Z}H$ for $I \geq 2$.

Lemma 3.28. *Let $f\colon X \to T$ be a contraction and suppose that D and E are $\mathbb{R}$-divisors on T and X, respectively, such that*

- *D is $\mathbb{R}$-Cartier,*
- *E is vertical, and*
- *$E \sim_{\mathbb{R}} f^*D$.*

Then there is a unique $\mathbb{R}$-divisor F on T such that

- *$F \sim_{\mathbb{R}} D$,*
- *in particular, F is also $\mathbb{R}$-Cartier, and*
- *$E = f^*F$.*

Proof. We can assume that $D = 0$. Indeed, if we replace E by $E - f^*D$ and D by 0, then the required F is $F - D \sim_{\mathbb{R}} 0$.

Thus, E is $\mathbb{R}$-principal and vertical. In other words, $E = \sum d_i(f_i)$, where all $d_i \in \mathbb{R}$ and all f_i are nonzero rational functions on X. We need to find a presentation in this form with functions $f_i = f^*g_i$, where the g_i are nonzero rational functions on T. Indeed, we can then take $F = \sum d_i(g_i)$.

Since E is vertical, a presentation of the required form is a presentation with $\mathbb{Q}$-linearly independent and nonzero (f_i) over the generic point of T, that is, in the horizontal components.

A presentation of this form exists by the following inductive procedure. Suppose that $\sum r_i(f_i) = 0$ over the generic point of T, where all $r_i \in \mathbb{Q}$ and one of them is nonzero, say, $r_0 \neq 0$. Then the same holds for some integral coefficients r_i. Hence, $(\prod f_i^{r_i}) = 0$ over the generic point of T and the rational function $g = \prod f_i^{r_i}$ does not have horizontal zeros or poles over the generic point of T, that is, 0 over the generic point of T. Thus, in our presentation, we can replace (f_0) by a rational linear combination of other divisors (f_i) and (g). But $(g) = 0$ over the generic point of T.

The uniqueness follows because f^* is injective since f is surjective. $\square$

Proof of Proposition 3.27. We first check that the extension $D_W^{\mathrm{sm}} = \mathcal{D}_W^{\mathrm{sm}}$ is also a $\mathbb{Q}$-divisor. Indeed, by definition $D_W^m = \mathcal{D}_W^m \sim_{\mathbb{R}} h^*H$. Thus, $D_W^m \sim_{\mathbb{R}} 0$ over the generic point of Y; again by definition, $(D_W^m)_X = (D_W^{\mathrm{sm}})_X = D^{\mathrm{sm}}$ outside $\operatorname{Supp} E$. On the other hand, over the generic point on Y, E is 0 and the irrational part of D_W^m is exceptional on X. It is unique over the generic point of Y by Addendum 3.4.2. Therefore, D_W^m is rational (a unique point given by rational linear equations, nonexceptional multiplicities on X, in a finite-dimensional rational linear space of divisors with given support and $\sim_{\mathbb{R}} 0$ over the generic point of Y) and $D_W^m \sim_{\mathbb{Q}} 0$ over the generic point of Y. In particular, D_W^m is vertical if D^m is.

Thus, over the generic point of any prime divisor in Y, the irrational part of D_W^m and $\operatorname{Supp} E$ is very exceptional on Y. By Lemma 3.22, it is uniquely determined by the linear equations with rational coefficients $D_W^m \cdot C_i = 0$ for general curves C_i of $E_i/h(E_i)$. Hence, it is rational and D_W^m is rational over such points.

Since $D_W^m \sim_{\mathbb{Q}} 0$ over the generic point of Y and E is vertical, we can suppose that the divisors D_W^m and D are themselves vertical/Y up to $\mathbb{Q}$-linear equivalence. By Lemma 3.28, we can replace H by an $\mathbb{R}$-linearly equivalent divisor such that $D_W^m = h^*H$ and it is also (numerically) ample/Z. Then H is also rational because h^* is $\mathbb{Q}$-linear. This is our choice of H. In particular, D_W^m and D_W^{sm} are also $\mathbb{Q}$-divisors.

Take I as the minimal natural number such that ID_W^m, or even ID_W^{sm} is vertical/Y up to $\sim$. The former vertical divisor is $ID_W^m = Ih^*H = h^*(IH)$. Then H and I satisfy the required conditions. Indeed, for any natural number i divisible by I, $iD + F \sim iD_W^m + iE = ih^*H + iE$ on W, where F is exceptional on X. This induces

$$iD \sim iD^m + iE = ic^*H + iE \qquad \text{on } X.$$

On the other hand, if this holds for any integer i, it holds for its nonnegative remainder r by I, and $r = 0$ by our choice of I since then $rD_W^m = rh^*H$ and rD_W^{sm} are vertical.

This implies that $\mathcal{R}_i = 0$ whenever $I \nmid i$. Thus, Addendum 3.27.1 follows by Corollary 3.25. In addition, $\mathcal{R}^{[I]}$ has a nonzero element in each $i \gg 0$ divisible by I because this holds for $\mathcal{R}_{Y/Z}H$. This gives Proposition 3.27(iv).

As a minimal natural number, I is unique. This implies the uniqueness of H up to linear equivalence $\sim$. $\square$

In Section 4, p. 119, we derive the following result from Truncation Principle 4.6:

Corollary 3.29 (cf. Stupid Example 3.16)**.** *Let H be an $\mathbb{R}$-divisor that is numerically ample/Z. Then H is a $\mathbb{Q}$-divisor if and only if its algebra $\mathcal{R}_{X/Z}H$ is f.g. Moreover, it is g.a.g. with the empty stable base locus, and with $E = 0$ whenever it is f.g.*

Proof of Theorem 3.18. Suppose that $\mathcal{R} = \mathcal{R}_{X/Z}D$ is f.g. and g.a.g. Then we can take $Y/Z \cong \operatorname{Proj}_Z \mathcal{R}$. It is well known that the correspondence c from X to Y/Z is then a rational 1-contraction by Lemma 3.19, and $D = D^m + D^e$, where, for some natural number $N \gg 0$,

$$D^m = \operatorname{Mov}(ND)/N, \qquad D^e = \operatorname{Fix}(ND)/N,$$

and

$$|ND| = |\operatorname{Mov}(ND)| + \operatorname{Fix}(ND)$$

is a decomposition of the linear system/Z into its *mobile* and *fixed* components. More precisely, this holds for any $N > 0$ for which the Nth component $\mathcal{R}_N = f_*\mathcal{O}_X(ND)$ generates the subalgebra

$$\mathcal{R}^N = \bigoplus_{i=0}^{\infty} \mathcal{R}_{iN}$$

(see Definition 4.3). G.a.g. implies that $E = D^e$ is exceptional on Y and very exceptional, again by Lemma 3.19. Indeed, in codimension 1, $\operatorname{Supp} E$ is exactly the locus where $\mathcal{R}_{X/X}ND$ is not generated by global sections. Then we apply Lemma 3.19 to a divisor $h^*c(D')$ for $D' \in |ND|$ on any model W/Z of X/Z that is regular/Y with a contraction $h\colon W \to Y/Z$. If $E = D^e$ is exceptional but not very exceptional on Y, there is an effective exceptional but not very exceptional $\mathbb{Q}$-divisor $E' \le E$. By Lemma 3.19, this contradicts f.g. The greatest property see in [34, 5.9, 5.11]. Thus, D is bss ample and D^{sm} is a $\mathbb{Q}$-divisor equal to D^m outside $\operatorname{Supp} D^e$.

Conversely, suppose that D is bss ample/Z and D^{sm} is a $\mathbb{Q}$-divisor. By Corollary 3.25, Proposition 3.27, and the description of its isomorphism, the divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}D = \mathcal{R}_{X/Z}(D-E)$ is isomorphic to its truncation $(\mathcal{R}_{Y/Z}H)^{[I]}$. On the other hand, the latter is f.g. by Corollary 3.29 because H is a $\mathbb{Q}$-divisor and is (numerically) ample/Z. Moreover, H is g.a.g. with the empty stable base locus on Y. Since E is an effective divisor on X and $\mathcal{R}_{X/Z}D = \mathcal{R}_{X/Z}(D-E)$, g.a.g. holds for D on X with exceptional stable base divisor E on X. □

Remark 3.30. By the proof of the theorem, every nontrivial f.g. algebra $\mathcal{R}_{X/Z}D$ gives a decomposition $D = D^m + E$, where D^m is b-semiample/Z, $E = D^e \ge 0$ is a stable base divisor, and $\mathcal{R}_{X/Z}D = \mathcal{R}_{X/Z}D^m$ (cf. Example 4.30). D^m is the *minimal divisor* with these properties and is, of course, a $\mathbb{Q}$-divisor.

It is not difficult to show that this is equivalent to a geometric decomposition $D = D^m + E$ for an $\mathbb{R}$-divisor D with $E \ge 0$ and with *maximal* b-semiample D^m, under the assumption that the maximal divisor D^m is a $\mathbb{Q}$-divisor. A decomposition of this form can be defined even if the maximum D^m is not a $\mathbb{Q}$-divisor; it is unique if it exists and can be considered as a *generalized Zariski decomposition*. Our decomposition differs from Zariski decompositions in the sense of Fujita or Cutkosky, Kawamata, and Moriwaki (see [25, Definitions 7-3-2 and 7-3-5]) because we replace b-nef by b-semiample. More precisely, our decomposition implies either of the latter whenever it exists. Details will appear in [34]. (Compare pseudo-decompositions in Example 4.30.)

In the case when D is bss ample/Z, E is very exceptional. However, in general, there is no numerical criterion for D^m to be maximal, e.g., for E to be exceptional or negative (cf. [43, Examples 1.1 and 1.2]). But this can be done in certain situations, namely, if

(1) D is big/Z (cf. Corollary 3.31 below), or
(2) (cf. Theorem 3.33 and Corollary 3.34 below) there is a boundary B on X such that $(X/Z, B)$ is a 0-*log pair*, that is,
  - X/Z is projective (actually, this is not necessary; see Theorem 3.33 below),
  - (X, B) is Klt, and
  - $K + B \equiv 0/Z$.

Moreover, we then expect the existence of the decompositions for all divisors $D \sim_{\mathbb{R}} D' \ge 0$ that are effective up to $\sim_{\mathbb{R}}$.

For example, (1) holds if X/Z is finite or birational; and, by the base point free theorem, (2) holds if X/Z has the structure of a weak log Fano contraction $(X/Z, B)$ with only Klt singularities [43, Conjecture 1.3, (WLF)]. Moreover, for weak log Fano contractions, we can replace D effective by D pseudo-effective, since one expects the cone of effective divisors to be polyhedral in this case (Batyrev [4]).

Corollary 3.31. *Suppose that D is big/Z. Then D is bss ample/Z and has a $\mathbb{Q}$-divisor D^{sm} if and only if $\mathcal{R}_{X/Z}D$ is f.g.*

In particular, the rationality of D^{sm} holds if D is rational.

Proof. Indeed, for each divisor D that is big/Z and with f.g. divisorial algebra $\mathcal{R} = \mathcal{R}_{X/Z}D$, the algebra $\mathcal{R}$ is big/Z (that is, $k(X)$ is finite/$k(\mathcal{R})$) and g.a.g. This follows by definition because the model Y/Z is birational to X/Z. Thus, we can omit the g.a.g. assumption in Theorem 3.18. □

The corollary applies, for example, to a birational contraction f.

Corollary 3.32. *Suppose that $f\colon X \to T$ is birational and D is a $\mathbb{Q}$-divisor on X. Then the D-flip exists if and only if $\mathcal{FR}_{X/T}D$ is f.g.*

If X/T is small, $\mathcal{FR}_{X/T}D = \mathcal{R}_{X/T}D$, so that the D-flip exists if and only if $\mathcal{R}_{X/T}D$ is f.g.

Proof. Immediate by Lemma 3.7, Example 3.15, and Corollary 3.31. Indeed, each divisor on X is big$/T$ since $f\colon X \to T$ is birational. $\square$

Theorem 3.33. *LMMP and the* log semiampleness (*see* [45, Conjecture 2.6]) *in dimension n imply the existence of both decompositions of Remark* 3.30(2) *in* $\dim X = n$, *for any divisor D that is effective up to $\sim_{\mathbb{R}}$. In particular, the algebra $\mathcal{R}_{X/Z}D$ is f.g. for any $\mathbb{Q}$-divisor D.*

Moreover, we can omit the log semiample assumption if D is big$/Z$ or D is b-semiample$/Z$. We can also omit the projectivity of X/Z.

Proof. This follows essentially because the LMMP is compatible with the decompositions. After a $\mathbb{Q}$-factorialization of X, we can assume that D is $\mathbb{R}$-Cartier.

By Corollary 3.5, we can also assume that D is effective and $(X, B + D)$ is still Klt. Moreover, the new D and D^{sm} are again $\mathbb{Q}$-divisors whenever the old D and D^{sm} are; and the old algebra $\mathcal{R}_{X/Z}D$ is a truncation of the new. Thus, it is enough to establish the theorem for effective D; the f.g. needs Truncation Principle 4.6.

For this, we apply LMMP to the pair $(X/Z, B + D)$. This is the same as the D-MMP [45, Section 5] since $K + B + D \equiv D/Z$ by Remark 3.30(2). Thus, if D is not nef, we have an (extremal) D-contraction $h\colon X \to Y/Z$ that is birational since D is effective. By LMMP, we can make a D-flip $c\colon X \dashrightarrow X^+/Z$, where $D = c^*D^+ + E$ with $E \geq 0$ by Lemma 3.22 (cf. Example 3.23). Lemma 3.22 also implies that the maximal D^m, if it exists, is $\leq c^*D^+$; and it exists and is equal to c^*D^{+m} if and only if it exists for D^+. By Example 3.23, the same holds for f.g. of the algebra $\mathcal{R}_{X/Z}D = \mathcal{R}_{X^+/Z}D^+$ and the rationality of D^{sm} because the divisorial $\operatorname{Supp} E$ is contractible on X^+.

By the termination in LMMP, we can suppose that D is nef. Then it is semiample by [45, Conjecture 2.6], and we have its bss decomposition by Corollary 3.8 with $D^m = D$ and $E = 0$. Thus, if $D^{\mathrm{sm}} = D$ is a $\mathbb{Q}$-divisor, then $\mathcal{R}_{X/Z}D$ is f.g. by Theorem 3.18. If D is big$/Z$ or b-semiample$/Z$, the log semiampleness for a $\mathbb{Q}$-divisor D [45, Conjecture 2.6] follows from [40] and [25, Remark 6-1-15, (3)] (by this remark, it is enough that $D \geq M$ where $M \neq 0$ is b-semiample), and then, for an $\mathbb{R}$-divisor D, from [45, Remark 6.23.5].

If we do not assume that X/Z is projective, we can establish bss ampleness starting from any projective log resolution of $(X/Z, B)$. Then $c\colon X \dashrightarrow Y/Z$ is the log canonical rational morphism, and H is defined by an Iitaka contraction from a log minimal model $(X_{\min}/Z, B_{\min})$, that is, $\mathcal{D}^m = \overline{K_{X_{\min}} + B_{\min}}$.

Finally, note that if D is not effective up to $\sim_{\mathbb{R}}$, then $\mathcal{R}_{X/Z}D = f_*\mathcal{O}_X$ or $\mathcal{O}_Z$ if $f\colon X \to Z$ is a contraction, is trivial and f.g., but not g.a.g. $\square$

Corollary 3.34. *In dimension $n \leq 3$, we can omit LMMP and log semiampleness as assumptions in Theorem* 3.33.

Proof. Immediate by Theorem 3.33 and [45, Theorems 2.7 and 5.2]. For a $\mathbb{Q}$-divisor D, log semiampleness is essentially due to Kawamata and Miyaoka (see [26]). $\square$

Thus, we expect, in particular, the following:

Conjecture 3.35. *Let $f\colon X \to X_\vee$ be a pl contraction with respect to $S = \sum S_i$. Then*

$(\mathrm{PLF})_n$ *the divisorial algebra $\mathcal{R}_{X/X_\vee}S$ is f.g.*

We denote this statement by $(\mathrm{PLF})_{n,d}$ *if f has core dimension $d = n - s$.*

On the other hand,

Corollary 3.36. $(\mathrm{PLF})_n$ *implies Induction Theorem* 1.4, $(\mathrm{PLF})_n^{\mathrm{small}}$, *and the existence of elementary pl flips in dimension n.*

Proof. Immediate by Corollary 3.32. □

In turn, we have the following result.

Theorem 3.37. $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *of Conjecture* 3.48 *implies* $(\mathrm{PLF})_{n,d}$ *of Conjecture* 3.35.

This is the main result of this section. We explain and prove it below, but we start with the following:

Proof of (RFA) assertion of Induction Theorem 1.4. Immediate by Corollary 3.36 and Theorem 3.37. □

Example 3.38. Let X be a projective variety with an ample reduced effective divisor S and H be an $\mathbb{R}$-divisor such that $S \sim rH$ for some natural number r and the divisorial $\operatorname{Supp} H \cap \operatorname{Supp} S = \varnothing$. Then it is known that H is a $\mathbb{Q}$-divisor and $\mathcal{R}_X H$ is f.g. (cf. Corollary 3.29). However, the essential ingredient here is not just Truncation Principle 4.6 but rather an induction on $n = \dim X$ as follows: restricting rational functions without poles on S to S defines a canonical homomorphism of graded k-algebras

$$\cdot_{|S} \colon \mathcal{R}_X H \to \mathcal{R}_S H_{|S}.$$

Moreover, to prove that $\mathcal{R}_X H$ is f.g., it is enough to prove that the image subalgebra $\mathcal{R} = \mathcal{R}_X H_{|S} \subset \mathcal{R}_S H_{|S}$ is f.g. (use Main Lemma 3.43 below).

On the other hand, by Serre vanishing, the truncation $\mathcal{R}^{[I]}$ of $\mathcal{R}$ is equal to that of $\mathcal{R}_S H_{|S}$. Thus, by Truncation Principle 4.6 it is enough to verify that $\mathcal{R}_S H_{|S}$ is f.g. This gives an induction using the base point free theorem for multiples of $H_{|S}$.

We use the same idea in the diametrically opposed situation, when $-S$ can be ample. However, as for pl contractions, this is only possible in the local case if X/Z is generically finite or birational. For a more general situation than the above example, in Main Lemma 3.43 below, the f.g. of $\mathcal{R}_{X/Z} D$ reduces to that of its restriction $\mathcal{R} = \mathcal{R}_{X/Z} D_{|S}$. It is a subalgebra of the divisorial algebra $\mathcal{R}_{S/Z} D_{|S}$. Truncation Principle 4.6 applies again, but now Serre vanishing does not work. Replacing it by Kawamata–Viehweg vanishing is a natural guess; the idea is right, but, rather than to divisorial algebras up to truncation, it leads us to the (FGA) algebras of Section 4.

Definition 3.39. A *normal ladder* (respectively, *normal irreducible ladder*) $\{S^i\}$ in X is a chain $S^0 = X \supset S^1 \supset S^2 \supset \ldots \supset S^s$, where

- each S^i is a normal (irreducible) proper subvariety of pure codimension i in X, and
- for all $j \le i$, the generic points of S^i are nonsingular points of S^j.

It would be sufficient in applications to treat only irreducible ladders, but we prove Main Lemma 3.43 without making this assumption.

Let D be an $\mathbb{R}$-Cartier divisor of X. A sequence $S_1, \ldots, S_s$ of reduced Weil divisors is *inductive* with respect to D if

- the subvarieties $S^i = S_1 \cap \ldots \cap S_i$ form a normal (irreducible) ladder with normal crossing at the generic points of each S^i;
- $\operatorname{Supp} D$ does not contain any components of S^i; and
- each $S_i \sim r_i D$ for some natural number $r_i \ge 1$.

The final condition means that, for each $i = 1, \ldots, s$, there exists a rational function $0 \ne t_i \in k(X)$ such that $(t_i) = S_i - r_i D$. In particular, the $r_i D$ must be integral divisors. Thus, D itself is a $\mathbb{Q}$-Cartier divisor, possibly not integral, and each S_i is also $\mathbb{Q}$-Cartier.

Note also that $t_i \in \mathcal{O}_X(r_i D)$ since $(t_i) + r_i D = S_i > 0$, so that $t_i \in \mathcal{R}_X D$ has degree r_i. Moreover, the rational function $t_{i|S^j} \in k(S^j)$ must be $\ne 0$ on each component of S^j for all $s \ge i > j \ge 0$

because, otherwise, a component of S^j lies in S_i, contradicting the assumption on a normal ladder; it gives the rational equivalence $S_{i|S^j} \sim r_i D_{|S^j}$. We call t_i a *translation function* for S_i of *degree* r_i.

Example 3.40. Let $(X/T, S+B)$ be a log pair such that

- X/T is a (birational) contraction;
- $S = \sum S_i$ is a sum of $s \geq 1$ prime Weil $\mathbb{Q}$-Cartier divisors $S_1, \dots, S_s$;
- the S_i with $t \leq i \leq s$ are *linearly proportional* to an integral divisor D, that is, each $S_i \sim r_i D$ for some natural number $r_i \geq 1$ (cf. 1.1(1) for pl contractions);
- $K+S+B$ is divisorially log terminal with $\lfloor B \rfloor = 0$; and
- $K+S+B$ is numerically negative/T.

(Here $t \geq 1$ is given: see Definition 3.47 for an application with $t=1$, and flips of type $(S{+}{-})$ in Section 11, p. 212 for one with $s=t=2$.)

This is a slight twist on the definition of pl contraction, which we use as follows: the restrictions $S_{i|S^{t-1}}$ for $t \leq i \leq s$ form an inductive sequence with respect to a divisor $D'_{|S^{t-1}}$, where $D' \sim D$. Moreover, the normal ladder S^j for $1 \leq j \leq s$ is *irreducible* and X/T induces a contraction $S^j/f(S^j)$ for each j. This follows by induction on s.

By [41, следствие 3.8], $S^1 = S_1$ is normal and irreducible/T, even formally so (that is, single branched). The fact that $K+S+B$ is numerically negative and the other assumptions imply that $K+S^1+B'$ is numerically negative for a boundary $S^1+B' \leq S+B$ with $\lfloor B' \rfloor = 0$. In other words, (X, S^1+B') is purely log terminal with the reduced divisor S^1 in the boundary. Then any linear system that defines the contraction X/T restricts surjectively to S^1 (cf. the proof of Lemma 3.6 in [41]) and, thus, induces a contraction $S^1/f(S^1)$; and $P \in f(S^1)$ since S_1 intersects the fibre $f^{-1}P$.

Each S_i with $i \geq 2$ gives a prime Weil divisor $S_{i|S^1} = S_i \cap S_1$ in S^1. Indeed, by [41, следствие 3.8] again, this last intersection is normal as a subvariety and transverse at generic points. In other words, it gives a normal ladder $S^1 \supset S_1 \cap S_i$ and $S_{i|S^1} = \sum T_j$ is a sum of prime Weil divisors. In fact, we have a single T_i. First, some T_j exists by Connectedness of LCS (Kollár and others [27, Theorem 17.4]) for $K+S+B-\varepsilon S'$, where $0 \ll \varepsilon < 1$ and $S' = \sum_{j \neq 1,i} S_j$. In other words, S_1 intersects any other S_i in $f^{-1}P$ since each S_i intersects $f^{-1}P$. Second, there is only one T_i for the same reasons; namely, $\bigcup T_i$ is a normal subvariety with disjoint components T_j and, by adjunction [41, (3.2.3)], $(K+S+B-\varepsilon S')_{|S^1} = K_{S^1} + B'' + \sum T_j$ is divisorially log terminal with $\lfloor B'' \rfloor = 0$, where the LCS is $\bigcup T_j$. Finally, each restriction $T_i = S_{i|S^1}$ is $\mathbb{Q}$-Cartier.

Thus, by adjunction, we preserve the situation for $T_i = S_{i|S^1}$ for $s \geq 2$ and for an appropriate choice of D. Since D is integral, the generic point of S^s is nonsingular in X and X/T is projective, and by the Moving Lemma we can assume that, up to linear equivalence, $\operatorname{Supp} D$ does not contain S^s or any other S^i with $i \geq 2$. The new contraction $S^j/f(S^j)$ may not be birational even if X/T is (cf. Example 3.45).

Note also that the definition of pl contractions does not always give the above situation since the S_i do not necessarily satisfy the linear proportionality. Indeed, the proportionality 1.1(1) only implies that there exists an integral divisor D such that each $S_i \sim_{\mathbb{Q}} r_i D$ for a natural number r_i. Fortunately, we can upgrade this to $\sim$ using the covering trick (cf. Lemma 3.51 and the proof of Theorem 3.37 below).

To generalize Example 3.38, we need to consider restrictions of divisorial algebras.

Definition 3.41. Let $Y \subset X$ be a subvariety and D be an $\mathbb{R}$-Cartier divisor such that

- $\operatorname{Supp} D$ does not contain the generic points of Y.

Then restricting functions to Y induces a *restriction* of divisorial algebras

$$\cdot_{|Y} \colon \mathcal{R}_{X/Z} D \to \mathcal{R}_{Y/Z} D_{|Y}$$

(see Remark 3.42 for the restriction of divisors). This map is an $\mathcal{O}_Z$-homomorphism of $\mathbb{N}$-graded algebras. In particular, it induces a restriction homomorphism $\cdot_{|Y}\colon \mathcal{L} \to \mathcal{R}_{Y/Z}D_{|Y}$ of any $\mathbb{N}$-graded $\mathcal{O}_Z$-subalgebra $\mathcal{L} \subset \mathcal{R}_{X/Z}D$ (functional in the terminology of Definition 4.1). Note that the image $\mathcal{L}_{|Y}$ of a sheaf subalgebra $\mathcal{L}$ is also a sheaf $\mathcal{O}_Z$- or $\mathcal{O}_{f(Y)}$-algebra which is locally generated by the restrictions of some sections from $\mathcal{L}$. In particular, the restriction $\mathcal{L}_{|Y}$ of the coherent $\mathcal{O}_Z$-subalgebra $\mathcal{L}$ is again coherent.

Remark 3.42. The above restriction is defined provided that functions in $\mathcal{R}_{X/Z}D$ have no poles at generic points of Y, that is, the support of positive components of D does not contain any generic point of Y. The restriction can, however, be zero; and this may happen even under our assumption on D. The important point in the definition is that if $D_{|Y}$ is defined, then each restricted nonzero function $f_{|Y}$ for $f \in \mathcal{R}_{X/Z}D$ belongs to a divisorial algebra in the same degree, so that its zeros and poles are controllable.

Restriction naturally generalizes to D that are not $\mathbb{R}$-Cartier if we assume that

(1) we restrict $\mathcal{L}$ either as a subalgebra for a divisorial algebra of $\mathbb{R}$-Cartier $D' \geq D$; or

(2) Y is the final step S^s of a normal ladder (or $_{|Y^\nu}$ on the normalization Y^ν for a nonnormal ladder).

See [41, §3] for restricting Weil $\mathbb{R}$-divisors to a normal divisor, and thus, at the same time, to its normalization. Note that a $\mathbb{Q}$-divisor restricts to a $\mathbb{Q}$-divisor, but an integral divisor may not restrict to a $\mathbb{Z}$-divisor (cf. [41, предложение 3.9]). However, an integral divisor restricts to an integral divisor along any nonsingular locus of X (the ambient space; cf. D in Definition 3.39 and Lemma 6.29 below). Inequalities between divisors such as $D_1 \geq D_2$ are preserved, and, in particular, an effective divisor remains effective. (This follows from Negativity 1.1 in [41].) This implies an inclusion $\mathcal{R}_{X/Z}D_{2|Y} \subset \mathcal{R}_{X/Z}D_{1|Y}$.

Main Lemma 3.43.

(SDA) *Let $\mathcal{L} \subset \mathcal{R}_{X/Z}D$ be a coherent $\mathcal{O}_Z$-subalgebra of a divisorial algebra* (*see Definition* 4.1);

(IND) *let $\{S_i \mid 1 \leq i \leq s\}$ be an inductive sequence with respect to D with translations t_i satisfying*

(TRL) *$t_i \in \mathcal{L}_{r_i}$ for each i and*

$$t_i^{-1}\colon \mathcal{L}_N(-S_i) \to \mathcal{L}_{N-r_i} \qquad \textit{for every } N \geq r_i;$$

that is, $at_i^{-1} \in \mathcal{L}_{N-r_i}$ for a function $a \in \mathcal{L}_N(-S_i)$, where

$$\mathcal{L}_j = \mathcal{L} \cap f_*\mathcal{O}_X(jD)$$

is the jth component of $\mathcal{L}$, and

$$\mathcal{L}_N(-S_i) = \mathcal{L}_N \cap f_*\mathcal{O}_X(ND - S_i) = \{a \in \mathcal{L}_N \mid (a) + ND - S_i \geq 0\}.$$

Then, for $Y = S^s$, the algebra $\mathcal{L}$ is f.g. near $f(Y)$ if and only if its restriction $\mathcal{L}_{|Y}$ is.

Addendum 3.43.1. *The required translations t_i exist if $\mathcal{L} = \mathcal{R}_{X/Z}D$ is divisorial.*

Proof. We only need to check that $\mathcal{L}$ is of finite type if $\mathcal{L}_{|Y}$ is, and, by induction on s, we only need to consider the divisorial case, that is, $s = 1$ and $Y = S = S_1 = S^1$ with a single translation $t = t_1$ of degree $r = r_1 \geq 1$. Indeed, for $s \geq 2$, the restrictions

$$\mathcal{L}_{|S^1}, \quad \{S_{i|S^1} \mid 2 \leq i \leq s\}, \quad D_{|S^1}, \quad \text{and} \quad t_{i|S^1} \text{ for } 2 \leq i \leq s$$

again satisfy the assumptions of the lemma (with the same r_i). Thus, $\mathcal{L}_{|S^1}$ is f.g. by induction.

The problem is local. Thus, we fix a point $P \in f(S) \subset Z$ and check that $\mathcal{L}$ is f.g. near P. Since the algebra $\mathcal{L}_{|S}$ has finite type near P, it has a system of homogeneous generators $s_1, \dots, s_m$. Recall that a homogeneous element of a graded algebra is an element in a homogeneous piece; in our case, $(\mathcal{L}_{|S})_j = \mathcal{L}_{j|S}$. The sections can be presented as restrictions $s_i = t_{i|S}$, where $t_i \in \mathcal{L}_j$. We claim that t, the sections t_i, and the generators of the $\mathcal{O}_Z$-modules $\mathcal{L}_j$ for $j < r$ generate $\mathcal{L}$ near P. We can take this last set to be finite since each $\mathcal{L}_j$ is coherent.

Indeed, for any homogeneous $a \in \mathcal{L}$ of degree $N \geq r$, we can find a polynomial $p(x_1, \dots, x_m) \in \mathcal{O}_{f(S),P}[x_1, \dots, x_m]$ such that $a_{|S} = p(s_1, \dots, s_m)$ and p is homogeneous, that is, all its monomials are homogeneous of weighted degree N, where $\deg x_i = \deg s_i$. The coefficients of p are in the local ring $\mathcal{O}_{f(S),P}$ of $P \in f(S)$. Thus, p can also be obtained as the restriction $p = q_{|S}$ of a similar polynomial $q(x_1, \dots, x_m) \in \mathcal{O}_{Z,P}[x_1, \dots, x_m]$ with coefficients in the local ring $\mathcal{O}_{Z,P}$. Then $a - q(t_1, \dots, t_m)$ vanishes on S and belongs to $\mathcal{L}_N(-S)$. This last conclusion holds because $S \not\subset \operatorname{Supp} D$. Hence, by (TRL), $b = t^{-1}(a - q(t_1, \dots, t_m)) \in \mathcal{L}_{N-r}$, and $a = q(t_1, \dots, t_m) + tb$, where b is homogeneous of degree $(N - r) < N$. Induction on N completes the proof.

For Addendum 3.43.1, we need to verify (TRL) for the divisorial algebra $\mathcal{L} = \mathcal{R}_{X/Z} D$ with any translations t_i. By Definition 3.39, $t_i \in \mathcal{R}_{X/Z} D$ is homogeneous of degree r_i. Moreover,

$$(a) + ND - S_i \geq 0 \qquad \text{for any } a \in \mathcal{L}_N(-S_i) = f_* \mathcal{O}_X(ND - S_i).$$

Thus, $a t_i^{-1}$ belongs to $\mathcal{L}_{N-r_i} = f_* \mathcal{O}_X((N - r_i)D)$:

$$(a t_i^{-1}) + (N - r_i)D = (a) - (t_i) + (N - r_i)D = (a) - S_i + r_i D + (N - r_i)D = (a) - S_i + ND \geq 0. \qquad \square$$

Remark 3.44. We have proved more, namely, that the kernel $\mathcal{I}$ of the restriction is f.g. as a subring (without 1) with (some *external*) generators from $\mathcal{L}$; if $1 \in \mathcal{I}$, all the generators lie in $\mathcal{I} = \mathcal{L}$ (cf. the next Example 3.45). Of course, if $A \twoheadrightarrow B$ is a surjection of algebras and its kernel is f.g. in this sense, then A is f.g. if and only if B is.

Example 3.45. Let $g\colon Y = S^s \to g(Y)$ be a contraction of fibre type, and suppose that D is numerically negative on general curves of its generic fibres. Then $\mathcal{L}_{|Y} = \mathcal{O}_{f(Y)}$ is trivial and needs 0 generators. Thus, $\mathcal{L}$ is f.g./Z. More precisely, it is generated by the translations t_i and generators of $\mathcal{L}_j$ with $j \leq \max\{r_i\}$.

If we replace condition (7) (see Subsection 1.1) for elementary flips by its opposite, that is, by the condition

- f is divisorial,

or, more generally, if we consider a divisorial pl contraction with S numerically negative/$X_\vee$, then $S = S_1 = S^1$ and $\mathcal{R}_{X/X_\vee} S = \mathcal{R}_{X/X_\vee} 0$ is f.g., that is, $(\mathrm{PLF})_{n,n-1}$ holds in this situation. This is, of course, well known. But from our point of view, $S \sim D$ with $S \not\subset \operatorname{Supp} D$ and D is numerically negative on $S/X_\vee$. Hence, $\mathcal{R}_{X/X_\vee} S$ is isomorphic to $\mathcal{L} = \mathcal{R}_{X/X_\vee} D$, which is generated by t with $(t) = S - D$ and $1 \in \mathcal{L}_0 = \mathcal{O}_{X_\vee}$.

Of course, Main Lemma 3.43 only proves f.g. near $f(S)$, but it holds outside $f(S)$ because $\mathcal{R}_{X/X_\vee} S = \mathcal{R}_{X/X_\vee} 0$ there. In general, *the lemma is entirely sufficient for local purposes* since $P \in f(S)$, as we assume throughout what follows.

Corollary 3.46. *Under the assumptions of Example* 3.40, *there exists $D' \sim D$ such that the algebra $\mathcal{L} = (\mathcal{R}_{X/T} D')_{|S^{t-1}}$ is f.g. if and only if $\mathcal{L}_{|S^s}$ is.*

Proof. Immediate by Main Lemma 3.43 and Example 3.40 if we take D such that $S^s \not\subset \operatorname{Supp} D'$. The restricted algebra $\mathcal{L}$ is an $\mathcal{O}_{f(S^{t-1})}$-subalgebra of $\mathcal{R}_{S^{t-1}/f(S^{t-1})} D'_{|S^{t-1}}$. The condition (TRL) on S^{t-1} is induced by the same condition on X by Addendum 3.43.1. $\square$

Definition 3.47. Under the assumptions of Example 3.40, suppose that X/T is birational, $t = 1$, and $S^s \not\subset \operatorname{Supp} D$. Then we say that the *restricted algebra* $\mathcal{L} = (\mathcal{R}_{X/T} D)_{|Y}$ is of type

$(\mathrm{RFA})_{n,d}$, where $Y = S^s$, $n = \dim X$, and $d = n - s = \dim Y$. If $Y/f(Y)$ is again birational, we say that $\mathcal{L}$ is of type $(\mathrm{RFA})_{n,d}(\mathrm{bir})$.

Conjecture 3.48.

$(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *Every algebra of type* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *is f.g.*

Corollary 3.49. $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *of Conjecture* 3.48 *implies*

$(\mathrm{RFA})_{n,d}$ *every algebra of type* $(\mathrm{RFA})_{n,d}$ *is f.g.*

Proof. In other words, $(\mathcal{R}_{X/T}D)_{|Y}$ is still f.g., even if $Y/f(Y)$ is not birational. Indeed, since X/T is birational but $Y = S^s/f(Y) = f(S^s)$ is not, it follows that, for some $1 \le i < s$, $S^i/f(S^i)$ is still birational but $E = S^{i+1}/f(S^{i+1})$ is not. Therefore, E is a Weil divisor of S^i and is exceptional on $f(S^{i+1})$. By our assumptions, it is $\mathbb{Q}$-Cartier and $\sim r_{i+1}D$. Thus, D is numerically negative on general curves of the generic fibre of $S^{i+1}/f(S^{i+1})$. Therefore, the algebras $(\mathcal{R}_{X/T}D)_{|S^{i+1}}$ and $(\mathcal{R}_{X/T}D)_{|Y}$ are trivial and f.g. by Example 3.45. □

3.50. The covering trick. Further applications need the covering trick.

Lemma 3.51. *Let* $\pi\colon \widetilde{X} \to X$ *be a finite cover that is etale in codimension* 1.

(1) *The pair* $(\widetilde{X}, \widetilde{D} = \pi^{-1}D)$ *is divisorially log terminal if* (X, D) *is*; *and* $(\widetilde{X}, \widetilde{D} = \pi^{-1}D)$ *is log canonical, respectively, Klt, or purely log terminal if and only if* (X, D) *is*;

(2) $\pi^{-1}D = \pi^*D$ *is* $\mathbb{Q}$*-Cartier* (*or* $\mathbb{R}$*-Cartier*) *if and only if* D *is*;

(3) $\pi^{-1}D \sim_{\mathbb{R}} \pi^{-1}D'$ *if and only if the same holds for* D *and* D';

(4) $\pi^{-1}D = \pi^*D$ *is bss ample/*Z (*or b-semiample, semiample, ample*) *if and only if* D/Z *is*; *and*

(5) *a pl contraction* $f\colon X \to X_\vee$ *induces a pl contraction* $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}_\vee$ *with* $\widetilde{S} = \pi^{-1}S$ *and finite* $\widetilde{X}_\vee/X_\vee$; *the same holds for* $f\colon X \to T$ *of Example* 3.40; *in addition,* f *has a pl flip or even a* D*-flip for any contraction* f *if and only if the same holds for* $\widetilde{f}$.

Proof. (1) is immediate from the pullback formula [41, 2.1] (cf. the proof of Corollary 2.2 in [41]). For the divisorial log terminality (which means that the log discrepancies are only 0 over normal crossing intersections of $\lfloor B \rfloor$), note that π is etale over the nonsingular points and preserves the nonsingularity of the log canonical centers and normal crossings of $\lfloor B \rfloor$ in their generic points. The nonsingularity in codimension 1 of irreducible components of $\lfloor \widetilde{B} \rfloor$ follows from [41, лемма 3.6, следствие 2.2].

For (2) and (3), see the proof of Corollary 2.2 in [41]. (4) is immediate by Definition 3.3 and the uniqueness of Proposition 3.4.

(5) follows by the connectedness arguments of Example 3.40 and again by [41, следствие 2.2]. They imply that $\widetilde{S}_i = \pi^{-1}S_i$ are again prime. The rational numbers $r_{i,j}$ and natural numbers r_i are the same on $\widetilde{X}$. The final statement on pl and D-flips follows from (4) (cf. [41, лемма 2.5]). □

Proof of Theorem 3.37. Let $f\colon X \to X_\vee$ be a pl contraction with respect to S. By Corollary 3.31, $\mathcal{R}_{X/X_\vee}S$ is f.g. if and only if S is bss ample. Thus, by Lemma 3.51, we can replace X by any finite cover that is etale in codimension 1.

On the other hand, by condition 1.1(1), the divisors S_i generate an Abelian subgroup of rank 1 in the group of integral Weil divisors up to $\sim_{\mathbb{Q}}$. It is torsion free. Hence, there is a generator of this subgroup that is the class up to $\sim_{\mathbb{Q}}$ of an integral Weil divisor D. In other words, each $S_i \sim_{\mathbb{Q}} r_iD$ for some natural number r_i; and $S \sim_{\mathbb{Q}} (\sum r_i)D$. By Corollary 3.5, $\mathcal{R}_{X/X_\vee}S$ is f.g. if and only if D is bss ample and, hence, if and only if $\mathcal{R}_{X/X_\vee}D$ is f.g.

After a finite cover that is etale in codimension 1, we can assume that each $S_i \sim r_iD$ for the same natural number r_i; and $S \sim (\sum r_i)D$. Indeed, if $S_i \sim_{\mathbb{Q}} r_iD$, then it defines a cyclic cover π

that is etale in codimension 1 such that $\pi^{-1}S_i \sim r_i\pi^{-1}D$ [41, конструкция 2.3]. Then we use induction on i. In addition, we can choose D up to $\sim$ such that $\operatorname{Supp} D$ does not contain all S^j for $j \geq 1$. We are now in the situation of Definition 3.47, and, by Main Lemma 3.43, $\mathcal{R} = \mathcal{R}_{X/X_\vee}D$ is f.g. if $\mathcal{L} = \mathcal{L}_{|S^s}$ is. This last algebra is of type $(\mathrm{RFA})_{n,d}$. Hence, if $S^s/f(S^s)$ is birational, then $\mathcal{L}$ is f.g. by our assumption. Otherwise, we use Corollary 3.49. $\square$

Corollary 3.52. $(\mathrm{PLF})_n$ *holds for a pl contraction* $X/X_\vee$ *if* $Y = S^i = E$ *or* $Y \subset f^{-1}P$, *where* E *is the exceptional locus for* $X/X_\vee$ *and* $f^{-1}P$ *is the fibre of* $X/X_\vee$ *over* P.

Proof. Immediate by the proof of Theorem 3.37 because $Y = S^s$ is a point whenever it is birational/$f(Y)$. But if Y is a point, any subalgebra of $\mathcal{R}_{Y/P}D = k_\bullet$ is f.g./k (and is quasi-isomorphic to a divisorial algebra whenever it is nontrivial). $\square$

For example, the latter case in the proof applies to Example 3.38 even if f is not a pl contraction and is not birational.

However, even if Y is a curve, a subalgebra $\mathcal{L} \subset \mathcal{R}_{Y/f(Y)}D$ need not be divisorial in general (cf. (1) and (2) in Example 4.18). Thus, further investigations are needed to specify restricted subalgebras. We do this in Section 4.

Example 3.53 (cf. flips of type (6.6.2) in [41])**.** Let $X/X_\vee$ be a pl contraction with respect to S such that

- S is numerically negative/$X_\vee$.

For example, this holds for an elementary pl contraction. Then $X/P \subset E \subset Y = S^s$, where $X/P = f^{-1}P$ is the fibre/P and E denotes the exceptional locus of f.

Indeed, $S_i \cdot C < 0$ for any curve C/P and for each S_i. Hence, if such a curve C/P exists, then $C \subset Y$, and $s \leq n-1$, where $n = \dim X$. More precisely,

(DPT) $s \leq n - \dim E \leq n - \dim X/P$. Moreover, $s = n - \dim E$ holds only if $E = Y$; and $\dim E = \dim X/P$ holds only if $E = X/P$.

Thus, if $E \neq \varnothing$ and $s = n - \dim E \leq n-1$, then $\mathcal{R}_{X/X_\vee}S$ is f.g. by Corollary 3.52. In particular, elementary pl flips exist under this assumption. For example, this holds if $s = n-1$ and f is not an isomorphism.

Another trivial case is when $f\colon X \to T$ is an *isomorphism*. Then any divisorial algebra $\mathcal{R}_{X/T}D$ is f.g. for a $\mathbb{Q}$-Cartier divisor D by Corollary 3.29. Our restriction arguments apply again to this case in the last Example 3.53 because the restrictions are surjective.

Example 3.54. The same arguments apply to any toric contraction $(X/T, S)$ provided that

- X is $\mathbb{Q}$-factorial, and
- X/T is birational and extremal, that is, $\rho(X/T) = 1$;

$S = \sum S_i$ denotes the invariant divisor, where the S_i are the invariant prime divisors and $1 \leq i \leq \dim X + 1 = n+1$ (cf. Theorem 6.4 and the conjecture after it in [43]). We order the S_i so that those with $1 \leq i \leq s$ are the only divisors that are numerically negative/T. It is known that the exceptional locus E of X/T is then also invariant and, thus, is of the form $E = \bigcap_{1\leq i\leq s} S_i$ with $s \leq n-1$, and one of the other divisors, say S_n, is numerically positive/T. Unfortunately, the contraction may not be pl with respect to $S^- = \sum_{1\leq i\leq s} S_i$ since the pair (X, S^-) may not be divisorially log terminal, having singularities along log centers. However, we can eliminate all these after a finite covering (even in the toric category) using the positive S_n and get a pl contraction. Then Example 3.53 implies that S^- is bss ample or, equivalently, that $\mathcal{R}_{X/T}S^-$ is f.g. This gives also a construction of toric flips (more general than in [39]), or it would be better to say flops. Finally, the termination of toric log flips implies the existence of all toric log flips.

This is the best we can do using divisorial algebras.

4. FINITELY GENERATED pbd ALGEBRAS

This section focuses on $\mathbb{N}$-graded $\mathcal{O}_Z$-algebras over a normal algebraic variety Z. We again consider the local case near $P \in Z$.

Definition 4.1. We say that an $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra $\mathcal{L} = \bigoplus_{i\geq 0} \mathcal{L}_i \subset k(X)_\bullet$ is a (coherent) *functional algebra* if each homogeneous piece $\mathcal{L}_i$ of degree i is a coherent $\mathcal{O}_Z$-submodule $\mathcal{L}_i \subset k(X)$, where X/Z is a proper morphism. We always assume that $\mathcal{L}_0 = f_*\mathcal{O}_X$ or $\mathcal{O}_Z$ if $f : X \to Z$ is a contraction.

In a slightly more abstract form, we have:

Proposition 4.2. *For any functional algebra $\mathcal{L}$,*

- *$\mathcal{L}$ is commutative;*
- *$\mathcal{L}$ is* integral, *that is, has no zero divisors;*
- *the homogeneous field of fractions of $\mathcal{L}$ has finite type over $k(Z)$; and*
- *each $\mathcal{L}_i$ is a coherent $\mathcal{O}_Z$-module, and $\mathcal{L}_0 = f_*\mathcal{O}_X$ is the integral closure of $\mathcal{O}_Z$ in the homogeneous field; thus, $\mathcal{L}_0 = \mathcal{O}_Z$ if $k(Z)$ is algebraically closed in the homogeneous field.*

Conversely, each $\mathbb{N}$-graded $\mathcal{O}_Z$-algebra with these properties is a functional algebra.

Proof–Explanation. The homogeneous field of fractions of an integral graded algebra $\mathcal{L}$ is a field L such that $\mathcal{L}$ is a graded subalgebra of $L_\bullet$; each element (function) of L is a fraction l/l', where $l, l' \in \mathcal{L}_i$ have the same degree i and $l' \neq 0$. For a functional algebra $\mathcal{L}$, it is a subfield of $k(X)$ and is of finite type over $k(Z)$. Thus, the properties of $\mathcal{L}$ are immediate from the definition.

Conversely, every integral graded algebra has a homogeneous field of fractions L. It has an inclusion $\mathcal{L} \hookrightarrow L_\bullet$ given by fractions:

$$\mathcal{L}_i \hookrightarrow L \quad \text{defined by} \quad l_i \mapsto \frac{l_i}{t^{i/d}} = \frac{l_i l_m^{i/d}}{l_n^{i/d}},$$

where, for a nontrivial algebra $\mathcal{L}$, $d \,|\, i$, $d = \gcd\{i \mid \mathcal{L}_i \neq 0\}$, and $t = l_n/l_m$ with $n - m = d$, $0 \neq l_n \in \mathcal{L}_n$, $0 \neq l_m \in \mathcal{L}_m$. For a trivial algebra, we take $t = 1$.

Since L has *finite type* over $k(Z)$, it follows that $L = k(X)$ for a normal algebraic variety X, and we can assume that X/Z is proper. Then $\mathcal{L}_0 = f_*\mathcal{O}_X$.

Note that any $\mathcal{O}_Z$-subalgebra of finite type of $k(X)$ is coherent since $\mathcal{O}_Z$ is Noetherian. We give a more explicit form soon (see Proposition 4.15). $\square$

If X/Z is birational, then a coherent $\mathcal{O}_Z$-submodule $\mathcal{L}_i \subset k(X) = k(Z)$ is locally a submodule of $\mathcal{O}_Z$ up to a nonzero multiple, and is thus known as a *fractional ideal* sheaf. For some classes of algebra, the coherent property is itself nontrivial and actually equivalent to *quasicoherent* (even slightly weaker, equivalent to locally generated; see Example 4.12 and Remark 4.20 below). In birational geometry, this problem frequently means that a divisorial sheaf is independent of a good or "sufficiently high" model (cf. Examples 4.48, 4.47 and Proposition 4.46; see also [14]).

Definition 4.3. Let $\mathcal{L} = \bigoplus \mathcal{L}_i$ be an $\mathbb{N}$-graded algebra and I be a natural number. The Ith *truncation* of $\mathcal{L}$ is the algebra $\mathcal{L}^{[I]} = \bigoplus_{I|i} \mathcal{L}_i$. Our convention is to give elements the same degree (although we sometimes follow the other tradition and give them degree i/I). A truncation of a functional algebra is again functional.

Two algebras are *quasi-isomorphic* if they have isomorphic truncations.

For example, two divisorial algebras $\mathcal{R}_{X/Z}D$ and $\mathcal{R}_{X/Z}D'$ are quasi-isomorphic if $D' \sim_\mathbb{Q} rD$ for $0 < r \in \mathbb{Q}$; the converse does not hold in general (cf. Example 4.12 and Proposition 4.15(8)).

Example 4.4. Let $(X/Z, B)$ be a Klt log pair with $\mathbb{Q}$-boundary B and relative log Kodaira dimension l (that is, its log canonical algebra $\mathcal{R}(X/Z, B)$ has homogeneous field of fractions of transcendence degree l over $k(Z)$). Then the effective adjunction of Fujino and Mori [12, Theorem 5.2] says that there exists another Klt log pair $(Y/Z, B_Y)$ of relative dimension l whose log canonical algebra $\mathcal{R}(Y/Z, B_Y)$ is quasi-isomorphic to $\mathcal{R}(X/Z, B)$. We do not expect isomorphism here in general (compare Kollár's remark on the stability properties of $f_*\omega^m_{X/Z}$ [28, p. 362, (ii)]). Thus, we usually consider log canonical algebras up to quasi-isomorphism. In particular, this allows us to reduce the f.g. of such algebras to the case when $K + B$ is big/Z [12, Corollary 5.3]. However, g.a.g. (Definition 3.17) is more subtle, especially, when $l \le 0$ (cf. Example 3.14).

Example 4.5. Let $\mathcal{L}$ be the graded k-algebra

$$\mathcal{L}_i = \begin{cases} kx^{i/2} & \text{for } i \text{ even,} \\ k\varepsilon_i & \text{for } i \text{ odd,} \end{cases} \qquad \text{with} \quad x^i x^j = x^{i+j}, \quad \varepsilon_i x^j = \varepsilon_i\varepsilon_j = 0.$$

Then $\mathcal{L}^{[2]}$ is isomorphic to $k[x]$, and so is f.g., in fact generated by $x = x^1$. But $\mathcal{L}$ itself is not f.g. Note that $\mathcal{L}$ is not integral, in particular, not functional.

Theorem 4.6 (Truncation Principle)**.** *Quasi-isomorphism preserves f.g. of functional algebras.*

Proof. It is enough to prove this for a truncation $\mathcal{L}^{[I]} \subset \mathcal{L}$. If $\mathcal{L}$ is f.g., it is clear that $\mathcal{L}^{[I]}$ is also f.g.: if $\mathcal{L}$ is generated by $s_1, \dots, s_m$, then $\mathcal{L}^{[I]}$ is generated by the monomials of degree I in these.

Conversely, assume that $\mathcal{L}^{[I]}$ is f.g. To prove that $\mathcal{L}$ is f.g., it is enough to prove that it is of finite type as a $\mathcal{L}^{[I]}$-module. Now

$$\mathcal{L} = \bigoplus_{0 \le r \le I-1} \mathcal{L}^r, \qquad \text{where} \quad \mathcal{L}^r = \bigoplus_{i \equiv r \mod I} \mathcal{L}_i,$$

and each $\mathcal{L}^r$ is isomorphic to a coherent $\mathcal{O}_Z$-submodule of $\mathcal{L}^{[I]}$: indeed, we can assume that $\mathcal{L}^r \ne 0$. Then there is some $0 \ne s \in \mathcal{L}^r$ of degree $i \equiv r$, and the multiplication $x \mapsto s^{I-1}x$ is an inclusion $\mathcal{L}^r \hookrightarrow \mathcal{L}^{[I]}$. Thus, $\mathcal{L}^r$ is of finite type by the Noetherian property. $\square$

Example 4.7 (blowup of an ideal)**.** Let $\mathcal{I} \subset k(Z)$ be a *fractional ideal*, that is, up to a multiple, a coherent $\mathcal{O}_Z$-ideal $\mathcal{I} \subset \mathcal{O}_Z$. It generates a functional algebra $\mathcal{S} = \mathcal{S}(\mathcal{I})$, with $\mathcal{S}_i = \mathcal{I}^i = \operatorname{im}\{\mathcal{I}^{\otimes i} \to \mathcal{O}_Z\} \subset \mathcal{O}_Z^{\otimes i} = \mathcal{O}_Z$ (see Example 3.12). This algebra is always f.g. and its projective spectrum $\sigma\colon \operatorname{Proj}_Z \mathcal{S} \to Z$ is the *blowup* of Z in $\mathcal{I}$ or in its subscheme. It is a projective birational morphism, possibly not small: any projective birational morphism to Z can be obtained in this form.

If $\mathcal{I} = \mathcal{O}_T(f(D))$ for a birational contraction $f\colon X \to T = Z$ as in Example 3.15, the corresponding tensor algebra $\mathcal{S}(\mathcal{I})$ is a subalgebra of the flipping algebra $\mathcal{FR}_{X/T}D = \mathcal{R}_{T/T}f(D)$, and is a proper subalgebra in many interesting cases. Indeed, for integral D, $\mathcal{FR}_{X/T}D = \mathcal{A}(\mathcal{I}) = \mathcal{S}(\mathcal{I})^{\vee\vee}$ is the reflexive hull of the tensor algebra (called the *symbolic power algebra* in [27, Remark 4.3]). If this last algebra is f.g., its Proj is a *small* contraction: namely, the D-flip X^+/T. However, if D is not Cartier on X^+, the blowup in $\mathcal{I}$ is bigger, and even divisorial if X^+ is $\mathbb{Q}$-factorial.

Blowups in ideals have other drawbacks: they can give nonnormal varieties, which are outside our category. This also concerns other functional algebras, and can be improved in terms of functional algebras.

Example 4.8 (integral closure)**.** Let $\mathcal{L}$ be a functional algebra. Then $\overline{\mathcal{L}} = \bigoplus \overline{\mathcal{L}}_i$ with

$$\overline{\mathcal{L}}_i = \left\{ a \in k(X) \;\middle|\; \begin{array}{l} a^m + l_1 a^{m-1} + \ldots + l_{m-1}a + l_m = 0 \\ \text{for some elements } l_j \in \mathcal{L}_i^j \end{array} \right\},$$

where $\mathcal{L}_i^j$ denotes the product $\mathcal{O}_Z$-submodule $\mathcal{L}_i \dots \mathcal{L}_i$ (j times) in $k(X)$ (or in any other field F), is also a functional algebra in $k(X)_\bullet$ (or in $F_\bullet$, that depends on F). We call it the *integral closure* of $\mathcal{L}$ in $k(X)$ (respectively, in F). We have $\overline{\overline{\mathcal{L}}} = \overline{\mathcal{L}}$. The proof is given in Proposition 4.15 below, together with another description of integral closure. We also prove that $\mathcal{L}$ is f.g. if and only if $\overline{\mathcal{L}}$ is; in this case, the projective spectrum morphism $\operatorname{Proj}_Z \overline{\mathcal{L}} \to \operatorname{Proj}_Z \mathcal{L}$ is the normalization. It is birational in the homogeneous field of fractions for $\mathcal{L}$ (see Corollary 4.17). We say that an algebra is *normal* if it is integrally closed in its homogeneous field of fractions. By Proposition 4.15, these are the pbd algebras (up to a truncation).

Example 4.9 (twist)**.** Let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra. It gives an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra $\mathcal{A}_\bullet$ with components $\mathcal{A}_i = \mathcal{A}$ for $i \geq 1$ and $\mathcal{A}_0 = \mathcal{O}_X$. We define its *twist* $\mathcal{A}[D]$ by a Cartier divisor D on X by setting $\mathcal{A}[D]_i = \mathcal{A}_i(iD) = \mathcal{A}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(iD)$. Since $\mathcal{A}$ is coherent, the twisted algebra is finite as a module over $\mathcal{R}_{X/X} D$. Moreover, $f_*\mathcal{A}[D]$ is also finite as a module over the algebra $\mathcal{R}_{X/Z} D = f_* \mathcal{R}_{X/X} D$ if D is ample/Z.

The algebra $f_*\mathcal{A}[D]$ is functional if $\mathcal{A}$ does not have torsion, that is, is a submodule of $k(Y)/k(X)$. For example, if $g\colon Y \to X$ is a proper morphism, then $g_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-subalgebra of $k(Y)$ with the natural multiplication. Its twist $g_*\mathcal{O}_Y[D]$ is the divisorial algebra $\mathcal{R}_{Y/X} g^* D$ by the Projection Formula.

However, the most important functional algebras for us are geometric, that is, associated with divisors.

Definition 4.10 (pbd algebra)**.** Let $\mathcal{D}_\bullet = (\mathcal{D}_i)_{i \in \mathbb{N}}$ be a system (sequence) of $\mathbb{R}$-b-divisors of X/Z such that

- $\mathcal{D}_i + \mathcal{D}_j \leq \mathcal{D}_{i+j}$;
- $\mathcal{D}_0 = 0$; and
- each $f_*\mathcal{O}_X(\mathcal{D}_i)$ is a coherent $\mathcal{O}_Z$-module,

where the sections of the $\mathcal{O}_X$-sheaf $\mathcal{O}_X(\mathcal{D}_i) = \mathcal{O}_X(\lfloor \mathcal{D}_i \rfloor)$ are

$$\Gamma(U, \mathcal{O}_X(\mathcal{D}_i)) = \{a \in k(X) \mid (a) + \mathcal{D}_i \geq 0 \text{ over } U\},$$

and the principal divisor (a) is considered as a b-divisor, namely, the Cartier completion of the ordinary principal divisor (a) [43, Example 1.1.1]. Since X/Z is proper, it is enough to assume that $\mathcal{O}_X(\mathcal{D}_i)$ is coherent. A sheaf of this form is a torsion-free sheaf of rank 1 (a fractional ideal sheaf) and is integrally closed. We call it a *b-divisorial* sheaf. Note that $f_*\mathcal{O}_X(\mathcal{D}_0) = f_*\mathcal{O}_X$. In the incoherent case, it makes sense to consider the maximal coherent subsheaf $\mathcal{O}_X(\mathcal{D}_i)^{\mathrm{coh}}$ contained in $\mathcal{O}_X(\mathcal{D}_i)$ (cf. Example 4.12 and the proof of Proposition 4.15 below); for example, this can be 0 as for $\mathcal{D}_i = -\infty$.

Then we can associate a functional $\mathcal{O}_Z$-algebra

$$\mathcal{R}_{X/Z}\mathcal{D}_\bullet = \mathcal{R}_f \mathcal{D}_\bullet \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} f_*\mathcal{O}_X(\mathcal{D}_n)$$

with the system, which we call a *pseudo-b-divisorial* algebra (*pbd algebra*).

Multiplication is well defined since it is for X/X: for $a \in \mathcal{O}_X(\mathcal{D}_i)$ and $b \in \mathcal{O}_X(\mathcal{D}_j)$, $ab \in \mathcal{O}_X(\mathcal{D}_{i+j})$ because $(ab) + \mathcal{D}_{i+j} \geq (a) + (b) + \mathcal{D}_i + \mathcal{D}_j$ (cf. Definition 3.10). The product of two coherent subsheaves is contained in the coherent subsheaf $\mathcal{O}_X(\mathcal{D}_{i+j})^{\mathrm{coh}}$.

This definition gives a functorial homomorphism $\mathcal{R}_{X/Z}(\cdot)$ from systems of b-divisors to functional algebras. It is compatible with truncations, where the *truncation* of $\mathcal{D}_\bullet$ is the system $\mathcal{D}_i^{[I]} = \mathcal{D}_{iI}$. To preserve degrees, we make the convention $\mathcal{D}_i^{[I]} = \mathcal{D}_i$ for $I \mid i$ and $\mathcal{D}_i^{[I]} = -\infty$ otherwise. We say that two systems having identical truncations are *similar*. We can replace identity by linear equivalence of truncations: two systems $\mathcal{D}_\bullet$ and $\mathcal{D}'_\bullet$ are *linearly equivalent* if $\mathcal{D}_i = \mathcal{D}'_i + i\overline{(a)}$ for some $0 \neq a \in k(X)$. Linear equivalence induces an isomorphism of algebras (cf. Lemma 3.24). This is multiplication by a^i on $\mathcal{R}_i$ and so the identity on the homogeneous field of fractions. We always assume this condition for isomorphisms of functional algebras (cf. Proposition 4.15(8)). This applies to general isomorphisms if we add isomorphisms of systems.

Corollary 4.11. *Quasi-isomorphism preserves f.g.* (*and g.a.g.*) *of a pbd algebra. In particular, a pbd algebra is f.g.* (*and/or g.a.g.*) *if and only if any truncation is*; *or, more generally, similar systems have the same f.g.* (*and g.a.g.*) *properties* (*with the same stable base divisor and locus*; *cf. Example* 4.29).

Moreover, if

- $\mathcal{D}_\bullet$ *is big*$/Z$ (*cf. Remark* 3.30(1)); *and*
- $\mathcal{R} = \mathcal{R}_{X/Z}\mathcal{D}_\bullet$ *is f.g.*,

then the algebra is g.a.g. (*cf. Corollary* 3.31).

Proof–Explanation. F.g. is immediate by Truncation Principle 4.6, and g.a.g. comes from the following definitions.

A pbd algebra $\mathcal{R} = \mathcal{R}_{X/Z}\mathcal{D}_\bullet$ is *globally almost generated* (g.a.g.) if there are a natural number N and sections of $\mathcal{O}_X(N\mathcal{D})$ that generate the algebra $\mathcal{R}_{X/X}\mathcal{D}_\bullet^{[N]}$ in codimension 1 except for a divisorial subset that is exceptional on $Y = \operatorname{Proj}_Z \mathcal{R}$. However, for pbd algebras, g.a.g. is not directly associated with the stable base locus (see Example 4.31 below).

A system $\mathcal{D}_\bullet$ is *big*$/Z$ if its algebra $\mathcal{R}$ is big, that is, $k(X)/F$ is a finite field extension, where F is the homogeneous field of fractions of $\mathcal{R}$; in particular, $F = k(Y)$, where $Y = \operatorname{Proj}_Z \mathcal{R}$ whenever $\mathcal{R}$ is f.g. $\square$

Example 4.12. Let $\mathcal{D}$ be an $\mathbb{R}$-b-divisor such that

- $\mathcal{O}_X(i\mathcal{D})$ is a coherent $\mathcal{O}_X$-module for all $i \in \mathbb{N}$.

Then the system $(\mathcal{D}_i = i\mathcal{D} \mid i \geq 0)$ satisfies the assumptions of Definition 4.10 and gives a *b-divisorial algebra* $\mathcal{R}_{X/Z}\mathcal{D} = \mathcal{R}_{X/Z}(i\mathcal{D} \mid i \geq 0)$. The divisorial algebra of Definition 3.10 for some b-divisor $\mathcal{D}$ with $\mathcal{D}_Y = D$ is a particular case. For example, if D is $\mathbb{R}$-Cartier on a model Y/Z of X/Z, we can take $\mathcal{D} = \overline{D}$ by Proposition 3.20.

Since each $\mathcal{O}_X(i\mathcal{D}) = \bigcap g_*\mathcal{O}_Y(i\mathcal{D}_Y)$, where the intersection runs over all models $g\colon Y \to X/Z$ of X/Z, coherence means that this intersection is also coherent; this condition usually fails. Since $\mathcal{O}_X(i\mathcal{D})$ is a subsheaf of the coherent sheaf $g_*\mathcal{O}_Y(i\mathcal{D}_Y)$, it is coherent if and only if it is quasicoherent, or even locally generated. This holds, for example, when the intersection *stabilizes*, that is, $\mathcal{O}_X(i\mathcal{D}) = g_*\mathcal{O}_Y(i\mathcal{D}_Y)$ over some (sufficiently high) model Y/X where $\mathcal{D}_Y$ is the trace of $\mathcal{D}$ on Y (cf. Examples 4.47 and 4.48 as well as Example 4.31).

For any $\mathcal{D}$, we can replace each $\mathcal{O}_X(i\mathcal{D})$ by its maximal coherent subsheaf $\mathcal{O}_X(i\mathcal{D})^{\mathrm{coh}}$; cf. Remark 6.15(4).

In general, we need infinitely many models, as in the more typical case of Example 4.14 below.

Thus, the difference between (pbd) b-divisorial and divisorial algebras is the minor one that our divisor is not on X but rather on a different model Y (or on different models X_i). However, divisorial is not so useful from the f.g. point of view (cf. Example 3.15).

Finally, note that, as for divisors, $\mathcal{D}' \sim_{\mathbb{Q}} r\mathcal{D}$ for some $0 < r \in \mathbb{Q}$ if and only if the corresponding systems $(i\mathcal{D})$ and $(i\mathcal{D}')$ are similar. The equivalences $\sim_*$ can be defined for b-divisors as for divisors (see Definition 3.26): we replace $*$-principal divisors (a) by their b-divisors $\overline{(a)}$. Thus, an equivalence $\mathcal{D}' \sim_{\mathbb{Q}} r\mathcal{D}$ with $0 < r \in \mathbb{Q}$ gives a quasi-isomorphism of the algebras $\mathcal{R}_{X/Z}\mathcal{D}$ and $\mathcal{R}_{X/Z}\mathcal{D}'$ provided that both are functional (cf. Corollary 3.5). For the converse, see Proposition 4.15(8).

Example 4.13. For a b-divisor $\mathcal{D}$ as in Example 4.12, the system $\mathcal{D}_i = \lfloor i\mathcal{D} \rfloor$ defines a pbd algebra that is actually equal to the b-divisorial algebra $\mathcal{R}_{X/Z}\mathcal{D}$. However, for fractional $\mathcal{D}$, the system $\mathcal{D}_i = \lceil i\mathcal{D} \rceil$ does not usually define a pbd algebra, because it may happen that $\lceil i\mathcal{D} \rceil + \lceil j\mathcal{D} \rceil > \lceil (i+j)\mathcal{D} \rceil$, so that multiplication is not defined. This type of rounding up appears later as an important tool to pass from saturation to f.g. (see Example 4.41 and Section 5 below).

Example 4.14. Let $(X_i/Z, D_i \mid i \in \mathbb{N})$ be a sequence of models X_i/Z of X/Z and their $\mathbb{R}$-Cartier divisors such that

- $\mathcal{D}_i + \mathcal{D}_j \leq \mathcal{D}_{i+j}$, where $\mathcal{D}_i = \overline{D}_i$.

Then the system $(\mathcal{D}_i)$ again satisfies the assumptions of Definition 4.10.

In fact, by Proposition 4.15 below, an integrally closed functional algebra $\mathcal{L}$ is always of this form, but is usually not b-divisorial. Moreover, we can assume that each D_i is Cartier with $\operatorname{Bs}|D_i| = \varnothing$ whenever $\mathcal{L}_i \neq 0$. A system with this property for all $\mathcal{D}_i$ or for some truncation is *birationally free* (b-free). In particular, any b-free system or its truncation is integral, that is, has integral $\mathcal{D}_i$ or D_i.

We now construct the inverse of $\mathcal{R}_{X/Z}(-)$, namely, the functorial homomorphism sending a functional algebra $\mathcal{L}$ to its *mobile system* $\operatorname{Mov}\mathcal{L} = \mathcal{M}_\bullet$. But first we make the convention either to accept $-\infty$ as a divisor and b-divisor, or to define $\mathcal{M}_\bullet$ only as a truncation, and only for nontrivial $\mathcal{L}$.

Proposition 4.15. *For each functional $\mathcal{O}_Z$-algebra $\mathcal{L}$ in $k(X)_\bullet$, there is a unique system $\mathcal{M}_\bullet$ of b-divisors of X/Z with the following properties*:

(1) *each $\mathcal{M}_i$ is b-free and, in particular, integral*;

(2) *$\mathcal{M}_\bullet$ satisfies the assumptions of Definition* 4.10 *and $\mathcal{L} \subset \mathcal{R}_{X/Z}\mathcal{M}_\bullet$*;

(3) *for any system $\mathcal{D}_\bullet$, if $\mathcal{L} \subset \mathcal{R}_{X/Z}\mathcal{D}_\bullet$, then $\mathcal{M}_\bullet \leq \mathcal{D}_\bullet$* (each $\mathcal{M}_i \leq \mathcal{D}_i$); *that is, $\mathcal{M}_\bullet$ is the minimal system for which inclusion* (2) *holds*;

(4) $\overline{\mathcal{L}} = \mathcal{R}_{X/Z}\mathcal{M}_\bullet$ *and* $\operatorname{Mov}\overline{\mathcal{L}} = \operatorname{Mov}\mathcal{L} = \mathcal{M}_\bullet$; *thus, $\mathcal{L} = \mathcal{R}_{X/Z}\mathcal{M}_\bullet$ if $\mathcal{L}$ is integrally closed in $k(X)/Z$*;

(5) *if $\mathcal{D}_\bullet$ is b-free, then* $\operatorname{Mov}\mathcal{R}_{X/Z}\mathcal{D}_\bullet = \mathcal{D}_\bullet$;

(6) *$\mathcal{L}$ is f.g. if and only if $\mathcal{R}_{X/Z}\mathcal{M}_\bullet$ is f.g.*;

(7) *$\mathcal{R}_{X/Z}\mathcal{M}_\bullet$ is g.a.g. with $E = 0$ on each model Y/Z of X/Z*; *and finally,*

(8) *quasi-isomorphic algebras give similar systems.*

For an incoherent functional algebra $\mathcal{L}$, we say that it has *bounded components* $\mathcal{L}_i$ if each $\mathcal{L}_i$ is a subsheaf of a functional coherent sheaf (see Remark 4.20). We can define the *mobile system* of an algebra $\mathcal{L}$ of this type to be that of its maximal coherent subalgebra $\mathcal{L}^{\mathrm{coh}}$ (cf. Definition 4.10 above and Remark 6.15(4) below; see also [14]).

Proof–Construction. We first construct $\mathcal{M}_i$ for each $\mathcal{L}_i \neq 0$ (if $\mathcal{L}_i = 0$, then $\mathcal{M}_i = -\infty$). Set

$$\begin{aligned}\mathcal{M}_i &= \sup\{-\overline{(s)} \mid 0 \neq s \in \mathcal{L}_i\} = -\inf\{\overline{(s)} \mid 0 \neq s \in \mathcal{L}_i\} \\ &= \text{max divisor of poles of } s \in \mathcal{L}_i.\end{aligned}$$

Here the sup and inf are taken componentwise, as for b-divisors (the same applies to max and min in what follows); in particular, the result is not necessarily a principal divisor. Since $\mathcal{L}_i \subset k(X)$ is a finite $\mathcal{O}_Z$-module, locally/Z, there is a finite set of generators $0 \neq s_i \in k(X)$ for $\mathcal{L}_i$, and the sup and inf are in fact the max and min, respectively, for b-divisors $\overline{(s_i)}$ considered as functions over prime b-divisors. In particular, it is a b-divisor, and its trace on each model of X/Z is an ordinary divisor.

Indeed, the analogous results for ordinary principal divisors imply

- if $s \in \mathcal{O}_Z \subset k(X)$, then $\overline{(s)} \geq 0$; and
- for any $s, s' \in k(X)$, $\overline{(s + s')} \geq \min\{\overline{(s)}, \overline{(s')}\}$.

Therefore, for any $0 \neq s \in \mathcal{L}_i$, since $s = \sum a_i s_i$ with all $a_i \in \mathcal{O}_Z$,

$$\overline{(s)} = \overline{\left(\sum a_i s_i\right)} \geq \min\{\overline{(a_i s_i)}\} = \min\{\overline{(a_i)} + \overline{(s_i)}\} \geq \min\{\overline{(s_i)}\}.$$

For each $0 \neq s \in \mathcal{L}_i$, we have $\mathcal{M}_i \geq -\overline{(s)}$ or $\overline{(s)} + \mathcal{M}_i \geq 0$, which gives the inclusion $\mathcal{L}_i \subset f_*\mathcal{O}_X(\mathcal{M}_i)$ of (2). However, for $\mathcal{R}_{X/Z}\mathcal{M}_\bullet$ to be functional, we need it to be coherent. For this, (1) is enough. Next, $\mathcal{M}_i$ is the minimal divisor for which the inclusion $\mathcal{L}_i \subset f_*\mathcal{O}_X(\mathcal{M}_i)$ holds, which proves (3). Indeed, if the same holds for $\mathcal{D}_i$, then $\mathcal{D}_i \geq -\overline{(s)}$ or $\mathcal{D}_i + \overline{(s)} \geq 0$ for any $0 \neq s \in \mathcal{L}_i$. Hence, $\mathcal{D}_i \geq \mathcal{M}_i$.

The trace $M_i = (\mathcal{M}_i)_Y$ on each model Y/Z of X/Z is *mobile*$/Z$, provided that $\mathcal{M}_i \neq -\infty$ or $\mathcal{L}_i \neq 0$. (In a certain sense this always holds.) This means that $\operatorname{Mov} M_i = M_i$ or, equivalently, the linear system $|M_i|$ does not have fixed components. This proves (7) with $E = 0$ on each model of X/Z. Moreover, $\mathcal{L}_i$ gives its nonempty linear subsystem L_i also without fixed components (in that sense any functional algebra $\mathcal{L}$ is g.a.g. with $E = 0$).

The linear system L_i only depends on $\mathcal{L}_i$: every element of

$$L_i = \{(s) + M_i \mid 0 \neq s \in \mathcal{L}_i\} \subset |M_i|$$

is the restriction to X_i of $\overline{(s)} + \mathcal{M}_i$, and the general element has no fixed components. Indeed, by the definitions, $\operatorname{Bs} L_i = \min L_i = 0$. If we take a model X_i/Z on which L_i is free (which exists by Hironaka), then M_i is also free on X_i, and $\mathcal{M}_i = \overline{M_i}$. Indeed, for any other model $g\colon X_i' \to X_i$ and general $D_i = (\mathcal{D}_i)_{X_i}$ in L_i, the divisor does not contain the center of any exceptional divisor, and its birational transform $D_i' = (\mathcal{D}_i)_{X_i'}$ does not contain any exceptional divisor. Thus,

$$D_i' = g^{-1}D_i = g^*D_i, \qquad \mathcal{D}_i = \overline{(s)} + \mathcal{M}_i = \overline{D}_i, \qquad \text{and} \qquad \mathcal{M}_i = \overline{M_i},$$

which completes the proof of (1). Hence, the components $\mathcal{L}_i \neq 0$ of functional algebras are equivalent to linear systems L_i without fixed components.

Now $\mathcal{M}_{i+j} \geq \mathcal{M}_i + \mathcal{M}_j$ for all i, j, which completes the proof of (2). Indeed, since $\mathcal{L}_{i+j} \supset \mathcal{L}_i\mathcal{L}_j$,

$$\begin{aligned}\mathcal{M}_{i+j} &= \sup\Big\{-\overline{(s)} \;\Big|\; 0 \neq s \in \mathcal{L}_{i+j}\Big\} \geq \sup\Big\{-\overline{(s)} \;\Big|\; 0 \neq s \in \mathcal{L}_i\mathcal{L}_j\Big\} \\ &\geq \sup\Big\{-\overline{(ss')} = -\overline{(s)} - \overline{(s')} \;\Big|\; 0 \neq s \in \mathcal{L}_i \text{ and } 0 \neq s' \in \mathcal{L}_j\Big\} \\ &= \sup\Big\{-\overline{(s)} \;\Big|\; 0 \neq s \in \mathcal{L}_i\Big\} + \sup\Big\{-\overline{(s')} \;\Big|\; 0 \neq s' \in \mathcal{L}_j\Big\} = \mathcal{M}_i + \mathcal{M}_j.\end{aligned}$$

Each $\overline{\mathcal{L}}_i = f_*\mathcal{O}_X(\mathcal{M}_i)$, which proves the first statement in (4). Indeed, $\overline{\mathcal{L}}_i \subset \overline{f_*\mathcal{O}_X(\mathcal{M}_i)} = f_*\mathcal{O}_X(\mathcal{M}_i)$ by (2). This last equation corresponds to the completeness of the linear system for $f_*\mathcal{O}_X(\mathcal{M}_i)$: if $0 \neq s \in k(X)$ such that $s^m + l_1 s^{m-1} + \ldots + l_{m-1}s + l_m = 0$ with all $l_j \in f_*\mathcal{O}_X(j\mathcal{M}_i)$ or, equivalently, $\overline{(l_j)} + j\mathcal{M}_i \geq 0$, then

$$\begin{aligned}m(\overline{(s)} + \mathcal{M}_i) &= \overline{(s^m)} + m\mathcal{M}_i = \overline{(l_1 s^{m-1} + \ldots + l_{m-1}s + l_m)} + m\mathcal{M}_i \\ &\geq \min\Big\{\overline{(l_1 s^{m-1})}, \ldots, \overline{(l_{m-1}s)}, \overline{(l_m)}\Big\} + m\mathcal{M}_i \\ &= \min\Big\{\overline{(l_1)} + \mathcal{M}_i + (m-1)(\overline{(s)} + \mathcal{M}_i), \ldots, \\ &\qquad \overline{(l_{m-1})} + (m-1)\mathcal{M}_i + (\overline{(s)} + \mathcal{M}_i), \overline{(l_m)} + m\mathcal{M}_i\Big\} \\ &\geq \min\Big\{(m-1)(\overline{(s)} + \mathcal{M}_i), \ldots, \overline{(s)} + \mathcal{M}_i\Big\},\end{aligned}$$

and $\overline{(s)} + \mathcal{M}_i \geq 0$. Hence, $s \in f_*\mathcal{O}_X(\mathcal{M}_i)$.

On the other hand, each element $s \in f_*\mathcal{O}_X(\mathcal{M}_i)$ is integral for $\mathcal{L}_i$, which gives the opposite inclusion $f_*\mathcal{O}_X(\mathcal{M}_i) \subset \overline{\mathcal{L}}_i$. The linear systems L_i and $|M_i|$ induce a morphism $g\colon \operatorname{Proj}_Z \mathcal{R}_{X/Z}\mathcal{M}_i = Y' \to \operatorname{Proj}_Z \bigoplus \mathcal{L}_i^j = Y$ that is finite because it is quasifinite and proper. Hence, the b-divisorial algebra $\mathcal{R}_{X/Z}\mathcal{M}_i$, isomorphic to $\mathcal{R}_{Y/Z}\mathcal{A}[D]$, is finite as a module over the product functional algebra

$\bigoplus \mathcal{L}_i^j \hookrightarrow \mathcal{R}_{Y/Z}D$ (under the above isomorphism) by Example 4.9, where $\mathcal{A} = g_*\mathcal{O}_{Y'} = h_*\mathcal{O}_{X_i}$, $h\colon X_i \to Y$, and $\mathcal{O}_Y(D) = \mathcal{O}_Y(1)$ or $\mathcal{M}_i \sim \overline{h^*D}$ with very ample D on Y/Z. Indeed, $\bigoplus \mathcal{L}_i^j \cong \mathcal{R}_{Y/Z}D$ for all degrees $i \gg 1$ because $\mathcal{L}_i$ is isomorphic to $e_*\mathcal{O}_Y(1)$, $e\colon Y \to Z$. In particular, each $s \in f_*\mathcal{O}_X(\mathcal{M}_i)$ is integral because, by the Noetherian property, the chain of $\bigoplus \mathcal{L}_i^j$-submodules

$$(1) \subset (1,s) \subset \ldots \subset (1,s,\ldots,s^{m-1}) = (1,s,\ldots,s^{m-1},s^m)$$

terminates.

We have $\operatorname{Mov}\overline{\mathcal{L}} \le \mathcal{M}_\bullet$ by (3) since $\mathcal{R}_{X/Z}\mathcal{M}_\bullet = \overline{\mathcal{L}}$. Conversely, $\mathcal{M}_\bullet \le \operatorname{Mov}\overline{\mathcal{L}}$ by (3) again because $\mathcal{L} \subset \overline{\mathcal{L}}$ implies $\mathcal{M}_\bullet = \operatorname{Mov}\mathcal{L} \le \operatorname{Mov}\overline{\mathcal{L}}$. This completes the proof of (4).

(5) follows by construction: a b-free divisor is the minimal one giving any linear system.

If $\mathcal{L}$ is f.g., by Truncation Principle 4.6, we can assume that it is generated by $\mathcal{L}_1$. In addition, we can assume that $\mathcal{R}_{X/Z}\mathcal{M}_1$ gives the normalization $\operatorname{Proj}_Z \mathcal{R}_{X/Z}\mathcal{M}_1$ of $Y = \operatorname{Proj}_Z \mathcal{L}$ in $k(X)$. Then $\overline{\mathcal{L}} = \mathcal{R}_{X/Z}\mathcal{M}_\bullet = \mathcal{R}_{X/Z}\mathcal{M}_1$ up to a finite number of components (by normality for high multiples of a very ample divisor). Hence, $\mathcal{R}_{X/Z}\mathcal{M}_\bullet$ is a finite module over $\mathcal{R}_{X/Z}\mathcal{M}_1$. But in turn, this last algebra is a finite $\mathcal{L}$-module by Example 4.9 and the above arguments. Hence, $\mathcal{R}_{X/Z}\mathcal{M}_\bullet$ is also a finite $\mathcal{L}$-module and is f.g. by the Hilbert basis theorem.

Conversely, suppose that $\mathcal{R}_{X/Z}\mathcal{M}_\bullet$ and $f_*\mathcal{O}_X(M_1)$ generate the algebra; in particular, $\mathcal{R}_{X/Z}\mathcal{M}_\bullet = \mathcal{R}_{X_1/Z}M_1$. Then each $\mathcal{L}_i$ defines a finite morphism $Y = \operatorname{Proj}_Z \mathcal{R}_{X_1/Z}M_1 \to Y_i = \operatorname{Proj}_Z \bigoplus \mathcal{L}_i^j$. Moreover, the inclusions $\mathcal{L}_i^j \subset \mathcal{L}_{ij}$ define finite morphisms $Y_{ij} \to Y_i$. However, this is only possible under a stabilization: $Y_{ij} \cong Y_i$ for some $i \gg 1$ and all $j \ge 1$. This implies that $\mathcal{L}^{[i]}$ is f.g. because $\mathcal{L}_i \cong e_*\mathcal{O}_{Y_i}(1)$ generates $\mathcal{L}_{ij}$ for all $j \gg 1$, and completes the proof of (6).

Finally, suppose that algebras $\mathcal{L}$ and $\mathcal{L}'$ are quasi-isomorphic, that is, isomorphic up to a truncation. Since we assume that the isomorphism induces the identity on their homogeneous fields of fractions, each nontrivial $\mathcal{L}_i$ is isomorphic to $\mathcal{L}'_i$ under a multiplication $t \mapsto s_i t$ for a unique $0 \ne s_i \in k(X)$. Since this is a homomorphism of algebras, $s_i s_j = s_{ij}$. Then we can find $s_d \in k(X)$ (even if $\mathcal{L}_d = 0$) such that each $s_{id} = s_d^i$, where d is the gcd of j with $\mathcal{L}_j \ne 0$. This implies the linear equivalence $\mathcal{M}_\bullet \sim \mathcal{M}'_\bullet = \operatorname{Mov}\mathcal{L}'$ and completes the proof of (8). ☐

On the way we have proved the following two corollaries:

Corollary 4.16. *$\mathcal{L}$ is f.g. if and only if its mobile system $\mathcal{M}_\bullet$ is b-divisorial up to a truncation: there exists a natural number I such that $\mathcal{M}_{iI} = i\mathcal{M}_I$ for all $i \ge 1$.*

Corollary 4.17. *If $\mathcal{L}$ or $\overline{\mathcal{L}}$ is f.g., the operation of taking projective spectrum induces the normalization $\operatorname{Proj}_Z \overline{\mathcal{L}} \to \operatorname{Proj}_Z \mathcal{L}$ of $\operatorname{Proj}_Z \mathcal{L}$ in $k(X)$. In particular, if it is birational, it is the normalization of $\operatorname{Proj}_Z \mathcal{L}$.*

By Proposition 4.15(6), it is enough to consider pbd algebras to establish the criterion of Theorem 4.28 for f.g. By Corollary 4.16, most pbd algebras are not f.g.

Example 4.18. Let C/C be a curve considered locally. In either of the following cases, the pbd algebra $\mathcal{R}_{C/C}D_\bullet$ is not f.g.:

(1) $D_\bullet = (-P, -P, \ldots, -P, \ldots)$;

(2) $D_\bullet = (P, 2^2P, \ldots, n^2P, \ldots)$.

The algebra in (2) is not f.g. because its mobile system grows too fast.

Definition 4.19. We say that a functional algebra $\mathcal{L}$ is *bounded* by a b-divisor $\mathcal{D}$ of X if $\mathcal{L}$ is a subalgebra of $\mathcal{R}_{X/Z}\mathcal{D}$. We will see in Proposition 4.22 that it is equivalent for $\mathcal{L}$ to be bounded by a divisor.

Remark 4.20. In particular, each component $\mathcal{L}_i$ is *bounded*, that is, a subsheaf of a b-divisorial sheaf $f_*\mathcal{O}_X(i\mathcal{D})$ (cf. Example 4.12), provided that $f_*\mathcal{O}_X(i\mathcal{D})$ is coherent, equivalently, locally bounded and generated.

Example 4.21 (restricted algebra)**.** Comparing the two Definitions 3.41 and 3.47, we see that in either the restriction to Y of the divisorial algebra $\mathcal{R}_{X/Z}D$ is bounded: $\mathcal{R}_{X/Z}D_{|Y} \subset \mathcal{R}_{Y/Z}D_{|Y}$. Moreover, in this inclusion, we can replace D by its b-divisor $\mathcal{D} = \overline{D}$, and $D_{|Y}$ by $\mathcal{D}_{|Y} = \overline{D}_{|Y}$. See 7.2 for fixed restriction of b-divisors.

In particular, each restricted algebra of type $(\mathrm{RFA})_{n,d}$ (Definition 3.47) is bounded. This is a good sign, but not surprising because it holds for any bounded algebra (cf. Corollary 4.27).

Proposition 4.22. *Let $\mathcal{L}$ be a functional algebra and $\mathcal{M}_\bullet = \operatorname{Mov}\mathcal{L}$ be its mobile system. Then $\mathcal{L}$ is bounded if and only if any one of the following holds*:

(1) *$\mathcal{L}$ is* divisorially bounded; *moreover, there exists a Cartier divisor D on X such that $\mathcal{L} \subset \mathcal{R}_{X/Z}D = \mathcal{R}_{X/Z}\overline{D}$ if X/Z is projective.*

(2) *The* characteristic system $(\mathcal{D}_i) = (\mathcal{M}_i/i)$ *is bounded*; *that is, there exists a b-divisor $\mathcal{D}$ such that all $\mathcal{D}_i \le \mathcal{D}$.*

(3) *The algebra is* convergent; *that is, the limit*

$$\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$$

exists, where we consider only $\mathcal{D}_i \ne -\infty$.

Addendum 4.22.1. *In* (3), *the limit $\mathcal{D}$ satisfies*

(MXD) *each $\mathcal{D}_i = \mathcal{M}_i/i \le \mathcal{D}$; and*

(BSD) *all the $\mathcal{D}_i$ are supported in a b-divisor, that is,* $\operatorname{Supp}\mathcal{D}_i \le \mathcal{S}$ *for a fixed reduced b-divisor $\mathcal{S}$.* (*Here we set* $\operatorname{Supp}(-\infty) = 0$.)

Thus, to say that an algebra is bounded restricts its characteristic divisors $\mathcal{D}_i$ in two ways: in multiplicities and in supports.

Lemma 4.23. *Let $\mathcal{M}$ be an $\mathbb{R}$-b-divisor that is* b-nef$/X$ *and H be an $\mathbb{R}$-Cartier divisor on X such that $\mathcal{M}_X \le H$. Then $\mathcal{M} \le \overline{H}$.*

In particular, this holds if $\mathcal{M}$ is b-semiample.

Proof–Explanation. In general, we say that an $\mathbb{R}$-b-divisor $\mathcal{M}$ is *b-nef/Z* (or *b-semipositive*) if it is $\mathbb{R}$-Cartier and nef, that is, there is a model Y/Z of X/Z such that

- $\mathcal{M}_Y$ is an $\mathbb{R}$-Cartier divisor on Y and is nef$/Z$, and
- $\mathcal{M} = \overline{M_Y}$.

b-semiample (see Section 3, after the proof of Corollary 3.5) implies b-nef; *numerically b-seminegative* is defined similarly.

For the lemma, we need to prove that $\overline{H} - \mathcal{M}$ is effective. This is immediate by Negativity Lemma 3.22 because $\overline{H} - \mathcal{M}$ is numerically b-seminegative$/X$ and $(\overline{H} - \mathcal{M})_X = H - \mathcal{M}_X \ge 0$. □

Lemma 4.24. *Let $\mathcal{M}_\bullet$ be a system of $\mathbb{R}$-b-divisors satisfying*

$$\mathcal{M}_i + \mathcal{M}_j \le \mathcal{M}_{i+j}$$

(*compare Example* 4.14). *Then the system $(\mathcal{D}_i) = (\mathcal{M}_i/i)$ has the following properties*:

- convexity $\mathcal{D}_{i+j} \ge (i\mathcal{D}_i + j\mathcal{D}_j)/(i+j)$ *for all i, j; thus,*
- arithmetic monotonicity $\mathcal{D}_{ij} \ge \mathcal{D}_i$; *and*
- convergence $\sup\{\mathcal{D}_i\} = \lim_{i\to\infty}\mathcal{D}_i$; *in particular, $\mathcal{D}_\bullet$ is convergent if and only if it is bounded from above. In this case, all the $\mathcal{D}_i$ have common support as in* (BSD) *of Addendum* 4.22.1.

Caution 4.25. We should really write $\limsup$ rather than $\lim$; thus, to be more precise, we make the convention that we *omit all* $\mathcal{D}_i = -\infty$ in the sequence! The limits are taken componentwise and are *not necessarily uniform* (cf. Remark 5.8(2)). The limit itself may not be a b-divisor (cf. (BSD) of Addendum 4.22.1).

If $\mathcal{M}_\bullet$ is a b-free system, that is, $\mathcal{M}_\bullet = \operatorname{Mov}\mathcal{L}$ for a pbd algebra $\mathcal{L} = \mathcal{R}_{X/Z}\mathcal{M}_\bullet$, then, by arithmetic monotonicity, $\mathcal{L}$ has subalgebras

$$\mathcal{L}^i = \mathcal{R}_{X/Z}(j\mathcal{M}_i/i)',$$

where $(j\mathcal{M}_i/i)'$ is the system $(j\mathcal{M}_i/i)$ with $j\mathcal{M}_i/i$ replaced by $-\infty$ whenever $i \nmid j$. The algebra is quasi-isomorphic to a (b-)divisorial algebra $\mathcal{R}_{X/Z}\mathcal{M}_i$ and is bounded by $\mathcal{R}_{X/Z}\mathcal{M}_i/i$. The total algebra $\mathcal{L}$ is an inductive limit $\lim_{i|j}\mathcal{L}^i$ for the inclusions $\mathcal{L}^i \subset \mathcal{L}^j$ with $i \mid j$. From this point of view, it is *pseudo-b-divisorial* but not b-divisorial since the limit of b-divisors does not always exist, and, if it exists under the convergence of the lemma, its algebra may be different from $\mathcal{L}$ (even up to quasi-isomorphism, cf. Example 4.18(1)), or worse, may not be defined (that is, not coherent). The only case when the inductive limit is well defined as a b-divisorial algebra up to a quasi-isomorphism is when it *stabilizes*, that is, $\mathcal{L}$ is quasi-isomorphic to $\mathcal{L}^i$ for some $i \gg 1$ (cf. Limiting Criterion 4.28 below). This means f.g.

Proof of Lemma 4.24. The convexity means

$$\mathcal{M}_{i+j} = (i+j)\mathcal{D}_{i+j} \geq i\mathcal{D}_i + j\mathcal{D}_j = \mathcal{M}_i + \mathcal{M}_j.$$

Thus, by induction on j,

$$\mathcal{D}_{ij} \geq (i(j-1)\mathcal{D}_{i(j-1)} + i\mathcal{D}_i)/ij \geq (i(j-1)\mathcal{D}_i + i\mathcal{D}_i)/ij = \mathcal{D}_i.$$

It is enough to prove the convergence componentwise, that is, for a numerical sequence $\mathcal{D}_i$. Moreover, it follows from the existence of $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$ and from its upper bound (cf. Proposition 4.22, (MXD)), that is, for all i,

$$\mathcal{D}_i \leq \mathcal{D},$$

where in the limit we consider only $\mathcal{D}_i \neq -\infty$, but the limit can be $+\infty$. The case when all $\mathcal{D}_i = -\infty$ is trivial.

Otherwise, by monotonicity, if d is the gcd of the indices i with $\mathcal{D}_i \neq -\infty$, then all $\mathcal{D}_j \neq -\infty$ with $d \mid j \gg 1$ and all $\mathcal{D}_j = -\infty$ with $d \nmid j$. Truncating by d, we can assume that $\mathcal{D}_i \neq -\infty$ for all $i \gg 1$.

Again by monotonicity, we have an increasing subsequence $\mathcal{D}_{ij}$ for $\mathcal{D}_i \neq -\infty$. It converges to $\mathcal{D}$ that is a real number if the sequence is bounded from above, or to $+\infty$. Thus, we have a convergent subsequence $\mathcal{D}_{ij} \to \mathcal{D}$.

This gives the required limit. For any $\varepsilon > 0$, there exists $j \geq 1$ such that $\mathcal{D}_j > \mathcal{D} - \varepsilon/2$ (or $> 2\varepsilon$ if $\mathcal{D} = +\infty$), and all $\mathcal{D}_i \neq -\infty$ with $i \geq j$. Then, for all $i \gg 1$, $i = jq + r$ with $j \leq r \leq 2j$, $0 < jq$, and

$$\mathcal{D}_i \geq \frac{jq}{i}\mathcal{D}_{jq} + \frac{r}{i}\mathcal{D}_r \geq \frac{jq}{i}\mathcal{D}_j + \frac{r}{i}\mathcal{D}_r > \frac{jq}{i}\left(\mathcal{D} - \frac{\varepsilon}{2}\right) + \frac{r}{i}\mathcal{D}_r > \mathcal{D} - \varepsilon$$

(or $> \frac{jq}{i}2\varepsilon + \frac{r}{i}\mathcal{D}_r > \varepsilon$). Thus, $\liminf_{i\to\infty}\mathcal{D}_i \geq \mathcal{D}$.

Since this holds for any convergent subsequence, $\lim_{i\to\infty}\mathcal{D}_i$ exists, is equal to $\mathcal{D}$, and by arithmetic monotonicity we have the inequality $\mathcal{D}_i \leq \mathcal{D}$.

Finally, to find the common support $\mathcal{S}$ of the b-divisors $\mathcal{D}_i$, we can again assume that $\mathcal{D}_i \neq -\infty$ for all $i \gg 1$, or say, for all $i \geq j \geq 1$. Then, as above, for all $i \gg 1$,

$$\mathcal{D}_i \geq \frac{jq}{i}\mathcal{D}_j + \frac{r}{i}\mathcal{D}_r,$$

where $j \leq r \leq 2j$. On the other hand, if the characteristic system is bounded by $\mathcal{D}$, then all $\mathcal{D}_i \leq \mathcal{D}$. Thus, for all $i \gg 1$, $\operatorname{Supp}\mathcal{D}_i$ is in the union of $\operatorname{Supp}\mathcal{D}$ and of all $\operatorname{Supp}\mathcal{D}_r$. (If $\mathcal{D} \geq \mathcal{D}_i \geq \mathcal{D}'$, then $\operatorname{Supp}\mathcal{D}_i$ is in the union of $\operatorname{Supp}\mathcal{D}$ and $\operatorname{Supp}\mathcal{D}'$.) Taking the union of $\operatorname{Supp}\mathcal{D}_i$ not included in the $i \gg 1$ gives $\mathcal{S}$. $\square$

Proof of Proposition 4.22. Let $\mathcal{L}$ be an algebra bounded by a b-divisor $\mathcal{D}$. Then, by Proposition 4.15(3), each $\mathcal{M}_i \le i\mathcal{D}$, which implies (2). Conversely, (2) together with Proposition 4.15(2) imply the inclusion $\mathcal{L} \subset \mathcal{R}_{X/Z}\mathcal{D}$, provided that $\mathcal{D}$ satisfies the coherence of Example 4.12. In fact, we can choose the required $\mathcal{D}$ as the completion $\overline{D}$, where D is a Cartier divisor on X such that $D \ge D_X$. Moreover, such D exists among the hyperplane sections of X/Z.

Thus, each $(\mathcal{M}_i)_X \le (i\mathcal{D})_X = i(\mathcal{D}_X) \le iD$. Hence, each $\mathcal{M}_i \le \overline{iD} = i\overline{D}$ by Lemma 4.23, which implies (1) and boundedness.

Finally, by Lemma 4.24, (2) is equivalent to (3) and Addendum 4.22.1 with a b-divisor $\mathcal{D}$. □

Corollary 4.26. *Quasi-isomorphism preserves boundedness of algebras. The corresponding* characteristic limit *satisfies $\mathcal{D}' \sim_{\mathbb{Q}} r\mathcal{D}$ for some positive $r \in \mathbb{Q}$* (cf. Corollary 3.5).

Proof. Immediate by Proposition 4.15(8) and Proposition 4.22(2),(3). □

Corollary 4.27. *The restriction of a bounded algebra is also bounded* (provided it is well defined).

Proof–Explanation. Immediate by Proposition 4.22(1) because $\mathcal{L}_{|Y} \subset \mathcal{R}_{Y/Z}D_{|Y}$. The existence of a Cartier divisor D such that $D_{|Y}$ is well defined follows from the proof of Proposition 4.22(1). Indeed, $\mathcal{L}_{|Y}$ is well defined if $Y \subset X$ is not contained in the poles of any $0 \ne s \in \mathcal{L}$. We then say that Y is *in general position* with respect to $\mathcal{L}$ or $\mathcal{L}$ is in general position with respect to Y (cf. Definitions 3.41 and 3.47). Equivalently, the positive component of each b-divisor $\mathcal{M}_i$ in $\operatorname{Mov}\mathcal{L}$ or of its characteristic system does not contain Y. The same should hold for the limit $\mathcal{D}$. Then we can find $D \ge \mathcal{D}_X$ such that its positive component does not contain Y. □

Note that the restriction $\mathcal{D}_{|Y}$ is usually not well defined even on a subvariety $Y = S^i$ of a normal ladder (see Definition 3.39), even if $\mathcal{L}$ is considered up to a quasi-isomorphism or the limit $\lim_{i\to\infty}\mathcal{D}_i = \mathcal{D}$ up to similarity. This is because $\mathcal{D}$ may have irrational multiplicities along Y (cf. Section 9, Corollary 9.20). However, in Mixed restriction 7.3, we introduce a certain type of restriction $_{\mathbin{\vdots}Y}$ for which $\lim_{i\to\infty}\mathcal{D}_{i\mathbin{\vdots}Y}$ exists, provided that $\mathcal{L}$ is bounded. The limit does not usually commute with the restriction (actually, it semicommutes by Lemma 4.23; that is, the limit of the restriction $\le$ the restriction of the limit; cf. the b-nef property of $\mathcal{D}^m$ in Example 4.30 below) and is formal but, nonetheless, useful (cf. Proposition 9.13, (BWQ)).

Theorem 4.28 (Limiting Criterion)**.** *A functional algebra $\mathcal{L}$ is f.g. if and only if the limit* $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$ stabilizes, *that is, $\mathcal{D} = \mathcal{D}_i$* for some $i \gg 1$.

Under either assumption, the limit $\mathcal{D}$ is a b-semiample $\mathbb{Q}$-divisor; *the algebra and its characteristic system are bounded.*

Proof. Immediate by Corollary 4.16 and the arithmetic monotonicity of Lemma 4.24. Note that stabilization is preserved under similarity of systems.

In general, stabilization does not mean that $\mathcal{D} = \mathcal{D}_i$ for all $i \gg 1$. However, this holds up to a truncation, essentially by the above monotonicity (cf. Corollary 5.13). □

By Example 4.18(1), for an algebra to be f.g., it is not sufficient that its limit $\mathcal{D}$ is a b-semiample $\mathbb{Q}$-divisor.

Example 4.29. Let $\mathcal{D}_\bullet$ be a *bounded* system of $\mathbb{R}$-b-divisors associated with a pbd algebra (see Definition 4.10); that is, there exists a b-divisor $\mathcal{D}'$ such that all $\mathcal{D}_i/i \le \mathcal{D}'$ (cf. Proposition 4.22(2)). Then, by Lemma 4.24, the characteristic system $\mathcal{D}_i/i$ has a finite limit $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i/i$. The pbd algebra is also bounded and has a characteristic limit $\mathcal{D}^m \overset{\text{def}}{=} \lim_{i\to\infty}\mathcal{M}_i/i \le \mathcal{D}$. Thus, we have a decomposition

$$\mathcal{D} = \mathcal{D}^m + \mathcal{D}^e,$$

where $\mathcal{D}^e$ is effective, the *fixed part* of $\mathcal{D}_\bullet$. We say that $\mathcal{D}^e$ is the *stable base b-divisor*; its support is the *stable base b-locus*. We call their traces on X the *stable base divisor* and *stable base locus*.

Example 4.30. We now discuss the case of b-divisorial algebras. Let $\mathcal{D}$ be an $\mathbb{R}$-b-divisor as in Example 4.12. Its *characteristic limit* is the characteristic limit of its algebra $\mathcal{R}_{X/Z}\mathcal{D}$, that is, $\mathcal{D}^m \stackrel{\text{def}}{=} \lim_{i\to\infty} \mathcal{D}_i$. This exists because $\mathcal{R}_{X/Z}\mathcal{D}$ is bounded by itself, and $\mathcal{D}^m \le \mathcal{D}$ by Proposition 4.15(3) (but not always $= \mathcal{D}$). Thus, we have a decomposition

$$\mathcal{D} = \mathcal{D}^m + \mathcal{D}^e,$$

where $\mathcal{D}^e = \mathcal{D} - \mathcal{D}^m \ge 0$, and, if algebra $\mathcal{R}_{X/Z}\mathcal{D}$ is nontrivial (e.g., $\mathcal{D}$ is effective up to $\sim_{\mathbb{Q}}$ or big$/Z$), then $\mathcal{D}^m$ is the *maximal pseudo-b-semiample* part (or b-divisor) of $\mathcal{D}$ (or *maximal pbs ample* part of $\mathcal{D}$ for short), and $\mathcal{D}^e$ is the *fixed part* of $\mathcal{D}$. Indeed, the *b-semiample part* of $\mathcal{D}$ is a b-semiample $\mathbb{R}$-b-divisor $\mathcal{D}' \le \mathcal{D}$, and $\mathcal{D}' = \mathcal{D}^m$ if $\mathcal{D}' \ge \mathcal{D}^m$. A *pseudo-b-semiample* divisor is a limit of these (see [34, разд. 8]). In particular, $\mathcal{D} = \mathcal{D}^m$ itself is pbs ample if $\mathcal{D}^e = 0$.

This generalizes the decompositions of Remark 3.30. Indeed, if $D = D^m + E$ is as in the remark, then $D^m = \mathcal{D}^m_X$ for some b-semiample $\mathbb{R}$-b-divisor $\mathcal{D}^m$. Thus, if we take $\mathcal{D} = \mathcal{D}^m + E$, then $D = \mathcal{D}_X$, $D^m = \mathcal{D}^m_X$, and $E = \mathcal{D}^e$.

In general, $\mathcal{D}^m$ is not always b-semiample. However, it is b-nef whenever it is b-$\mathbb{R}$-*Cartier*, i.e., $\mathcal{D}^m = \overline{D^m}$ for an $\mathbb{R}$-Cartier divisor D^m on a model Y/Z of X/Z. This follows from the formal fact that, on a good model Y, the mixed restriction $\mathcal{D}^m{}_{\restriction C}$ to a generic curve is $\ge \lim_{i\to\infty}(\mathcal{D}_{i\restriction C})$, although it may not commute with the limit (cf. Lemma 4.23).

If $\mathcal{R}_{X/Z}\mathcal{D}$ is f.g., then $\mathcal{D}^m$ is b-semiample by Limiting Criterion 4.28 (cf. Remark 3.30); moreover, $\mathcal{D}^m$ is then a $\mathbb{Q}$-b-divisor. However, g.a.g. can fail even if $\mathcal{D}$ is big$/Z$ because Lemma 3.19 does not hold for any prime b-divisor E (see Example 4.31 below). The latter is related to saturation, as we discuss below.

We do not expect f.g. for an arbitrary b-divisor $\mathcal{D}$, even in the situation of Remark 3.30(2) (cf. Theorem 3.33). This can again be attributed to saturation (cf. Conjecture 4.39 and Example 4.35 below).

Example 4.31. Let D be a nonzero effective Weil $\mathbb{R}$-divisor on X considered as a b-divisor and $\dim X = n \ge 2$. Then the system iD satisfies the coherent assumptions of Example 4.12 with $\mathcal{O}_X(iD) = \mathcal{O}_X$ for all i but does not stabilize. That is, for any model $g\colon Y \to X$ of X, $g_*\mathcal{O}_Y(iD_Y) \ne \mathcal{O}_X = \mathcal{O}_X(iD)$ for $i \gg 0$. Thus, D has mobile part $D^m = 0$ and fixed part $D^e = D$; the algebra is f.g. and g.(a.)g., but with nontrivial stable base divisor.

Proof of Corollary 3.29. Immediate by Corollary 4.11 for $\mathbb{Q}$-divisors; and g.a.g. holds with empty base divisorial locus or $E = 0$ (cf. Proposition 4.15(7)).

If D is an $\mathbb{R}$-divisor that is ample$/Z$ but not a $\mathbb{Q}$-divisor, then $\mathcal{R}_{X/Z}D$ is never f.g. (cf. Stupid Example 3.16). Indeed, since $\mathcal{D} = \overline{D}$ is (b-semi)ample and $\mathcal{R}_{X/Z}\mathcal{D} = \mathcal{R}_{X/Z}D$ is f.g., $\mathcal{D} = \mathcal{D}^m$ is a $\mathbb{Q}$-divisor by Example 4.30. □

4.32. Saturation of linear systems. Saturation can sometimes be explained in terms of sheaves and their algebras. However, in the situations where we need it in an essential way, a better explanation (and arguably the only possible explanation) uses divisors and linear systems. Let D be an $\mathbb{R}$-Weil divisor on X/Z. We recall that its linear system is the set

$$|D| = |D|_{X/Z} = \{D' \mid D' \ge 0 \text{ and } D' \sim D/Z\}.$$

In its local version over a point $P \in Z$, $|D|$ is defined up to the equivalence relation that identifies divisors D' that are equal over a neighborhood of P. Most important for us is the decomposition of D into mobile and fixed parts (see Conventions 1.14).

Similarly, for any $\mathbb{R}$-b-divisor, we can define $\mathbb{R}$-b-divisors $\operatorname{Mov}\mathcal{D}$ and $\operatorname{Fix}\mathcal{D}$, the *mobile* and *fixed* parts (or *components*) of $\mathcal{D}$ (or of its linear system $|\mathcal{D}|$, or sheaf $f_*\mathcal{O}_X(\mathcal{D}) \subset k(X)$); for the definition, see the construction of Proposition 4.15. By definition, $(\operatorname{Mov}\mathcal{D})_Y \le \operatorname{Mov}(\mathcal{D}_Y)$ on any model Y/Z

of X/Z. However, if the sheaf $\mathcal{O}_X(\mathcal{D})$ is coherent and stabilizes on passing to higher models as in Example 4.12, then, by Proposition 4.15(1),(4), we have the following stabilization on a *sufficiently high* resolution X_{hr}: there is a model $g\colon X_{\mathrm{hr}}\to Z$ of X/Z such that $f_*\mathcal{O}_X(\mathcal{D})=g_*\mathcal{O}_{X_{\mathrm{hr}}}(M)$, where

$$M=(\operatorname{Mov}\mathcal{D})_{X_{\mathrm{hr}}}=\operatorname{Mov}(\mathcal{D}_{X_{\mathrm{hr}}}),\qquad \operatorname{Mov}\mathcal{D}=\overline{M},\qquad \operatorname{Bs}|M|=\varnothing,$$

$$\text{and}\qquad |M|+\operatorname{Fix}\mathcal{D}_{X_{\mathrm{hr}}}=|\mathcal{D}_{X_{\mathrm{hr}}}|=|\mathcal{D}|_{X_{\mathrm{hr}}}.$$

(However, see Example 4.31.)

Definition 4.33. Let D and C be $\mathbb{R}$-divisors on X. We say that D is *saturated* with respect to C or C-*saturated* if $M=\operatorname{Mov}\lceil D+C\rceil\le D$. By convention, $\operatorname{Supp}(-\infty)=\varnothing$, and the condition holds if $|\lceil D+C\rceil|=\varnothing$, i.e., $M=-\infty$. There is also nothing to check if $\lceil D+C\rceil\le D$.

We can attribute the same saturation to the linear system $|D|$, or even to any (incomplete) linear system, as well as to a functional sheaf $\mathcal{F}\subset k(X)$. In this last case, we take $D=\operatorname{Mov}\mathcal{F}$. Moreover, if $\mathcal{F}=f_*\mathcal{O}_X(D)$, then C-saturation of D implies the same for $\mathcal{F}$ (cf. Remark 4.34(1) below).

For $\mathbb{R}$-*b-divisors* $\mathcal{D}$ and $\mathcal{C}$, the *saturation* of $\mathcal{D}$ with respect to $\mathcal{C}$ or $\mathcal{C}$-saturation means the $C=\mathcal{C}_{X_{\mathrm{hr}}}$-*saturation* of $D=\mathcal{D}_{X_{\mathrm{hr}}}$ on *any sufficiently high models* X_{hr}/Z of X/Z. This means that there is a model Y/Z of X/Z such that saturation holds for any model X_{hr}/Z that is also $/Y$.

For asymptotic saturation, we consider a system $\mathcal{D}_\bullet$ of $\mathbb{R}$-b-divisors (e.g., a characteristic system $\mathcal{D}_i=\mathcal{M}_i/i$). This system is *asymptotically saturated* with respect to $\mathcal{C}$ or *asymptotically* $\mathcal{C}$-*saturated* if there exists a natural number I, the *saturation index*, such that, for all natural numbers i,j with $I\mid i,j$ and on any sufficiently high model X_{hr}/Z, the decomposition into mobile and fixed parts satisfies

$$\operatorname{Mov}(\lceil j\mathcal{D}_i+\mathcal{C}\rceil_{X_{\mathrm{hr}}})\le j(\mathcal{D}_j)_{X_{\mathrm{hr}}}$$

(cf. Remarks 4.34(2),(3) below).

A functional algebra is *asymptotically* $\mathcal{C}$-*saturated* if this holds for its characteristic system $\mathcal{D}_\bullet$. Moreover, a pbd algebra associated with a system $\mathcal{M}_\bullet$ is asymptotically $\mathcal{C}$-saturated if this holds for its *characteristic* system $(\mathcal{D}_i=\mathcal{M}_i/i)$ (cf. Remark 4.34(1)).

Finally, we say that an $\mathbb{R}$-b-divisor $\mathcal{D}$ is *asymptotically* $\mathcal{C}$-*saturated* if this holds for its characteristic system $(\mathcal{D}_i=\mathcal{D})$ (cf. Example 4.12). Even if $\mathcal{D}=\overline{D}$ is a Cartier b-divisor, we still need to run through all sufficiently high models X_{hr}/Z (cf. Example 4.35 below). The point is that we have no universal X_{hr}/Z unless the b-divisorial algebra of $\mathcal{D}$ is f.g. (cf. Remark 4.34(2)).

Remarks 4.34. (1) It is enough to verify the inequality

$$M=\operatorname{Mov}\lceil D+C\rceil\le D$$

over $\lceil D+C\rceil>D$, that is, to check that $\operatorname{mult}_{D_i}M\le\operatorname{mult}_{D_i}D$ for any prime divisors D_i with $\lceil\operatorname{mult}_{D_i}D+\operatorname{mult}_{D_i}C\rceil>\operatorname{mult}_{D_i}D$. The same holds for the other saturations.

This is equivalent to $\operatorname{Mov}\lceil D+C\rceil\le\operatorname{Mov}D$. In most applications, D is mobile, or even free. Then $\operatorname{Mov}D=D$ and saturation is equivalent to $\operatorname{Mov}(D+\lceil C\rceil)=\operatorname{Mov}\lceil D+C\rceil\le\operatorname{Mov}D=D$. Thus, if in addition $\lceil C\rceil\ge0$, then $D\le\operatorname{Mov}(D+\lceil C\rceil)\le D$, and $\operatorname{Mov}(D+\lceil C\rceil)=D$ or, in terms of linear systems,

$$|D+\lceil C\rceil|=|D|+\lceil C\rceil,$$

where $\lceil C\rceil=\operatorname{Fix}(D+\lceil C\rceil)$.

(2) In general, $(\operatorname{Mov}\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil)_{X_{\mathrm{hr}}}\le\operatorname{Mov}(\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\mathrm{hr}}})$, but equality does not necessarily hold. In important cases, equality holds on any sufficiently high model X_{hr}; this means that

$\operatorname{Mov}(\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\mathrm{hr}}})$ stabilizes on *all* sufficiently high models, but only for the given i,j (cf. Examples 4.35 and 4.47 below). In applications, however, we need *some* sufficiently high model X_{hr} on which, say, some divisors have normal crossing, and some b-divisors are Cartier (cf. the proof of Proposition 6.26). But we prefer to consider all sufficiently high models X_{hr} because, in most cases, it is easy to verify saturation for all such X_{hr} by stabilization (cf. Proposition 4.46 below).

The mobile part decreases on passing to higher models. Thus, if $\mathcal{D}_j$ is b-$\mathbb{R}$-Cartier over a model Y/Z of X/Z, that is, $\mathcal{D}_j=\overline{D_j}$ for an $\mathbb{R}$-Cartier divisor D_j on Y, then we do not have to worry about *any* sufficiently high model if saturation holds on *some* model (see Lemma 6.36), for example, if $\mathcal{D}_j$ is b-semiample or $\mathcal{M}_j=j\mathcal{D}_j$ is b-free/Z. This holds for any functional algebra.

(3) By remark (1), the inequality in saturation is equal to the inequality $\operatorname{Mov}\lceil D+C\rceil\le\operatorname{Mov}D$ for the mobile part and, in turn (cf. Proposition 4.15(3)), to the inclusion

$$f_*\mathcal{O}_X(\lceil D+C\rceil)\subset f_*\mathcal{O}_X(D).$$

The same thing holds for other types of saturation.

If $\mathcal{D}_\bullet$ is the characteristic system of a functional algebra with mobile system $\mathcal{M}_\bullet$ and $i\mid j$, then, as in (1) above, asymptotic saturation means that

$$M=\operatorname{Mov}(jD_i+\lceil C\rceil)=\operatorname{Mov}\lceil jD_i+C\rceil\le\operatorname{Mov}jD_j=jD_j$$

with $C=\mathcal{C}_Y$, $D_i=(\mathcal{D}_i)_Y$, $D_j=(\mathcal{D}_j)_Y$ on any sufficiently high model $X_{\mathrm{hr}}=Y/X$, where $iD_i=\operatorname{Mov}(\mathcal{M}_i)_Y$, $jD_j=\operatorname{Mov}(\mathcal{M}_j)_Y$ and both linear systems $|iD_i|$ and $|jD_j|$ are free (cf. Proposition 4.15(1)). Thus, in terms of linear systems,

$$|jD_i+\lceil C\rceil|=|M|+F,$$

where $M=\operatorname{Mov}(jD_i+\lceil C\rceil)\le M_j=jD_j$ and $F=\operatorname{Fix}(jD_i+\lceil C\rceil)$. This can be interpreted in terms of sheaves: namely,

$$f_*\mathcal{O}_Y(M_i)^{\otimes j/i}\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(\lceil C\rceil)=f_*\mathcal{O}_Y(jD_i+\lceil C\rceil)\subset f_*\mathcal{O}_Y(jD_j)=f_*\mathcal{O}_Y(M_j),$$

where $M_i=iD_i$ and $\lceil C\rceil$ is assumed to be Cartier. However, in most applications, we consider $i\gg j$, and then asymptotic saturation has no sheaf theoretic interpretation (cf. Example 4.35 below) because the integral parts depend on divisors up to a linear equivalence, not up to $\mathbb{Q}$-linear equivalence (compare Lemma 6.30 below).

(4) Again, if $\mathcal{D}_\bullet$ is the characteristic system of a functional algebra, then the arithmetic monotonicity of Lemma 4.24 gives the inequalities

$$\operatorname{Mov}(\lceil j(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\mathrm{hr}}})\le j(\mathcal{D}_j)_{X_{\mathrm{hr}}}\le j(\mathcal{D}_l)_{X_{\mathrm{hr}}}\le j(\mathcal{D})_{X_{\mathrm{hr}}}$$

with any $j\mid l$ and limiting divisor $\mathcal{D}$. In fact, this last inequality for $\mathcal{D}$ is the most important (cf. Example 4.41).

(5) For $j=i$, the asymptotic saturation inequality implies

$$\operatorname{Mov}(\lceil\mathcal{M}_i+\mathcal{C}\rceil_{X_{\mathrm{hr}}})=\operatorname{Mov}(\lceil i(\mathcal{D}_i)+\mathcal{C}\rceil_{X_{\mathrm{hr}}})\le i(\mathcal{D}_i)_{X_{\mathrm{hr}}}=(\mathcal{M}_i)_{X_{\mathrm{hr}}},$$

which means that $\mathcal{M}_i=i\mathcal{D}_i$ is $\mathcal{C}$-saturated. The same holds for each component $\mathcal{L}_i$ of a functional algebra provided that $\mathcal{L}$ is asymptotically saturated (cf. the proofs of Theorem 6.19(2), p. 163 and Theorem 9.9, p. 197).

(6) In general, asymptotic $\mathcal{C}$-saturation does not imply that $\mathcal{D}_\bullet$ is bounded (cf. Example 5.14); but this holds for a functional algebra and its characteristic system. By Lemma 4.23, it is enough to

establish this on any model Y/Z of X/Z, for example, on X itself. Then $\mathcal{D}_\bullet$ is convergent and has other properties (cf. $(\mathrm{FGA})_n$ in Definition 4.38 and Conjecture 4.39 below). However, we always consider bounded algebras, as they appear in our applications.

(7) Note also that *similarity* of characteristic systems is induced by similarity of $(\mathcal{M}_i = i\mathcal{D}_i)$; e.g.,

- $\mathcal{D}_\bullet \sim \mathcal{D}'_\bullet$, where $\mathcal{D}_\bullet = \mathcal{D}'_\bullet + \overline{(a)}$; and
- truncation means $\mathcal{D}_\bullet^{[I]} = I(\mathcal{D}_{iI})$, that is, *each* $\mathcal{D}_i^{[I]} = I\mathcal{D}_{iI}$.

Similarity preserves asymptotic $\mathcal{C}$-saturation for any system $\mathcal{D}_\bullet$ (see Proposition 9.17 below).

We discuss other properties of saturation below (see Propositions 4.42, 4.50, Lemma 4.44, and their addenda, Propositions 6.32, 6.34, and 9.17, Lemmas 6.36, 9.16, and 9.23, Corollary 9.19). In most applications, the saturation is *very exceptional*, i.e., $\mathcal{C}$ or $\lceil \mathcal{C} \rceil$ are exceptional on X.

Example 4.35 (exceptional saturation)**.** We say that the saturation of Definition 4.33 is *exceptional*/X if it holds for any $\mathcal{C}$ that is exceptional on X. Exceptional saturation is the extension from Weil divisor to Cartier b-divisor that gives the equality $\mathcal{O}_X(D) = \mathcal{O}_X(\mathcal{D})$. It usually means that D is also exceptionally saturated as a b-divisor with $\operatorname{Mov} D = D$. In dimension 1, the saturation means nothing; cf. Example 4.41, even with a stronger saturation.

In particular, an *integral* divisor D on X/X is exceptionally saturated; for this, since $\mathcal{C}$ is quite arbitrary, we only need to check the inequality

$$\operatorname{Mov}\lceil D_{\mathrm{hr}} + \mathcal{C}_{X_{\mathrm{hr}}} \rceil = \operatorname{Mov}(D_{\mathrm{hr}} + \lceil \mathcal{C}_{X_{\mathrm{hr}}} \rceil) \le D_{\mathrm{hr}}$$

for a prime divisor D that is not exceptional on X. On X_{hr}, we can add any effective exceptional divisor over X to D. This is usually taken for granted in classical algebraic geometry when we work with Cartier divisors (cf. Lemmas 3.19 and 4.23). Indeed, if D is Cartier, its completion $\mathcal{D} = \overline{D}$ is saturated: $\mathcal{O}_X(D) = \mathcal{O}_X(\mathcal{D})$ and $(\operatorname{Mov} \mathcal{D})_X = \operatorname{Mov} D = D$. This does not hold if D is considered as a Weil b-divisor (cf. Example 4.31 where $D \ne 0$ but $\operatorname{Mov} D = D^m = 0$).

The sheaf $\mathcal{O}_X(D)$ is also saturated; for a divisorial sheaf, this means that

$$\mathcal{O}_X(D)^{\vee\vee} = \mathcal{O}_X(D)$$

(or [37, Proposition 2, (iv)]). The divisorial algebra $\mathcal{R}_{X/X} D$ is also exceptionally asymptotically saturated with $I = 1$. This means that $\mathcal{O}_X(iD) = \mathcal{O}_X(i\mathcal{D})$ for each i, where $\mathcal{D}$ is the characteristic limit of this algebra; for an $\mathbb{R}$-divisor D, this holds for some $I \ge 1$ if and only if D is a $\mathbb{Q}$-divisor (cf. Stupid Example 3.16 and Theorem 3.18). Thus, it fails for most $\mathbb{R}$-divisors D, and asymptotic saturation is closely related to rationality.

This also holds for X/Z, and we can explain exceptional saturation for D/Z and its algebra $\mathcal{R}_{X/Z} D$ in the same style in terms of double dual. However, for divisorial functional sheaves, it has three different versions, namely: e.g., for birational X/Z,

$$(f_*\mathcal{O}_X(D))^{\vee\vee} \supset (f_*(\mathcal{O}_X(D)^\vee))^\vee \supset f_*(\mathcal{O}_X(D)^{\vee\vee}) = f_*\mathcal{O}_X(D).$$

In general, for a functional subsheaf $\mathcal{L} \subset k(X)$, we can take any of these versions. We use the *maximal saturation* $(f_*\mathcal{L})^{\vee\vee} = \mathcal{L}$ (independent of X) or the *minimal saturation* $f_*(\mathcal{L}^{\vee\vee}) = \mathcal{L}$, where we take the double dual for $\mathcal{O}_X$. Thus, the minimal saturation depends on the choice of X. However, the maximal saturation corresponds to *very exceptional* saturation on divisors; that is, we can add any divisors that are very exceptional on Z. Usually, $\mathcal{R}_{X/Z} D$ is not maximally or very exceptionally saturated, say, if X/Z is birational, because there is a difference between divisorial and b-divisorial sheaves and algebras. Minimal saturation agrees with exceptional asymptotic saturation/X, and it holds for any integral or $\mathbb{Q}$-divisor D (only integral in the case of sheaves).

This does not hold in general for b-divisorial, pbd-divisorial, or functional algebras $\mathcal{L} \subset k(X)_\bullet$. But in the cases we are interested in, minimal saturation and exceptional asymptotic saturation hold *over a different birational model* Y/Z of X/Z (for this, see prediction models, triples, Addendum 5.12.2, and Conjecture 6.14, (BIR)):

- the flipping algebra of Example 3.15 with $Y = X^+ = \operatorname{Proj}_T \mathcal{FR}_{X/T}D$ and the $\mathbb{Q}$-divisor D (here $I = 1$ if D is integral);
- the (pre-restriction) algebra $\mathcal{R}_{X/T}D$ with $Y = \operatorname{Proj}_T \mathcal{R}_{X/T}D$ and $I = 1$ of Example 3.40 (cf. proof of Induction Theorem 1.4, p. 134);
- any b-divisorial algebra as in Example 4.12 with a Cartier $\mathbb{Q}$-b-divisor $\mathcal{D}$, on a model Y over which $\mathcal{D} = \overline{D}$ for a $\mathbb{Q}$-Cartier divisor D on Y; and
- any f.g. functional algebra $\mathcal{L}$ on any model of $Y/\operatorname{Proj}_Z \mathcal{L}$, by Corollary 4.16.

In this sense, asymptotic saturation is a necessary condition for f.g., but not a sufficient condition since f.g. also needs certain conditions on X/Z (cf. Remark 3.30(2), Theorem 3.33, Conjectures 3.35 and 4.39, Proposition 4.42, and Remark 4.40(2)). However, asymptotic saturation may help in finding a model Y on which we expect f.g. to hold. Indeed, if $\mathcal{L}$ is exceptionally asymptotically saturated/Y and f.g., then Proj defines a rational contraction of X to $\operatorname{Proj}_Z \mathcal{L}$ that is a rational 1-contraction on Y; in particular, apart from a contractible divisorial subset, Y and $\operatorname{Proj}_Z \mathcal{L}$ are isomorphic in codimension 1 if $\mathcal{L}$ is big. In the case of a pbd algebra, it gives the g.a.g. of $\mathcal{D}^m$ or even $\mathcal{D}_\bullet$ over Y (cf. bss ampleness in Theorem 3.18, where the $\mathbb{Q}$-divisor condition for D^{sm} can be replaced by exceptional asymptotic saturation/X; see an example after Proposition 4.54); this generalizes Zariski decomposition (cf. Example 4.30).

Unfortunately, saturation conditions are no good for induction on the dimension. It is well known that the usual saturation is not preserved under restrictions that are not surjective; cf. reasons (2) and (3) in our motivation for introducing (log) canonical saturations below. The same applies to our saturations for an arbitrary $\mathcal{C}$.

Definition 4.36. A $\mathcal{C}$-saturation is *log canonical*/X or, more precisely, log canonical over (X, B), if $\mathcal{C} = \mathcal{A} = \mathcal{A}^X = \mathcal{A}(X, B)$ is the *discrepancy* b-divisor of the pair (X, B) for some $\mathbb{R}$-divisor B on X. In the divisorial case, we take $C = \mathcal{A}_X = -B$.

We abbreviate log canonical asymptotic saturation to *lca saturation.*

Caution 4.37. For another pair (Y, B_Y) with a model Y/Z of X/Z, in general $\mathcal{A}^Y = \mathcal{A}(Y, B_Y) \neq \mathcal{A}$ and $\mathcal{B}^Y = \mathcal{B}(Y, B_Y) \neq \mathcal{B}$. Thus, the log canonical saturation can be different over (Y, B_Y).

As usual, $\mathcal{A}_Y$ denotes the restriction divisor of the b-divisor $\mathcal{A}$ on a model Y of X. In general, $\mathcal{B}_Y = B^Y \neq B_Y$. The latter is the codiscrepancy of (Y, B_Y) on Y, and $\mathcal{B}_Y = B^Y = B_Y$ if and only if (Y, B_Y) is a *crepant* birational transform of (X, B) (cf. Definition 6.9, (CRP)). Then lc and lca saturations are the same over both models.

By Example 4.18(1), for a nondivisorial algebra, saturation on its own is not enough for f.g. But we hope that asymptotic saturation works better, especially lca saturation (cf. Conjecture 4.39 and Example 4.41 below). In view of the applications we have in mind, there are four reasons for this:

(1) the restriction of a flip may be a nonflip, e.g., a semistable flip restricted on irreducible $Y = D = D_1$, with $a = 1$ [42, рис. 4(a)] (cf. also the proof of Special Termination 2.3);

(2) log canonical saturation is compatible with restrictions by Kawamata–Viehweg vanishing (cf. Proposition 4.50);

(3) $\mathcal{A}$ is sufficiently effective: e.g., for a Klt pair (X, B), $\lceil \mathcal{A} \rceil \geq 0$ is exceptional and, at the same time, grows at least linearly with blowups (cf. Proposition 4.46, (CGR), and (LGD) for prediction models in Section 5 below); and

(4) $\mathcal{A}$-saturation is well defined (cf. Example 4.47 below).

Definition 4.38. Suppose that a pair $(X/T, B)$ satisfies

(i) X/T is a contraction;

(ii) (X, B) is Klt;

(iii) $-(K + B)$ is nef and big/T; and, finally,

(iv) $\dim X = n$.

Conditions (ii) and (iii) mean that $(X/T, B)$ is a weak log Fano contraction (cf. (WLF) in Proposition 4.42).

Then we say that a functional $\mathcal{O}_T$-algebra $\mathcal{L} \subset k(X)_\bullet$ or its characteristic system $\mathcal{D}_\bullet$ is *of type* $(\mathrm{FGA})_n$ if

$(\mathrm{FGA})_n$ the algebra is bounded and *lca saturated* over $(X/T, B)$.

It is of type $(\mathrm{FGA})_n(\mathrm{bir})$ if X/T is birational.

Conjecture 4.39. *An algebra of type* $(\mathrm{FGA})_n$ *is f.g.*/T.

By Limiting Criterion 4.28, an equivalent condition is the stabilization of $\lim_{i\to\infty} \mathcal{D}_i$ for the characteristic system $\mathcal{D}_\bullet$ of $\mathcal{L}$. Thus, the system stabilizes if

(LIM) there is a limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ (which is finite for nontrivial algebra); and

(LCA) it is lca saturated.

However, we prefer to work with b-divisors (cf. Remark 4.40(5)). Note for this that the system $\mathcal{D}_\bullet$ also satisfies

(MXD) each $\mathcal{D}_i = \mathcal{M}_i/i \le \mathcal{D}$ by Addendum 4.22.1;

(LBF) each $\mathcal{M}_i = i\mathcal{D}_i$ is b-free by Proposition 4.15(1); and

(AMN) *arithmetic monotonicity*: for any $j \mid i$, we have $\mathcal{D}_i \ge \mathcal{D}_j$ by Lemma 4.24.

Moreover, one can expect that LMMP in dimension $m \le n$ implies $(\mathrm{FGA})_n$ (cf. Theorem 3.33).

By Corollary 1.5, the conjecture would imply the existence of log flips in dimension n modulo LMMP in dimension $m \le n-1$ (cf. Remark 4.40(6)). This would be an inductive construction of log flips.

Remark 4.40. Some of the assumptions in the conjecture can be relaxed or modified:

(1) We can replace the contraction X/T by any proper morphism X/Z of normal algebraic varieties or normal algebraic spaces (certainly), and probably also normal analytic spaces (use the Stein factorization of $X \to Z$ [10, Pt. II, Ch. 2, 3.7]). (Compare Lemma 10.15 and its proof.)

(2) We expect that Conjecture 4.39 also holds for the 0-log pairs $(X/Z, B)$ of Remark 3.30(2) (cf. Theorem 3.33).

(3) Moreover, Conjecture 4.39 may hold for pairs $(X/Z, B)$ with $\mathcal{D}_X = D = K + B + H$, where H is a nef and big/Z $\mathbb{R}$-divisor; or the same on some crepant model $(Y/Z, B_Y)$ of $(X/Z, B)$ for $(\mathcal{D}_i)_X$ with $i \gg 0$. Possibly, $K + B + H$ can be weakened just to log canonical divisor or its positive multiple on a good (e.g., Klt) model, and even up to numerical equivalence $\equiv /Z$ if it is big (cf. Example 4.49 below). Note that, in a similar situation, the proof of base point free uses induction for log canonical divisors of this type, rather than nef divisors on Fano contractions [40, §2].

(4) Everything possibly works for more general gradings, e.g., $\mathbb{N}^r$-gradings (cf. Cox's rings and [13, Conjecture 2.14]).

(5) It can also be generalized to worse singularities, that is, when (X, B) is not Klt, B is not a boundary, and X is nonnormal, or not even seminormal. However, we still need to assume that B is effective and that the algebra $\mathcal{L}$ behaves well (f.g.) or the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ stabilizes near the locus of log canonical singularities $\mathrm{LCS}(K + B)$ (see [41, определение 3.14], Example 4.41, and Corollary 6.42 below).

(6) The restriction of a general flip or a pl flip is not necessarily a flip. However, we expect that $(\mathrm{FGA})_{n-1}(\mathrm{bir}) = (\mathrm{RFA})_{n,n-1}(\mathrm{bir})$. For algebras, this means that each normal algebra $\mathcal{L}$ of type $(\mathrm{FGA})_{n-1}(\mathrm{bir})$ up to a quasi-isomorphism is a restricted algebra of type $(\mathrm{RFA})_{n,n-1}(\mathrm{bir})$ as in Definition 3.47 (cf. proof of Induction Theorem 1.4, p. 134). Then the existence of n-fold flips would be equivalent to $(\mathrm{FGA})_{n-1}(\mathrm{bir})$.

(7) Finally, if $\mathcal{D}$ is big, the *stable* crepant model $(X_{\mathrm{st}}/T, B_{\mathrm{st}})$ of $(X/T, B)$ in the conjecture with $X_{\mathrm{st}}/T = \mathrm{Proj}_T\,\mathcal{L}$ is isomorphic in codimension 1 to a crepant weak log Fano model of $(X/T, B)$ (cf. the bss ampleness, Addendum 5.12.2, Example 5.27, and Remark 6.23, as well as Addendum 6.26.2). For given $(X/T, B)$, the family of such X_{st}/T should be finite according to Batyrev [4] (cf. canonical confinement and triples in (BIG) of Conjecture 6.14 below). Moreover, if we replace Klt by the more precise property of ε-log terminality, then, for all such $(X/T, B)$ and under the big condition, according to the conjectures of Alexeev and the Borisovs, X_{st} should be bounded in modulus (cf. [1, 0.4, (1)] and [5, c. 134, теорема]). One expects the same to hold even without the big condition (cf. the proof of Theorem 6.19(2), Conjecture 6.14, (CCS), and Remark 6.15(8) below).

Example 4.41 (a Pythagorean dream)**.** Suppose that $X = C$ is a normal algebraic curve and $B = \sum b_m P_m$ is a Klt $\mathbb{R}$-boundary; in other words, each $b_i \in [0,1)$. Then any lca saturated/(C, B) algebra $\mathcal{L}$ is f.g., or the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ stabilizes. Note that in this case $D = \sum d_m P_m = \mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ is a limit of divisors $D_i = \mathcal{D}_i = \sum d_{m,i} P_{m,i}$. Equivalently, each $d_m = \lim_{i\to\infty} d_{m,i}$ with $d_{m,i} \in \mathbb{Q}$. In particular, stabilization means that each d_m is *also* $\in \mathbb{Q}$. This is the point.

All the divisors D and D_i are supported in a finite subset of C by Lemma 4.24. Thus, the limit takes place for the vector of coefficients in $\mathbb{R}^n$ (for a divisor supported in F, see $\mathfrak{D}_F$ on p. 141).

On the other hand, the discrepancy $\sum a_m P_m = \mathcal{A} = A = -B \le 0$. Moreover, by Klt, all $a_m = -b_m \in [0, -1)$. In the one-dimensional case, any $X_{\mathrm{hr}} = C_{\mathrm{hr}} = C$. After truncating $\mathcal{L}$, we can assume that the saturation has index $I = 1$. Thus, lca saturation gives the inequality

$$\mathrm{Mov}\lceil jD_i + A\rceil \le jD_j \qquad \text{for all natural numbers } i, j.$$

In our situation, we can assume that $\mathrm{Mov}\lceil jD_i + A\rceil = \lceil jD_i + A\rceil$ for all $i, j \gg 1$.

By RR on curves, this holds unless C is complete, $Z = \mathrm{pt.}$, and $\deg D \le 0$. Moreover, then $\deg D = 0$ because we consider the limit only for nonempty linear systems $|M_i| = |iD_i|$. Thus, $\deg D_i \ge 0$. Hence, $D = 0$, and we have stabilization by (MXD) (see Addendum 4.22.1). The stable model in this case is $C_{\mathrm{st}} = \mathrm{Proj}_Z\,\mathcal{L} = \mathrm{pt.}/Z = \mathrm{pt.}$

Otherwise, we have the inequality $\lceil jD_i + A\rceil \le jD_j$ or, componentwise, for each P_m and all $i, j \gg 1$,

$$\lceil jd_{m,i} + a_m\rceil \le jd_{m,j}.$$

Again by (MXD), this gives, for each P_m and all $j \gg 1$,

$$\lceil jd_m + a_m\rceil = \lceil jd_{m,i} + a_m\rceil \le jd_{m,j} \le jd_m,$$

when we take $i \gg j$! By Diophantine geometry, for any real number $a = a_m > -1$, any real number $d = d_m$ satisfying

$$\lceil jd + a\rceil \le jd$$

for all natural numbers $j \gg 1$ is rational. (See Cassels [7, Lemma 1A]; take the best upper approximations, for $\theta = d$; cf. proof of Approximation Lemma 5.15 below. See also Kronecker's theorem.) This explains also that positive $a = a_m$ are also not good (cf. Remark 4.40(5)). However, A with $\lceil A\rceil = 0$, if even the total $A = 0$, does have an effect!

Since d_m is rational, for any $j \gg 1$ with integral jd_m we get stabilization

$$\lceil jd_m + a_m\rceil = jd_{m,j} = jd_m.$$

By (AMN), this implies stabilization: each $d_{m,i} = d_m$ for all i divisible by some j. Finally, since the divisors have bounded supports, we get stabilization by a truncation of $D_\bullet$. In particular, this establishes $(\mathrm{FGA})_1$.

By these arguments, we can generalize f.g. for any effective B assuming that the limit stabilizes near the $\mathrm{LCS}(K+B) = \{P_m \mid b_m \geq 1\}$; that is, $d_m = \lim_{i\to\infty} d_{m,i}$ stabilizes at each such point P_m.

For nonnormal C, stabilization near bad singularities means that, up to a similarity, all D_i and D are nonsingular, that is, are supported in nonsingular points of C.

To establish even $(\mathrm{FGA})_2$ is not so simple; this is discussed below after some preparations (see the proof of (FGA) in Main Theorem 1.7, p. 174). Now we turn to the proof of Induction Theorem 1.4. For this, we need some results on saturations.

Proposition 4.42. *For any effective $\mathbb{R}$-divisor B such that $K+B$ is $\mathbb{R}$-Cartier, any exceptional saturation/X implies lca saturation/(X,B).*

Thus, $(\mathrm{FGA})_n$ *implies f.g. of any algebra $\mathcal{R}_{X/T}D$ with a $\mathbb{Q}$-divisor D, provided that*

(WLF) *$(X/T,B)$ is a* weak log Fano contraction, *that is, (X,B) is Klt and $-(K+B)$ is nef and big/T*

(*cf. Theorem* 3.33 *and Remark* 4.40(2)).

Addendum 4.42.1. *Any exceptional saturation/X implies the corresponding saturation for $\mathcal{A}' = \mathcal{A}(X,B) + \mathcal{E}$, where $\mathcal{E}$ is a reduced b-divisor such that*

- *$\mathcal{E}_X$ is supported over* $\operatorname{Supp}\mathcal{A}_X = \operatorname{Supp} B$, *and*
- *the $\mathcal{A}$-saturation is* integral *over $\mathcal{E}$.*

Definition 4.43. The fact that asymptotic $\mathcal{C}$-saturation is *integral* over $\mathcal{E}$ means that each $j\mathcal{D}_i + \mathcal{C}$ is integral over $\mathcal{E}$.

We also need its weak version (see (εA$'$S) in Proposition 9.13). The asymptotic $\mathcal{C}$-saturation that is *integral weak* over $\mathcal{E}$ requires only the inequalities in Definition 4.33 under the assumption that $j\mathcal{D}_i + \mathcal{C}$ is integral over $\mathcal{E}$.

The same applies to other types of saturations for $\mathcal{D} + \mathcal{C}$. The weak assumption is void when $\mathcal{E} = 0$, but the saturation is void when all the inequalities do not satisfy the integral weak assumption.

The proposition applies, in particular, to a flipping algebra $\mathcal{FR}_{X/T}D$. Models of algebras having exceptional asymptotic saturations are obtained as models for bss ample divisors by a rational 1-contraction of X. The former $(\mathrm{FGA})_n$ algebras are more general, and their models can blow up some divisors exceptional on X (cf. Remark 4.40(7)). This is the main difference between the former (FGA) approach and the latter Zariski approach to f.g. algebras.

Lemma 4.44. *Let $C_1 \geq C_2$ be divisors and $\mathcal{C}_1 \geq \mathcal{C}_2$ be b-divisors on X/T. Then C_1-saturation implies C_2-saturation, $\mathcal{C}_1$-saturation implies $\mathcal{C}_2$-saturation, and asymptotic $\mathcal{C}_1$-saturation implies asymptotic $\mathcal{C}_2$-saturation.*

Addendum 4.44.1. *Asymptotic $\mathcal{C}_1$-saturation implies asymptotic $\mathcal{C}_2'$-saturation with $\mathcal{C}_2' = \mathcal{C}_2 + \mathcal{E}$, where $\mathcal{E}$ is a reduced b-divisor such that*

- *$\mathcal{C}_1 > \mathcal{C}_2$ over $\mathcal{E}$ (that is, $>$ over each b-prime component of $\mathcal{E}$); and*
- *the $\mathcal{C}_2$-saturation is integral over $\mathcal{E}$.*

For the integral weak asymptotic $\mathcal{C}_2'$-saturation, the inequality $\mathcal{C}_1 > \mathcal{C}_2$ over $\mathcal{E}$ is sufficient. The same holds for the other types of saturation.

Proof. We use two facts:

- $C_1 \geq C_2$ implies that $\lceil D + C_1 \rceil \geq \lceil D + C_2 \rceil$; and
- $C_1 \geq C_2$ implies the inclusion $|\lceil D + C_1 \rceil| \supset |\lceil D + C_2 \rceil| + E$ with $E = \lceil D + C_1 \rceil - \lceil D + C_2 \rceil \geq 0$.

This last fact implies that if $|\lceil D+C_2\rceil|\neq\varnothing$, then $|\lceil D+C_1\rceil|\neq\varnothing$ and $D\geq \operatorname{Mov}\lceil D+C_1\rceil\geq \operatorname{Mov}\lceil D+C_2\rceil$. This gives the first statement.

The b-divisorial case and asymptotic saturation follow from the same estimate. $\mathcal{C}_2'$-saturation follows from two facts:

- we can replace $\mathcal{C}_2$ by $\mathcal{C}_2+\sum\varepsilon_i E_i$ with $0<\varepsilon_i\ll 1$ by the lemma; and
- the effect of this is equivalent to replacing $\mathcal{C}_2$ by $\mathcal{C}_2'$.

This last fact follows from equations $\lceil d+\varepsilon_i\rceil = d+1=\lceil d+1\rceil$ for any integer d. □

Proof–Explanation of Proposition 4.42. By any saturation in the statement, we mean any type of saturation in Definition 4.33: e.g., for a divisor, for a b-divisor, etc.

Since B is effective, the codiscrepancy $\mathcal{B}=\mathcal{B}(X,B)$ is *also* effective up to components exceptional on X. Thus, there is an exceptional b-divisor $\mathcal{C}$ such that $\mathcal{C}\geq\mathcal{A}=-\mathcal{B}$; in the divisorial case, we take $C=\mathcal{C}_X=0$. Then the proposition follows immediately from Lemma 4.44, and Addendum 4.42.1 follows from Addendum 4.44.1. The statement on $\mathcal{R}_{X/T}D$ then follows from Example 4.35. □

Now we apply this to the following situation.

Example 4.45. Let $(X/T,B)$ and $(X/T,B^+)$ be two log pairs such that

- (X,B) is Klt;
- (X,B^+) is log canonical;
- $B^+\geq B$, or equivalently,
$$A=\mathcal{A}_X\geq A^+=\mathcal{A}^+_X,$$
where $\mathcal{A}^+=\mathcal{A}(X,B^+)$; and
- $\mathcal{A}^+$-saturation is integral over the prime b-divisors where (X,B^+) is *exactly* log canonical, that is, the discrepancy is -1.

Then lca saturation/(X,B), e.g., of Proposition 4.42, implies asymptotic $\mathcal{A}'$-saturation with $\mathcal{A}'=\mathcal{A}^++\mathcal{E}$, where $\mathcal{E}=\sum E_i$ is a sum of prime b-divisors with discrepancy -1 for (X,B^+). Integral saturation for $\mathcal{A}^+$ means that each divisor $jD_{i,\mathrm{hr}}=j(\mathcal{D}_i)_{X_{\mathrm{hr}}}$ is integral over $\operatorname{LCS}(X_{\mathrm{hr}},B^+_{\mathrm{hr}})$ (for crepant $(X_{\mathrm{hr}},B^+_{\mathrm{hr}})$). For example, this holds when each $\mathcal{D}_i$ is in general position with respect to each log canonical center of (X,B^+) (see (GNP) in Proposition 4.50 below).

Indeed, by monotonicity [41, (1.3.3)], we have $\mathcal{B}^+=\mathcal{B}(X,B^+)\geq\mathcal{B}$, or equivalently, $\mathcal{A}\geq\mathcal{A}^+$; and $\mathcal{A}>\mathcal{A}^+$ wherever the discrepancy of (X,B^+) is -1. Hence, lca saturation implies $\mathcal{A}^+$- and $\mathcal{A}'$-saturations by Lemma 4.44 and Addendum 4.44.1, respectively.

In general, the b-divisorial sheaf $\mathcal{O}_X(\mathcal{D})$ may not be a coherent sheaf (see Definition 4.10 and Examples 4.12–4.14). In our applications, this usually means also a *stabilization*, that is, the equality for inclusions $\mathcal{O}_Y(\mathcal{D}_Y)\supseteq\mathcal{O}_X(\mathcal{D})$ on a sufficiently high model Y/X (see Example 4.12). For example, this holds for any $\mathbb{R}$-Cartier divisor $\mathcal{D}$ by Proposition 3.20 over any model Y, where $\mathcal{D}=\overline{D}$ for some $\mathbb{R}$-Cartier divisor D on Y; the same holds for $\lfloor\mathcal{D}\rfloor$, but usually not for $\lceil\mathcal{D}\rceil$ (cf. Example 4.13). However, stabilization and even coherence are sometimes unimportant (cf. the remark on $\leq$ instead of $=$ in the proof of Proposition 4.50, p. 131, as well as the independence of the upper bound in the proof of Theorem 5.12, p. 146).

Proposition 4.46. *Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier b-divisor on X, $\mathcal{E}$ be a finite reduced b-divisor, and $\mathcal{C}$ be an $\mathbb{R}$-b-divisor on X such that*

(INO) *$\mathcal{D}$ and $\mathcal{C}$ (or just $\mathcal{D}+\mathcal{C}$) are* integral over $\operatorname{Supp}\mathcal{E}$; *and*

(CGR) $\mathcal{C}$ grows canonically: $\mathcal{C}_{X_{\mathrm{hr}}}\geq g^*(\mathcal{C}_Y)+\sum a_iE_i$ *for any further resolution $g\colon X_{\mathrm{hr}}\to Y$ of some (sufficiently high) resolution Y of X, where $a_i=a(Y,0,E_i)$ are the standard discrepancies.*

Then the sheaf $\mathcal{O}_X(\lceil\mathcal{D}+\mathcal{C}+\mathcal{E}\rceil)$ is coherent and stabilizes (see Example 4.12).

Proof. Take a sufficiently high resolution Y such that

- $\mathcal{D} = \overline{D}$ for an $\mathbb{R}$-divisor D on Y;
- (CGR) is satisfied for Y;
- $\mathcal{C}_Y$ has normal crossing support on Y (a property of all b-divisors on some resolution);
- moreover, the supports of all divisors under consideration have normal crossings; and
- $\mathcal{E} = \mathcal{E}_Y$ is a divisor on Y whose prime components are all pairwise disjoint.

It is enough to verify that, for any other model $g\colon X_{\mathrm{hr}} \to Y$,

$$g^*(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y) \le \lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_{X_{\mathrm{hr}}}$$

because then

$$\mathcal{O}_Y(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y) \subset g_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil (\mathcal{D} + \mathcal{C} + \mathcal{E})_{X_{\mathrm{hr}}} \rceil)$$

by Proposition 3.20, which even gives the equality as well as the stabilization and the coherence of $\mathcal{O}_X(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil)$.

Set

$$\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y = (\mathcal{D} + \mathcal{C} + \mathcal{E})_Y + F,$$

where $F = \sum f_i E_i$ is the *fractional* part, that is, all $f_i \in [0, 1)$. Note that $(Y, \mathcal{E} + F)$ is purely log terminal by (INO), the fact that the $\mathcal{E}$ are disjoint, and normal crossings.

Thus, by (INO), because $\mathcal{D}$ is Cartier, and by (CGR), we get the inequality

$$\begin{aligned} g^*(\lceil \mathcal{D} + \mathcal{C} + \mathcal{E} \rceil_Y) &= g^*(\mathcal{D}_Y + \mathcal{C}_Y + \mathcal{E}_Y) + g^*F = g^*(\mathcal{D}_Y) + g^*(\mathcal{C}_Y) + g^*(\mathcal{E} + F) \\ &\le \mathcal{D}_{X_{\mathrm{hr}}} + \mathcal{C}_{X_{\mathrm{hr}}} + g^*(\mathcal{E} + F) - \sum a_i E_i \\ &= (\mathcal{D} + \mathcal{C} + \mathcal{E})_{X_{\mathrm{hr}}} + g^*(\mathcal{E} + F) - \mathcal{E} - \sum a_i E_i. \end{aligned}$$

Since we need to verify the above inequality for integral parts over the exceptional E_i, it is enough to verify that

$$\operatorname{mult}_{E_i}\Bigl(g^*(\mathcal{E} + F) - \mathcal{E} - \sum a_i E_i\Bigr) = \operatorname{mult}_{E_i}\Bigl(g^*(\mathcal{E} + F) - \sum a_i E_i\Bigr) < 1$$

for each exceptional E_i. (For any integer i and real numbers $f < 1$ and r, the inequality $i \le r + f$ implies $i \le \lceil r \rceil$.) But this means that $(Y, \mathcal{E} + F)$ is purely log terminal: $\mathcal{A}(Y, \mathcal{E} + F) = \mathcal{A}(Y, 0) - g^*(\mathcal{E} + F)$ has each exceptional multiplicity > -1. $\square$

Example 4.47. A discrepancy divisor $\mathcal{C} = \mathcal{A} = \mathcal{A}(X, B)$ is the typical example when (CGR) in Proposition 4.46 holds. Indeed,

$$\begin{aligned} \mathcal{A}_{X_{\mathrm{hr}}} &= K_{X_{\mathrm{hr}}} - (\overline{K + B})_{X_{\mathrm{hr}}} = K_{X_{\mathrm{hr}}} - g^*(\overline{K + B_Y}) = K_{X_{\mathrm{hr}}} - g^*(K_Y - \mathcal{A}_Y) \\ &= g^*\mathcal{A}_Y + K_{X_{\mathrm{hr}}} - \overline{K}_Y = g^*\mathcal{A}_Y + \mathcal{A}(Y, 0)_{X_{\mathrm{hr}}}. \end{aligned}$$

Thus, for $\mathcal{C} = \mathcal{A}$ in Proposition 4.46, we get that $\mathcal{O}_X(\lceil \mathcal{D} + \mathcal{A} + \mathcal{E} \rceil)$ is coherent.

In particular, for $\mathcal{D} = 0$ and $\mathcal{E} = 0$, the fractional ideal $\mathcal{O}_X(\lceil \mathcal{A} \rceil)$ is always coherent and stabilizes; the fractional ideal sheaf

$$J_X = J(X, B) = \mathcal{O}_X(\lceil \mathcal{A} \rceil) = h_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{A} \rceil_{X_{\mathrm{hr}}})$$

and the functional sheaf

$$J_Z = f_*J(X, B) = f_*\mathcal{O}_X(\lceil \mathcal{A} \rceil) = (f \circ h)_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{A} \rceil_{X_{\mathrm{hr}}})$$

are independent of the sufficiently high resolution $h\colon X_{\mathrm{hr}} \to X$. If B is effective, J_X is the well-known ideal sheaf of $\mathrm{LCS}(X, B)$.

Example 4.48. Since $\mathcal{K} = \overline{K_Y} + \mathcal{A}(Y,0)$ over any $\mathbb{Q}$-Gorenstein model Y/Z of X/Z, the proposition again implies that $\mathcal{O}_X(\mathcal{K}) = \mathcal{O}_X(\lceil\mathcal{K}\rceil)$ is coherent and stabilizes. This is the well-known invariance of regular differential forms. The same applies to each natural multiple $m\mathcal{K}$, and the coherent property holds for all m on a universal model, e.g., a nonsingular one. This is also known for log b-divisors $\mathcal{K}+\mathcal{B}$, $\lceil\mathcal{K}+\mathcal{B}\rceil = \mathcal{K}+\lceil\mathcal{B}\rceil$ and their multiples: $m(\mathcal{K}+\mathcal{B})$,

$$m\lceil\mathcal{K}+\mathcal{B}\rceil = m(\mathcal{K}+\lceil\mathcal{B}\rceil), \qquad \text{and} \qquad \lceil m(\mathcal{K}+\mathcal{B})\rceil = m(\mathcal{K}+\lceil m\mathcal{B}\rceil/m),$$

where $\mathcal{B}$ is a b-boundary [45, Example 1.1.2]. However, the corresponding coherent sheaves may be different and the proposition may not apply directly (see Example 3.14 and the case with $\mathcal{E}=0$ in [14]).

The algebra $\mathcal{R}_{X/Z}(\mathcal{K}+\mathcal{B})$ is log canonical over some model. The same is true for the b-divisors $m(\mathcal{K}+\mathcal{B})$, $m\lceil\mathcal{K}+\mathcal{B}\rceil$, and $\lceil m(\mathcal{K}+\mathcal{B})\rceil$; their algebras up to a quasi-isomorphism correspond to the b-divisors $\mathcal{B}$, $\lceil\mathcal{B}\rceil$, and $\lceil m\mathcal{B}\rceil/m$, respectively.

We can also replace $\lceil\cdot\rceil$ by $\lfloor\cdot\rfloor$. Moreover, the system $\mathcal{D}_i = \lfloor i(\mathcal{K}+\mathcal{B})\rfloor$ defines a pbd algebra. But it is again log canonical $\mathcal{R}_{X/Z}(\mathcal{K}+\mathcal{B})$ (cf. Example 4.13).

Example 4.49. A bounded pbd algebra is called *pseudo*-log canonical if its limit $\mathcal{D} = \lim_{i\to\infty}\mathcal{D}_i$ satisfies $\mathcal{D}_Y \sim_{\mathbb{R}} r(K_Y+B_Y)/Z$ (or $= r(K_Y+B_Y)$ under an appropriate choice of K_Y up to $\sim_{\mathbb{R}}$), where r is a positive real and $(Y/Z, B_Y)$ is a log canonical pair that is a model of X/Z with some boundary B_Y. In particular, an algebra of this form is *log canonically bounded* when $r=1$. By the LMMP and [45, Log semiampleness conjecture 2.6], we expect that each log canonical algebra (see Example 3.14) is also pseudo-log. We expect the converse to hold under lca saturation (see Remark 4.40(3)). In addition, if (Y, B_Y) is Klt and $(Y/Z, B_Y)$ is of general type, then one can replace $\sim_{\mathbb{R}} /Z$ by $\equiv /Z$.

Each nontrivial algebra of (FGA) type is pseudo-log canonical because, after a complement, we can assume that $(X, B^+ + \varepsilon\mathcal{D}_X)$ with $K_X + B^+ \sim_{\mathbb{R}} 0$ and $\mathcal{D}_X \geq 0$ up to $\sim_{\mathbb{R}}$ (cf. the proof of Theorem 3.33) is log canonical, even Klt. One may hope that a similar conjecture on f.g. for pseudo-log canonical algebras has a better induction than for (FGA) ones. In the latter case, we use the following property of lca saturation.

Proposition 4.50. *Let $(X/Z, B)$ be a pair with B an $\mathbb{R}$-divisor, Y be a prime divisor of X, and $\mathcal{D}_\bullet$ be a system of $\mathbb{R}$-b-divisors such that*

- (GLF) *$(X/Z, B)$ is a* general log Fano map: *$-(K+B)$ is nef and big/Z;*
- (LCC) *Y is a* log canonical center: *$a(X,B,Y) = -1$;*
- (ASA′) *the system $\mathcal{D}_\bullet$ is asymptotically saturated with respect to $\mathcal{A}' = \mathcal{A}(X,B)+Y$;*
- (BNF) *each $\mathcal{D}_i$ is b-nef, in particular, if*
- (BSA) *each $\mathcal{D}_i$ is* b-semiample: *$\mathcal{D}_i = \overline{D_i}$ for a semiample $\mathbb{R}$-divisor D_i on a model W/Z of X/Z* (cf. Lemma 3.7)*; and*
- (GNP) *$\mathcal{D}_\bullet$ is in general position with respect to Y: for each $\mathcal{D}_i$, $Y \not\subset \operatorname{Supp}\mathcal{D}_i$.*

Then the restriction $\mathcal{D}_{\bullet\,\lceil Y^\nu}$ is lca saturated over (Y^ν, B_{Y^ν}), where $\nu\colon Y^\nu \to Y$ is the normalization of Y, $\mathcal{D}_{\bullet\,\lceil Y^\nu}$ is the fixed restriction (*see the proof and Fixed restriction* 7.2 *below*)*, and $B_{Y^\nu} = (B-Y)_{Y^\nu}$ is the different of $B-Y$* (see [41, §3] and [27, Proposition 16.5]).

Addendum 4.50.1. *$\mathcal{L} \subset k(X)_\bullet$ be a functional algebra such that*

- (ASA′) *$\mathcal{L}$ is asymptotically saturated with respect to $\mathcal{A}'$; and*
- (GNP) *$\mathcal{L}$ is* in general position *with respect to Y, as is the mobile system $\mathcal{N}_\bullet$.*

Then $\mathcal{L}_{|Y^\nu}$ is lca saturated/(Y^ν, B_{Y^ν}). If, in addition, (X,B) is purely log terminal, then $\mathcal{L}_{|Y^\nu}$ is normal/$f(Y)$ if $\mathcal{L}/Z$ is.

Addendum 4.50.2. *In general, under* (ASA$'$) *and* (GNP), *if $\mathcal{L}/Z$ is normal, then $\mathcal{L}_{|Y^\nu}$ is normal*/$f(Y)$ with respect *to the fractional ideal $J = J_{Y^\nu}$* (*that we consider as a b-divisorial sheaf*/Y^ν; *see Example* 4.47); *that is, for each mobile component $\mathcal{M}_i$ of $\mathcal{L}_{|Y^\nu}$ on any sufficiently high model* $Y_{\mathrm{hr}}/f(Y)$ *of* $Y^\nu/f(Y)$, *the complete linear system*

$$|J_{Y_{\mathrm{hr}}}(\mathcal{M}_i)_{Y_{\mathrm{hr}}}| = \big|(\mathcal{L}_{|Y^\nu})[J]_i\big|_{Y_{\mathrm{hr}}}$$

is associated with the ith component of the twisted fractional ideal $\mathcal{L}_{|Y^\nu}[J] = J(\mathcal{L}_{|Y^\nu}) \cap \mathcal{L}_{|Y^\nu}$ in $\mathcal{L}_{|Y^\nu}$, where $J(\mathcal{L}_{|Y^\nu})_i = J_{Y_{\mathrm{hr}}} \otimes (\mathcal{L}_{|Y^\nu})_i = g_* J_{Y_{\mathrm{hr}}}(\mathcal{M}_i)$ for $g\colon Y_{\mathrm{hr}} \to Y^\nu$. Hence, $\mathcal{L}_{|Y^\nu}$ satisfies

(NOR) *the* normal property: *if $\mathcal{L}_{|Y^\nu}$ is f.g., then its* $\mathrm{Proj}_{f(Y)}(\mathcal{L}_{|Y^\nu})$ *is a normal* (*geometrically but not projectively*) *algebraic variety.*

For a log canonical center Y of higher codimension, we need the divisorial part of an adjunction formula [23, 36].

Addendum 4.50.3. *If we assume only that Y is an exceptionally log terminal center* [43, Definition 1.5] (*compare* (ELT) *in Addendum* 9.21.1 *below*), *$B \geq 0$ near the general point of E, and $\mathcal{A}' = \mathcal{A}(X,B) + E$, where E is a prime b-divisor with $a(X,B,E) = -1$ and* $\operatorname{center}_X E = Y$, *then lca saturation holds over Y^ν with the above $\mathcal{A}(Y^\nu, B_{Y^\nu})$ replaced by* divisorial $\mathcal{A}_{\mathrm{div}}$ (*see the proof below*). *Of course, we assume that $\mathcal{D}_{\bullet}{}_{\upharpoonright Y^\nu}$ or $\mathcal{L}_{|Y^\nu}$ are well defined*: *e.g., each $\mathcal{D}_i = 0$ over the generic point of Y and is b-semiample* (*see also Fixed restriction* 7.2 *below*).

Proof. Take a log resolution W/Z of $(X/Z,B)$ that is sufficiently high over X and over Y^ν, and take any natural numbers i and j. Let $Y_{\mathrm{hr}} \subset W$ be the birational transform of Y on W. By construction, Y_{hr} is normal, and we have a decomposition $Y_{\mathrm{hr}} \to Y^\nu \to Y$, where $\nu\colon Y^\nu \to Y$ is the normalization. Then the restriction

$$|\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil| \xrightarrow{|_{Y_{\mathrm{hr}}}} |\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil_{|Y_{\mathrm{hr}}}| = |\lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}}\rceil|$$

is surjective, where $\mathcal{A}^{Y^\nu} = \mathcal{A}(Y^\nu, B_{Y^\nu})$. Indeed, by adjunction, $K_{Y^\nu} + B_{Y^\nu} = (K+B)_{|Y^\nu}$ (see [41, 3.1] and [27, Proposition 16.5]). On the other hand, by definition, $\mathcal{A} = \mathcal{A}(X,B) = \mathcal{K} - \overline{K+B}$ and $\mathcal{A}^{Y^\nu} = \mathcal{K}^{Y^\nu} - \overline{K_{Y^\nu} + B_{Y^\nu}}$, where $\mathcal{K}$ and $\mathcal{K}^{Y^\nu}$ are canonical b-divisors of X and Y^ν, respectively [45, Example 1.1.3], that is, $\mathcal{K}_W = K_W$ and $\mathcal{K}^{Y^\nu}_{Y^\nu} = K_{Y^\nu}$. Hence,

$$\begin{aligned}\mathcal{A}'_{W|Y_{\mathrm{hr}}} = (\mathcal{A}_W + Y_{\mathrm{hr}})_{|Y_{\mathrm{hr}}} = \big(K_W + Y_{\mathrm{hr}} - (\overline{K+B})_W\big)_{|Y_{\mathrm{hr}}} &= (K_W + Y_{\mathrm{hr}})_{|Y_{\mathrm{hr}}} - \big(\overline{(K+B)_{|Y^\nu}}\big)_{Y_{\mathrm{hr}}}\\ &= K_{Y_{\mathrm{hr}}} - (\overline{K_{Y^\nu} + B_{Y^\nu}})_{Y_{\mathrm{hr}}} = (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}},\end{aligned}$$

where ${}_{|Y^\nu} = \nu^* \circ {}_{|Y}$. (In terms of fixed restrictions, this amounts to $\mathcal{A}'{}_{\upharpoonright Y^\nu} = \mathcal{A}^{Y^\nu}$; see 7.2 below.)

By normal crossings of supports in the following formula, the roundup $\lceil\cdot\rceil$ commutes with restriction:

$$\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil_{|Y_{\mathrm{hr}}} = \lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + \mathcal{A}'_{W|Y_{\mathrm{hr}}}\rceil = \lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^\nu})_{Y_{\mathrm{hr}}}\rceil.$$

For surjectivity, it is enough to establish the vanishing

$$R^1 h_* \mathcal{O}_W(\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil - Y_{\mathrm{hr}}) = 0,$$

where $h\colon W \to Z$. It is natural to use the Kawamata–Viehweg vanishing for this:

$$\begin{aligned}\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil - Y_{\mathrm{hr}} &= \lceil j(\mathcal{D}_i)_W + \mathcal{A}_W + Y_{\mathrm{hr}}\rceil - Y_{\mathrm{hr}}\\ &= \lceil j(\mathcal{D}_i)_W + \mathcal{A}_W\rceil = \lceil j(\mathcal{D}_i)_W + K_W - r^*(K+B)\rceil\\ &= K_W + \lceil j(\mathcal{D}_i)_W - r^*(K+B)\rceil,\end{aligned}$$

because Y_{hr} and K_W are integral and $\mathcal{A}_W = K_W - r^*(K+B)$, where $r\colon W \to X$ is the resolution. Now we get the required vanishing since $(\mathcal{D}_i)_W$ is nef/Z by (BNF) and $-(K+B)$ is nef and big/Z by (GLF). The former holds on W over any model where $\mathcal{D}_i = \overline{D}_i$ for nef D_i. (Vanishing holds for $\mathbb{R}$-divisors. However, in our applications, we only need $\mathbb{Q}$-divisors, e.g., for Addendum 4.50.1.)

Thus, by the surjectivity of the above restriction,

$$|\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil| \neq \varnothing \qquad \text{provided that} \qquad |\lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^{\nu}})_{Y_{\mathrm{hr}}}\rceil| \neq \varnothing.$$

Moreover, in this case, by Hironaka and because $\mathcal{O}_X(\lceil j(\mathcal{D}_i) + \mathcal{A}'\rceil)$ is coherent by Proposition 4.46 and Example 4.47, we can assume that

$$\mathrm{Bs}\,|\mathrm{Mov}\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil| = \varnothing$$

on a sufficiently high resolution W (cf. the proof of Proposition 4.15(1)) and both Mov stabilize. Note now that, by definition, $(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} = (\mathcal{D}_{i\shortmid Y^{\nu}})_{Y_{\mathrm{hr}}}$. Therefore, surjectivity gives

$$\mathrm{Mov}\lceil j(\mathcal{D}_i)_{W|Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^{\nu}})_{Y_{\mathrm{hr}}}\rceil = (\mathrm{Mov}\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil)_{|Y_{\mathrm{hr}}}$$

(in general, only $\leq$ holds), and

$$\mathrm{Mov}(\lceil j\mathcal{D}_{i\shortmid Y^{\nu}} + \mathcal{A}^{Y^{\nu}}\rceil_{Y_{\mathrm{hr}}}) \leq j(\mathcal{D}_j)_{W|Y_{\mathrm{hr}}} = j(\mathcal{D}_{j\shortmid Y^{\nu}})_{Y_{\mathrm{hr}}}$$

provided that

$$\mathrm{Mov}\lceil j(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil \leq j(\mathcal{D}_j)_W.$$

The previous equality holds by the definition of fixed restrictions of b-divisors (see 7.2 below) if Y_{hr} lies over a model such that $\mathcal{D}_i = \overline{D}_i$ for semiample D_i. This implies lca saturation for $\mathcal{D}_{\bullet\shortmid Y^{\nu}}$ and on any model/Y_{hr} by (ASA$'$) with the same index I.

Addendum 4.50.1 follows from the proposition applied to the characteristic system $\mathcal{D}_{\bullet}$ of $\mathcal{L}$. Indeed, the characteristic system of $\mathcal{L}_{|Y^{\nu}}$ is the restriction $\mathcal{D}_{\bullet\shortmid Y^{\nu}}$ (cf. Fixed restriction 7.2), and (BSA) holds by Proposition 4.15(1). The normality of restricted algebras follows from the surjectivity of mobile parts when $j = i$.

Indeed, (GLF) and the connectedness of $\mathrm{LCS}(W, B_W = \mathcal{B}_W)/Z$ (Kollár and others [27, Theorem 17.4]) imply that, locally/Z near the connected component of fibre/P intersecting Y, $\mathcal{A}$ has only one divisor Y with $a(X,B,Y) = -1$ if (X,B) is purely log terminal. Then $\lceil\mathcal{A}'\rceil \geq 0$, but $\mathcal{N}_i = i\mathcal{D}_i$ is integral as a b-free b-divisor in Proposition 4.15(1). Hence, $|\lceil i(\mathcal{D}_i)_W + \mathcal{A}'_W\rceil| = |(\mathcal{N}_i)_W| + \lceil\mathcal{A}'_W\rceil$ by saturation (cf. Remarks 4.34(1),(3)).

On the other hand, by the adjunction of [41, (3.2.3)], $(Y^{\nu}, B_{Y^{\nu}})$ is Klt. Thus, $\lceil\mathcal{A}^{Y^{\nu}}\rceil \geq 0$ and

$$i(\mathcal{D}_{i\shortmid Y^{\nu}})_{Y_{\mathrm{hr}}} = (\mathcal{N}_{i\shortmid Y^{\nu}})_{Y_{\mathrm{hr}}} = (\mathcal{N}_i)_{W|Y_{\mathrm{hr}}} = (\mathcal{M}_i)_{Y_{\mathrm{hr}}}$$

is integral and mobile, where $\mathcal{N}_{i\shortmid Y^{\nu}} = \mathcal{M}_i = (\mathrm{Mov}\,\mathcal{L}_{|Y^{\nu}})_i$. Thus, again,

$$|\lceil i\mathcal{D}_{i\shortmid Y_{\mathrm{hr}}} + (\mathcal{A}^{Y^{\nu}})\rceil_{Y_{\mathrm{hr}}}| = |(\mathcal{M}_i)_{Y_{\mathrm{hr}}}| + \lceil\mathcal{A}^{Y^{\nu}}\rceil_{Y_{\mathrm{hr}}}.$$

Thus, we get the surjectivity of the restriction

$$|(\mathcal{N}_i)_W| \xrightarrow{|_{Y_{\mathrm{hr}}}} |(\mathcal{M}_i)_{Y_{\mathrm{hr}}}|,$$

and, by Proposition 4.15(4), this implies that $\mathcal{L}_{|Y^{\nu}}$ is normal if $\mathcal{L}$ is. Finally, note that $Y = Y^{\nu}$ is normal itself by [41, лемма 3.6].

If (X, B) is not purely log terminal, the last surjectivity does not hold. However, the above argument gives by definition the surjectivity

$$|J'(\mathcal{N}_i)_W| \xrightarrow{|_{Y_{\mathrm{hr}}}} |J_{Y_{\mathrm{hr}}}(\mathcal{M}_i)_{Y_{\mathrm{hr}}}|,$$

where $J' = \mathcal{O}_W(\lceil \mathcal{A}' \rceil)$ and $J'(\mathcal{N}_i) = \mathcal{O}_W(\mathcal{N}_i + \lceil \mathcal{A}' \rceil)$ for the crepant model (W, B_W). Thus, by (ASA$'$) for $j = i$ and Proposition 4.15(4),

$$|J'(\mathcal{N}_i)| = |\overline{\mathcal{L}}[J']_i|_W.$$

The left-hand side denotes the complete linear system of $(\mathcal{N}_i + \lceil \mathcal{A}' \rceil)_W$, and the right-hand side its complete subsystem with the mobile part in $|\mathcal{N}_i| = |\operatorname{Mov} \overline{\mathcal{L}}_i|$. In terms of rational functions, this means that the subsystem corresponds to the functions in

$$\overline{\mathcal{L}}[J']_i = J'(\overline{\mathcal{L}}_i) \cap \overline{\mathcal{L}_i} = J'(\mathcal{N}_i) \cap \overline{\mathcal{L}_i} = J'(\mathcal{N}_i) \cap f_* \mathcal{O}_W(\mathcal{N}_i)$$

(cf. $\mathcal{L}_N(-S_i)$ in Main Lemma 3.43, (TRL), and the twist in Example 4.9). Now when $\mathcal{L}$ is normal, $\mathcal{L} = \overline{\mathcal{L}}$ by Proposition 4.15(4). Thus, $|J'(\mathcal{N}_i)| = |\mathcal{L}[J']_i|_W$. This is normality with respect to J' on X/Z (conversely, normality implies (ASA$'$)). Then surjectivity implies the required normality of Addendum 4.50.2 with respect to J on $Y^\nu / f(Y)$.

The normal property (NOR) of Addendum 4.50.2 follows from the following fact: for any ideal sheaf J^- and any ample divisor H on an algebraic variety X/Z, $J^-(NH)$ with $N \gg 0$ is very ample/Z. We apply this to the model of $\operatorname{Proj}_{f(Y)} \overline{\mathcal{L}_{|Y^\nu}}$. The normality condition and the projection formula then imply that $\mathcal{L}_{|Y^\nu}$ has at least sections vanishing on J^-, where J^- is given by the negative part of the b-divisor $\lceil \mathcal{A}^{Y^\nu} \rceil$ (base conditions on sections).

Note that $J' = \mathcal{O}_W$ and $J = \mathcal{O}_{Y^\nu}$ if (X, B) is purely log terminal and (Y^ν, B_{Y^ν}) is Klt. This gives the usual normality of Addendum 4.50.1.

In Addendum 4.50.3, we are done if $Y = E$ is divisorial. Otherwise, on the resolution, we replace Y_{hr} by E and lca saturation for $\mathcal{D}_{\bullet |E}$ over $(E/Y^\nu, B_E)$, where (E, B_E) is the adjunction of (W, B_W) as described above (note that B_E and B_W are not necessarily boundaries or subboundaries). On the other hand, Y has a canonical b-divisor $\mathcal{B}_{\mathrm{div}}$, the *divisorial adjunction b-boundary* (again not necessarily even a subboundary) such that

(boundary), (BP) $\mathcal{B}_{\mathrm{div}}$ has the boundary property; that is, we have

$$\mathcal{B}_{\mathrm{div}} = \mathcal{B}(Y_{\mathrm{hr}}, B_{\mathrm{div}} = (\mathcal{B}_{\mathrm{div}})_{Y_{\mathrm{hr}}}) = \mathcal{B}^{Y_{\mathrm{hr}}}$$

on a sufficiently high model Y_{hr}/Y. It is equivalent to saying that the *divisorial discrepancy* b-divisor $\mathcal{A}_{\mathrm{div}} = -\mathcal{B}_{\mathrm{div}}$ behaves as a discrepancy. Moreover, the following also hold:

(monotonicity) $(l\mathcal{A}^E)_E \geq f_E^*(l\mathcal{A}_{\mathrm{div}})_{Y_{\mathrm{hr}}}$, where l means that we take log discrepancies, assuming that Y_{hr} has mild singularities (at worst log canonical), $(l\mathcal{A}_{\mathrm{div}})_{Y_{\mathrm{hr}}}$ is $\mathbb{R}$-Cartier (e.g., Y is nonsingular), and the induced projection $f_E\colon E \to Y_{\mathrm{hr}}$ is regular; and

(semiadditivity) if we replace B by $B + D$, where D is an $\mathbb{R}$-Cartier divisor in general position with Y, then $\mathcal{B}_{\mathrm{div}}$ should be replaced by $\mathcal{B}_{\mathrm{div}} + \overline{D}_{|Y}$; equivalently,

(anti-semiadditivity) $\mathcal{A}_{\mathrm{div}}$ should be replaced by $\mathcal{A}_{\mathrm{div}} - \overline{D}_{|Y}$.

If (X, B) is exceptionally log terminal [43, Definition 1.5] and $B \geq 0$ near E, then all this holds over $Y_{\mathrm{hr}} = Y^\nu = Y$, and this is known as the divisorial part of adjunction. In general, by the semiadditivity, we can assume that $B \geq 0$. Then we can use a log resolution and semiadditivity once again to reduce to the divisorial part of adjunction. (Here we do not need the boundary property (BP). It follows from LMMP at least in $\dim W \leq 4$.[5])

[5]Recently F. Ambro proved (BP) in any dimension [3].

Question 4.51. Can we omit the assumption that $B \geq 0$ near the general point of Y?

Now we are ready to verify the asymptotic saturation of $(\mathcal{D}_\bullet)_{|Y^\nu}$ with respect to $\mathcal{A}_{\mathrm{div}}$. By lca saturation over (E, B_E), which is already proved, and the monotonicity (cf. (CGR) and the proof of Proposition 4.46), we have

$$\begin{aligned} f_E^* \operatorname{Mov}\lceil j\mathcal{D}_{i|Y^\nu} + \mathcal{A}_{\mathrm{div}}\rceil_{Y_{\mathrm{hr}}} &= f_E^* \operatorname{Mov}\Big((j\mathcal{D}_{i|Y^\nu} + \mathcal{A}_{\mathrm{div}})_{Y_{\mathrm{hr}}} + \sum e_m E_m\Big) \\ &= f_E^* \operatorname{Mov}\Big((j\mathcal{D}_{i|Y^\nu} + l\mathcal{A}_{\mathrm{div}})_{Y_{\mathrm{hr}}} + \sum (e_m - 1) E_m\Big) \\ &\leq \operatorname{Mov} f_E^*\Big((j\mathcal{D}_{i|Y^\nu} + l\mathcal{A}_{\mathrm{div}})_{Y_{\mathrm{hr}}} + \sum (e_i - 1) E_i\Big) \\ &\leq \operatorname{Mov}\Big((j\mathcal{D}_{i|E} + l\mathcal{A}^E)_E + \sum r_{m'}(e_{m'} - 1) E_{m'}\Big) \\ &= \operatorname{Mov}\Big((j\mathcal{D}_{i|E} + \mathcal{A}^E)_E + \sum (r_{m'}(e_{m'} - 1) + 1) E_{m'}\Big), \end{aligned}$$

and, since all $r_{m'}(e_{m'} - 1) + 1 < 1$, Mov is integral, and Fix is effective,

$$\begin{aligned} &\leq \operatorname{Mov}\lceil j\mathcal{D}_{i|E} + (\mathcal{A}^E)\rceil_E \leq (j\mathcal{D}_{j|E})_E \\ &= f_E^*(j\mathcal{D}_{j|Y^\nu})_{Y_{\mathrm{hr}}}, \end{aligned}$$

where E_i and E_i' are, respectively, on Y_{hr} and E, all $0 \leq e_i < 1$, and all $0 < r_{m'} = \operatorname{mult}_{E_{m'}} f_E^* E_m$ for all $E_{m'}/E_m$. The sums are finite even if we add an extra redundant divisor E_i, the fractional and branching places. We also assume that $j\mathcal{D}_{i|Y^\nu}$ is $\mathbb{R}$-Cartier over Y_{hr}. This gives the required saturation in Addendum 4.50.3. □

Example 4.52. Consider a pl contraction $X/X_\vee$ and assume that

- $X/X_\vee$ satisfies the inductive assumptions of Definition 3.39 with respect to a divisor D on X; and
- $\mathcal{L}$ is a functional algebra bounded by D and satisfying the other assumptions of Main Lemma 3.43.

We say that such an algebra $\mathcal{L}$ is of type $(\mathrm{FGA})_n^{\mathrm{pl}}$ if, in addition,

- $\mathcal{L}$ is lca saturated over $(X, B + S - \varepsilon S)$ for some $\varepsilon > 0$; and
- $\dim X = n$.

As in Conjecture 4.39, we expect such algebras to be f.g. More precisely,

$$(\mathrm{FGA})_d \;\Rightarrow\; (\mathrm{FGA})_n^{\mathrm{pl}},$$

where $d = \dim Y$ is the dimension of the irreducible normal variety $Y = S^s = \bigcap S_i$ (cf. Example 3.40 with $t = 1$ and $s = n - d$). The implication means that the first conjectural finite generation implies the second.

Indeed, by Example 4.45 with $B^+ = B + S - \varepsilon S'$ and $S' = \sum_{i \geq 2} S_i$, lca saturation of $\mathcal{L}$ implies asymptotic saturation of $\mathcal{L}$ for $\mathcal{A}' = \mathcal{A}^+ + S_1$. Thus, by Addendum 4.50.1, this implies lca saturation/(S_1, B_{S_1}) with the different $B_{S_1} = (B^+ - S_1)_{S_1} = (B + S' - \varepsilon S')_{S_1}$. Since $K + B + S$ is numerically negative/$X_\vee$, (GLF) holds for $B^+/X_\vee$ if $0 < \varepsilon \ll 1$. Because (X, B) is divisorially log terminal, (X, B^+) is purely log terminal for such ε and S_1 is the only log canonical center, with $a(X, B^+, S_1) = -1$. The algebra $\mathcal{L}$ is in general position to S_1 by our choice of D (cf. Definition 3.39 and Main Lemma 3.43).

Then, by Example 3.40, we can use induction on d because

$$B_{S_1} = (B + S' - \varepsilon S')_{S_1} = (B + S')_{S_1} - \varepsilon(S'_{|S_1})$$

by semiadditivity [41, (3.2.1)]. The pair $(S_1/f(S_1), (B + S')_{S_1})$ satisfies the same assumption as

$(X/X_\vee, B)$ by Example 3.40; in particular,

$$(S_1, (B+S')_{S_1} - \varepsilon(S'_{|S_1}))$$

is Klt, and the restriction $\mathcal{L}_{|S_1}$ is of type $(\mathrm{FGA})_{n-1}^{\mathrm{pl}}$.

Hence, by induction, $\mathcal{L}_{|Y}$ is lca saturated/(Y, B_Y), where B_Y corresponds to the successive adjunction. Moreover, (Y, B_Y) is Klt. Thus, by $(\mathrm{FGA})_d$, the algebra $\mathcal{L}_{|Y}$ is f.g. Then $\mathcal{L}$ is f.g. by Main Lemma 3.43 (of course, locally near $P \in X_\vee$). In addition, $\mathcal{L}_{|Y}$ is normal if $\mathcal{L}$ is.

This is a generalization of Induction Theorem 1.4 for (FGA) algebras (cf. its proof below).

If, for $d = n$ and for big $\mathcal{L}$, we knew the equivalence

$$(\mathrm{FGA})_n \;\Leftrightarrow\; (\mathrm{FGA})_n^{\mathrm{pl}},$$

this would imply the existence of log flips in dimension n by induction on n (cf. Reduction Theorem 1.2 and Conjecture 4.39).

In Section 6, as an application of the example, we prove $(\mathrm{FGA})_3^{\mathrm{pl}}$ (cf. Corollary 6.44 and Remark 11.8 below).

Now we are ready to prove the main result of this section.

Corollary 4.53. $(\mathrm{FGA})_d \Rightarrow (\mathrm{RFA})_{n,d}$, *and small pl flips of core dimension d exist.*

Proof–Explanation. We verify that a restricted algebra $\mathcal{L} = (\mathcal{R}_{X/T}D)_{|Y}$ of type $(\mathrm{RFA})_{n,d}$ satisfies $(\mathrm{FGA})_d$; the required f.g. then follows from Main Lemma 3.43. Indeed, the bounded algebra $\mathcal{R}_{X/T}D$ is exceptionally asymptotically saturated/X by Example 4.35. Thus, by Proposition 4.42, $\mathcal{R}_{X/T}D$ is lca saturated/$(X, B+S-\varepsilon S)$ for any $\varepsilon \in [0,1]$. In addition, by Addendum 3.43.1, the algebra $\mathcal{R}_{X/T}D$ is of type $(\mathrm{FGA})_n^{\mathrm{pl}}$. Thus, by Example 4.52, the restriction $\mathcal{L}$ is of type $(\mathrm{FGA})_d$.

The existence of pl flips follows from Induction Theorem 1.4. □

Proof of Induction Theorem 1.4 for (FGA). Immediate by the corollary and by case (RFA) in Section 3 or by Theorem 3.37. Moreover, we only need $(\mathrm{FGA})_d(\mathrm{bir})$ as for (RFA). □

As a first application of Induction Theorem 1.4 and its proof, by Examples 4.41, 4.52 and by Corollary 4.53, we get the f.g. of algebras of type $(\mathrm{FGA})_n^{\mathrm{pl}}$ and, by Truncation Principle 4.6, $(\mathrm{PLF})_n$ with $d = 1$ (see Conjecture 3.35). By Reduction Theorem 1.2, this gives all log flips in dimension 2, which is well known since they are divisorial contractions. However, the same method also allows us to construct log flips for seminormal surfaces and these in fact may not be contractions (cf. modifications of surfaces D of the semistable models for D [42]).

In higher dimensions, pl flips with $d = 1$ are also trivial since they are negative on Y (cf. Example 3.53). Nonetheless, $(\mathrm{FGA})_1$ also gives f.g. for $\mathcal{L}$ in $(\mathrm{FGA})_n$ over the generic points $P \in T$ when $\dim f^{-1}P = 1$ and over the generic points of the prime divisors $D \subset T$ when X/T is birational (cf. condition (BED) in Proposition 4.54 and in Theorem 9.9). This follows by induction on n after taking general hyperplane sections.

Proposition 4.54. *Over codimension-d points,* $(\mathrm{FGA})_n$ *follows from* $(\mathrm{FGA})_d$. *Thus, by induction on n, it is enough to prove* $(\mathrm{FGA})_n$ *over closed points in T assuming, after a truncation, that $\mathcal{L}$ locally has a stabilized characteristic system*:

(BED) *for all i, $\mathcal{D} = \mathcal{D}_i$ outside $f^{-1}P$ over T*, that is, over $T \setminus P$.

The same holds for (RFA) *in place of* (FGA).

For example, $\mathcal{L} = \mathcal{R}_{X/T}D$ has (BED) in codimension 1 when D is integral and X/T is birational. For a pl contraction $X/T = X/X_\vee$, after a truncation, this holds even in codimension 2 since X is normal and $\mathbb{Q}$-factorial in codimension 2. (Any flipping algebra satisfies this over $X = T$ if X is $\mathbb{Q}$-factorial in codimension 2, e.g., has rational singularities.) By $(\mathrm{FGA})_1$ and the proposition, this holds, but only in codimension 1, for any $(\mathrm{FGA})_n$ algebra when X/T is birational. In Corollary 6.43, this is established in codimension 2 on a similar basis.

Proof of Proposition 4.54. Take a general hyperplane section H of T and a point $P \in T$ and $Y = f^{-1}H$. Then $\mathcal{L}_{|Y}$ is of type $(\mathrm{FGA})_{n-1}$. The lca saturation$/(Y/H, B_{|Y})$ follows from Addendum 4.50.1 with $X/Z = X/T$ and $B := B + H$, where (ASA$'$) holds by Example 4.45. Then local f.g. of $\mathcal{L}_{|Y}$ near $H \cap P$ implies local f.g. of $\mathcal{L}$ near P by general properties of restrictions and the definition of stabilization. Then we use induction on n.

For (RFA), note that the restriction $_{|Y}$ also preserves (RFA). $\square$

We remark finally that, in the proof of $(\mathrm{FGA})_1$, it was very important that every prime divisor, and every effective divisor of rather high degree up to linear equivalence, was nonsingular, that is, each is a reduced, nonsingular subvariety (cf. Example 6.24 below). In Section 5, we explain to what extent we need this property ("prediction models" in Definition 5.10), and in Section 6, we state when we may expect this property to hold ("triples" and Conjecture 6.14 below).

5. SATURATION AND DESCENT

5.1. Descent of divisors. In this section, we consider the following problem, the key to our construction of pl flips. Given a morphism $f\colon Y \to X$ of algebraic varieties and an $\mathbb{R}$-Cartier divisor D on Y, can we find an $\mathbb{R}$-Cartier divisor D_X on X for which $D = f^*D_X$? If so, we say that D_X is the *descent* of D; this is thus a descent problem. We solve this problem under very special circumstances, when our divisor is a limit or a rather high element in a sequence of divisors on birational models of X. Thus, in fact, we consider this problem for b-divisors (see [15] and [45]). Therefore, we can replace D by its Cartier completion $\mathcal{D} = \overline{D}$ or some other $\mathbb{R}$-b-divisor, and Y by some birational model. The problem is then to know whether $\mathcal{D} = \overline{\mathcal{D}_X}$ holds.

Remark 5.2. If we replace the equality of divisors by linear equivalence, we get a different problem, that is better stated in terms of sheaves: $\mathcal{O}_Y(\mathcal{D}) = f^*\mathcal{O}_X(\overline{\mathcal{D}_X})$. This is equivalent to the first problem if Y/X is birational and $\mathcal{D}_X$ is Cartier; otherwise, the second is quite different and more flexible. However, we can relax the problem with equality $=$ down to the problem with linear equivalence $\sim$ by considering $\mathcal{D}$ up to linear equivalence: $\mathcal{D}' \sim \mathcal{D}$ and $\mathcal{D}'$ has a descent (cf. Lemma 3.28). It is even reasonable to consider $\sim_{\mathbb{Q}}$, $\sim_{\mathbb{R}}$, or $\equiv$ (cf. bss and b-semiampleness above, and the case of triples in Lemma 8.12 below).

The obstruction to the problem is easy to find.

5.3. Descent data. The descent of $\mathcal{D}$ on X exists if and only if $\mathcal{E} = \mathcal{E}(\mathcal{D}) := \overline{\mathcal{D}_X} - \mathcal{D}$ is 0 as a b-divisor. The b-divisor $\mathcal{E}$ is the *descent data* of $\mathcal{D}$ *over* X.

Proposition 5.4 (properties of descent data)**.** *Let $\mathcal{D}$ be an $\mathbb{R}$-b-divisor of X. Then*

(EXI) *the descent data $\mathcal{E}$ of $\mathcal{D}$ over X* exists *if $\mathcal{D}_X$ is $\mathbb{R}$-Cartier, in particular, if X is $\mathbb{Q}$-factorial*;

(EXC) $\mathcal{E}/X$ *is* exceptional *on* X;

(ADD) additivity: $\mathcal{E}(\mathcal{D} + \mathcal{D}') = \mathcal{E}(\mathcal{D}) + \mathcal{E}(\mathcal{D}')/X$;

(HOM) homogeneity: $\mathcal{E}(r\mathcal{D}) = r\mathcal{E}(\mathcal{D})/X$ *for any real number* r;

(DEP) *the descent data $\mathcal{E}$ only depends on $\mathcal{D}$* up to $\mathbb{R}$-linear equivalence, *or even only up to numerical equivalence$/X$*; *and*

(EFF) $\mathcal{E}$ *is* effective *if $\mathcal{D}$ is b-semiample, or even b-nef$/X$.*

Thus, for $\mathcal{D}$ as in (EFF), if we know that $\mathcal{E} \le 0$, then $\mathcal{E} = 0$ and the descent exists.

Proof. (EXI), (EXC), (ADD), and (HOM) follow directly from the definitions; (EXC) means that $\mathcal{E}_X = 0$. Assertion (DEP) for $\sim_{\mathbb{R}}$ follows because the descent data for a principal $\mathbb{R}$-b-divisor is trivial. In turn, by (ADD) and (HOM), this follows from the same fact for principal b-divisors:

$$\mathcal{E}(\overline{(f)}) = \overline{\overline{(f)}_X} - \overline{(f)} = \overline{(f)} - \overline{(f)} = 0.$$

For numerical equivalence, by (ADD) and (HOM), we can apply (EFF) to the difference.

For (EFF), it is enough to check that $E_Y = f^*\mathcal{D}_X - \mathcal{D}_Y$ is effective for any resolution $f\colon Y \to X$. For b-semiample $\mathcal{D}$, this follows from (DEP) and the fact that f^*D is effective if D is. For a b-divisor $\mathcal{D}$ that is b-nef$/X$, we get this by Lemma 4.23 or by [43, Negativity 2.15]. □

For a general divisor D, as well as for a general b-divisor $\mathcal{D}$, descent may not exist even on a sufficiently high model. However, we are interested in two related cases: $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$, where each $\mathcal{D}_i$ is the Cartier completion of some $\mathbb{R}$-Cartier divisor D_i on a model Y_i/X; that is, $\mathcal{D}_i = \overline{D}_i$ (cf. Example 4.14). Thus, each $\mathcal{D}_i$ has a descent D_i on Y_i. We also expect that, under certain conditions, the b-divisor $\mathcal{D}$ may also have a descent, or even that all the $\mathcal{D}_i$ have a descent on one model, and $\mathcal{D}$ together with them.

5.5. Asymptotic descent problem. Suppose that $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ is a limit of b-divisors on X/Z. Find a model Y/Z of X/Z such that

- $\mathcal{D} = \overline{(\mathcal{D})_Y}$;
- the limit *stabilizes*, that is, $\mathcal{D}_i = \mathcal{D}$ for some $i \gg 0$; in particular,
- for every such i, also $\mathcal{D}_i = \overline{(\mathcal{D}_i)_Y} = \mathcal{D}$.

Thus, in this case, infinitely many $\mathcal{D}_i$ have a descent on a single model Y (cf. Remark 5.6(4)). Our approach gives a *strict* asymptotic descent; that is, stabilization holds for some $i \gg 0$ after any truncation of $\mathcal{D}_\bullet$. But, in fact, in applications, we get a *complete* asymptotic descent; that is, stabilization holds for all i after a truncation of $\mathcal{D}_\bullet$ (cf. Theorem 5.12 versus Corollary 5.13).

The sheaf version of these conditions is related to f.g. and is given in our standard Example 5.27 below.

The *minimal* assumptions on the system $\mathcal{D}_\bullet$ are the following:

(FDS) *finite divisorial support*: the divisors $(\mathcal{D}_i)_X$ are supported in one reduced divisor F; and

(MXD) as in Addendum 4.22.1: each $\mathcal{D}_i \le \mathcal{D}$.

However, these assumptions plus the two standard ones (saturation and boundedness) are scarcely enough for partial *rationality* of $\mathcal{D}$ (cf. Examples 4.41, 5.14, and Theorem 5.12). For descent, we need more. The *additional* assumption on the system $\mathcal{D}_\bullet$ is

(BNF) each $\mathcal{D}_i$ is *b-nef$/X$* in the sense of Lemma 4.23; in particular, this includes

(CAR) each $\mathcal{D}_i$ is $\mathbb{R}$-*Cartier*: $\mathcal{D}_i = \overline{D}_i$ for the $\mathbb{R}$-Cartier divisor $D_i = (\mathcal{D}_i)_{X_i}$ over some model X_i/Z of X/Z (cf. Example 4.14).

In the above assumptions, *each* $\mathcal{D}_i$ means *up to a truncation*.

Remarks 5.6. (1) As in Remark 4.34(7), similarity of characteristic type preserves the minimal assumptions, the additional assumptions, and the asymptotic descent itself.

(2) In our applications, b-nef follows from b-semiample; and, usually, each $\mathcal{D}_i$ is $\mathbb{Q}$-Cartier (cf. Example 5.27 below).

(3) Since every modification blows up at most a finite number of divisors, assertion (FDS) is birationally invariant: if it holds on one model of X, then it also holds on any other. It is equivalent to (BSD) in Addendum 4.22.1 but is weaker than the following condition:

(FSP) *finite support*: there exists a proper subvariety $S \subset X$ such that all $\mathcal{D}_i$ are supported over S.

However, under (BNF), which holds in most of our applications, if each $\mathcal{D}_i \ge 0$, the two conditions are equivalent by Lemma 10.9 below.

(4) A limit of $\mathbb{R}$-Cartier b-divisors is not necessarily $\mathbb{R}$-Cartier in general. For example, this is usually the case for the limit of the characteristic system of an infinitely generated functional algebra. However, under (FDS), if (CAR) holds on a single model $Y = X_i$ for infinitely many i, the limit $\mathcal{D}$ is also $\mathbb{R}$-Cartier/Y. By definition, it is enough to establish this (locally) for principal divisors. These are $\mathbb{R}$-Cartier/Y because $\mathbb{R}$-principal divisors with support in a fixed reduced divisor form an $\mathbb{R}$-vector subspace (defined/$\mathbb{Q}$) in the space of all $\mathbb{R}$-divisors with given support.

We can resolve the asymptotic descent problem for $\mathcal{D}_\bullet$ under asymptotic saturation and the following confinement.

Definition 5.7 (confined descent data)**.** Suppose that $\mathcal{C}$ and $\mathcal{D}$ are two $\mathbb{R}$-b-divisors of X. We say that the descent data $\mathcal{E}$ of $\mathcal{D}$ *over* X is *confined* by $\mathcal{C}$ if

- the data $\mathcal{E} = \overline{\mathcal{D}_X} - \mathcal{D}$ is *defined*, in particular, $\mathcal{D}_X$ is $\mathbb{R}$-Cartier, and
- $\mathcal{E} \le \mathcal{C}$ *over* Y, that is, $\operatorname{mult}_{E_i} \mathcal{E} \le \operatorname{mult}_{E_i} \mathcal{C}$ for each prime b-divisor E_i (see [15]) that is *exceptional* on X.

Now let $\mathcal{D}_\bullet$ be a system of $\mathbb{R}$-b-divisors of X, for example, the sequence of a limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$. The system $\mathcal{D}_\bullet$ is *asymptotically confined* by $\mathcal{C}$ if there exists a sequence of positive real numbers r_i for some $i \gg 0$ such that

- $\lim_{i\to\infty} r_i = +\infty$, and
- for each r_i, the descent data $\mathcal{E}_i = \overline{(\mathcal{D}_i)_X} - \mathcal{D}_i$ over X is confined by $\mathcal{C}/r_i$, or equivalently, $r_i\mathcal{E}_i \le \mathcal{C}$ over Y.

We say that the asymptotic confinement is *strict* if we can choose a subsequence of real numbers r_i with the stated property for any truncation of $\mathcal{D}_\bullet$; the subsequence of $\mathcal{D}_\bullet$ corresponding to the r_i will be called *strictly infinite*.

Remarks 5.8. (1) Under (FDS), $\lim_{i\to\infty} \mathcal{E}_i = \mathcal{E} = \overline{\mathcal{D}_X} - \mathcal{D}$ is the descent data for $\mathcal{D}$ over X. Indeed, $\mathcal{D}_X$ is also $\mathbb{R}$-Cartier by Remark 5.6(4), and passing to the limit commutes with restriction $(\cdot)_X$ and Cartier completion $\overline{(\cdot)}$.

(2) If, in addition, the sequence of the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ satisfies (BNF), then each $\mathcal{E}_i \ge 0$ by (EFF) in Proposition 5.4, and thus $\mathcal{E} \ge 0$. On the other hand, for chosen i, we have $\mathcal{E}_i \le \mathcal{C}/r_i$ over X, so that $\mathcal{E} = \lim_{i\to\infty} \mathcal{E}_i \le 0$. (This means that the limits $\mathcal{E} = \lim_{i\to\infty} \mathcal{E}_i$ and $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ are *uniform with respect* to $\mathcal{C}$.) Hence, $\mathcal{E} = 0$ and $\mathcal{D} = \overline{\mathcal{D}_X}$ has descent $\mathcal{D}_X$ on X. Thus, the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ has asymptotic descent if and only if it stabilizes; and the latter descent is also on X.

(3) Asymptotic confinement is a necessary condition for the existence of asymptotic descent on X, namely, for any $\mathcal{C} \ge 0$ and any sequence of r_i with i such that $\mathcal{E}_i = 0$.

(4) Similarity of characteristic type as in Remark 4.34(7) preserves strict asymptotic confinement. For the truncation $I\mathcal{D}_{iI}$, we can take the same $\mathcal{C}$ and new real numbers $r_{i/I} := r_i/I$, whereas we can take the same (truncated) real numbers $r_{i/I} := r_i$ for the usual truncation $\mathcal{D}_{iI}$.

(5) Similarly, by Proposition 5.4, (HOM), multiplication by any positive real number q preserves (strict) asymptotic confinement: we can take positive reals $r_i := r_i/q$.

(6) It is also useful to consider asymptotic confinement *over a sequence* $(X_i, \mathcal{C}_i)$, where now $\mathcal{E}_i$ is the descent data of $\mathcal{D}_i$ over X_i, $r_i\mathcal{E}_i \le \mathcal{C}_i$, and $\mathcal{D}_i$, $\mathcal{C}_i$ are b-divisors of X_i. The most important case for us is *canonical* asymptotic confinement, where $\mathcal{C}_i = \mathcal{A}(X_i, B_i)$ for a sequence of log pairs (X_i, B_i) (see Addendum 6.8.2 below), say, of crepant models of a given $(X/Z, B)$ with the same $\mathcal{C} = \mathcal{A} = \mathcal{A}(X, B) = \mathcal{A}(X_i, B_i)$ (see the proof of Proposition 9.13 below).

Remark 5.8(2) is very close to the stabilization required for f.g. The main difference is in a *universal* descent; that is,

- infinitely many $\mathcal{D}_i$ have descent on the single model X.

To establish this, we need asymptotic $\mathcal{C}$-saturation (cf. Theorem 5.12). In addition, asymptotic $\mathcal{C}$-saturation gives

- *rationality*: $\mathcal{D}$ is a $\mathbb{Q}$-divisor.

Both of the properties are essential for f.g. of functional algebras (cf. Limiting Criterion 4.28 above and Example 5.27 below).

For b-divisors, the choice of an appropriate model for descent is crucial (cf. Example 5.14, triples, and Conjecture 6.14). A criterion for an appropriate model is asymptotic confinement over it (cf. Remarks 5.8(2),(3)).

5.9. Choice of Y in the asymptotic descent. In our approach to the descent problem (cf. Theorems 5.12, 6.19(3), and 8.23 below), it is easier to present a b-divisor by a semiample $\mathbb{R}$-Cartier divisor than by any other. Because, by definition, semiample gives some model and an $\mathbb{R}$-Cartier divisor on it! Thus, we need a model Y/Z of X/Z such that

(SAM) $\mathcal{D}_Y$ is *semiample*$/Z$, in particular, nef$/Z$.

Semiampleness$/Z$ means the semiampleness of $\mathbb{R}$-divisors [45, Definition 2.5]. If $D = \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor, this condition is equivalent to *eventually free*, that is, $\operatorname{Bs}|ND| = \varnothing$ for some natural number N.

We also need an $\mathbb{R}$-divisor $\mathcal{C}$ on X, a reduced $\mathbb{Q}$-Cartier divisor F on Y, and a positive real number γ such that

(EEF) *exceptional effectiveness*: $\mathcal{C} \geq 0/Y$, that is, $\operatorname{mult}_{E_i} \mathcal{C} \geq 0$ in each prime b-divisor E_i that is exceptional$/Y$; and

(LGD) *linear growth for divisor*: in each prime b-divisor P_i,

$$\operatorname{mult}_{P_i} \mathcal{C} > -1 + \gamma \operatorname{mult}_{P_i} F,$$

or, equivalently,

(AEF) $\mathcal{C} - \gamma\overline{F}$ is *almost effective*: $\lceil \mathcal{C} - \gamma\overline{F} \rceil \geq 0$.

The latter implies that $\mathcal{C}$ is also almost effective, i.e., $\lceil \mathcal{C} \rceil \geq 0$, whereas (EEF) states more$/Y$, namely, that $\mathcal{C}$ is exceptionally effective$/Y$.

Definition 5.10. We say that $(Y/Z, \mathcal{C}, F, \gamma)$ is a *prediction model* for the asymptotic descent problem if Y, $\mathcal{C}$, F, and γ exist and satisfy the following conditions:

(SAC) the descent data for $\mathcal{D}_\bullet$ is *strictly asymptotically confined*$/Y$ by $\mathcal{C}$; and

(UAD) F includes the supports of the irrational part of $\mathcal{D}_Y$, and the *divisorial* locus where the *strict asymptotic descent* is *unknown* (*not given*) (cf. Remark 5.11(3)); that is, for any prime divisor P_j in this locus of Y, $\operatorname{mult}_{P_j} \mathcal{D}$ is irrational, or after any truncation $\operatorname{mult}_{P_j} \mathcal{D}$ is an accumulation point of $\operatorname{mult}_{P_j} \mathcal{D}_i$.

Under (FDS), by Definition 5.7 and Remark 5.6(4), (SAC) implies (SAM).

Remarks 5.11. (1) In applications under (BNF), confinement implies that

$$\beta\mathcal{C} \geq \mathcal{E}_i \geq 0/Y$$

for some natural number i and positive real β (cf. Proposition 5.4, (EFF), and Example 5.27). Thus, $\mathcal{C} \geq 0/Y$ as in (EEF).

(2) If $\mathcal{C} \geq \varepsilon\overline{F}/Y$ for some $\varepsilon > 0$ and $\lceil \mathcal{C}_Y \rceil \geq 0$, then (EEF) and (LGD) hold for some $0 < \gamma \ll \varepsilon$.

(3) For simplicity, we can assume in (LGD) that, for *any* F, there exists a positive γ for which it holds. In fact, we need a bounded family of F, or just a single F in most applications (compare the triples of Definition 6.9 below). The minimal assumption on F is (UAD), but it is enough that F contains the support of the fractional part for every divisor $(\mathcal{D}_i)_Y$. (See (FDS) in the minimal assumptions above and Examples 5.27, 8.21 below.) We need this assumption even though *a posteriori* $F = 0$, as in Theorem 5.12.

(4) Similarity of the characteristic type as in Remark 4.34(7) preserves (SAC) and the other properties for the same prediction model. For (UAD), we can use the invariance of the irrational part of $\mathcal{D}$ under similarity (see Remark 5.8(4) above and Lemma 6.30 below).

Now we are ready to state the main result of the section.

Theorem 5.12. *Let $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ be an asymptotic descent problem such that*

- *the problem has a prediction model $(Y/Z, \mathcal{C}, F, \gamma)$;*
- *$\mathcal{D}_\bullet$ satisfies the minimal assumptions of* 5.5: (FDS) on Y (maybe, for some reduced divisor $\neq F$) *and* (MXD); *and*
- *$\mathcal{D}_\bullet$ is asymptotically $\mathcal{C}$-saturated.*

Then $\mathcal{D}_Y$ is a $\mathbb{Q}$-divisor, and strict stabilization *holds on Y: after any truncation, $(\mathcal{D}_i)_Y = \mathcal{D}_Y$ for some $i \gg 0$.*

Addendum 5.12.1. *If* (BNF) *also holds* (*on an infinite subsystem in $\mathcal{D}_\bullet$*), *then the strict asymptotic descent problem has a solution on Y.*

Addendum 5.12.2. *Moreover, if we assume that $\mathcal{D}$ is big/Z and* (CGR) *of Proposition* 4.46 *also holds, then $\mathcal{D}_Y$* (*or $\mathcal{D}$*) *contracts all prime divisors P_i on Y with* $\operatorname{mult}_{P_i} \mathcal{C} > 0$. *Thus, $\mathcal{C}_{X_{\rm st}} \leq 0$ on the* stable *model*

$$X_{\rm st}/Z = \operatorname{Proj}_Z \mathcal{R}_Z \overline{\mathcal{D}_Y} = \operatorname{Proj}_Z \mathcal{R}_Z \mathcal{D}$$

(*see Conjecture* 6.14 *and Addendum* 6.26.2).

The main part of the theorem is a rationality theorem (see Theorem 3.18 and Example 4.41), while Addenda 5.12.1 and 5.12.2 treat the descent and the structure of the stable model.

Corollary 5.13. *Under arithmetic monotonicity* (*assertion* (AMN) *of Conjecture* 4.39), complete stabilization *on Y holds, i.e., for all i after a truncation of $\mathcal{D}_\bullet$. More precisely, there exists a stabilization index I such that $(\mathcal{D}_i)_Y = \mathcal{D}_Y$ for all $I \mid i \gg 0$.*

Addendum 5.13.1. *Again under arithmetic monotonicity* (AMN) *and assumption* (BNF), complete stabilization *holds for $\mathcal{D}_\bullet$. More precisely, there exists a stabilization index I such that $\mathcal{D}_i = \mathcal{D} = \overline{\mathcal{D}_Y}$ for all $I \mid i \gg 0$.*

Proof. Immediate by the theorem together with Addendum 5.12.1, (MXD), and (AMN). □

The proof of the theorem also implies its nonnormal and rather singular generalization that we discuss at the end of the section (see Corollaries 5.21 and 5.23 below). Finally, we use this in the proof of (RFA) of Main Theorem 1.7, in particular, for 4-fold log flips (see Corollaries 6.42, 10.14 and Theorem 6.45 below).

Example 5.14. Let $Y \to X$ be a divisorial contraction with an exceptional Cartier divisor E. For example, it might be a blowup of a smooth locus in X. Take a system $\mathcal{D}_i = d_i\overline{E}$, where d_i is a sequence of real numbers with $\lim_{i\to\infty} d_i = d$. If all $d_i \leq d$ and $d > 0$, then $\mathcal{D}_\bullet$ satisfies all the conditions of Theorem 5.12 on X, except for (BNF). The descent problem has a solution on Y, but it does not stabilize and may not be rational. Nonetheless, for a prediction model X (for example, $(X/X, 0, 0, 1)$), $\mathcal{D}_X = 0$ is always rational as in the main part of the theorem.

Note that the same holds even for $d = +\infty$.

Lemma 5.15 (Diophantine approximation). *Let $\mathfrak{Q} \subset \mathbb{R}^N = \mathbb{Z}^N \otimes_{\mathbb{R}} \mathbb{R}$ be a rational polyhedron* (*not necessarily convex*) *and $D = (d_1, \dots, d_N) \in \mathfrak{Q}$ be a point in it. Then there exist positive real numbers r* (*depending only on N*) *and ε, an infinite set of natural numbers m, and a sequence of rational points $Q_m \in \mathfrak{Q}$ such that*

- *each mQ_m is* integral: *$mQ_m \in \mathbb{Z}^N$,*
- *$\|D - Q_m\| < \varepsilon/m^{1+r}$.*

Moreover, if D is irrational, then there exist $D' \in \mathfrak{Q}$ and a sequence $Q_m = (q_{m,1}, \dots, q_{m,N})$ as above for D' such that

- *$D' \le D$; but*
- *$q_{m,i} > d_i$ at least in one coordinate* (*the ith*); *and*
- *$q_{m,i} = d_i$ for the rational coordinates d_i.*

Notation 5.16. We write $\|(x_1, \dots, x_N)\| = \max\{|x_i|\}$ for the *maximal absolute value* norm in $\mathbb{R}^N$; it induces the usual real topology of $\mathbb{R}^N$. The inequality

$$D' = (d'_1, \dots, d'_N) \le D = (d_1, \dots, d_N)$$

means that each $d'_i \le d_i$.

By the proof below, we can take any $0 < r < 1/N$. For a weaker approximation with r about $(1/2)^N$, we can take $D' = D$ (for this and other directed approximations, see Borisov and Shokurov [6, разд. 1]). In Section 8, we use similar but nondirected approximations.

Proof of Lemma 5.15. *Step* 1. We can assume that D is irrational; that is, it has an irrational coordinate d_i. Otherwise, we take $Q_m = D$ for m such that mQ_m are integral, and take any r and $\varepsilon > 0$. In particular, $N \ge 1$ if D is irrational.

Step 2. We can assume that each coordinate d_i of D is irrational and, moreover, that D does not belong to any proper rational affine subspace H. (If d_i is rational, then $D \in H = \{x_i = d_i\}$.) Indeed, if $D \in H$, then there is a rational affine embedding $A\colon \mathbb{R}^M \hookrightarrow \mathbb{R}^N$ with

- $A(\mathbb{R}^M) = H$, and
- $M \le N - 1$.

Note that $\mathfrak{P} = A^{-1}\mathfrak{Q} \subset \mathbb{R}^m$ is also a rational polyhedron with $C = A^{-1}D \in \mathfrak{P}$. Thus, if $P_m \in \mathfrak{P}$ is a sequence with the required property for C, then $Q_{Im} = A(P_m)$ gives the required sequence for $D = A(C) \in \mathfrak{Q}$, where I is an *index* for A, that is, IA is integral. The set of natural numbers m is replaced by Im (truncation). Hence, $ImQ_m = (IA)(mP_m) \in \mathbb{Z}^N$, and

$$\|D - Q_{Im}\| = \|A(C - P_m)\| < \|A\|\varepsilon/m^{1+r} = \|A\|I^{1+r}\varepsilon/(Im)^{1+r};$$

thus, we can take $\varepsilon := \|A\|I^{1+r}\varepsilon$ for Q_{Im}, where $\|A\| = \max\|Ax\|/\|x\|$, for $x \ne 0$, is the usual norm of the linear part of A. However, we need to replace the conditions on D' by geometric ones; namely, for a polyhedral cone $\mathfrak{D}$ with the single vertex D,

- $D' \in \mathfrak{D}$; but
- each $Q_m \notin \mathfrak{D}$; and
- D' is still in each rational face of $\mathfrak{D}$.

For example, the original cone $\mathfrak{D}$ is given by the inequalities $x_i \le d_i$. But the cone $\mathfrak{C} = A^{-1}\mathfrak{D}$ with C as its only vertex is more general. Nonetheless, if $C' \in \mathfrak{C}$ has a required sequence P_m, then Q_{Im} satisfies the lemma with $D' = A(C')$. Note that each rational face of $\mathfrak{D}$ gives a (nonempty) rational face of $\mathfrak{C}$ (because C belongs to it).

Step 3. We can assume that $\dim \mathfrak{Q} = N$ near D and D is in the interior of $\mathfrak{Q}$. Thus, we can assume that $\mathfrak{Q} = \mathbb{R}^N$ locally near D. Otherwise, there is a rational face $H \subset \mathbb{R}^N$ of $\mathfrak{Q}$ such that $D \in H$. Then we can reduce the lemma to H as in Step 2. Therefore, by induction on N, we get the interior.

The required approximations then exist for any $0 < r < 1/N$ and $\varepsilon \geq N/(N+1)$ by Cassels [7, Ch. I, Theorem VII], where $N \geq 1$ is the dimension of the approximation space $\mathbb{R}^N$.

Step 4. By the arguments of Step 3, we can assume that the cone $\mathfrak{D}$ has no rational faces.

Finally, the last statement of the lemma requires a more accurate choice of m and Q_m, namely, $Q_m \notin \mathfrak{D}$. Otherwise, there is an infinite subsequence $Q_m \in \mathfrak{D}$. Their affine span H is rational and contains D since $D = \lim_{m\to\infty} Q_m$. Thus, by Step 2, $H = \mathbb{R}^N$, and the cone $\mathfrak{D}$ is *big*; that is, $\dim \mathfrak{D} = N$. Since $N \geq 1$, there is a rational line L in $\mathbb{R}^N$ such that the polyhedron $\mathfrak{D} \cap L = [D', \infty)$ is a ray and its vertex D' is close to D and irrational on L. By Step 2, we can reduce the last statement of the lemma to L with D'. In the latter case, we use continued fractions for the best approximations in L from outside $[D', \infty)$ [7, Ch. I, Theorem II]. $\square$

We apply the lemma to a polyhedron in an $\mathbb{R}$-vector space

$$\mathfrak{D}_F = \bigoplus_{i=1}^{N} \mathbb{R}P_i$$

of $\mathbb{R}$-divisors on X supported in F, where $F = \sum_{i=1}^N P_i$ is a reduced divisor on X. The multiplicities of divisors define a canonical isomorphism/$\mathbb{Q}$ of the space with $\mathbb{R}^N$. It induces the well-known topology in $\mathfrak{D}_F$ and norm, the *maximal absolute value* of the multiplicities in prime divisors (or in prime b-divisors in the case of $\mathbb{R}$-b-divisors). In particular, the limits of *divisors* (but not b-divisors) are limits in the norm (cf. Caution 4.25). The isomorphism transforms the ordering of divisors into $\leq$ in $\mathbb{R}^N$.

In the space, we take a polyhedron $\mathfrak{D}$ of semiample/Z divisors under the following extra assumption.

Definition 5.17. For a given set of divisors D, e.g., for a cone of divisors in Example 5.18 below,

(BND) semiampleness is *bounded* (*effective*) if there exists a natural number M such that, for some $N \leq M$, all $|N'ID|$ with $N' \geq N$ are free, where I is the index of D

(compare triples and Lemma 8.12(4) below). Of course, this *only applies* to $\mathbb{Q}$-divisors in the set.

Example 5.18. Any rational polyhedral cone $\mathfrak{D}$ generated by a finite set of divisors D_i that are semiample/Z satisfies (BND) of Definition 5.17. Indeed, we can assume that

- each D_i is a $\mathbb{Q}$-divisor;
- F is the common support of all divisors D_i; and
- the cone is simplicial.

Thus, each $D \in \mathfrak{D}$ has a unique presentation $D = \sum r_i D_i$ with *coordinates* $r_i \in \mathbb{R}$, each $r_i \geq 0$, and D is rational if and only if each r_i is rational. There is a natural number j such that each rational jID has integer coordinates r_i, where I is the index of D. Then we can take M in (BND) such that M is the minimal natural number for which every $|MD_i|$ is free.

Corollary 5.19 (Diophantine approximation). *Let D be an $\mathbb{R}$-divisor on X that is semiample/Z. Then there exist positive real numbers r and ε, an infinite set of natural numbers m, and a sequence of $\mathbb{Q}$-divisors Q_m on X such that*

- *each $|mQ_m|$ is free, and,* in particular, each Q_m is semiample;

- *each* $\operatorname{Supp}(D - Q_m) \subset \operatorname{Supp} D_{\mathrm{irr}}$, *where* D_{irr} *is the irrational part of* D; *that is, the approximation concerns only the irrational multiplicities of* D;
- $\|D - Q_m\| < \varepsilon/m^{1+r}$.

Moreover, if D is not a $\mathbb{Q}$-divisor, then there exist an $\mathbb{R}$-divisor D' and a sequence of $\mathbb{Q}$-divisors Q_m on X as above for D' such that

- $\operatorname{Supp}(D - D') \subset \operatorname{Supp} D_{\mathrm{irr}}$;
- $D' \le D$; *but*
- $\operatorname{mult}_{P_i} Q_m > \operatorname{mult}_{P_i} D$ *at least in one prime divisor* P_i *on* X; *and*
- $\operatorname{mult}_{P_i} Q_m = \operatorname{mult}_{P_i} D$ *for the rational multiplicities* $\operatorname{mult}_{P_i} D$.

Addendum 5.19.1. *For any ideal sheaf J on X, we can also assume the following* freedom with respect *to* J:

- *each subsystem* $|J(mQ_m)| \subset |mQ_m|$ *is* free over $T \setminus c(\operatorname{Supp} J)$, *where* $c\colon X \to T/Z$ *is a contraction given by* D, *that is,* $c(\operatorname{Bs}|J(mQ_m)|) \subset c(\operatorname{Supp} J)$.

The last statement generalizes to the nonnormal and nonreduced case in the style of Corollary 5.21 below. We can also take $D' = D$; for this, see Borisov and Shokurov [6, следствие 2.1, добавление 2.1.1].

Proof. To apply Lemma 5.15, we include D in a rational polyhedron $\mathfrak{Q}$.

Since D is semiample/Z,

(NBH) each divisor in a *neighborhood* of D is (nef and) semiample/Z;

moreover, (BND) holds in it. Indeed, $D \sim_{\mathbb{R}} c^*H$, where

- $c\colon X \to T/Z$ is the contraction in Addendum 5.19.1; and
- H is an $\mathbb{R}$-divisor on T that is numerically ample/Z.

In other words, $D = c^*H + \sum r_i(f_i)$, where $r_i \in \mathbb{R}$ and $0 \ne f_i \in k(X)$. Let F be a reduced divisor on X that contains the supports of divisors (f_i) and c^*H. Take any rational polyhedral cone $\mathfrak{H}$ consisting of $\mathbb{R}$-divisors $H \in \mathfrak{H}$ on T that are numerically ample/Z, with $\operatorname{Supp} c^*\mathfrak{H} \subset F$. Then the rational polyhedral cone

$$\mathfrak{D} = c^*\mathfrak{H} \oplus \Big(\bigoplus \mathbb{R}(f_i)\Big) \subset \mathfrak{D}_F$$

is generated by semiample divisors because each divisor that is numerically ample/Z is semiample/Z by the Kleiman criterion, and each principal divisor is also semiample/Z. By Example 5.18, it satisfies (BND) with some natural number M. (In addition, we can suppose that $\mathfrak{H}$ and $\mathfrak{D}$ are respective neighborhoods of H and D in their linear spans/$\mathbb{Q}$.) By construction, $D \in \mathfrak{D}$.

To preserve the rational multiplicities (coordinates) under the approximation, set $\mathfrak{Q} = \mathfrak{D} \cap Q$, where Q is the subspace of $\mathfrak{D}_F$ given by the system of affine equations/$\mathbb{Q}$: $\operatorname{mult}_{P_i} x = \operatorname{mult}_{P_i} D$, provided that the latter multiplicity is rational. By construction,

- $\mathfrak{Q}$ is a rational polyhedron; and
- $D \in \mathfrak{Q}$.

Then Lemma 5.15 gives real numbers r and ε and a sequence of $\mathbb{Q}$-divisors $Q_m \in \mathfrak{Q}$. They satisfy the corollary except for $|mQ_m|$ free. However, mQ_m is an *integral* divisor, jmQ_m is Cartier for some j depending only on $\mathfrak{Q}$, and $|MmQ_m|$ has no base points by Example 5.18 with $M := jM$. Thus, as in Step 2 in the proof of Lemma 5.15, we can replace Q_m by $Q_{Mm} := Q_m$ and take $\varepsilon := M^{1+r}\varepsilon$.

The addendum follows from its ample case. The splitting into $c^*\mathfrak{H}$ and $\mathbb{R}$-principal parts is given over $\mathbb{Q}$. Thus, again as in Step 2 of the proof of Lemma 5.15, we can assume that $mQ_m \sim mc^*Q'_m$, where $Q'_m \in \mathfrak{H}$ is a $\mathbb{Q}$-divisor and mQ'_m is also integral. Since the subsystem $|J(mQ_m)|$ in $|mQ_m|$ depends only on J and the complete system itself on mQ_m up to $\sim$, the freedom with respect to J is preserved up to $\sim$, and we need to establish it for $mc^*Q'_m$. On the other hand, J coherent implies that there is an ideal sheaf J_T on T such that

- $J_T = \mathcal{O}_T(-H')$ for some effective very ample divisor on T/Z;
- $c^*|J_T(mQ'_m)| \subset |J(mc^*Q'_m)|$;
- $c(\operatorname{Supp} J) \subset \operatorname{Supp} J_T$; and
- $c(\operatorname{Supp} J) = \bigcap \operatorname{Supp} J_T$ for such H' up to $\sim /Z$.

Then it is enough to establish the addendum for ample mQ'_m with respect to J_T, which follows from the fact that $mQ'_m - H'$ is ample and free for an appropriate m because we can assume that the generators of $\mathfrak{H}$ satisfy the same property: $MD - H'$ is very ample for any integral ample divisor D in $\mathfrak{H}$ (cf. the end of the proof of Addendum 4.50.2, p. 132). □

However, to apply the corollary to b-divisors, we need the following result.

Lemma 5.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be two b-divisors of X, F be a reduced divisor, and $\alpha \le 1$ and τ be positive real numbers such that*

- *F is $\mathbb{Q}$-Cartier and*
$$\operatorname{mult}_{P_i} \alpha\mathcal{C} > -\alpha + \tau \operatorname{mult}_{P_i} F; \tag{5.20.1}$$
in each prime b-divisor P_i,
- *the descent data $\mathcal{E}$ of $\mathcal{D}$ over X is confined by $(1-\alpha)\mathcal{C}$,* that is,
$$\mathcal{E} \le (1-\alpha)\mathcal{C}/X; \tag{5.20.2}$$
and
- *$\mathcal{D}_X = D_{\mathrm{bf}} + D_{\mathrm{fr}}$, where $|D_{\mathrm{bf}}|$ is free,* $\operatorname{Supp} D_{\mathrm{fr}} \subset F$, *and $\|D_{\mathrm{fr}}\| < \tau$.*

Then $\operatorname{Mov}\lceil \mathcal{D} + \mathcal{C}\rceil/Z$ *satisfies the following estimate*: *on any model W/X,*
$$\operatorname{Mov}\lceil \mathcal{D} + \mathcal{C}\rceil_W \ge (\overline{D_{\mathrm{bf}}})_W;$$
in particular, the nonvanishing $|\lceil \mathcal{D} + \mathcal{C}\rceil_W| \ne \varnothing/Z$ holds.

Addendum 5.20.1. *If an ideal sheaf J on X is given, then*
$$\operatorname{Mov} J(\lceil \mathcal{D} + \mathcal{C}\rceil_W) \ge (\overline{D_{\mathrm{bf}}})_W$$
over $X \setminus \operatorname{Bs}|J(D_{\mathrm{bf}})|$; *in particular, the nonvanishing $|\lceil \mathcal{D} + \mathcal{C}\rceil_W| \ne \varnothing/Z$ holds if* $X \ne \operatorname{Bs}|J(D_{\mathrm{bf}})|$; *the base points are considered as for the subsystem $|J(D_{\mathrm{bf}})|$ in $|D_{\mathrm{bf}}|$.*

Note that $\mathcal{D}_X$ is $\mathbb{R}$-Cartier since its descent data is confined (cf. Definition 5.7).

Proof. By the definition of descent data, $\mathcal{D} = \overline{\mathcal{D}_X} - \mathcal{E}$. Then, by the decomposition of $\mathcal{D}_X$,
$$\mathcal{D} = \overline{D_{\mathrm{bf}} + D_{\mathrm{fr}}} - \mathcal{E} = \overline{D_{\mathrm{bf}}} + \overline{D_{\mathrm{fr}}} - \mathcal{E},$$
and, hence, $\mathcal{D}+\mathcal{C} = \overline{D_{\mathrm{bf}}} + \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E}$. Since $\overline{D_{\mathrm{bf}}}$ is integral, it is enough to verify that $\lceil \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E}\rceil \ge 0$ (cf. the proof of Proposition 4.46), or that $\overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E}$ is almost effective (cf. (AEF) in 5.9).

We make it into two steps. First, on X. Since $0 < \alpha \le 1$, (5.20.1) implies
$$\lceil \mathcal{C}_X - \tau F\rceil \ge 0. \tag{5.20.3}$$
Indeed, (5.20.1) times α^{-1} gives (5.20.3) with τ/α instead of τ. Since $F \ge 0$ and $1/\alpha \ge 1$, we can replace τ/α by τ.

Since $\operatorname{Supp} D_{\mathrm{fr}} \subset F$, F is reduced and $\|D_{\mathrm{fr}}\| < \tau$; hence, $F = \sum P_i$ and $D_{\mathrm{fr}} = \sum d_{\mathrm{fr},i} P_i$ with each $|d_{\mathrm{fr},i}| < \tau$. Thus, each $d_{\mathrm{fr},i} > -\tau = -\tau \operatorname{mult}_{D_i} F$ for P_i in F. In other words, $D_{\mathrm{fr}} \geq -\tau F$. Therefore,

$$(\overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E})_X = D_{\mathrm{fr}} + \mathcal{C}_X \geq \mathcal{C}_X - \tau F,$$

and, by (5.20.3), $\lceil \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \rceil_X \geq 0$ on X.

Second, we work on a model$/X$. Note that $D_{\mathrm{fr}} \geq -\tau F$ implies $\overline{D_{\mathrm{fr}}} \geq -\tau \overline{F}$. Therefore, by (5.20.2),

$$\overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \geq \mathcal{C} - \tau \overline{F} - (1-\alpha)\mathcal{C} = \alpha \mathcal{C} - \tau \overline{F}/X.$$

Hence, by (5.20.1), $\lceil \overline{D_{\mathrm{fr}}} + \mathcal{C} - \mathcal{E} \rceil \geq 0$ over X because $-\alpha \geq -1$. Finally, the nonvanishing follows since $|D_{\mathrm{bf}}|$ is free.

We have also proved the inclusion

$$\mathcal{O}_W(g^* D_{\mathrm{bf}}) = \mathcal{O}_W((\overline{D_{\mathrm{bf}}})_W) \subset \mathcal{O}_W(\lceil \mathcal{D} + \mathcal{C} \rceil).$$

Thus, by the projection formula,

$$J(D_{\mathrm{bf}}) = J \otimes g_* \mathcal{O}_W(g^* D_{\mathrm{bf}}) \subset J \otimes g_* \mathcal{O}_W(\lceil \mathcal{D} + \mathcal{C} \rceil_W) = J(\lceil \mathcal{D} + \mathcal{C} \rceil_W),$$

where $g\colon W \to X/Z$, and $|J(D_{\mathrm{bf}})| \subset |J(\lceil \mathcal{D} + \mathcal{C} \rceil_W)|$. The latter implies

$$\operatorname{Mov} J(\lceil \mathcal{D} + \mathcal{C} \rceil_W) \geq \operatorname{Mov}((\overline{D_{\mathrm{bf}}})_W)$$

over $X \setminus \operatorname{Bs} |J(D_{\mathrm{bf}})|$ but $\operatorname{Mov}((\overline{D_{\mathrm{bf}}})_W) = (\overline{D}_{\mathrm{bf}})_W = g^* D_{\mathrm{bf}}$. This completes the proof of the addendum. $\square$

Proof of Theorem 5.12. After a truncation of $\mathcal{D}_\bullet$, we can assume that the saturation index is $I = 1$ and (UAD) holds without truncation. The conditions of the theorem are preserved by Remarks 5.8(4), 5.11(4), 5.6(1), and 4.34(7). To prove strict asymptotic descent, we can do it after any truncation.

First, we verify that $D := \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor. Thus, suppose that D is not a $\mathbb{Q}$-divisor. Then we find i and j such that, on any model $W/Y/Z$ of X/Z,

$$\operatorname{Mov} \lceil j\mathcal{D}_i + \mathcal{C} \rceil_W \not\leq j\mathcal{D}_W,$$

and the mobile part is finite; the latter means that the *nonvanishing* result $|\lceil j\mathcal{D}_i + \mathcal{C} \rceil_W| \neq \varnothing$ holds$/Z$. The $\not\leq$ contradicts asymptotic $\mathcal{C}$-saturation. Indeed, by (MXD), this implies that the mobile part is $\not\leq j(\mathcal{D}_j)_W$ for such i and j on any sufficiently high model $W = X_{\mathrm{hr}}$.

An estimate for the mobile part can be obtained from Lemma 5.20 for $\mathcal{D} = j\mathcal{D}_i'$ constructed below and F in the prediction model with some j and $i \gg 0$ on any W/Y.

By Corollary 5.19 and (SAM), for D, there exist positive real numbers r and ε, an $\mathbb{R}$-divisor D' on Y, an infinite set of natural numbers m, and a sequence of $\mathbb{Q}$-divisors Q_m on Y such that

- each $|mQ_m|$ is free;
- each $\operatorname{Supp}(mD' - mQ_m) \subset \operatorname{Supp}(mD_{\mathrm{irr}}) \subset F$ by (UAD) for the prediction model;
- $\|mD' - mQ_m\| < \varepsilon/m^r$;
- $D' \leq D$; but
- $mQ_m \not\leq mD$; and
- $mQ_m = mD' = mD$ in the rational multiplicities of D.

It is enough to verify that, for some $j = m$ and $i \gg 0$, on any model $W/Y/Z$ of X/Z,

$$\operatorname{Mov}\lceil j\mathcal{D}'_i + \mathcal{C}\rceil_W \geq m(\overline{Q_m})_W,$$

where $\mathcal{D}'_i = \mathcal{D}_i - \overline{E}$, and $E = D - D' \geq 0$ is an effective divisor on Y by construction. Since $\overline{E}$ is also effective, for the same j and $i \gg 0$, on any model $W/Y/Z$ of X/Z, we have

$$\operatorname{Mov}\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W = \operatorname{Mov}\lceil j\mathcal{D}'_i + j\overline{E} + \mathcal{C}\rceil_W \geq \operatorname{Mov}\lceil j\mathcal{D}'_i + \mathcal{C}\rceil_W \geq m(\overline{Q_m})_W.$$

On the other hand, $m(\overline{Q_m})_W \not\leq m\mathcal{D}_W = j\mathcal{D}_W$, even on $W = Y$.

Note that the b-divisors $\mathcal{D}'_i$ share most of their properties with the b-divisors $\mathcal{D}_i$: e.g.,

- (FDS) and (UAD) hold for $\mathcal{D}'_\bullet$; moreover, each $\operatorname{Supp}(j\mathcal{D}'_i - j\mathcal{D}')_Y = \operatorname{Supp}(\mathcal{D}_i - \mathcal{D})_Y \subset F$, where $\mathcal{D}' = \mathcal{D} - \overline{E}$; usually, in (FDS), we need a reduced divisor $\geq F$;
- $\lim_{i\to\infty} \mathcal{D}'_i = \mathcal{D}'$, whereas $\mathcal{D}'_Y = D'$; and
- (SAC) holds for $\mathcal{D}'_\bullet$ with the same i because $\mathcal{E}(\overline{E}) = 0$ and $\mathcal{E}(\mathcal{D}'_i) = \mathcal{E}(\mathcal{D}_i - \overline{E}) = \mathcal{E}(\mathcal{D}_i)/Y$ by (EFF) and (ADD) in Proposition 5.4.

Now take any positive $\alpha < 1$ and set $\tau = \alpha\gamma$. Then (LGD) for the prediction model (multiplied by α) implies (5.20.1). Take any $j = m \gg 0$ such that $\varepsilon/m^r \leq \tau$. Then $\mathcal{D}_Y = mD' = D_{\mathrm{bf}} + D_{\mathrm{fr}}$ satisfies the assumptions for the decomposition in Lemma 5.20, where

$$\mathcal{D} := m\mathcal{D}', \qquad D_{\mathrm{bf}} = mQ_m, \qquad \text{and} \qquad D_{\mathrm{fr}} = mD' - mQ_m.$$

Indeed, by construction, $\|D_{\mathrm{fr}}\| < \varepsilon/m^r \leq \tau$ and $\operatorname{Supp} D_{\mathrm{fr}} \subset F$.

By (FDS), the limit

$$\lim_{i\to\infty} (\mathcal{D}'_i)_Y = \mathcal{D}'_Y = D'$$

takes place on Y in the norm $\|\cdot\|$. Thus, by (UAD), for such $j = m$ and *any* $i \gg 0$, the decomposition $\mathcal{D}_Y = j(\mathcal{D}'_i)_Y = D_{\mathrm{bf}} + D_{\mathrm{fr}}$ satisfies the assumptions of Lemma 5.20, where $D_{\mathrm{bf}} = mQ_m$ is still the same but $\mathcal{D} := j\mathcal{D}'_i$ and

$$D_{\mathrm{fr}} := j(\mathcal{D}'_i)_Y - mQ_m = j(\mathcal{D}_i)_Y - jE - mQ_m$$

with $\|D_{\mathrm{fr}}\| < \tau$ and $\operatorname{Supp} D_{\mathrm{fr}} \subset F$.

The bound (5.20.2) in the lemma holds for *some* $i \gg 0$. More precisely, take any $i \gg 0$ with $m/r_i \leq (1-\alpha)$; the latter is > 0. Then (5.20.2) holds for the i. Indeed, by (EEF) in the choice of Y, $m\mathcal{C}/r_i \leq (1-\alpha)\mathcal{C}/Y$. Thus, by (ADD) in Proposition 5.4 and by the definition of confinement,

$$\mathcal{E}(\mathcal{D}) = \mathcal{E}(j\mathcal{D}'_i) = m\mathcal{E}(\mathcal{D}'_i) = m\mathcal{E}(\mathcal{D}_i) = m\mathcal{E}_i \leq m\mathcal{C}/r_i \leq (1-\alpha)\mathcal{C}/Y.$$

Hence, by Lemma 5.20,

$$\operatorname{Mov}\lceil j\mathcal{D}'_i + \mathcal{C}\rceil_W = \operatorname{Mov}\lceil \mathcal{D} + \mathcal{C}\rceil_W \geq (\overline{D_{\mathrm{bf}}})_W = m(\overline{Q_m})_W.$$

This completes the proof that D is a $\mathbb{Q}$-divisor.

Second, we check that $\lim_{i\to\infty} \mathcal{D}_i = \mathcal{D}$ stabilizes on Y. Using Corollary 5.19 again, but this time in the trivial case when $D := \mathcal{D}_Y$ is a $\mathbb{Q}$-divisor, the above arguments do not give a contradiction. But the asymptotic saturation gives that, for some $j = m$ (with integral $m\mathcal{D}_Y$) and some $i \gg 0$, on any $W/Y/X$,

$$j(\mathcal{D}_j)_W \geq \operatorname{Mov}\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W \geq m(\overline{Q_m})_W,$$

whereas $m(\overline{Q_m})_Y = mQ_m = mD = m\mathcal{D}_Y$ because the approximation is trivial. Hence, $\overline{Q_m} \le \mathcal{D}_j$ for such $j = m \gg 0$, and $\mathcal{D}_Y = Q_m \le (\mathcal{D}_j)_Y$ on Y. Thus, (MXD) gives the stabilization of the limit on Y, namely, $(\mathcal{D}_j)_Y = \mathcal{D}_Y$.

Third, we assume (BNF) and prove the stabilization in Addendum 5.12.1. It is enough to verify that $\mathcal{E}(\mathcal{D}) = 0/Y$. Indeed, then, for some $j = m \gg 0$, $\overline{Q_m} = \mathcal{D}$ and $\mathcal{D}_j \ge \overline{Q_m} = \mathcal{D}$. Hence, (MXD) gives stabilization, namely, $\mathcal{D}_j = \mathcal{D}$.

By Remarks 5.8(1),(2), under (FDS) and (SAC),

$$\mathcal{E}(\mathcal{D}) = \mathcal{E}\Bigl(\lim_{i\to\infty} \mathcal{D}_i\Bigr) = \lim_{i\to\infty} \mathcal{E}(\mathcal{D}_i) \le \lim_{i\to\infty} \mathcal{C}/r_i = \mathcal{C}/\lim_{i\to\infty} r_i = 0/Y.$$

On the other hand, under (BNF), $\mathcal{E}(\mathcal{D}) \ge 0/Y$ by Remarks 5.8(1),(2) once again. Therefore, $\mathcal{E}(\mathcal{D}) = 0/Y$.

Finally, we also assume (CGR) and verify Addendum 5.12.2, that is, $D := \mathcal{D}_Y$ contracts every prime divisor P_i on Y with $\operatorname{mult}_{P_i} \mathcal{C} > 0$. (Equivalently, $D_{|P_i}$ is semiample but not big on every such P_i. For b-divisors P_i that are exceptional on Y, the same follows from the above descent.) Here we use the arguments of Reid [37, Proposition 1.2]. By our assumptions, $\delta P_i \le \mathcal{C}$ for some $\delta > 0$. By the asymptotic descent Addendum 5.12.1 and by Reid (or by Addendum 3.19.1), if $\mathcal{D}$ is big$/Z$ and $\operatorname{mult}_{P_i} \mathcal{C} > 0$, on any model $W/Y/Z$ of X/Z and for all $j = m \gg 0$ such that $|mD|$ is free, we have the following lower bound:

$$\operatorname{Mov}(j(\mathcal{D}_j)_W + P_i) = \operatorname{Mov}(j\mathcal{D}_W + P_i) = j\mathcal{D}_W + P_i = j(\mathcal{D}_j)_W + P_i.$$

This contradicts asymptotic $\mathcal{C}$-saturation whenever the upper bound

$$\operatorname{Mov}\lceil j\mathcal{D}_j + \mathcal{C}\rceil_W \le j(\mathcal{D}_j)_W$$

with $i = j$ holds in asymptotic saturation over some W (cf. Remarks 4.34(2),(5)) that does not depend on j. More precisely, the above upper bound holds on any X_{hr}/W and for any $j = m$. Indeed, since $j\mathcal{D}_j = j\mathcal{D}$ is integral,

$$j(\mathcal{D}_j)_W + P_i \le \lceil j(\mathcal{D}_j)_W + \delta P_i\rceil \le \lceil j\mathcal{D}_j + \mathcal{C}\rceil_W$$

and

$$j(\mathcal{D}_j)_W + P_i = \operatorname{Mov}(j(\mathcal{D}_j)_W + P_i) \le \operatorname{Mov}\lceil j\mathcal{D}_j + \mathcal{C}\rceil_W,$$

which contradicts asymptotic $\mathcal{C}$-saturation.

The independence of the upper bound from X_{hr}/W follows from Proposition 4.46 for $\mathcal{D} := j\mathcal{D}_j$ and $\mathcal{E} := 0$. Note only that the bound for some $j = m$ implies the same bound with the same W for any other natural number $j := m'$ under our assumptions. Indeed, we can assume that $m' \ge m$ and $|(m' - m)D|$ is free. Then

$$\lceil m'\mathcal{D}_{m'} + \mathcal{C}\rceil = \lceil m'\mathcal{D} + \mathcal{C}\rceil = (m' - m)\mathcal{D} + \lceil m\mathcal{D}_m + \mathcal{C}\rceil.$$

Thus, by the projection formula (see a standard text on algebraic geometry), if

$$\mathcal{O}_W(\lceil m\mathcal{D}_m + \mathcal{C}\rceil_W) = \mathcal{O}_W(\lceil m\mathcal{D}_m + \mathcal{C}\rceil),$$

the same holds for other natural numbers m' on W because $(m' - m)\mathcal{D} = \overline{(m' - m)D}$ is a Cartier b-divisor$/Y$ (cf. b-free in Example 4.14). $\square$

Our construction actually proves more:

Corollary 5.21. *Under the assumptions of Theorem* 5.12, *suppose that J is a sheaf of ideals on the prediction model Y such that*

(RST) $\mathcal{D}_Y$ *is* rational, *and* complete stabilization *holds with an index I on Y over* $c(\operatorname{Supp} J)$, *where $c\colon Y \to T/Z$ is a contraction given by $\mathcal{D}_Y$*; *that is, each $(\mathcal{D}_i)_Y = \mathcal{D}_Y$ with $I \mid i$ and is rational over* $c(\operatorname{Supp} J)$; *and*

(JAS) asymptotic $\mathcal{C}$-saturation with respect to J *holds*/Z (*rather than asymptotic $\mathcal{C}$-saturation*), *that is,*

$$g_*J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_{X_{\mathrm{hr}}}) \subset (g \circ h)_*\mathcal{O}_{X_{\mathrm{hr}}}(j(\mathcal{D}_j)_{X_{\mathrm{hr}}});$$

here we write $J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_{X_{\mathrm{hr}}}) = J \otimes h_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_{X_{\mathrm{hr}}})$, *where $h\colon X_{\mathrm{hr}} \to Y$ and $g\colon Y \to Z$.*

Then the conclusions of Theorem 5.12 *and its addenda hold, where Addendum* 5.12.2 *concerns only prime P_i not in* $\operatorname{Supp} J$ *and $\mathcal{C}_{X_{\mathrm{st}}} \le 0$ over $X_{\mathrm{st}} \setminus \operatorname{Supp} J_{\mathrm{st}}$ with the image J_{st} of J on X_{st}.*

Moreover, this holds for nonnormal Y (*and X*) *if* $\operatorname{Supp} J$ *includes the nonnormal singularities of Y, where J can be replaced by any nontrivial coherent functional sheaf/Y, and the support is taken only for the common zeros* (*fixed points*) *of its sections.*

This can also be extended to the nonreduced case where the nonreduced locus of Y is in $\operatorname{Supp} J$. Note that if $J = \mathcal{O}_Y$, then this gives exactly Theorem 5.12 with its addenda. Indeed, then Y is reduced and normal, (RST) is empty, and (JAS) is equal to asymptotic $\mathcal{C}$-saturation (see Remark 4.34(3)).

Addendum 5.21.1. *For all $\mathcal{D}_i$ that are b-nef/Y and $\mathbb{Q}$-divisor $(\mathcal{D}_i)_Y$ on Y, we can modify* (RST) *into*

(RST)$'$ $(\mathcal{D}_i)_Y$ *is* rational *and* stabilization $\mathcal{D}_i = \mathcal{D} = \overline{\mathcal{D}_Y}$ *holds on Y over* $\operatorname{Supp} J \subset Y$.

In particular, this holds for the finite b-divisors $\mathcal{D}_i$ of the characteristic system for any functional algebra when all the $\mathcal{D}_i'$ stabilize over $\operatorname{Supp} J \subset Y$ (*see Conjecture* 5.26 *and the log singular case in Example* 5.27, *p.* 152).

Proof–Explanation. We can use the same proof as that of Theorem 5.12 with the following modifications. First, we need an additional truncation to satisfy (RST) with $I = 1$.

Second, instead of contradicting asymptotic $\mathcal{C}$-saturation, we aim to contradict the J-version of asymptotic $\mathcal{C}$-saturation as in (JAS), namely, that

$$g_*J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W) \not\subset (g \circ h)_*\mathcal{O}_W((j\mathcal{D}_j)_W),$$

where $h\colon W \to Y/Z$ for an arbitrary model W versus X_{hr}, or, equivalently (see Remark 4.34(3)),

$$\operatorname{Mov} J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W) \not\le (j\mathcal{D}_j)_W.$$

By (MXD), the latter follows from the estimate

$$\operatorname{Mov} J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W) \not\le j\mathcal{D}_W.$$

Third, instead of Corollary 5.19, we use its Addendum 5.19.1, which only gives freedom with respect to J. Note also that, by (RST), the irrational multiplicities of $D = \mathcal{D}_Y$ are over $T \setminus c(\operatorname{Supp} J)$, and so $mQ_m \not\le D$ over $T \setminus c(\operatorname{Supp} J)$.

Fourth, this time we can prove that, for some $j = m$ and $i \gg 0$,

$$\operatorname{Mov} J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_W) \ge m(\overline{Q_m})_W,$$

but only over $T \setminus c(\operatorname{Supp} J)$. This is enough to give a contradiction. This can be done as in the proof of the theorem with Addendum 5.20.1 instead of Lemma 5.20, where the same inequalities work (at least) over $T \setminus c(\operatorname{Supp} J)$ with $\operatorname{Mov} J(\lceil\cdot\rceil_W)$ instead of $\operatorname{Mov}\lceil\cdot\rceil_W$. This proves the rationality and stabilization on Y. (Actually, a more accurate approximation allows us to prove the rationality and stabilization for D over $T \setminus c(\operatorname{Supp} J)$ without (RST).)

Addendum 5.12.1 can be done in this case in the same way as without J, and the same goes for Addendum 5.12.2 outside $\operatorname{Supp} J$.

If Y is nonnormal, we can replace it by its normalization $\nu\colon Y^\nu \to Y$. In general, if we replace J by the functional sheaf J^ν over Y^ν generated by J, (JAS) might not be preserved. However, since each nontrivial functional sheaf J on Y has an invertible subsheaf of ideals, we can first replace J by a subsheaf with the property that it preserves (JAS) on Y and on its normalization, by definition of $J(\lceil j\mathcal{D}_i + \mathcal{C}\rceil_{X_{\mathrm{hr}}})$ and the projection formula. We can also assume that the support of the new J includes the old one plus a *mobile* divisorial set, in particular, the one that does not include all components of F over $T \setminus c(\operatorname{Supp} J)$ for an appropriate choice.

Then (RST) is preserved by the following argument: if Y is nonnormal, the main difficulty is defining $\mathcal{D}_Y$. For a nonnormal divisorial point, there is no reasonable choice to extend $\mathcal{D}_Y$. In view of (RST), we can assume that each b-divisor $\mathcal{D}$ is $\mathbb{R}$-Cartier over the nonnormal locus, that is,

(EXT) $\mathcal{D}_Y$ has an extension as an $\mathbb{R}$-Cartier divisor to the nonnormal locus such that $\mathcal{D} = \overline{\mathcal{D}_Y}$ along this locus.

Note that such an extension is unique if exists. Then (RST) over the nonnormal locus means that, in addition, each $\mathcal{D}_i = \mathcal{D}$ and is rational over it. Thus, (RST) is preserved on Y^ν over the nonnormal locus and so over $\operatorname{Supp} J^\nu$, by our choice of J^ν and the finiteness of $\operatorname{Supp}\mathcal{D}_Y$. In addition, we need to continue the normalization of a nonnormal prediction model and explain what it means. Under (EXT), the descent data $\mathcal{E}(\mathcal{D})$ is well defined/Y and $= 0$ over the nonnormal locus. We assume the same for each $\mathcal{D}_i$ as a condition for the existence of descent data (cf. Definition 5.7). In particular, (SAC) over the nonnormal locus follows from (EEF). As for F, we assume that it is *nonsingular* reduced $\mathbb{Q}$-Cartier, that is, is supported *divisorially* in the nonsingular and normal locus of Y. Thus, we define F^ν on the normalized prediction model as ν^*F on it; we take the same γ. This converts the normalization Y^ν into a prediction model.

Now we can apply Theorem 5.12 with its addenda in the normal case, and this implies the nonnormal case.

Finally, to prove Addendum 5.21.1, we need to derive (RST) from (RST)$'$. More precisely, if $\mathcal{D}_i$ is b-nef/Y and $(\mathcal{D}_i)_Y$ is a $\mathbb{Q}$-divisor such that $\mathcal{D}_i = \mathcal{D} = \overline{\mathcal{D}_Y}$ over $\operatorname{Supp} J$, then $(\mathcal{D}_i)_Y = (\mathcal{D})_Y$ holds over $c(\operatorname{Supp} J)$ (in fact, over a neighborhood of $c(\operatorname{Supp} J)$). Indeed, the $\mathbb{R}$-Cartier divisor $G = (\mathcal{D}_i - \mathcal{D})_Y \le 0$ by (MXD). Thus, by Lemma 4.23 and (RST)$'$,

$$\mathcal{D}_i - \mathcal{D} = \mathcal{D}_i - \overline{\mathcal{D}_Y} \le \overline{(\mathcal{D}_i)_Y} - \overline{\mathcal{D}_Y} = \overline{G} \le 0 \qquad \text{over } \operatorname{Supp} J,$$

and $\mathcal{D}_i - \overline{\mathcal{D}_Y} \le 0$ over Y. Since $\mathcal{D}_i - \overline{\mathcal{D}_Y}$ is b-nef/Y and the fibres of c are connected, the inequality is strictly < 0 everywhere over a point $P \in c(\operatorname{Supp} J)$ with $c^{-1}P$ intersecting negative components of G. Thus, by (RST)$'$, $G = 0$ over a neighborhood of $c(\operatorname{Supp} J)$, which is (RST). □

Example 5.22 (cf. Example 4.41)**.** Let $(C/\mathrm{pt.}, B)$ be a nonsingular curve with three distinct points p, q, and $B = r$ such that $n(p-q)+p \not\sim r$ for any integer n. Take all $\mathcal{D}_i = \mathcal{D} = D = \alpha(p-q)$ with real α. Then the trivial limit $\lim_{i\to\infty}\mathcal{D}_i = \mathcal{D}$ satisfies the assumption of Corollary 5.21 with $J = \mathcal{O}_C(-r)$ (see Example 5.25 below for such a choice) and (RST)$'$ instead of (RST). (As a prediction model, we take $(C/\mathrm{pt.}, 0, 0, 1)$.) Indeed, $J\lceil jD\rceil = \mathcal{O}_C(\lceil j\alpha(p-q) - r\rceil)$ does not have global sections because $\lceil j\alpha(p-q) - r\rceil \le n(p-q) + p - r \not\sim 0$, where n is $\lfloor j\alpha\rfloor$. However, α can be irrational.

Corollary 5.23. *Corollary* 5.13 *and its addendum hold under the changes of Corollary* 5.21 *in Theorem* 5.12.

Proof. Immediate by Corollary 5.21. □

Remark 5.24. In Theorem 5.12 and its addenda, corollaries, and generalizations, we can replace one prediction model by a finite set $(Y_i/Z, \mathcal{C}, F_i, \gamma)$ of them. However, (SAC) has a sequence of r_i with confinement over the corresponding Y_i. Then, on one of these prediction models, (SAC) holds for a strictly infinite subsequence of r_i, which gives the case of a single prediction model. Actually, we can replace the finite set by a set of prediction models that are bounded in the following sense (for simplicity, they are all models versus deformations of X/Z with the same b-divisor $\mathcal{C}$): each $(Y_i/Z, \mathcal{C}, F_i, \gamma)$ corresponds to a closed point in an algebraic family of prediction models $(Y_u/Z, \mathcal{C}, F_u, \gamma)$ for $u \in U$. As above, we can consider a single model; now we can assume that the family is irreducible. But an irreducible family of birational models is birationally a single model over a function field that can have infinitely many special points corresponding to elements of $(Y_i/Z, \mathcal{C}, F_i, \gamma)$. (Note that the family might have no constant resolution.) Thus, $\lim_{i\to\infty} \mathcal{D}_i = \mathcal{D}$ and $\mathcal{C}$ are defined as constant b-divisors with respect to this family. But this does not hold for the descent data because $\mathcal{D}_{Y_u}$ may vary for the family. Conditions (FDS), (EEF), and (LGD) in Theorem 5.12 are assumed to hold over the whole family (we recommend the reader to state this precisely). However, (MXD), (SAC), (BNF), and (CGR) hold (at least) birationally over the generic point (see the proof of Theorem 6.19(3) and Remark 6.15(8) below). Asymptotic $\mathcal{C}$-saturation also holds over the generic point because it holds at each special point as a birational property over a *sufficiently high* model. In applications, the latter needs just a good resolution as in the proof of Proposition 4.46 (for example, cf. the proof of Proposition 9.13 below). Thus, we again get the case of a single model over the generic point. Then the rationality and stabilization over the generic point imply, by Theorem 5.12, the same on most prediction models in our family. Likewise for its addenda, corollaries, and generalizations.

Example 5.25. Let $(X/Z, B)$ be a *general* log pair with a discrepancy divisor $\mathcal{A} = \mathcal{A}(X, B)$. For this, we only need that $K + B$ is $\mathbb{R}$-Cartier. Then, by Example 4.47, the fractional ideal sheaf $J = J_X = h_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{A} \rceil_{X_{\mathrm{hr}}})$ is coherent and independent of the sufficiently high model $h\colon X_{\mathrm{hr}} \to X$. If $(X/Z, B)$ is *generalized*, that is, $B \geq 0$, then J is the well-known ideal sheaf for $\mathrm{LCS}(X, B)$, and it can be given as $J = J_X = h_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{A}^- \rceil_{X_{\mathrm{hr}}})$, where $\mathcal{A}^-$ corresponds to the log nonpositive part of $\mathcal{A}$, that is, to the multiplicities ≤ -1. In this case, any $\mathcal{A}$-saturation implies the same type of $\mathcal{C}$-saturation with respect to J/Z, where $\mathcal{C} = \mathcal{A}^+ = \mathcal{A} - \mathcal{A}^-$ corresponds the log positive part of $\mathcal{A}$; in particular, the lca saturation gives (JAS) of Corollary 5.21. Indeed, by Remark 4.34(3), the saturation $\mathrm{Mov}\lceil \mathcal{D} + \mathcal{A} \rceil_{X_{\mathrm{hr}}} \leq \mathcal{D}_{X_{\mathrm{hr}}}$ is equivalent to the inclusion $(f \circ h)_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{D} + \mathcal{A} \rceil_{X_{\mathrm{hr}}}) \subset (f \circ h)_*\mathcal{O}_{X_{\mathrm{hr}}}(\mathcal{D}_{X_{\mathrm{hr}}})$. On the other hand, if $\mathcal{D}$ is integral over $\mathrm{Supp}\,\mathcal{A}^-$, in particular, over the LCS, the decomposition $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$ implies the inclusions

$$\begin{aligned} J(\lceil \mathcal{D} + \mathcal{C} \rceil_{X_{\mathrm{hr}}}) &\subset h_*h^*J \otimes h_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{D} + \mathcal{C} \rceil_{X_{\mathrm{hr}}}) \subset h_*(h^*J \otimes \mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{D} + \mathcal{C} \rceil_{X_{\mathrm{hr}}})) \\ &\subset h_*(\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{A}^- \rceil_{X_{\mathrm{hr}}}) \otimes \mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{D} + \mathcal{A}^+ \rceil_{X_{\mathrm{hr}}})) \\ &\subset h_*\mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{D} + \mathcal{A} \rceil_{X_{\mathrm{hr}}}); \end{aligned}$$

this, after taking f_*, implies (JAS) for $\mathcal{D} = j\mathcal{D}_i$ by the above inclusions under the same integral condition.

This also applies to the *generalized* nonnormal and nonreduced case, where a generalized log canonical divisor ω instead of $K + B$ is given (see [2, Definition 4.1 and Example 4.3.1]):

- ω is $\mathbb{R}$-Cartier;
- *adjunction*: a reduction and normalization $\nu^*\omega \sim_{\mathbb{R}} K + B$ for a general (X^ν, B); and
- $B \geq 0$.

We also assume that $\omega \sim K + B$ at normal and reduced points. In applications, B will also be unique over nonnormal points. In addition, a closed subscheme $X_{-\infty} \subset X$ is given. In the normal case, $X \setminus X_{-\infty}$ is the largest open subset on which (X, B) is log canonical. Thus, $X_{-\infty} \subset \mathrm{LCS}(X, B)$.

Conjecture 5.26. *The f.g. conjecture for* $(\mathrm{FGA})_n$ *algebras has an important generalization to* $(\mathrm{FGA})_n^*$ *algebras, where* $*$ *means the presence of log singularities. Namely, f.g. for a functional* $\mathcal{O}_T$*-algebra* $\mathcal{L} \subset k(X)_\bullet$ *such that*

- $(X/T, B)$ *is a generalized log Fano contraction, that is, we replace Klt in Definition* 4.38(ii) *by the condition that* $K + B$ *is* $\mathbb{R}$*-Cartier and* $B \geq 0$;
- $-(K + B)$ *is ample*/T *in a neighborhood of the* $\mathrm{LCS}(X, B)$;
- *the algebra* $\mathcal{L}$ *is bounded and lca saturated over* $(X/T, B)$; *and*
- $\mathcal{L}$ *is* ample on the LCS/T and stabilizes over a neighborhood *of the* LCS. *That is, in this neighborhood, up to similarity of characteristic systems, each* $\mathcal{D}_i = \overline{H}$, *where* $\mathcal{D}_\bullet$ *is the characteristic system of* $\mathcal{L}$ *and* H *is a* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor that is ample on* LCS/T.

Actually, we only need to assume ampleness over $X_{-\infty}$, *but stabilization holds for Cartier semiample* H *over a neighborhood of the* LCS (*see the log singular case in our standard Example* 5.27); *we possibly need weaker conditions* (*cf. Corollary* 6.42).

$(\mathrm{FGA})_1^*$ is proved in Example 4.41, and $(\mathrm{FGA})_2^*$ in Corollary 6.42 below. Now we are ready to explain what we have to do to prove (FGA) or (FGA)* and how to apply it to flips (cf. Corollary 1.5).

Example 5.27. Let $\mathcal{L}$ be an algebra of type (FGA) over $(X/T, B)$ (see Definition 4.38) and $\mathcal{D}_\bullet$ be its characteristic system. We expect that such $\mathcal{L}$ is always f.g. By Theorem 4.28, this is equivalent to an affirmative solution of the problem of asymptotic descent for the limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ on some model $Y/\operatorname{Proj}_T \mathcal{L}/T$ of X/T, where $\mathcal{D}$ is b-semiample/T and is a semiample/Y/T $\mathbb{Q}$-b-divisor. This last condition means that $D = \mathcal{D}_Y$ is a $\mathbb{Q}$-Cartier divisor that is semiample/T and $\mathcal{D} = \overline{D}$.

By Addendum 4.22.1 and Proposition 4.15(1), respectively, the system $\mathcal{D}_\bullet$ satisfies the minimal and additional assumptions in Theorem 5.12. Indeed, (BSD) in Addendum 4.22.1 implies (FDS). Moreover, by (FGA) of Definition 4.38, asymptotic saturation holds for $\mathcal{D}_\bullet$ with $\mathcal{C} = \mathcal{A} = \mathcal{A}(X, B)$, that is, lca saturation holds. By Example 4.47, (CGR) holds for the chosen $\mathcal{C} = \mathcal{A}$.

Thus, the main obstacle to apply Theorem 5.12 is choosing a prediction model $(Y/Z = T, \mathcal{C}, F, \gamma)$. Our choice of $\mathcal{C} = \mathcal{A}$ is canonical. On the other hand, because (X, B) is Klt, on any model Y/T of X/T and for any $\mathbb{Q}$-Cartier divisor F on Y, linear growth (LGD) of 5.9 holds for some $\gamma > 0$ (cf. Example 8.21 below). In particular, (FDS) implies (UAD) on any Y/T for some F (cf. Remarks 5.6(3) and 5.11(3)).

However, to satisfy the effectiveness (EEF), we need to take a model Y/T on which every prime b-divisor P_i with nonpositive discrepancy $a_i = d_i = d(P_i, B, X) = \operatorname{mult}_{P_i} \mathcal{A}$ (see [45, Example 1.1.4]) is blown up, that is, P_i is a divisor on Y. Again, since (X, B) is Klt, such models exist by [45, Lemma 1.6]. In addition, we can assume that Y is $\mathbb{Q}$-factorial.

Using LMMP, we can make a more accurate choice of Y, where infinitely many $D_i = (\mathcal{D}_i)_Y$ are nef/T and even semiample/T. This implies (SAM) for $D = \mathcal{D}_Y = \lim_{i\to\infty} D_i$ on such Y/T (cf. Remark 5.6(4)) and indicates where one can choose a prediction model (cf. triples in (MOD) of Conjecture 6.14 and Corollary 6.40 below).

Note that the limit D is only nef/T, in particular, $\mathbb{R}$-Cartier by Remark 5.6(4). But, in general, nef/T does not imply semiample/T; this is the difference between numerical and linear geometries. However, by the log semiampleness conjecture [45, Conjecture 2.6] (proved in dimension ≤ 3 [45, Theorem 2.7]), for an effective divisor D on X (or it would be enough for D to be effective up to $\sim_{\mathbb{R}}$), nef is equivalent to semiample if

(0LP) X/T is a 0-log pair for some boundary B as in Remark 3.30(2); in particular, if $(X/T, B)$ has an $\mathbb{R}$-complement (cf. (EC) in Conjecture 1.3 in [43] and [35]).

For example, by [41, предложение 5.5], this holds if

(WLF) $(X/T, B)$ is a weak log Fano contraction as in Proposition 4.42.

Moreover, when X/T is projective, we do not need the log semiampleness conjecture if we assume (WLF); and (WLF) also implies

(RPF) the effective cone $\overline{\mathrm{NE}}(X/T)$ is *rationally polyhedral* and has *contractible* and *flippable birational* faces/T (cf. (RPC) in Inductive Theorem 2.3 in [43]).

We assume X/T projective in the remainder of this section; this holds, for example, if $(X/T, B)$ is a Fano contraction, that is, $-(K+B)$ is ample/T. Or X/T projective can be achieved by the contraction by $-(K+B)$. If X/T is projective, we can increase B to B' such that $(X/T, B)$ is a Klt Fano contraction; see Lemma 9.7 below.

In addition, (WLF) for some boundary and (RPF) are preserved by modifications in any face of $\overline{\mathrm{NE}}(X/T)$, that is, under the D-flips of any birational contraction of X/T for which D is negative. Indeed, let $X \dashrightarrow X^+/T$ be such a modification, H be an ample/T divisor on X^+, and H^- be its birational transform on X. Then there exist $\varepsilon > 0$ and a boundary $B' \geq B$ such that $(X/T, B' + \varepsilon H^-)$ is a 0-log pair. The modification is a flop, $(X^+/T, (B')^+ + \varepsilon H)$ is again a 0-log pair, and $(X^+/T, (B')^+)$ is a (weak) log Fano contraction. Therefore, it again satisfies (RPF).

Since $(X/T, B)$ satisfies (WLF), under LMMP, we can apply this to modify X/T to make a divisor semiample (cf. Theorem 3.33). For any effective $\mathbb{R}$-Cartier divisor D (even up to $\sim_{\mathbb{R}}$; compare the effectiveness of bss divisors explained before Proposition 3.4), there is a model Y/T of X/T where the birational transform (composition of the above modifications in faces) of D is nef/T, and so, by (RPF), is semiample/T. Moreover, if

(SA1) D is semiample in *codimension* 1, that is, for any effective divisor E on X, there is an effective $D' \sim_{\mathbb{R}} D$ with disjoint $\operatorname{Supp} E$ and $\operatorname{Supp} D'$; hence, such D is nef on general curves/T of any divisor in X,

then we can take the model Y/T isomorphic to X/T in codimension 1.

Indeed, if D is nef, we take $Y = X$ and we are done. Otherwise, by (RPF), there is a contraction $X \to Z/T$ on which D is negative. It is birational because $D \geq 0$. Moreover, it is small under (SA1). Thus, we can modify X/T by D-flips, and the D-MMP gives the required model Y/T. Assuming (SA1), the modification is small. The required termination is as in the proof of Theorem 3.33, that is, by LMMP for $(X/T, B' + \varepsilon D)$, where $(X/T, B')$ is a 0-log pair.

Note that the use of LMMP is justified by our approach (cf. Corollary 1.5 and the remark at the end of Conjecture 4.39).

Since f.g. depends on $\mathcal{L}$ only up to quasi-isomorphism, or equivalently, stabilization depends on $\mathcal{D}_\bullet$ only up to similarity, we can assume that all $\mathcal{D}_i$ are effective. Moreover, by (LBF) in Conjecture 4.39, we can assume (SA1) for $D := D_i := (\mathcal{D}_i)_X$. Thus, if D_i is $\mathbb{Q}$-Cartier, we can construct a model Y_i/T such that

- Y_i is isomorphic to X in codimension 1; and
- D_i is nef, and even semiample/T on Y_i by (RPF).

After taking a complement $(X/T, B')$, by the conjecture of Alexeev [1, 0.4, (1)] and the Borisovs [5, с. 134, теорема] (compare Remark 4.40(7)), we expect that only a finite number of models Y_i/T satisfy the above conditions. The same would follow from Kawamata's conjecture on the finiteness of log minimal models [20]. However, for our purposes, it is sufficient to have LMMP by (RPF) in the proof of the Second Main Theorem [45, Theorem 6.20] with prime b-divisors $D_i = P_i$ of $\operatorname{Supp} F$ in (FDS) and P_i that are exceptional on X with $a_i = \operatorname{mult}_{P_i} \mathcal{A}(X, B) \leq 0$. Then the equivalence of weakly log canonical models $(Y_i/T, B' + \varepsilon D_i)$ means that the divisors D_i are nef on both models. This gives a model Y/T with (SAM) and infinitely many semiample $D_i = (\mathcal{D}_i)_Y$.

To secure the other properties of a prediction model, e.g., (EEF), we need to make a crepant terminal (in codimension ≥ 2) resolution of $(Y/T, B')$, or of $(X/T, B')$, or even of (X, B) before the modifications. We can also assume that the resolution is $\mathbb{Q}$-factorial. The existence of such a resolution follows from [45, Theorem 3.1] again up to the LMMP. (Apply it to (X, B') with $B' \geq B$ and such that the new discrepancies $a_i := \operatorname{mult}_{P_i} \mathcal{A}(X, B')$ are negative for all exceptional P_i with old $a_i := \operatorname{mult}_{P_i} \mathcal{A}(X, B) \leq 0$.)

This satisfies all the requirements for a prediction model, except for confinement (SAC)! This may actually fail (cf. Addendum 6.26.2 and Example 6.38). However, we expect that it holds for Y/T provided that $\mathcal{D}$ is big (see Conjecture 6.14 and Proposition 6.26). Moreover, this is the only possible choice by Addendum 5.12.2 and Example 6.39.

Finally, note that the same approach with the following modifications works in the log singular case (FGA)*. Instead of LMMP, we use the log singular $\mathbb{Q}$-version LMMP* due to Ambro [2, Theorem 5.10] when B is a $\mathbb{Q}$-divisor; each $\mathcal{D}_i$ is a $\mathbb{Q}$-b-divisor. This time, the modifications do not change a neighborhood of $\mathrm{LCS}(X, B)$. Indeed, according to stabilization $\mathcal{D}_i = \overline{H}$ in a neighborhood of the LCS and b-semiampleness of $\mathcal{D}_i$, the exceptional locus does not intersect the LCS. Thus, we can make a log flip as in the usual LMMP. Log termination holds for the same reasons (the log singularities have a log resolution). We take $\mathcal{C}$ and J on a prediction model by Example 5.25. Then we can use Corollary 5.21, its addendum and Corollary 5.23 instead of Theorem 5.12, and Corollary 5.13 and their addenda. Since $\mathcal{C}$ has a lot of 0s over the LCS, to satisfy (LGD), F in (UAD) should be quite accurate, namely, $\operatorname{Supp} F$ should be disjoint from the LCS. This can be done either

- by $\mathbb{Q}$-factorialization of F; or
- if Y/T is projective, by augmenting F to a Cartier divisor by a free divisor in a neighborhood of the LCS, which gives $F = 0$ in this neighborhood, as for a free divisor; or
- by replacing F by an effective $\mathbb{R}$-Cartier divisor with the same support as F.

The final method is the most natural when F is the minimal divisor for (UAD) (cf. Remark 5.11(3)), but this requires a modification of the estimates in Lemma 5.20. Again, this satisfies all the requirements for a prediction model, except for confinement (SAC)!

Thus, the main difficulty in constructing pl flips is confinement (SAC). We start to attack it in the next section where we propose a conjecture to resolve the difficulty and solve it in dimension 2. This is enough for 3-fold log flips, for (FGA)* and 4-fold log flips.

6. CANONICAL CONFINEMENT AND SATURATION

We start with another interpretation of confinement and asymptotic confinement in the *canonical* case, that is, when we take $\mathcal{C}$ to be the discrepancy b-divisor $\mathcal{C} = \mathcal{A} = \mathcal{A}(X, B)$ (cf. Example 5.27, Definition 6.4, and Proposition 6.8).

Lemma 6.1. *Let (X, B) be a log pair, $\mathcal{D}$ be an $\mathbb{R}$-b-divisor, and $\{\eta\}$ be a set of scheme-theoretic points of X such that*

- *the divisors $K + B$ and $\mathcal{D}_X$ of X are $\mathbb{R}$-Cartier;*
- *each η has $\operatorname{codim} \eta \geq 2$ in X; and*
- *$\mathcal{D} \geq 0$ over each η,* that is, $\operatorname{mult}_{E_i} \mathcal{D} \geq 0$ in each prime b-divisor E_i with $\operatorname{center}_X E_i = \eta$.

Then, for any real number $c \geq 0$ such that $(X, B + c\mathcal{D}_X)$ is canonical in each η, we have

$$c\mathcal{E} \leq \mathcal{A}$$

over each η, where $\mathcal{E} = \overline{\mathcal{D}_X} - \mathcal{D}$ is the descent data of $\mathcal{D}$ over X (see 5.3) and $\mathcal{A} = \mathcal{A}(X, B)$ is the discrepancy b-divisor of (X, B).

We say that inequalities of this type hold *over* X (or hold/X) if $\{\eta\}$ is the set of all codimension ≥ 2 points of X (cf. (EEF) in 5.9). When $\dim X = 1$, the inequalities always hold/X because $\{\eta\} = \varnothing$: there are no exceptional divisors.

Remark 6.2. It is easy to generalize the lemma to the situation when

- the η have arbitrary codimension; and
- canonical η is replaced by ε-log canonical or ε-log terminal (where ε may depend on η).

Then the inequality $c\mathcal{E} \leq \mathcal{A}$ should be replaced by

$$\operatorname{mult}_{P_i} \mathcal{A} \geq \varepsilon - 1 + c \operatorname{mult}_{P_i} \mathcal{E}$$

for each prime b-divisor P_i with $\operatorname{center}_X P_i = \eta$ in the ε-log canonical case (respectively, $>$ in the ε-log terminal case; cf. (LGD) in 5.9), or $\mathcal{A}$ should be replaced by its ε-log version if it is defined as a *b-divisor* (cf. [45, Example 1.1.4]).

In the lemma, $\varepsilon = 1$. The log canonical and log terminal cases have $\varepsilon = 0$ (see Example 8.21).

The converse of the lemma usually fails, essentially because $\mathcal{A}$ and $\mathcal{E}$ are independent of $\mathcal{D}$ up to $\sim_{\mathbb{R}}$ by (DEP) in Proposition 5.4, whereas canonical *depends* on $\mathcal{D}$.

Example 6.3 (see Addendum 6.40.1)**.** Let D be an $\mathbb{R}$-Cartier divisor on X and $\mathcal{D} = \overline{D}$. Then $\mathcal{D}_X = D$ and $\mathcal{E} = 0$, so that $0 = c\mathcal{E} \leq \mathcal{A}/X$ if and only if (X, B) is canonical in codimension ≥ 2. Of course, this does not imply that $(X, B + c\mathcal{D}_X)$ is always canonical in codimension ≥ 2, e.g., when $c > 1$, D is effective Cartier passing through η, and $\dim X \geq 2$ (cf. Example 6.7(3)).

However, if $\mathcal{D} = \mathcal{D}_X$, then, by the proof of Lemma 6.1, $c\mathcal{E} \leq \mathcal{A}$ over each η if and only if $K + B + c\mathcal{D}_X$ is canonical in each η of Lemma 6.1. Indeed, then $\mathcal{K} = \mathcal{K} + c\mathcal{D}$ over each η.

Proof of Lemma 6.1. By definition, $\mathcal{K} = \overline{K + B} + \mathcal{A}$ and $\mathcal{D} = \overline{\mathcal{D}_X} - \mathcal{E}$ (cf. the proof of Proposition 4.50, p. 130). Thus, for each η, since $\mathcal{D} \geq 0/\eta$,

$$\mathcal{K} \leq \mathcal{K} + c\mathcal{D} = \overline{K + B + c\mathcal{D}_X} + \mathcal{A} - c\mathcal{E}/\eta$$

and $\mathcal{K} - \overline{K + B + c\mathcal{D}_X} \leq \mathcal{A} - c\mathcal{E}/\eta$. Therefore, if $K + B + c\mathcal{D}_X$ is canonical over η, then, by definition, $\mathcal{A}(X, B + c\mathcal{D}_X) = \mathcal{K} - \overline{K + B + c\mathcal{D}_X} \geq 0/\eta$ as the discrepancy b-divisor. Hence, $\mathcal{A} - c\mathcal{E} \geq 0$ and $c\mathcal{E} \leq \mathcal{A}/\eta$. $\square$

Definition 6.4. Let $\mathfrak{D}$ be a set of (effective) $\mathbb{R}$-b-divisors and (X, B) be a log pair such that

- B may not be a boundary, but
- $K + B$ and $\mathcal{D}_X$ for every $\mathcal{D} \in \mathfrak{D}$ are $\mathbb{R}$-Cartier.

We say that $\mathfrak{D}$ has *canonically confined singularities*, or simply that it is *canonically confined* on (X, B), if there exists a real number $c > 0$ such that each pair $(X, B + c\mathcal{D}_X)$ is canonical in codimension ≥ 2. More precisely, the family is *confined* by c (from *below*).

If we replace (X, B) by a family (X_i, B_i) of log pairs, with the X_i models of X (usually crepant models; cf. (CRP) in Definition 6.9), $\mathfrak{D}$ *confined on the family* means that, for each $\mathcal{D} \in \mathfrak{D}$, there is a pair (X_i, B_i) on which $\mathcal{D}$ is confined by the *same* c. Thus, c is *uniform*.

Remarks 6.5. (1) We define ε-log canonical (terminal) confinement of $\mathfrak{D}$ on (X, B) at a point η or on a family of points (cf. Remark 6.2) in the same way. Thus, if $\varepsilon = 1$ and $\operatorname{codim} \eta \geq 2$, we get again canonical (terminal) confinement; for $\varepsilon = 0$, we get log canonical (terminal) confinement (cf. Example 8.21).

(2) If $\mathfrak{D}$ is confined, we define its *threshold* as $\sup\{c\}$. This is a more precise characteristic of the singularities of $\mathfrak{D}$.

Definition 6.6. A set $\mathfrak{S}$ of reduced divisors $S = \sum P_i$ on X is *bounded* if its elements are bounded as reduced subvarieties of X (cf. the proof of Theorem 8.4, p. 180 and E in Fixed restriction 7.2). More generally, a *bounded set of divisors* is a set of divisors $\mathfrak{D}$ such that all $S = \operatorname{Supp} D$ of $D \in \mathfrak{D}$ are *bounded*, that is, belong to a bounded family of reduced divisors $\mathfrak{S}$. In this case, we say that $\mathfrak{D}$ is *bounded* by $\mathfrak{S}$ and D is bounded by S, respectively. For b-divisors, this also has the flavor of bounded in moduli (see Examples 6.7); that is, such divisors belong to a bounded algebraic family of b-divisors, where the family can be defined over birational models of X.

A stricter condition, *bounded with multiplicities* on Y, includes boundedness of the multiplicities[6] for each $\mathcal{D}_Y$, where Y is a model of X.

Examples 6.7 (trivial)**.** (1) If $\mathfrak{D}$ is a finite or bounded set of b-divisors $\mathcal{D}$, including *multiplicities* on X, and (X, B) is terminal in codimension ≥ 2, then $\mathfrak{D}$ is confined on (X, B) (cf. the proof of Lemma 8.22).

(2) If each b-divisor $\mathcal{D}$ of $\mathfrak{D}$ is the Cartier completion $\mathcal{D} = \overline{D}$ of a sufficiently general element in a free linear system and (X, B) is terminal in codimension ≥ 2, then $\mathfrak{D}$ is also confined on (X, B).

(3) If each b-divisor of $\mathfrak{D}$ is sufficiently general and b-free, that is, as in (2) on some model Y/X of X, and (X, B) is Klt, then $\mathfrak{D}$ is confined on a family (X_i, B_i) that consists of the crepant models $(X_i/X, B_i)$ terminal in codimension ≥ 2. Note that $c = 1 + \min \mathcal{A}$ is the minimal *log* discrepancy of (X, B) (cf. Example 6.25). In particular, $c = 1$ when $(X, B = 0)$ is canonical.

Now we show how to apply canonical confinement to get asymptotic confinement.

Proposition 6.8. *Under the assumptions of Definition* 6.4, *suppose also that*

(CMD) *each model* $(X_i/Z, B_i)$ *is a* crepant model *of* $(X/Z, B)$ (*cf. Example* 6.7(3)); *and*

(EFF) *each* $\mathcal{D} \in \mathfrak{D}$ *is effective.*

Then $\mathfrak{D}$ canonically confined by c on the family (X_i, B_i) implies that, for each $\mathcal{D} \in \mathfrak{D}$, there exists a pair (X_i, B_i) such that the descent data $c\mathcal{E}$ of $c\mathcal{D}$ over X_i is confined by $\mathcal{A}/X_i$, where $\mathcal{A} = \mathcal{A}(X, B) = \mathcal{A}(X_i, B_i)$ is the discrepancy b-divisor.

Addendum 6.8.1. *Let $\mathcal{D}_i$ be a sequence of b-divisors and p_i be a sequence of positive real numbers such that*

- $\lim_{i\to\infty} p_i = +\infty$; *and*
- *each* $p_i\mathcal{D}_i \sim_{\mathbb{R}} \mathcal{M}_i \in \mathfrak{D}$ (*or even just numerically equivalent*), *where*
- *all the $\mathcal{M}_i$ are canonically confined by c* on a common model $(Y = X_j, B_Y = B_j)$ *in the family.*

Then the descent data for the sequence $\mathcal{D}_i$ is asymptotically confined by $\mathcal{A}/Y$.

Addendum 6.8.2. *The same confinement holds over the sequence* $(X_i, \mathcal{A})$ (*in fact, even in the discrepant case*; *see Remark* 5.8(6)) *if we weaken the last condition to*

- *all the $\mathcal{M}_i$ are canonically confined by c* on a sequence (X_i, B_i) *in our family.*

Proof. The confinement $c\mathcal{E} \leq \mathcal{A}/X_i$ follows from Lemma 6.1 for $(X, B) := (X_i, B_i)$, for the set $\{\eta\}$ of points of codimension ≥ 2, and for the given canonical confinement c, where (X_i, B_i) itself corresponds to $\mathcal{D}$ under the canonical confinement. By (HOM) of Proposition 5.4, $\mathcal{E}(c\mathcal{D}) = c\mathcal{E}(\mathcal{D})/X := X_i$.

Since $c > 0$, each $r_i = cp_i > 0$ and also $\lim_{i\to\infty} cp_i = +\infty$. On the other hand,

$$r_i\mathcal{E}_i = cp_i\mathcal{E}_i = c\mathcal{E}(p_i\mathcal{D}_i) = c\mathcal{E}(\mathcal{M}_i)/Y$$

by Proposition 5.4, (DEP) and (HOM). Hence, the given canonical confinement for $\mathfrak{D}$ and for $\mathcal{M}_i$ implies that $r_i\mathcal{E}_i = c\mathcal{E}(\mathcal{M}_i) \leq \mathcal{A}/Y$, which means asymptotic confinement. □

[6] That is, the multiplicities on Y are in a fixed segment $[a, b] \subset \mathbb{R}$.

We define desirable pairs (X_i, B_i) of confining families in terms of triples.

Definition 6.9. A *triple* $(X/T/Z, B, \mathfrak{F})$ includes

- a log pair $(X/Z, B)$ for which $K + B$ is $\mathbb{R}$-Cartier but B may not be a boundary;
- a contraction $g\colon X \to T/Z$; and
- a bounded set $\mathfrak{F}$ of reduced divisors on X.

Let $(X/Z, B)$ be a weak log Fano contraction (see (WLF) in Proposition 4.42). A *desirable triple* for $(X/Z, B)$ is a triple $(Y/T/Z, B_Y, \mathfrak{F})$ such that

(CRP) $(Y/Z, B_Y)$ is a crepant model of $(X/Z, B)$;

(QFC) Y is $\mathbb{Q}$-factorial;

(TER) (Y, B_Y) is terminal in codimension ≥ 2; and

(RPC) the nef cone $\overline{\mathrm{NE}}(T/Z)$ is *rationally polyhedral* and has *contractible* faces$/T$ (cf. [43, Inductive Theorem 2.3]).

Let $\mathcal{M}$ be a b-free b-divisor. A triple $(Y/T/Z, B_Y, \mathfrak{F})$ is a *desirable triple* for $\mathcal{M}$ if, in addition,

- $\mathcal{M}_Y = g^*M$, where M is nef and big on T/Z; and
- $\kappa(X/Z, \mathcal{M}) = \dim T/Z$, where κ is the *Iitaka dimension*$/Z$.

Remarks 6.10. (1) T/Z is a contraction if X/Z is.

(2) We sometimes need to weaken the assumptions, even for desirable triples; e.g., X/T may be an incomplete morphism of normal varieties, with log singularities omitted, or (X, B) may not be Klt. Moreover, X itself may be nonnormal and nonreduced; for example, $(X/Z, B)$ could be a generalized log Fano contraction (as in Conjecture 5.26), or even just a morphism. In the latter case, we modify the conditions as follows:

(CRP)* $(Y/Z, B_Y)$ may also be a nonnormal and nonreduced *crepant* model of $(X/Z, B)$; that is, both have a common crepant normal model; in applications, $(X/Z, B)$ is isomorphic to $(Y/Z, B_Y)$ outside a neighborhood of $\mathrm{LCS}(X, B)$ that includes the non-Klt singularities;

(QFC)* Y is $\mathbb{Q}$-factorial outside LCS (the image of $\mathrm{LCS}(X, B)$, or the LCS of (Y, B_Y) itself); that is, a divisor that is $\mathbb{R}$-Cartier in a neighborhood of the LCS is so everywhere;

(TER)* (Y, B_Y) is terminal in codimension ≥ 2 in each closed center outside the LCS; and

(RPC)* the effective curves subcone of $\overline{\mathrm{NE}}(T/Z)$ having only faces defining contractions that embed $T_{-\infty} = g(Y_{-\infty}) \subset g(\mathrm{LCS})$ into their target space (that is, ample$/Z$ in a neighborhood of $g(\mathrm{LCS})$) is *rationally polyhedral* and has *contractible* faces$/Z$ (cf. Ambro [2, Theorems 5.6 and 5.10]).

Such a triple $(Y/T/Z, B_Y, \mathfrak{F})$ is a *desirable triple* for $\mathcal{M}$ if, in addition,

- M in Definition 6.9 is ample on $c(\mathrm{LCS})/Z$; and
- $\mathcal{M} = \overline{\mathcal{M}_Y} = \overline{g^*M}$ over a neighborhood of the LCS.

See (CCS)* in Conjecture 6.14 and Addendum 6.19.1 below.

(3) It might be better to spread the boundedness of $\mathfrak{F}$ into the boundedness of a family of *simple* triples $(X/T/Z, B, F)$ having a 1-element set $\{F\} \subset \mathfrak{F}$ for $F \in \mathfrak{F}$. We say that a family of triples is *bounded* to mean bounded in moduli (cf. (7′) in the proof of Theorem 8.4).

(4) (CRP) in the definition means that $\mathcal{A}(Y, B_Y) = \mathcal{A}(X, B) = \mathcal{A}$ (cf. Caution 4.37).

(5) We could possibly drop (QFC), but then we can only confine singularities of a divisor $\mathcal{D}$ having $\mathbb{R}$-Cartier restriction $\mathcal{D}_Y$ (cf. Lemma 6.1 versus Lemma 8.12). This holds in many applications. For example, $\mathcal{M}_Y$ is $\mathbb{Q}$-Cartier on a desirable triple for $\mathcal{M}$ by definition.

(6) The divisor B_Y in (TER) need not be a boundary. However, since (X, B) is Klt, B_Y is a *Klt subboundary*; that is, each of its multiplicities $b_i < 1$.

(7) (RPC) is intermediate among the following conditions on T/Z that we list in order of increasing strength:

(LSA) the limit of any semiample sequence of $\mathbb{R}$-divisors on T/Z is a semiample $\mathbb{R}$-divisor on T/Z, and we assume that M is *semiample* in the definition of desirable triple (cf. Remark 5.6(4));

(NSA) each divisor M on T that is nef/Z is semiample;

(0LP) T/Z is a 0-log pair for some boundary B_T as in Remark 3.30(2); in particular, (NSA) then holds for each effective divisor, and we can assume that $\mathcal{M}$ is effective, and any limit of effective divisors is also effective under (MXD);

(RPC) as in Definition 6.9 (cf. Lemma 8.12(2) below); and

(WLF) $(T/Z, B_T)$ is a weak log Fano contraction or morphism for some boundary B_T.

For the main application of (CCS) below (cf. Theorem 6.19), (LSA) is enough. But for approximations, we need at least (RPC) (cf. Theorem 8.15 below). In addition, we prefer to work with notions defined up to numerical equivalence rather than up to linear equivalence.

The reader may wonder what role the set of divisors $\mathfrak{F}$ plays in triples. In fact, we need very special sets $\mathfrak{F}$ that we call *standard*. First, standard families are bounded in typical examples, such as the fixed and fractional parts that appear in Corollary 6.40 and in Proposition 9.15. Second, they generalize the canonical confinement conjecture (cf. (GCC) in Conjecture 6.14). They are also needed for birational rigidity (see the proof of Theorem 6.19(3)). Finally, they are crucial in our construction of 4-fold flips (cf. Theorem 6.45).

Definition 6.11. Let $(X/Z, B)$ be a log pair. We say that a set $\mathfrak{F}$ is *standard* on $(X/Z, B)$ if it contains only standard divisors. A *standard divisor* S on $(X/Z, B)$ is a reduced divisor $S = \operatorname{Supp} \mathcal{D}_X$, where $\mathcal{D}$ is an *effective* $\mathbb{R}$-b-divisor such that

(SEF) $\mathcal{D}$ is *strictly effective*/X; that is, it has $\operatorname{mult}_{E_i} \mathcal{D} \geq 0$ in each exceptional/X prime b-divisor E_i and > 0 *over* $\operatorname{Supp} \mathcal{D}_X$, where the support is considered as a *divisorial subvariety* (cf. (EEF) in 5.9); and

(STD) 0 is saturated with respect to $\mathcal{A} + \mathcal{D}$ on any sufficiently high model X_{hr}/Z of X/Z.

The *standard set* $\mathfrak{S}$ is the set of all standard divisors. In general, it may be unbounded. However, by (SSB) in Conjecture 6.14, we expect that $\mathfrak{S}$ is bounded when $(X/Z, B)$ is a weak log Fano contraction. In particular, then $(X/X/Z, B, \mathfrak{S})$ is a triple. For any other triple $(Y/T/Z, B_Y, \mathfrak{S}_Y)$, the *induced standard set* $\mathfrak{S}_Y$ is defined as the log birational transform of the set $\mathfrak{S}$ of X.

A desirable triple for $(X/Z, B)$ with the *induced set* $\mathfrak{S}_Y$ is a desirable triple $(Y/T/Z, B_Y, \mathfrak{S}_Y)$ with the induced standard set.

Caution 6.12. It may happen that $\mathfrak{S}_Y$ itself is actually *nonstandard* (cf. Remark 6.13(2)).

Remarks 6.13. (1) If D is an effective $\mathbb{R}$-Cartier divisor on X, then $\overline{D}$ satisfies (SEF).

(2) The induced set $\mathfrak{S}_Y$ is well defined, namely, it is bounded when $\mathfrak{S}$ is bounded. Indeed, the log birational transform of $S \in \mathfrak{S}$ is $g^{-1}S + \sum E_i$, where each E_i is a prime divisor on Y that is exceptional on X. Note that $\sum E_i$ is finite and fixed for any such transform.

In general, $\mathfrak{S}_Y$ is not standard even up to the additive log contribution $\sum E_i$. Let X be a del Pezzo surface having two -1-curves E_1 and E_2 that intersect transversally in a single point P, for example, two intersecting lines on a cubic surface in $\mathbb{P}^3$. Then E_1 and E_2 belong to the standard set $\mathfrak{S}(X/Z = \text{pt.}, 0)$, but $E_1 + E_2$ does not (cf. Example 6.38). For the blowup $g \colon Y \to X$ in P, the blowup E_3 of P (the log transform of 0) is standard. The birational transform $g^{-1}E_i$ of each E_i

is also standard (in addition to their log transforms $g^{-1}E_i + E_3$). But now $g^{-1}E_1 + g^{-1}E_2$ and $g^{-1}E_1 + g^{-1}E_2 + E_3$ are also standard since $g^{-1}E_1 + g^{-1}E_2 + a(E_3)E_3 = g^{-1}E_1 + g^{-1}E_2 + E_3$ has trivial mobile part on Y.

Nonetheless, the difference between $\mathfrak{S}_Y$ and $\mathfrak{S}(Y/Z, B_Y)$ is not great (cf. (GFC) in Conjecture 6.14).

Other facts about standard sets and equivalent forms of the standard property (SEF)+(STD) are given below in Proposition 6.34. In general, a confining family (X_i, B_i) may itself be unbounded (cf. Example 6.7(3)), and an infinite subset in $\mathfrak{D}$ may be unbounded on any bounded subfamily of (X_i, B_i). However, we need and expect the following:

Conjecture 6.14 (canonical confinement)**.** *We assume that $(X/Z, B)$ is a weak log Fano contraction with a contraction X/Z. We expect the following sets to be bounded. First,*

(SSB) the standard set $\mathfrak{S} = \mathfrak{S}(X/Z, B)$ of $(X/Z, B)$ is bounded.

In particular, this includes boundedness of the prime components of the standard divisors. Moreover, we expect that

(PRM) the prime standard set $\mathfrak{P} = \mathfrak{P}(X/Z, B)$ is bounded; *here $\mathfrak{P}$ is the set of* prime standard divisors $P = \operatorname{Supp} \mathcal{D}_X$ *on X with $\mathcal{D} \geq 0$ and $\mathcal{D}$ satisfying saturation* (STD) *of Definition* 6.11.

The third set is the linearly fixed primes for the (SAT) set $\mathfrak{D} = \mathfrak{D}(X/Z, B)$ *of* effective $\mathbb{R}$-*b-divisors $\mathcal{D}$ under saturation*:

(SAT) $\mathcal{D}$ *is* log canonically saturated/(X, B) *on any sufficiently high model X_{hr}/Z of X/Z* (*cf. Definition* 4.36); *that is, it is $\mathcal{A}$-saturated, where $\mathcal{A} = \mathcal{A}(X, B)$ is the discrepancy b-divisor.*

In most cases, $\mathfrak{D}$ itself is unbounded, even on X. However, we expect that

(PFC) *the set of* prime linearly fixed components, *that is, the set of components of* $\operatorname{Supp}(\operatorname{Fix} \mathcal{D})_X$ *for $\mathcal{D} \in \mathfrak{D}$ on X* (*cf. Remark* 6.15(4)) *is* bounded.

(PFC) *is equivalent to* (PRM)*. Actually, the set is equal to $\mathfrak{P}$* (*cf. Theorem* 6.19(1) *and its proof*)*. By definition,* (PRM) *and* (PFC) *hold on any desirable triple $(Y/T/Z, B_Y, \mathfrak{F})$ if they hold on $(X/Z, B)$. Indeed, $\mathfrak{P}(Y/Z, B_Y)$ is the birational transform of $\mathfrak{P}$ plus some E_i on Y that are exceptional on X. Moreover, we expect that*

(GFC) *each of the conjectures* (PRM), (PFC), *and* (SSB) *holds on any general log Fano contraction $(X/Z, B)$* (*see* (GLF) *in Proposition* 4.50) *with a* Klt *subboundary B for* (SSB).

Each $\mathcal{D} \in \mathfrak{D}$ has the form $\mathcal{D} = \mathcal{M} + \mathcal{F}$ with b-free $\mathcal{M} \geq 0$, $\mathcal{M} = \operatorname{Mov} \mathcal{D} \in |\operatorname{Mov} \mathcal{D}|$, and fixed $\mathcal{F} = \operatorname{Fix} \mathcal{D} \geq 0$. This splits $\mathfrak{D}$ as a partial *sum* (*cf. Remark* 6.15(6))

$$\mathfrak{D}(X/Z, B) \subset \mathfrak{M}(X/Z, B) \oplus \mathfrak{F}(X/Z, B),$$

where $\mathfrak{M} = \mathfrak{M}(X/Z, B)$ and $\mathfrak{F} = \mathfrak{F}(X/Z, B)$ are the subsets *of b-free and fixed* (*with $\mathcal{M} = 0$*) *b-divisors in $\mathfrak{D}$. Indeed, under Klt,* (SAT) *is equivalent to*

(SAF) $\mathcal{M}$ *is* saturated *with respect to $\mathcal{A} + \mathcal{F}$ on any sufficiently high model X_{hr}/Z of X/Z by Proposition* 6.34(2);

and, in turn, by Proposition 6.34(3), (SAF) *is equivalent to*

(STD) *as in Definition* 6.11 *for* fixed $\mathcal{D} := \mathcal{F}$.

Also, $\mathcal{F}$ satisfies (SAT) *because* $\operatorname{Mov}\lceil \mathcal{F} + \mathcal{A} \rceil \leq 0 \leq \mathcal{F}$, *and so does $\mathcal{M}$ by Lemma* 4.44 *with $\mathcal{C}_1 = \mathcal{A} + \mathcal{F}$ and $\mathcal{C}_2 = \mathcal{A}$. We use the splitting $\mathcal{D} = \mathcal{M} + \mathcal{F}$ to confine the singularities of the sum in terms of its two summands. Moreover, we can consider any such sum* (*cf. Remark* 6.15(6)).

Indeed, $\mathcal{M}$ *and* $\mathcal{F}\ge 0$, *and, on each desirable triple* $(Y/T/Z, B_Y, \mathfrak{F})$ *of* $(X/Z, B)$, $\mathcal{D}_Y \sim \mathcal{M}_Y + \mathcal{F}_Y$ *with a* $\mathbb{Q}$*-Cartier integral Weil divisor* $\mathcal{M}_Y \ge 0$ *and an* $\mathbb{R}$*-Cartier* $\mathcal{F}_Y \ge 0$. *Thus, because the canonical property is convex* (*cf.* [41, (1.3.1)]), *it is enough to prove that* $\mathcal{D}_Y \sim \mathcal{M}_Y + \mathcal{F}_Y$ *is canonically confined for each summand. If* $\mathcal{F}$ *is confined by* c_f *from below and* $\mathcal{M}$ *by* c_m, *then* $\mathcal{D}$ *is confined by any weighted combination of them, e.g., by the half-sum* $c = (c_m + c_f)/2$.

On any desirable triple $(Y/T/Z, B_Y, \mathfrak{F})$, *by* (PRM) *and Example* 6.7(1), *each irreducible component* f_iP_i *of* $\mathcal{F}_Y$ (*with prime* P_i) *has canonically confined singularities if the multiplicities* f_i *are bounded. For the whole* $\mathcal{F}$, *we need more, namely, boundedness of the number of irreducible components, which is equivalent to* (SSB) *and holds for* $\mathcal{D} := \mathcal{F}$ *under condition* (SEF)/Y *of Definition* 6.11, *or for the log transforms in* $\mathfrak{S}_Y$ *from* $(X/Z, B)$ (*see Remark* 6.13(2)).

The mobile part $\mathcal{M}$ *behaves worse* (*cf. Remark* 6.15(7)):

- *canonical confinement from below holds for* some *effective* $\mathcal{M}$ (*usually meaning* sufficiently general); *and*
- *on some triples.*

With a certain degree of optimism, *we expect that*

(CCS) *there is an* algebraic (*throughout the paper*; *cf. Remark* 8.16) bounded *family of desirable triples* $(Y/W/T, B_Y, \mathfrak{F})$ *on which* $\mathfrak{M} = \mathfrak{M}(X/Z, B)$ *has* canonically confined singularities *up to* $\sim$, *where* $\mathfrak{M}$ *is the set of b-free b-divisors satisfying log canonical saturation* (SAT). *More precisely, for each* $\mathcal{M} \in \mathfrak{M}$, *there are* $\mathcal{D} \in |\mathcal{M}|$ *and a desirable triple* $(Y/W/T, B_Y, \mathfrak{F})$ *for* $\mathcal{M}$ *in the family such that* $\mathcal{D}$ *is canonically confined on* (Y, B_Y).

Thus, jointly,

(GCC) canonical confinement *for some* $\mathcal{D} + \mathcal{F} \sim \mathcal{M} + \mathcal{F}$ *over the family*

is equivalent to (CCS) + (SSB). *But the latter needs extra assumptions on* $\mathcal{F}$, *such as* (SEF) *of Definition* 6.11. *Without it,* (GCC) *means* (CCS)+(PRM).

In addition, we expect (*cf. Remark* 6.15(8)) *that*

(BIG) *for* big $\mathcal{M}$, *the subfamily of desirable triples is finite*; *in particular,*

(BIR) *for* birational X/Z, *the whole family of desirable triples can be taken to be finite*; *and, more generally,*

(MOD) *among* birationally equivalent $\mathcal{M}$, *that is,* $\mathcal{M}$ *that define birationally the same contractions* $Y \to T/Z$ (*see Definition* 6.20), *the subfamily of desirable triples* $\mathcal{M}$ *is finite*; *in addition,* $\dim T/Z = \kappa(X/Z, \mathcal{M})$ *is the Iitaka dimension of* $\mathcal{M}/Z$.

However, to be more realistic, *we can restrict ourselves to very special subsets in* $\mathfrak{M}$ *and impose further conditions. We consider mobile systems* $\mathcal{M}_\bullet \subset \mathfrak{M}$ *at least under* asymptotic saturation (LCA); *e.g.,*

(CCS)(fga) *condition* (CCS) *for* $\mathcal{M}_\bullet = \operatorname{Mov}\mathcal{L}$ *for each individual functional algebra* $\mathcal{L}$ *in Definition* 4.38; *and*

(CCS)(rfa) (CCS) *for each algebra in Definition* 3.47.

Finally, if $(X/Z, B)$ *is a generalized log Fano contraction* (*see Conjecture* 5.26), *we consider only b-free* $\mathcal{M} \in \mathfrak{M} = \mathfrak{M}(X/Z, B)$ *that are ample on the* $\operatorname{LCS}(X, B)/Z$, *with* $\mathcal{M} = \overline{\mathcal{M}_X}$ *over a neighborhood of the* LCS, *and* $|\mathcal{M}|_X$ *is free in a neighborhood of the* LCS, *where* $\mathcal{M}$ *satisfies the same log canonical saturation* (SAT) (*but now* $\mathcal{A}$ *may be quite ineffective*). *Thus, we concentrate only on singularities outside such a neighborhood and modify* (CCS) *to*

(CCS)* *there is an algebraic bounded family of desirable triples* (*see Remark* 6.10(2)) *on which* $\mathfrak{M} = \mathfrak{M}(X/Z, B)$ *has* canonically confined singularities *over* $Y \setminus \operatorname{LCS}$ *up to* $\sim$. *More* *precisely, for each* $\mathcal{M} \in \mathfrak{M}$, *there are* $\mathcal{D} \in |\mathcal{M}|$ *and a desirable triple* $(Y/W/T, B_Y, \mathfrak{F})$ *for* $\mathcal{M}$ *in the family such that* $\mathcal{D}$ *is canonically confined on* $(Y \setminus \mathrm{LCS}, B_Y)$.

Perhaps, we can weaken the ampleness on the LCS *as in Conjecture* 5.26 (*cf. also* (GEN) *of Remark* 6.15(9) *and Addendum* 6.40.2 *below*).

As above, we can consider (CCS)*(fga) *for algebras of* (FGA)* *type. The conditions we really need are* (CCS)(rfa),(bir) *and the same with* * (*see the notation below and Conjecture* 3.48).

Remarks 6.15. (1) We can expect generalizations of the conjectures when X/Z is only a proper morphism of normal varieties but possibly not a contraction, or X/Z has other extra structures, e.g., a group action, a morphism X/W, etc. The latter relates, in particular, to the nonlocal case X/W with a proper morphism $W \to Z$ and *local*/Z, as we always assume.

(2) (PRM): drop (SEF) in Definition 6.11 but assume that $S = P$ is prime.

(3) We can drop the effective condition for $\mathcal{D} \in \mathfrak{D}$ in (PFC). Indeed, if $|\mathcal{D}| = \varnothing$, the prime component of the base locus is a single prime variety X that we can add to $\mathfrak{P}$. Otherwise, $D \geq 0$ up to $\sim$ that preserves saturation (SAT) and the fixed components. Perhaps, we can also add other prime *maximal* fixed centers (that is, maximal under inclusion). In dimension 0, boundedness is obvious over an algebraically closed field. One can expect something like (SSB), or boundedness of the entire base locus (cf. Addenda 6.26.1 and 6.26.2).

Again, in (CCS) and (GCC), we need the effectiveness of $\mathcal{M}$ and $\mathcal{M}+\mathcal{F}$ (respectively, up to $\sim$). Otherwise, $\mathcal{M} = -\infty$ is canonically confined on any model. Thus, we can also drop effectiveness.

(4) Any $\mathbb{R}$-b-divisor $\mathcal{D}$ has a decomposition $\mathcal{D} = \mathcal{M} + \mathcal{F}$ with a b-free mobile part $\mathcal{M} = \operatorname{Mov}\mathcal{D}$ and a *fixed part* $\mathcal{F} = \operatorname{Fix}\mathcal{D} \geq 0$. We explained this in Proposition 4.15(1),(3) when $\mathcal{O}_Z(\mathcal{D})$ is coherent. In general, there is a *maximal coherent subsheaf*

$$\mathcal{O}_Z(\mathcal{M}) = \mathcal{O}_Z(\mathcal{D})^{\mathrm{coh}} \subset \mathcal{O}_Z(\mathcal{D}) \subset \mathcal{O}_Z(\mathcal{D}_X) \subset k(X)$$

(because the coherent sheaf $\mathcal{O}_Z(\mathcal{D}_X)$ is Noetherian). Note that, in the *global* case with $Z = $ pt., each $\mathcal{O}_Z(\mathcal{D})$ is coherent. See also [14].

An $\mathbb{R}$-b-divisor $\mathcal{D}$ is effective up to $\sim$ if and only if $\mathcal{M} \neq -\infty$, or equivalently, $|\mathcal{M}| \neq \varnothing$, or $\mathcal{O}_Z(\mathcal{M}) = \mathcal{O}_Z(\mathcal{D})^{\mathrm{coh}} \neq 0$. If $\mathcal{D} \geq 0$, then $\mathcal{O}_Z(\mathcal{D})^{\mathrm{coh}} \neq 0$ and $\mathcal{D} \geq \mathcal{F} \geq 0$. In the same way, $\mathcal{D} \geq 0$ is *mobile* if and only if $|\mathcal{M}| \neq |0|$, or equivalently, $\mathcal{O}_Z(\mathcal{M}) = \mathcal{O}_Z(\mathcal{D})^{\mathrm{coh}}$ has > 1 generators (is nonprincipal).

(5) For general log Fano contractions in (GLF) of Proposition 4.50, we drop any properties of B. In general, (SSB), (PRM), and (PFC) do not hold for contractions that are not general log Fano contractions, even for rational varieties X. For example, let $(X/\mathrm{pt.}, 0)$ be a sufficiently high blowup of $\mathbb{P}^2$, e.g., such that the set of exceptional (= contractible) curves on Y is infinite and unbounded. Then $\mathfrak{S}$ and $\mathfrak{P}$ include at least the exceptional curves C (take $\mathcal{D} = \overline{C}$ for (SSB)) that are linearly fixed. We expect that (PRM) and (PFC) hold on any general log Fano contraction but for (SSB) we need Klt; in particular, B is a subboundary (cf. bad singularities in remark (9) below).

(6) We say that $\mathfrak{D}$ splits only as a *partial sum* in Conjecture 6.14 because, for sufficiently ample $\mathcal{M}$, $\operatorname{Mov}\lceil \mathcal{M} + \mathcal{F} + \mathcal{A} \rceil$ can certainly be $> \mathcal{M}$, which gives a *different* splitting.

(7) Write $m = \mathrm{mld}(X, B)$ for the minimal log discrepancy of (X, B). If some element $\mathcal{D} \in |\mathcal{M}|$ is canonically confined by c with $0 \leq c \leq m$, then so is the general $\mathcal{D}' \in |\mathcal{M}|$. (By Proposition 9.17 below, $\mathcal{D}$ and $\mathcal{D}' \sim \mathcal{M}$ also satisfy (SAT).) Hence, (CCS) for some $\mathcal{D}$ implies (CCS) for general $\mathcal{D}$, in fact with the same c. Indeed, $\mathcal{D}$ canonically confined on (X, B) means that $(X, B + c\mathcal{D}_X)$ has canonical singularities in codimension ≥ 2. Equivalently, on *any* crepant model $(Y/X/Z, B)$, the crepant subboundary

$$B_Y := \mathcal{B}(X, B + c\mathcal{D}_X)_Y = \mathcal{B}(X, B)_Y + (\overline{c\mathcal{D}_X})_Y$$

has only nonpositive multiplicities in divisors that are exceptional on X, and (Y, B_Y) is canonical in codimension ≥ 2. Note that, for any other $\mathcal{D}'$, if $B'_Y := \mathcal{B}(X, B + c\mathcal{D}'_X)_Y$ is a subboundary on Y such that $B'_Y = cL + F$, where $F \leq B_Y$ in the divisors that are exceptional on X, and $\operatorname{Bs}|L| = \varnothing$, then, by monotonicity (cf. [41, (1.3.3)]), for sufficiently general L (that is, $\mathcal{D}'$), $(X, B + c\mathcal{D}'_X)$ is also canonical in codimension ≥ 2. This holds on *some* good model Y/Z, where $|\mathcal{D}'| = \overline{|\mathcal{D}'_Y|}$, with $\operatorname{Bs}|\mathcal{D}'_Y| = \varnothing$, and $L = \mathcal{D}'_Y = 0/X$. Indeed, for such fixed model and for a general $\mathcal{D}'$,

$$B'_Y = \mathcal{B}(X, B)_Y + (\overline{c\mathcal{D}'_X})_Y \qquad \text{and}$$
$$(\overline{c\mathcal{D}'_X})_Y = c(\overline{\mathcal{D}'_X})_Y \leq c(\overline{\mathcal{D}_X})_Y = (\overline{c\mathcal{D}_X})_Y/X.$$

This can also be established by inversion of adjunction (cf. after Proposition 5.13 in [41]).

Klt implies that (X, B) has minimal log discrepancy $\operatorname{mld}(X, B) > 0$. On the other hand, $c \leq \operatorname{mld}$ on $\mathfrak{M}$ by Example 6.7(3).

(8) We could perhaps expect more for the triples $(Y/T/Z, B_Y, \mathfrak{F})$ that are required for the conjecture, e.g.,

- each model Y/Z is a contraction *over* a crepant terminal resolution of $(X/Z, B)$ and is thus a rational 1-contraction over X/Z. That is, it is a 1-contraction from a model over the resolution, and the boundedness of triples means essentially the boundedness of blowups in such a resolution; and
- in (BIG), Y/Z is isomorphic in codimension 1 to a crepant terminal resolution of $(X/Z, B)$ (cf. Remark 4.40(7)).

The discussion in Example 5.27 sheds light on the finiteness in (BIG). In fact, each of the finiteness assertions in (BIG), (BIR), (MOD) and (CCS)(fga), (CCS)(rfa) amounts to a *single* triple (cf. Remark 5.24). In general, the desirable family may be bounded but not finite (cf. Addendum 6.26.2 and Remark 6.28(3)); however, it has only a finite number of irreducible components. Note that, by the definition of such a family, each special $\mathcal{M}_u$ extends to $\mathcal{M}$ over the family or its irreducible component that is desirable over the generic point for $\mathcal{M}$. In addition, $(\mathcal{M}_u)_{Y_u}$ gives a family of $\mathbb{Q}$-Cartier divisors obtained from the b-divisor $\mathcal{M}$ over the generic point. The inversion of adjunction from the special fibre (Y_u, B_{Y_u}) gives canonical confinement with the same c for $\mathcal{M}$ on the triple (at least) over the generic point. For the latter, we need to assume that the family is *smooth* in the following sense: the parameter space is nonsingular and each fibre is smooth at its generic point. For any scheme-theoretic point in the parameter space, the family restricted to its generic point satisfies the same properties. In applications, this allows us to derive finiteness from boundedness of triples (see the proof of Theorem 6.19(3) below).

To prove (CCS) in general, it is enough to consider a *desirable* triple $(X/T/Z, B, \mathfrak{F})$ with the subset $\mathfrak{N} = \mathfrak{N}(X/T/Z, B)$ of $\mathcal{M} \in \mathfrak{M}$ satisfying the following conditions:

(NEF) $\mathcal{M}_X$ is nef$/Z$ and

(SFB) $\mathcal{M}_X$ is *supported in fibres* of X/T of *Iitaka dimension*$/Z$ $\kappa(X/Z, \mathcal{M}) = \dim T/Z$.

In particular, both hold when the triple is desirable for $\mathcal{M}$ (conversely, $\mathcal{M}$ is desirable in codimension 1 on T; thus, if T is $\mathbb{Q}$-factorial, this always holds, and X does not have divisors that are truly exceptional on T; this rarely happens). Then, to prove that $\mathfrak{N}$ is *canonically confined* on (X, B) up to $\sim$, we can additionally assume that

(IRR) $\mathcal{M}_X$ is *irreducible*

except for the case of a pencil, when any boundedness and confinement of singularities are known, in particular, canonical confinement. Since $\mathcal{M}_X$ is integral and by the cone property (RPC), divisors for which the triple is desirable correspond up to $\sim$ to integral points in the rational cone dual to the Kleiman–Mori cone $\overline{\mathrm{NE}}(T/Z)$. As in Truncation Principle 4.6, it is enough to establish canonical confinement

- for a *truncated* subcone: for $\mathcal{M}$ with a rather high *index of height* on X, namely, for some natural number I, every such $\mathcal{M}_X \sim \sum h_i g^* H_i$, where each H_i is nef and big on T/Z, and $I \mid \sum h_i$, and
- for *bounded* heights $\sum h_i$.

We hope that, for some $I \gg 0$, the former $\mathcal{M}$ are *freef* over X, that is, $\mathcal{M} = \overline{\mathcal{M}_X}$ and $\mathrm{Bs}\,|\mathcal{M}_X| = \varnothing$. Perhaps, in the latter case, bounded height implies confined singularities by reduction to lower dimension of T. Finally, for general elements in $\mathfrak{N}$, we can use the birational case after a localization over non-$\mathbb{Q}$-factorial points of T or assume that T is $\mathbb{Q}$-factorial.

Example 5.27 explains how to find such triples and to secure their finiteness if $\mathcal{M}$ is big and $T = X$. This shows, in particular, why we expect Conjecture 6.14 to hold in full. However, for the (FGA) conjecture, we only need boundedness of triples (cf. the proof of Theorem 6.19(3)).

Canonical confinement really requires the terminal property (TER) of desirable triples (cf. remark (11) below). For example, if $\mathcal{M}_i = i\overline{H}$ over a terminal resolution Y/X of $(X/Z, B)$ with ample H/Z, then $\mathcal{M}_\bullet \subset \mathfrak{M}$, but $\mathcal{M}_\bullet$ has confined singularities on X only when $X = Y$.

Note also that we need fibred triples for which X/T is not birational (cf. Example 6.38 and the proof of Corollary 6.40 below).

(9) Possible important generalizations concern bad singularities of (X, B); e.g., we could weaken the Klt condition even further than in (CCS)* in Conjecture 6.14 to admit nonnormal and nonreduced singularities, but still assume that $B \geq 0$. As above, we need extra assumptions to ensure good behavior on bad singularities, namely,

(GEN) on a universal normal crepant model $(Y/Z, B_Y)$ of the contraction $(X/Z, \omega)$, each $\mathcal{D}_Y$ is *free* on $\mathrm{LCS}(Y, B_Y)/Z$, but possibly *ample* on $X_{-\infty}/Z$.

Thus, again, we do care about canonical singularities outside LCS.

(10) Another generalization concerns the numerical conditions of Fano type (GLF) (of Proposition 4.50); we can replace it by

(ADJ) $\mathcal{D}_X \equiv (K + B + D)/T$, where D is a sum $D = F + H$ of *effective* F plus *nef and big* H/T, or some version in the style of Remark 4.40(3).

(11) We expect similar results in the ε-log category, for example, ε-log canonical confinement. But this requires appropriate restrictions, e.g., ε-log saturation with respect to ε-log discrepancy. For triples, (TER) can be replaced by ε-log terminality in codimension ≥ 2.

Question 6.16. Can we replace the condition that $\mathfrak{M}$ is b-free by the b-nef assertion of Lemma 4.23?

Philosophy 6.17. Saturation (SAT) improves the properties of any $\mathcal{D}$ up to $\sim$, even if it is not b-free (by analogy with regularizing solutions of elliptic differential equations, compare [10, Weyl's lemma]). On the other hand, by (DEP) in Proposition 5.4 and by Proposition 6.8, we can confine the descent data for $\mathcal{M}$ in terms of the canonical confinement of an improved divisor. The same applies when $p\mathcal{D} \sim_{\mathbb{R}} \mathcal{M}$ with real number $p > 0$ and $\mathcal{M}$ has good canonical singularities (cf. Addenda 6.8.1 and 6.8.2), for example, by Conjecture 6.14.

Notation 6.18. We use the following specifications:

- $(\cdot)_n$ means $(\cdot)$ with $\dim X = n$;
- $(\cdot)(X/Z, B)$ means $(\cdot)$ for $(X/Z, B)$;
- $(\cdot)$(big) means that $(\cdot)$ concerns only b-divisors that are big/Z;
- $(\cdot)$(bir) means that X/Z is birational in $(\cdot)$;

- $(\cdot)$(fga) means that we consider $(\cdot)$ only on a subset $\mathcal{M}_\bullet = \operatorname{Mov}\mathcal{L}$ for an algebra $\mathcal{L}$ of type (FGA) in Definition 4.38;
- $(\cdot)$(gl) means that $Z = \mathrm{pt.}$, that is, X is global in $(\cdot)$; and
- $(\cdot)$(rfa) means that we consider $(\cdot)$ only on a subset $\mathcal{M}_\bullet = \operatorname{Mov}\mathcal{L}$ for an algebra $\mathcal{L}$ of type (RFA) in Definition 3.47.

We apply these mainly to (CCS) of Conjecture 6.14. For example,

- $(\mathrm{CCS})_n$ means (CCS) for all $(X/T, B)$ with $\dim X = n$;
- $(\mathrm{CCS})_n(\mathrm{fga}), (\mathrm{bir})$ means (CCS) for any system $\mathcal{M}_\bullet$ of type (FGA) with birational X/Z and with $\dim X = n$; and
- $(\mathrm{CCS})_n(\mathrm{bir}), (\mathrm{gl}) = \varnothing$ for $n \geq 1$.

The main result of this section is

Theorem 6.19. (1) (PFC) = (PRM) (*equality of sets*).

(2) $(\mathrm{CCS})_n$ *implies the same with any specifications* (big), (bir), (gl), (fga), *and* (rfa).

(3) $(\mathrm{CCS})(X/Z, B)$ *implies* $(\mathrm{FGA})(X/Z, B)$, *and* $(\mathrm{CCS})(\mathrm{fga})(X/Z, B)$ *is equivalent to* $(\mathrm{FGA})(X/Z, B)$.

(4) $(\mathrm{CCS})_n(\mathrm{fga})$ *is equivalent to* $(\mathrm{FGA})_n$.

Addendum 6.19.1. *Moreover, in the above statements, we can replace conditions* (CCS) *and* (FGA) *simultaneously and respectively by their singular modifications* $(\mathrm{CCS})^*$ *and* $(\mathrm{FGA})^*$.

We start by clarifying (MOD) of Conjecture 6.14.

Definition 6.20. Two contractions $X_1 \to Y_1$ and $X_2 \to Y_2$ are *birationally equivalent* if they fit in a commutative diagram with birational equivalences $X_1 \dashrightarrow X_2$ and $Y_1 \dashrightarrow Y_2$. After some resolution of X_1 and X_2 (compare the graph of a rational map in [10, p. 213]), we can assume that $X_1 = X_2$ and the contractions are isomorphic over nonempty open subsets $U_i \subset Y_i$.

Example 6.21. Suppose that X_1 and X_2 are birationally equivalent. Then

(1) any two big contractions are birationally equivalent;

(2) any two contractions to pt. are birationally equivalent; and

(3) two contractions over a curve are birationally equivalent if and only if each defines the same pencil.

Lemma 6.22. *Let $\mathcal{D}_1 \geq \mathcal{D}_2$ be two $\mathbb{R}$-b-divisors on X/Z. Then*

(1) *the Iitaka dimensions of $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy $\kappa(X/Z, \mathcal{D}_1) \geq \kappa(X/Z, \mathcal{D}_2)$, with equality if and only if they are birationally equivalent*;

(2) *if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are b-nef, then $\nu(X/Z, \mathcal{D}_1) \geq \nu(X/Z, \mathcal{D}_2)$, and equality holds if and only if*

$$\nu(X/Z, \mathcal{D}_2|_{(\mathcal{D}_1 - \mathcal{D}_2)}) \leq \nu(X/Z, \mathcal{D}_2) - 1;$$

(3) *if $\mathcal{D}_1$ and $\mathcal{D}_2$ are both b-semiample, $\nu(X/Z, \mathcal{D}_1) \geq \nu(X/Z, \mathcal{D}_2)$, with equality if and only if they are birationally equivalent.*

Proof. (1) Immediate from the definitions.

(2) Taking an appropriate model of X/Z, we can assume that each $\mathcal{D}_i = \overline{D}_i$ for a nef/Z divisor $D_i = (\mathcal{D}_i)_X$. By definition, each $\nu(\mathcal{D}_i) = \nu(D_i)$. Since $D_1 \geq D_2$, it is enough to check that $\nu(D_1) \geq \nu(D_2)$. Indeed, $D_1 = D_2 + F$ with $F \geq 0$. Taking a general point of Z, we can assume that $Z = \mathrm{pt.}$ and each $\nu(X/Z, D_i) = \nu(D_i)$.

It is enough to consider the case $\nu(D_1) \le \nu = \nu(D_2)$. Thus, for some cycle C_ν of dimension ν, we have $C_\nu D_2^\nu > 0$. Equivalently, the same holds for mobile cycles $H^{n-\nu}$, where $n = \dim X$ and H is a hyperplane section of X, because $H^{n-\nu}$ is rationally equivalent to C_ν plus an effective cycle. Taking a hyperplane section gives an induction on the dimension $n \ge \nu$ with divisors $D_{i|H}$. Hence, we can assume that $\nu = n$. Then

$$D_1^n = D_2 D_1^{n-1} + F D_1^{n-1} \ge D_2 D_1^{n-1} \ge \ldots \ge D_2^{n-1} D_1 = D_2^n + D_2^{n-1} F \ge D_2^n > 0.$$

This also means that $\nu(D_1) = \nu$. In addition,

$$\nu(D_{2|(D_1 - D_2)}) = \nu(D_{2|F}) \le \dim F \le n - 1 = \nu - 1.$$

(3) Immediate by (1) because $\nu(X/Z, \mathcal{D}) = \kappa(X/Z, \mathcal{D})$ for any b-semiample divisor $\mathcal{D}$. $\square$

Proof of Theorem 6.19. (1) Compare the proof of Proposition 6.34(3). For $\mathcal{D} \ge 0$, Klt implies that

$$\operatorname{Mov}\lceil \mathcal{A} + \mathcal{D} \rceil \ge \operatorname{Mov} \mathcal{D} \ge 0.$$

Hence, (PRM) for $P = \operatorname{Supp} \mathcal{D}_X$ implies (PFC) for $\mathcal{D}$ with prime $P = \operatorname{Supp}(\operatorname{Fix} \mathcal{D})_X$ because $\operatorname{Fix} \mathcal{D} = \mathcal{D}$. Conversely, for any $\mathcal{D} = \mathcal{M} + \mathcal{F} \ge 0$ under (SAT) and with $\mathcal{M} = \operatorname{Mov} \mathcal{D}$ and $\mathcal{F} = \operatorname{Fix} \mathcal{D}$,

$$\mathcal{M} + \operatorname{Mov}\lceil \mathcal{A} + \mathcal{F} \rceil \le \operatorname{Mov}(\mathcal{M} + \lceil \mathcal{A} + \mathcal{F} \rceil) = \operatorname{Mov}\lceil \mathcal{D} + \mathcal{A} \rceil \le \operatorname{Mov} \mathcal{D} = \mathcal{M}$$

since it is $\le \mathcal{D}$ (see Remark 4.34(3)). Thus, $\operatorname{Mov}\lceil \mathcal{A} + \mathcal{F} \rceil \le 0$ and, actually, $= 0$ by Klt again. This means (STD) of Definition 6.11 for $\mathcal{D} := \mathcal{F}$. By Lemma 4.44, the same holds for each prime component $\mathcal{D} := (\operatorname{mult}_P \mathcal{F})P$; that is, $P \in \mathfrak{P}$ when $\operatorname{mult}_P \mathcal{F} > 0$. For (PRM), we do not need (SEF) of Definition 6.11 (and prime standard divisors might not be prime divisors of the standard set $\mathfrak{S}$).

(2) Immediate from the definitions, except for (fga) and (rfa). In these cases, we only need to check that $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L} \subset \mathfrak{M}$. More precisely, this only concerns effective $\mathcal{M}_i$ up to $\sim$. Indeed, each $\mathcal{M}_i$ satisfies (SAT) by (LCA) and Remark 4.34(5).

(3) Let $\mathcal{M}_\bullet = \operatorname{Mov} \mathcal{L}$ be a mobile system on $(X/Z, B)$ for an algebra $\mathcal{L}$ of type (FGA). By Limiting Criterion 4.28, f.g. of $\mathcal{L}$ is equivalent to stabilization of its characteristic system, that is, of the limit $\mathcal{D} = \lim_{i \to \infty} \mathcal{D}_i$ with $\mathcal{D}_i = \mathcal{M}_i / i$. In turn, stabilization is equivalent to the asymptotic descent problem. To solve the problem, we apply Theorem 5.12.

First, we choose a prediction model (Definition 5.10). By Lemma 6.22(3) and the arithmetic monotonicity in Lemma 4.24, the numerical dimension $\nu(X/Z, \mathcal{D}_i)$ stabilizes. After a truncation, we can assume that each $\mathcal{D}_i$ has $\nu(X/Z, \mathcal{D}_i) = \nu$, the Iitaka dimension of $\mathcal{L}$. Moreover, up to a similarity, corresponding to a quasi-isomorphism of algebras by Proposition 4.15(8), we can assume that $\nu \ge 0$ and the entire $\mathcal{M}_\bullet \subset \mathfrak{M}$. Otherwise, $\mathcal{L} = \mathcal{O}_Z \oplus 0_\bullet$ and is f.g. In addition to the effectiveness of each $\mathcal{M}_i$, we use the saturation (LCA) of (FGA), as in (2) above.

Since, by Lemma 6.22(3), the b-divisors $\mathcal{M}_i / Z$ are birationally equivalent, by (MOD) there is a finite family of desirable triples $(Y/T/Z, B_Y, \mathfrak{F})$ *for* them. (We can actually replace it by a single triple; compare the arguments below.) Moreover, this can be in the *strict* form: each of the triples has an *infinite* subset of $\mathcal{M}_i$ for itself, and the same holds after any truncation. Indeed, we can discard all triples in our finite family that do not satisfy the essential infiniteness and replace $\mathcal{M}_\bullet$ by a truncation for which the corresponding $\mathcal{M}_i$ has been discarded with triples. Then we take a truncation such that some triple does not satisfy the essential infiniteness, etc. Finally, we get a family with the strict property.

Now any remaining triple $(Y/T/Z, B_Y, \mathfrak{F})$ predicts $(Y/Z, \mathcal{C} = \mathcal{A}, F, \gamma)$ for the asymptotic descent problem $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$. If we consider only i for which this is a desirable triple, then (SAM) for $\mathcal{D}_Y = \lim_{i\to\infty}(\mathcal{D}_i)_Y$ follows from the cone property (RPC) because, by our choice of b-divisors $\mathcal{D}_i$, each $(\mathcal{D}_i)_Y$ is semiample$/Z$ (cf. (LSA) in Remark 6.10(7) and the discussion in Example 5.27). The b-discrepancy $\mathcal{C} = \mathcal{A}$ is exceptionally effective (EEF) because the triples satisfy the terminal property (TER). The boundedness of algebra $\mathcal{L}$ and (BSD) of Addendum 4.22.1 imply that there is a reduced divisor F that contains the support of each $(\mathcal{D}_i)_Y$. Thus, (UAD) holds for F (cf. Remark 5.11(3)). On the other hand, the growth (LGD) for F and some $\gamma > 0$ follows from (QFC), (CRP), and (X, B) Klt.

The property of prediction models that is hard to establish is confinement (SAC) of Definition 5.10; by Addendum 6.8.1, this follows from (CCS)(fga)$(X/Z, B)$ with the common model $(Y_i/Z, B_i) = (Y/Z, B_Y)$ for $\mathcal{D}_i$ with i for which this is a desirable triple and with $p_i = i$. Indeed, the descent data for $\mathcal{D}_\bullet$ is asymptotically confined by $\mathcal{A}/Y$. This confinement is strict because each truncation leaves infinitely many b-divisors $\mathcal{D}_i$ for the required triple in the descent problem depending on the choice of the triple.

Secondly, $\mathcal{D}_\bullet$ satisfies the minimal assumptions (FDS) by (BSD) and (MXD) by Addendum 4.22.1 and the additional assumption (BNF) because each $\mathcal{M}_i$ is b-free.

Thirdly, as we know, (LCA) in (FGA) implies the required saturation. Thus, by Addendum 5.12.1, the asymptotic descent problem has a solution on Y. Thus, Addendum 5.13.1 gives the required stabilization.

Finally, we can use boundedness of triples instead of finiteness in (MOD) in the descent of Theorem 5.12, its corollaries and addenda. Indeed, we can construct a family or a prediction model over a functional field as in Remark 5.24. (Actually, this means that we choose again a single prediction model, but over a functional field.) For this, we consider a Zariski closed subset in the moduli of triples for (CCS) that has a Zariski dense strictly infinite subset of special (closed) points corresponding to $\mathcal{M}_i$ according to (CCS). The subset can be chosen by the Noetherian property and truncations (cf. the above choice of prediction model under (MOD)). Then we take any of its irreducible components W. It also has a dense subset with the same property. This functional triple gives the required prediction model. It is birationally equivalent (over the function field of W) to a constant family. For this, we include in $\mathfrak{F}$ the poles and zeros of rational function that generate $k(X)$; this induces $\mathfrak{F}_Y$ on the triples. The isomorphism induces birationally $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ with $\mathcal{D}_i = \mathcal{M}_i/i$ for the chosen family of triples$/W$ that satisfies (FDS) and birationally (MXD), (BNF), (AMN), and (LCA). Thus, as above, we only need confinement (SAC). To secure this, choose an open subset $U \subset W$ such that

- F in (FDS) is (in $\mathfrak{F}$ and) deforms universally$/U$; that is, $F_u = F_{|Y_u}$ and $\mathfrak{D}_{F_u} = \mathfrak{D}_{F|Y_u}$; and
- the birational equivalence of Y/U with the constant family $X \times U/U$ is well defined in each fibre$/U$; that is, it is regular for an open subset$/U$ that has nonempty intersection with each fibre of Y/U.

(SAC) holds for i corresponding to special points in U. This set of $\mathcal{D}_i$ is strictly infinite because its points are dense and preserves this property after truncation according to the construction. To verify (SAC) birationally$/W$, we can use again Addendum 6.8.1. For this, we find $\mathcal{M}'_i \sim \mathcal{M}_i/W$ such that $(\mathcal{M}'_i)_Y$ satisfies (CCS) over the general point of W. By our choice of U, the birational equivalence of Y/U with the constant family $Y_i \times U/U$ is well defined too. As above, by (LCA) and (CCS), we have $\mathcal{M}'_i \sim \mathcal{M}_i$ in the special fibre Y_i/U. We extend it as constant with $\sim /U$. On Y/U, this induces $(\mathcal{M}'_i)_Y \sim (\mathcal{M}_i)_Y/U$. On the other hand, $(\mathcal{M}_i)_Y \in \mathfrak{D}_F$, $(\mathcal{M}_i)_{Y|Y_i} = (\mathcal{M}_i)_{Y_i}$, and $(\mathcal{M}'_i)_{Y|Y_i} = (\mathcal{M}'_i)_{Y_i}$. The latter is canonically confined. Canonical confinement at any special point implies the same over the generic point by inversion of adjunction; this implies (SAC) at least over the generic point by Addendum 6.8.1 and so birationally. *A posteriori* stabilization holds over an open subset in W.

Conversely, the stabilization in Limiting Criterion 4.28 gives (CCS) for a truncation of $\mathcal{M}_\bullet$. Then, by Truncation Principle 4.6, the same holds for any system $\mathcal{M}_\bullet$ (cf. Remark 6.15(7)).

(4) Immediate from (3).

Addendum 6.19.1 can be proven in the same way with the following modifications:

- Replace usual triples by triples as in Remark 6.10(2).
- Replace (CCS) by (CCS)*.
- Replace (MOD) by its singular version (MOD)* or use arguments similar to the above.
- Replace Theorem 5.12, its corollaries and addenda by Corollary 5.21, its addendum and corollary. By Example 5.25, lca saturation implies (JAS) with $\mathcal{C} = \mathcal{A}^+$ on any triple for $\mathcal{M}_i$. After a truncation, each $j\mathcal{D}_i$ is integral over $\mathcal{A}^-$ by the definition of LCS and stabilization on triples below. Indeed, each b-prime P_i in $\operatorname{Supp}\mathcal{A}^-$ has $\operatorname{center}_Y P_i$ in the LCS. Thus, $j\mathcal{D}_i = j\mathcal{D}_j$ is integral over the generic point of $\operatorname{center}_Y P_i$ due to the stabilization of the limit near the LCS.
- Stabilization in a neighborhood of the LCS holds on any desirable triple for a strictly infinite subset $\mathcal{M}_i$. Indeed, by the stabilization on the LCS on the triple, $\mathbb{Q}$-Cartier property of corresponding $\mathcal{D}_i$, (AMN), and Lemma 10.9, $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ stabilizes in a neighborhood of the LCS. Thus, after a truncation, (AMN) again gives the complete stabilization in this neighborhood. The latter also implies the following:
- (RST)$'$ holds for all $\mathcal{D}_i$ on any desirable triple for a strictly infinite subset $\mathcal{M}_i$.
- F in (UAD) is disjoint from the $\operatorname{LCS}(Y, B_Y)$ by stabilization near the LCS. Thus, it is $\mathbb{Q}$-Cartier by (QFC)*.
- Addendum 6.8.1 is used over the same F only at terminal points by (TER)*, where $\mathcal{C} = \mathcal{A}^+ = \mathcal{A}$. To fulfill (EEF), we need to replace the exceptional/Y multiplicities < 0 in $\mathcal{C} = \mathcal{A}^+$ by 0. By (TER)* and the last stabilization, it does not effect the (JAS). Or we can replace (EEF) by (EEF)*, where we take into account only the exceptional b-prime divisors with centers not intersecting the LCS. $\square$

Remark 6.23. (CGR) of Proposition 4.46 holds for $\mathcal{C} = \mathcal{A}$ by Example 4.47. Hence, by Addendum 5.12.2, $\mathcal{A}_{X_{\mathrm{st}}} \le 0$ on the stable model X_{st}/Z, which proves Remark 4.40(7). Indeed, by Klt, the crepant boundary B_{st} is actually a boundary and $(X_{\mathrm{st}}, B_{\mathrm{st}})$ is Klt. On the other hand, by the improved (BIG) in Remark 6.15(8), it is dominated by a rational 1-contraction of a crepant weak log Fano contraction for $(X/Z, B)$. Thus, $-(K_{X_{\mathrm{st}}} + B_{\mathrm{st}})/Z$ is big and b-semiample. Hence, by LMMP, $(X_{\mathrm{st}}, B_{\mathrm{st}})$ is isomorphic in codimension 1 to a weak log Fano contraction, with the same boundary, on which $-(K_{X_{\mathrm{st}}} + B_{\mathrm{st}})$ is semiample = nef and big (cf. Example 5.27).

Similar facts hold for (FGA)* by Corollary 5.21 when $\mathcal{L}$ is ample in a neighborhood of the LCS.

The following examples illustrate Conjecture 6.14.

Example 6.24 (global curve)**.** Let $(X/Z = \mathrm{pt.}, B)$ be a complete log curve X with B arbitrary (reduced but possibly reducible). Then it is terminal in codimension ≥ 2. The b-divisors are divisors $\mathcal{D}$ on the normalization of X, that is, the space of a triple of the required form. Conjecture 6.14 holds in this case. Note that $\mathcal{D}_X$ is well defined for nonsingular divisors D (with $\operatorname{Supp} D \subset \operatorname{NonSing} X$), and $\mathcal{D} = \mathcal{D}_X = D$. Set $D = M + F \ge 0$ with $M = \operatorname{Mov} D$ and $F = \operatorname{Fix} D$. Then M satisfies (CCS) with $c = 1$, which does not need any saturation. (Even (SAT) with $\deg B \gg 0$ can exclude some M.) But the boundedness of F follows from the saturation (SAT) and RR. In addition, the multiplicities of F are bounded. For example, if $B = 0$, then $\lceil F \rceil \le g - 1$, where $g = g(X)$ is the arithmetic genus of X.

Example 6.25 (two-dimensional birational case)**.** Let $f\colon X \to Z$ be a birational contraction of a normal surface X. As we will soon see (cf. the proof of (FGA) in Main Theorem 1.7 at the end of this section), this is the key point in our construction of 3-fold log flips. Except for confinement (CCS), Conjecture 6.14 holds for *any* birational contraction with finite $\mathfrak{S}(X/Z,B)$, $\mathfrak{P}(X/Z,B)$ and $\operatorname{Supp}\mathfrak{F}(X/Z,B)_X$ supported in divisors of X that are exceptional on Z, and in the fractional part of B.

Suppose first that $B=0$, X is nonsingular, and X/Z is cohomologically rational, that is, Z has rational singularities. We claim that $(X/X/Z,0,\mathfrak{F})$ is a desirable triple in (CCS) for any $\mathcal{M}\in\mathfrak{M}=\mathfrak{M}(X/Z,B)$ (and any bounded set of reduced divisors $\mathfrak{F}$).

Indeed, $\mathcal{A}=\mathcal{A}(X,0)=\lceil\mathcal{A}\rceil\geq\mathcal{E}\geq 0$, where $\mathcal{E}$ is the reduced b-divisor of all prime b-divisors that are exceptional on X, the *support of the exceptional locus*. By the rationality of X/Z, on any model Y/Z of X/Z, each Cartier divisor M that is nef/Z is free/X (essentially due to M. Artin; see [38, p. 105, Lemma]). In particular, this holds for any integral $M\geq 0$ for which $\operatorname{Supp}M$ has no exceptional components of Y/Z. In addition, if C is a -1-curve on Y/X and $M\cdot C\geq 1$ (intersects any M up to $\sim$), then $M+C$ is also free, and

$$\begin{aligned}\operatorname{Mov}\lceil M+\mathcal{A}_Y\rceil &= \operatorname{Mov}(M+C+\mathcal{A}_Y-C)\geq \operatorname{Mov}(M+C)+\operatorname{Mov}(\mathcal{A}_Y-C)\\ &\geq M+C>M\end{aligned}$$

because $\mathcal{A}_Y-C\geq 0$. Hence, no b-free $\mathcal{M}=\overline{M}\in\mathfrak{M}$ since it does not satisfy (SAT). Conversely, by (SAT), every $\mathcal{M}\in\mathfrak{M}$ descends to X as a free divisor/Z. Thus, we can take $c=1$ in (CCS) of Conjecture 6.14. This explains the role of saturation.

Indeed, if Y/X is a minimal resolution of the base locus for $|\mathcal{M}|_X$, then a -1-curve C as above exists over the locus. If $Y:=X_{\mathrm{hr}}$ is also a sufficiently high resolution for (SAT) and $M=\mathcal{M}_Y$ is sufficiently general in $|\mathcal{M}_Y|=|\mathcal{M}|_Y$, then we replace C by $\overline{C}_Y\leq\mathcal{A}_Y$. This again contradicts (SAT). Thus, $\mathcal{M}$ is free on X/Z.

Moreover, the same holds for $(X/Z,B)$ under (TER). But in this case we may have $c<1$, namely, $c=\min\{1-b_i\}$.

If $(X/Z,B)$ is Klt, then (CCS) holds on a crepant terminal resolution $(Y/X,B_Y)$. Since $(X/Z,B)$ is not assumed to be a weak log Fano contraction, Remark 6.15(10) holds, even without (ADJ).

Another approach that we discuss below reduces our problem to freedom in dimension 1 (cf. Example 6.24). It is not so effective and, moreover, has some unpleasant new features (see Remark 6.28(2)), but this finally leads to 4-fold flips.

The main techniques to establish Conjecture 6.14 in dimension 2 are contained in the following proposition.

Proposition 6.26 (general two-dimensional case)**.** *Let $(X/Z,B)$ be a log pair with a surface X and $\mathcal{D}\geq 0$ be an* integral *b-divisor such that*

- (GLF) *$(X/Z,B)$ is a general log Fano contraction as in Proposition* 4.50 *with arbitrary B that is not even assumed to be a boundary*;
- (RIR) *on any model Y/Z of X/Z, the general $\mathcal{D}_Y\in|\mathcal{D}|_Y$ is* reduced and irreducible; *and*
- (SAT) *$\mathcal{D}\in\mathfrak{D}=\mathfrak{D}(X/Z,B)$*; that is, *$\mathcal{D}$* is saturated with respect to *$\mathcal{A}=\mathcal{A}(X,B)$.*

Then, except for a bounded set of complete divisors $\mathcal{D}_X$, $|\mathcal{D}|_X$ is free/Z outside $\operatorname{CS}(X/Z,B)$ *on X, where* $\operatorname{CS}(X/Z,B)$ *denotes the* locus of canonical singularities. *In other words, $(X/Z,B)$ satisfies* (TER) *outside* $\operatorname{CS}(X/Z,B)$.

In dimension 2, $\operatorname{CS}(X/Z,B)$ is the union of the $\operatorname{LCS}(X,B)$ and the finite set of closed points at which (X,B) is not terminal. In higher dimensions, these points may not be closed, and we take their closure. Note that a terminal resolution is a crepant model with $\mathrm{CS}=\varnothing$.

Commentary. We only need two essential cases:

- $\mathcal{D}$ is b-free and does not consist of ≥ 2 copies of a pencil; or
- $|\mathcal{D}| = \{P\}$ is a fixed prime b-divisor.

By (RIR), $|\mathcal{D}|_X$ free/Z somewhere on X means that $\mathcal{D}$ has a descent $\mathcal{D}_X$ over the locus. (In the proposition, outside CS.) From this point of view, "sufficiently general" $\mathcal{D}_Y$ means, in particular, that $\mathcal{D}_Y$ has minimal multiplicities in every divisor that is exceptional on X, e.g., 0 for b-free $\mathcal{D}$.

Addendum 6.26.1. *If the assumptions* (RIR) *and* (SAT) *in the proposition are replaced, respectively, by*

(RED) *on any model Y/Z of X/Z, a general $\mathcal{D}_Y \in |\mathcal{D}|_Y$ is* reduced; *and*

(SA′T) *$\mathcal{D}$ is saturated with respect to $\mathcal{A}' = \mathcal{A}(X,B) + \sum E_i$, where b-divisors E_i are the primes over* integral components of *$\mathcal{A}$ that are exceptional on X*,

then any two prime components (or even any two branches) $D_1 \neq D_2$ of general elements $\mathcal{D}_X \in |\mathcal{D}|_X$ intersect each other only at points $Q \in X$ at which $K+B$ is not Cartier or in $\mathrm{LCS}(X,B)$, *with a finite set of exceptions for D_1* or *D_2. More precisely, any set of exceptions to this can only include D_i with $(K+B)\cdot D_i = 0$ (in particular, D_i is complete and, hence, lies over P). Thus, these exceptions D_i belong to a bounded family of reduced divisors $\mathfrak{F}$.*

Moreover, we can replace (SA′T) *by a weaker condition (cf. Proposition* 6.34(4) *and Remark* 6.35(2) *below)*

(Sa′T) *as in* (SA′T), *but we assume that the primes E_i are over* $\mathrm{Bs}\,|\mathcal{D}|_X$.

Addendum 6.26.2. *Suppose, in addition, that*

- *(X,B) is Klt and*
- *$\mathcal{D}$ is b-free.*

Then $|\mathcal{D}|_X$ has at most one base point on X outside $\mathrm{CS}(X,B)$; *in particular, it has at most one base point on a terminal resolution of $(X/Z,B)$. In addition, $|\mathcal{D}|_X$ is free in* CS *after the point is blown up. Moreover, any pencil $\mathcal{D}$ is elliptic, that is, its general member is a curve of geometric genus* 0.

Corollary 6.27. *Locally/Z, under the assumptions of Proposition* 6.26, *the singularities of a general $\mathcal{D}_X \in |\mathcal{D}|_X$ are bounded and confined outside* CS, *in particular, outside* $\mathrm{LCS}(X,B)$ *on a terminal resolution.*

Proof. The singularities of $\mathcal{D}_X$ are bounded and confined for any free $|\mathcal{D}|_X$. The same holds for any bounded set of divisors $\mathcal{D}_X$ (Example 6.7(1)). □

Remarks 6.28. (1) (RIR) and (1) in the proof of Proposition 6.26 below (but not (2)) hold when $\mathcal{D}$ is b-free and big. For surfaces, the latter means that $\mathcal{D} \neq 0$ and is not a pencil.

(2) Sometimes, $\mathrm{Bs}\,|\mathcal{D}|_X \neq \varnothing$, and this requires that $c < 1$ even if (X,B) is terminal in codimension 2 (see Example 6.38 below).

(3) However, for b-free $\mathcal{D}$, we expect only a *finite* set of linear systems $|\mathcal{D}|$ having base points outside CS.

For example, one verifies this finiteness when $(X/Z = \mathrm{pt.}, B)$ is a del Pezzo surface with a standard boundary B and of nonexceptional type [43].

Lemma 6.29 (lifting $\mathbb{R}$-Cartier divisors). *f^*D is defined up to $\sim$ under the following conditions*:

- *D is an $\mathbb{R}$-Cartier divisor defined up to $\sim$*; *and*
- *for any morphism $f\colon X \to Y$ such that $f(X)$ is not contained in the fractional prime components of D, $f(X) \not\subset \mathrm{Sing}\, X$, and Y/Z is projective or $f(X)$ has only divisorial components.*

Addendum 6.29.1 (functoriality). *For $g \circ f \colon X \to Y \to Z$,*

$$(g \circ f)^* D = f^*(g^* D).$$

Proof. Since D is given up to $\sim$, we can assume that f and D are in *general position*; that is, $\operatorname{Supp} f(X) \cap \operatorname{Supp} D$ is proper in each irreducible component of $f(X)$. If Y/Z is nonprojective, we can use Chow's lemma and $\sim$ induced from a projective model of Y over Y. □

Lemma 6.30 (invariance of integral part under $\sim$). *If $D \sim D'$, then*

$$\lfloor D \rfloor \sim \lfloor D' \rfloor \qquad \text{and} \qquad \lceil D \rceil \sim \lceil D' \rceil.$$

Thus, the same holds for the fractional parts.

Proof. $F = D - D'$ is principal, in particular, integral, and

$$\lfloor D \rfloor - \lfloor D' \rfloor = \lfloor F + D' \rfloor - \lfloor D' \rfloor = F = \lceil D \rceil - \lceil D' \rceil$$

is also principal. □

Lemma 6.31. *Let D be a divisor on a complete nonsingular curve C with* $\deg D \geq d$. *Then* $\deg \lceil D \rceil \geq \lceil d \rceil$.

Proof. We need to check that if $\{d_i\}$ is a set of real numbers such that $\sum d_i \geq d$, then $\sum \lceil d_i \rceil \geq \lceil d \rceil$. We can apply this to the set of multiplicities of divisor $D = \sum d_i P_i$ on C.

Indeed, each $\lceil d_i \rceil \geq d_i$ (and $=$ holds only if d_i is integral). The sum of these gives

$$\sum \lceil d_i \rceil \geq \sum d_i \geq d$$

and the required inequality because $\sum \lceil d_i \rceil$ is integral and any integer $\geq d$ is $\geq \lceil d \rceil$. □

Proof of Proposition 6.26. We restrict to $D = \mathcal{D}_X$ after a resolution. Up to a bounded set of divisors D (or even a finite set), we can assume that

(1) $\operatorname{Supp} D \cap \operatorname{Supp} B$ is *small*,

that is, the intersection does not contain any divisors.

We consider a sufficiently high resolution $g \colon Y \to X$, where saturation (SAT) holds. Since $\sim$ preserves (SAT) (cf. Remark 4.34(7)), we can assume by (RIR) that $D := \mathcal{D}_Y$ is general, irreducible, and reduced. Let $g_D \colon D \to X$ be the induced morphism. After an additional resolution, we can also assume that Y is a log resolution for $(X, B + D)$. In particular, D is nonsingular.

The restriction

$$|D + \lceil \mathcal{A}_Y \rceil| \dashrightarrow |(D + \lceil \mathcal{A}_Y \rceil)_{|D}| = |K_D + \lceil -g_D^*(K + B) \rceil|$$

of relative linear systems $/Z$ is surjective. Indeed, we can use Lemma 6.29 because D is reduced by (RIR) and $K + B$ is integral on D by (1). Note also that $\mathcal{A}_Y \sim K_Y - g^*(K + B)$, where $\lceil \mathcal{A}_Y \rceil \sim K_Y + \lceil -g^*(K + B) \rceil$ by Lemma 6.30. Thus, $D + \lceil \mathcal{A}_Y \rceil \sim D + K_Y + \lceil -g^*(K + B) \rceil$ and, by adjunction, normal crossings on Y, and Addendum 6.29.1, we have

$$(D + \lceil \mathcal{A}_Y \rceil)_{|D} \sim (D + K_Y)_{|D} + \lceil -g_D^*(K + B) \rceil = K_D + \lceil -g_D^*(K + B) \rceil.$$

Also, we use Kawamata–Viehweg vanishing: $R^1 h_* \mathcal{O}_Y(\lceil \mathcal{A}_Y \rceil) = 0$ with $h = f \circ g \colon Y \to Z$ because $\lceil \mathcal{A}_Y \rceil \sim K_Y + \lceil -g^*(K + B) \rceil$ and (GLF).

By (SAT),

$$\operatorname{Mov}(D + \lceil \mathcal{A}_Y \rceil) = \operatorname{Mov} \lceil \mathcal{D} + \mathcal{A} \rceil_Y \leq \mathcal{D}_Y = D.$$

If $|\mathcal{D}|_X$ has a base point $Q \in X/Z$ outside CS, then $\operatorname{Fix}(D + \lceil \mathcal{A}_Y \rceil) \geq \lceil \mathcal{A}_Y \rceil \geq E_i$ for an exceptional divisor E_i of discrepancy $a_i = a(X, B, E_i) > 0$ and such that E_i intersects D on Y. Hence,

$$\operatorname{Fix}((D + \lceil \mathcal{A}_Y \rceil)_{|D}) \geq E_{i|D} > 0 \qquad \text{and} \qquad \operatorname{Bs}|(D + \lceil \mathcal{A}_Y \rceil)_{|D}| \neq \varnothing;$$

i.e., the restriction is not free by the above surjection. Hence, D is complete. Otherwise, $|(D + \lceil \mathcal{A}_Y \rceil)_{|D}|$ is always free.

Thus, D is complete and $|(D + \mathcal{A}_Y)_{|D}| = |K_D + \lceil -g_D^*(K + B) \rceil|$ must have base points or be $\varnothing$. This is impossible if $\deg B' > 1$, where $B' = -g_D^*(K + B)$, because then, by Lemma 6.31, $\deg \lceil B' \rceil > 1$ and, hence, ≥ 2. However, for such a divisor B', the linear system $|K_D + \lceil B' \rceil|$ is free. Hence, each D having fixed points on X outside CS should have bounded degree on X/Z; namely, $-(K + B) \cdot D \leq 1$. Such divisors are bounded. Indeed, we need only consider complete divisors D/Z. In particular, they include the contracted divisors D of X:

(2) $-(K + B)$ is not big$/Z$ on D;

this adds a finite set of divisors with discrepancy 0 to $\operatorname{Supp} B$; cf. (GLF) above. $\square$

Proposition 6.32. *If $D_1 \geq D_2$, then D_1 C-saturated implies D_2 C-saturated over prime divisors at which $D_1 = D_2$. The same holds in the b-divisorial versions.*

Proof–Explanation. Saturation over prime divisors at which $D_1 = D_2$ (or over any other set of divisors) means that $\operatorname{Mov} \lceil D_2 + C \rceil \leq D_2$ over this set of divisors; in other words, the inequality concerns only the prime divisors P_i in the set; that is, in our situation, $\operatorname{mult}_{P_i} D_1 = \operatorname{mult}_{P_i} D_2$. Thus, the proposition is immediate from the definition. $\square$

Proof of Addendum 6.26.1. Suppose that $K + B$ is Cartier and canonical at a point $Q \in D_1 \cap D_2$. Then $K + B$ Cartier implies that each D_i lies over an integral component of B. Again by (1) in the proof of Proposition 6.26, we assume either that $(K + B) \cdot D_i < 0$ for $i = 1, 2$ or that D_i is *not complete*. Then we verify that D is not $\mathcal{A}'$-saturated on any sufficiently high Y/X. Since $-(K + B)$ is nef and big$/Z$, there exists only a finite number of complete D_i with $(K + B) \cdot D_i = 0$. Thus, except finitely many possibilities for D_1 or D_2, the contradiction with $\mathcal{A}'$-saturation means that Q is not Cartier or not canonical.

By (RED) and Remark 4.34(1) (cf. the proof of Proposition 6.34(2)), we can replace D by its components that are not exceptional on X or, in other words, omit the divisors that are exceptional on X (because they are fixed in $|\mathcal{D}|_Y)$. Then, by (RED) and Proposition 6.32, it is enough to check that the *new* $D := D_1 + D_2$ is not $\mathcal{A}'$-saturated on any sufficiently high Y/X *over* $D_1 \cup D_2$ and some exceptional prime b-divisors E_i/Q. We also assume that each $g(D_i)$ passes through Q and that either $(K + B) \cdot D_i < 0$ or D_i is not complete. We derive a contradiction with $\mathcal{A}'$-saturation on any log resolution $g\colon Y \to X/Z$ for $B + D$.

On Y, we put together a chain of curves (maybe after additional blowups) $C = D_1 \cup (\bigcup E_i) \cup D_2$ with only nodal singularities, where E_i are the exceptional curves over Q and D_i are the end curves. We check that $|C + \lceil \mathcal{A}_Y \rceil| = |\lceil D + \mathcal{A}_Y + \sum E_i \rceil|$ does not have base points in a neighborhood of $\bigcup E_i$ on Y/Z. Since $\mathcal{A}_Y + \sum E_i \leq \mathcal{A}'_Y$ even under (Sa$'$T), by Lemma 4.44, this contradicts $\mathcal{A}'$-saturation for D on Y/X:

$$\operatorname{Mov} \lceil D + \mathcal{A}_Y + \sum E_i \rceil \geq D + \sum E_i > D$$

over each E_i. Note that $\mathcal{A}_Y \geq 0$ and is integral over Q because Q is canonical for $K + B$.

As in the proof of Proposition 6.26, for the restriction to D, we get a surjective restriction$/Z$ on C:

$$|C + \lceil \mathcal{A}_Y \rceil| \dashrightarrow |(C + \lceil \mathcal{A}_Y \rceil)_{|C}| = |K_C + \lceil -g_C^*(K + B) \rceil|$$

with $g_C\colon C \to X/T$. Indeed, here we can use Lemma 6.29 because C is reduced and $K + B_Y$ is integral on C (since $K + B$ is Cartier at Q). As above, by Lemma 6.30,

$$C + \lceil \mathcal{A}_Y \rceil \sim C + K_Y + \lceil -g^*(K + B) \rceil.$$

Thus, by Lemma 6.30, adjunction, and normal crossings on Y together with Addendum 6.29.1, we get

$$(C + \lceil \mathcal{A}_Y \rceil)_{|C} \sim (C + K_Y)_{|C} + \lceil -g_C^*(K+B) \rceil = K_C + \lceil -g_C^*(K+B) \rceil.$$

Now, as before, Kawamata–Viehweg vanishing gives $R^1 h_* \mathcal{O}_Y(\lceil \mathcal{A}_Y \rceil) = 0$.

The construction gives a Cartier divisor $L = (C + \lceil \mathcal{A}_Y \rceil)_{|C} \sim K_C + M/Z$ such that

- M is nef on C/Z;
- each $M_{|E_i} \sim 0$ and $\deg(K_C)_{|E_i} \geq 0$; and
- for each D_i, either $\deg M_{|D_i} \geq 1$ or D_i is not complete.

Indeed, $M = \lceil g^*(-K-B) \rceil_{|C} = \lceil -g_C^*(K+B) \rceil$ by the normal crossings and by Addendum 6.29.1, and we have

- $-(K+B)$ is nef on X/Z;
- each $K+B$ is Cartier in Q; and
- for each D_i, either $(K+B) \cdot D_i < 0$ or D_i is not complete.

Then $\operatorname{Bs}|L| = \varnothing$ near $\bigcup E_i$, and this implies that $|C + \lceil \mathcal{A}_Y \rceil|$ is free near $\bigcup E_i$ on Y/T. Indeed, L is nef/Q and, by the rationality of Q, $|L|$ is free over a neighborhood of Q (cf. Example 6.25). Thus, we only need to glue it with each $|L|_{|D_i}$. It is enough to have surjectivity $\mathcal{O}_{D_i}(L_{|D_i}) \to \mathcal{O}_{B_{D_i}}$, where, by adjunction (or the formula for canonical divisor),

$$L_{|D_i} \sim (K_C + M)_{|D_i} = K_{D_i} + B_{D_i} + M_{|D_i}$$

and B_{D_i} is the reduced boundary (gluing points). Surjectivity follows by Kodaira vanishing if D_i is complete, or for reasons of dimension otherwise. □

Question 6.33. State and prove a semi-log canonical version of this final *local* freedom result in higher dimensions (compare Fujita's conjecture).

Proof of Addendum 6.26.2. Since $K+B$ is Klt, $\lceil \mathcal{A}_Y \rceil \geq 0$. Thus, because D is b-free, in the proof of Proposition 6.26 we have an effective Cartier divisor $L \in |K_D + \lceil B' \rceil|$, this time with $\deg B' > 0$. Thus, L has at most a single base point Q of multiplicity 1 (by RR and [10, Riemann's formula]). Moreover, L has trivial mobile part (0) and has Q as a base point only if $K_D \equiv 0$ and $\deg L = 1$, that is, when D belongs to an elliptic pencil. Since $\deg L = 1$, we only need one blowup outside CS to resolve the base point, namely, the blowup in Q. □

Proposition 6.34. *Let $(X/Z, B)$ be a Klt pair. Then the following holds.*

(1) *For any crepant birational contraction $g\colon X \to Y/Z$, we have*

$$\mathfrak{S}_Y = g(\mathfrak{S}) = \{g(S), S \in \mathfrak{S}\} \supset \mathfrak{S}(Y/Z, g(B) = B_Y).$$

(2) (SAT) = (SAF) *in Conjecture* 6.14 *with the fixed $\mathcal{F}$.*

(3) (SAF) = (STD) *in Definition* 6.11 *for any $\mathcal{F} \geq 0$.*

(4) *The standard set $\mathfrak{S} = \mathfrak{S}(X/T, B)$ has the following equivalent description up to a component over the fractional part of B* (*see below*):

(Sa′F) $S = \operatorname{Supp}(\operatorname{Fix} \mathcal{D})_X$ *for some $\mathcal{D} \geq 0$ satisfying* (Sa′T) *of Addendum* 6.26.1; *and includes*

(SA′F) *a reduced divisor F is contained in the* substandard *set* $\mathfrak{S}' = \mathfrak{S}'(X/Z, B)$ *if* $\operatorname{Supp} F$ *lies only over integral components of B, and some b-free b-divisor $\mathcal{M}$ is saturated for $\mathcal{A}' + F$ on* some *sufficiently high model X_{hr}/X. $\mathcal{A}' = \mathcal{A} + \sum E_i$ with the exceptional divisors E_i on X over integral components of $\mathcal{A}$.*

Remarks 6.35. (1) *Up to a component over the fractional part of* B means that each $S \in \mathfrak{S}$ is $S' + F$, where S' under (Sa$'$F) and F is over the fractional part. Thus, the boundedness of the two sets are equivalent conditions.

(2) (SA$'$F) and (SA$'$T) may look more appealing than (Sa$'$F) and (Sa$'$T), respectively, because in them $\mathcal{A}'$ is universal, that is, independent of $\mathcal{D}$. However, in applications, we need (Sa$'$F) and (Sa$'$T), and if B in the proposition is a boundary, then they are equivalent to (SA$'$F) and (SA$'$T) (see the proof of Lemma 9.16 below).

(3) F is considered as a b-divisor.

Lemma 6.36. *Let* $\mathcal{C}$ *and* $\mathcal{M}$ *be b-divisors with* $\mathcal{M}$ *b-*$\mathbb{R}$*-Cartier. Then* $\mathcal{C}$*-saturation of* $\mathcal{M}$ *on* some *sufficiently high model of* X/Z *is equivalent to* $\mathcal{C}$*-saturation on* every *sufficiently high model.*

Proof. Indeed, $\operatorname{Mov}(\lceil \mathcal{M}+\mathcal{C}\rceil_Y)$ decreases for higher models X_{hr}/Y:

$$\operatorname{Mov}(\lceil \mathcal{M}+\mathcal{C}\rceil_{X_{\mathrm{hr}}})_Y \le \operatorname{Mov}(\lceil \mathcal{M}+\mathcal{C}\rceil_Y),$$

because $g_{\mathrm{hr}*}\colon \mathcal{O}_{X_{\mathrm{hr}}}(\lceil \mathcal{M}+\mathcal{C}\rceil_{X_{\mathrm{hr}}}) \subset g_*\mathcal{O}_Y(\lceil \mathcal{M}+\mathcal{C}\rceil_Y)$, where $g\colon Y\to Z$ and $g_{\mathrm{hr}}\colon X_{\mathrm{hr}}\to Z$. On the other hand, if Y/Z is a model of X/Z over which $\mathcal{M}=\overline{\mathcal{M}_Y}$, then, by Proposition 3.20, $g_*\mathcal{O}_Y(\mathcal{M}_Y)$ and $\operatorname{Mov}(\mathcal{M}_Y)$ are preserved on X_{hr}. Thus, X_{hr} satisfies saturation (cf. Remark 4.34(3)). $\square$

Proof of Proposition 6.34. (1) For each $s \in \mathfrak{S}(Y/Z, B_Y)$, the log transform S of s is an element of $\mathfrak{S}(X/Z, B)$. Thus, $s = g(S)$.

(2) By definition, $\operatorname{Mov}\lceil \mathcal{M}+\mathcal{F}+\mathcal{A}\rceil \le \mathcal{M}$. This means (SAF). Conversely, assuming Klt, we have

$$\lceil \mathcal{A}+\mathcal{F}\rceil \ge 0, \qquad \operatorname{Mov}\lceil \mathcal{M}+\mathcal{F}+\mathcal{A}\rceil \ge \mathcal{M},$$

and $=\mathcal{M}$ by (SAF). On the other hand, since $\mathcal{F}\ge 0$, $\mathcal{M}\le \mathcal{M}+\mathcal{F}$, and this gives (SAT) for $\mathcal{M}+\mathcal{F}$.

(3) $\mathcal{L} = \operatorname{Mov}\lceil \mathcal{A}+\mathcal{F}\rceil$. Again, $\lceil \mathcal{A}+\mathcal{F}\rceil \ge 0$ and $\mathcal{L}\ge 0$. Since $\mathcal{M}$ is b-free, by (SAF),

$$\mathcal{M} \ge \operatorname{Mov}\lceil \mathcal{M}+\mathcal{F}+\mathcal{A}\rceil \ge \operatorname{Mov}\mathcal{M} + \operatorname{Mov}\lceil \mathcal{F}+\mathcal{A}\rceil = \mathcal{M}+\mathcal{L},$$

and $\mathcal{L}\le 0$. Thus, $\mathcal{L}=0$, which means (STD) for $\mathcal{D}=\mathcal{F}$. Conversely, (STD) for $\mathcal{D}$ means (SAF) for $\mathcal{M}=0$ and $\mathcal{F}=\mathcal{D}$.

(4) If $S = \operatorname{Supp}\mathcal{D}_X \in \mathfrak{S}$, then, by Lemma 4.44, (STD) implies that 0 is saturated with respect to $\lceil S'+\mathcal{A}'\rceil \le \lceil \mathcal{A}+\mathcal{D}\rceil$, where S is fixed and S' is as in Remark 6.35(1). The inequality follows from (SEF) since $\operatorname{Bs}|S'|_X = S' \subset S$. Thus, (Sa$'$F) holds for $\mathcal{D}=S'$.

Conversely, if $\mathcal{D}$ satisfies (Sa$'$F), then, as in (2) and (3) above, $\mathcal{F}+\sum E_i + E'$ satisfies (STD) and (SEF), where $\mathcal{F} = \operatorname{Fix}\mathcal{D}$ and $E' = \sum \varepsilon_i E_i$ with $0<\varepsilon_i \ll 1$ for the exceptional divisors over the *fractional* components of $\mathcal{A}$. Note that E' does not affect (Sa$'$F) (cf. Addendum 4.44.1):

$$\lceil \mathcal{F}+\mathcal{A}+\sum E_i + E'\rceil = \lceil \mathcal{F}+\mathcal{A}'\rceil,$$

but (SEF) holds because $\mathcal{F}_X \subset \operatorname{Bs}|\mathcal{D}|_X$. Thus, $\operatorname{Supp}\mathcal{F}_X \in \mathfrak{S}$.

By Lemma 4.44, (SA$'$F) implies (Sa$'$F) for $\mathcal{D}=\mathcal{M}+F$, where $\operatorname{Fix}\mathcal{D}=F$ and $\operatorname{Supp}F = F\in\mathfrak{S}$. In (SA$'$F), we can replace "for some model" by "for every model" by Lemma 6.36 with $\mathcal{C}=\mathcal{A}'$. $\square$

Corollary 6.37. *Let* $(X/T, B)$ *be as in Proposition* 6.26, *and suppose that it is Klt* (*but* B *may only be a subboundary*). *Then the standard set* $\mathfrak{S}=\mathfrak{S}(X/Z,B)$ *of Definition* 6.11 *is bounded.*

In particular, $\mathfrak{S}'$ *is also bounded in* (SA$'$F).

If B *is a boundary,* $\mathfrak{S}$ *bounded is equivalent to* $\mathfrak{S}'$ *bounded* (*see Remarks* 6.35(1),(2)).

Proof. By Proposition 6.34(3),(4) and Remark 6.35(1), it is enough to establish boundedness for a *fixed reduced* $D = \sum D_i = \mathcal{D}_X = \mathcal{D}$ (that is, $\operatorname{Mov} D = 0$) that satisfies (Sa$'$F) and, in particular, (SA′F) with $\mathcal{L} = 0$ and $F = D$; we can also assume that each component $D_i \not\subset \operatorname{Supp} B$.

Since F is fixed and satisfies (Sa′F), by Lemma 4.44, each component D_i is $\mathcal{A}'$-saturated and even $\mathcal{A}$-saturated (cf. Proposition 6.34(3)). The same holds for any reduced $0 \le D' \le D$. In particular, these components are complete by Proposition 6.26 applied to a terminal resolution of $(X/T, B)$. Moreover, the prime components D_i belong to a bounded set. Thus, we need to verify that the number of components in D is bounded. It is enough to consider only the global case with $Z = \mathrm{pt}$.

Boundedness of the D_i implies that they have bounded intersections. Thus, any two components can be disconnected by a bounded number of resolutions. Suppose that the set of D is unbounded. We derive from this a contradiction with the boundedness of resolutions to disconnect two components D_i and D_j. (Equivalently, the intersection numbers $D_i \cdot D_j$ are bounded.)

Since the number of components is unbounded, there exists a set of divisors D containing unboundedly many algebraically equivalent irreducible components D_i; clearly, $D_i^2 \ge 0$, for otherwise there can only be one such D_i. If $D_i^2 = 0$, the divisors D_i are all fibres of a fibration $X \to C$ over a curve C. Then, for sufficiently many such components, there is an unbounded number of nonsingular fibres $0 < D' = \sum D_i \le D$ and $\sum D_i$ is free, which contradicts $\mathcal{A}$-saturation. Moreover, $D := D'$ is never sufficiently general on any model Y/Z of X/Z, that is, general in the free linear system $|D|$ on Y/Z. Note that two b-free divisors have the same 0 multiplicities in any fixed finite set of prime b-divisors. Indeed, then $K + B_Y + \overline{D}$ is again Klt. Hence, $\overline{D} \le \lceil D + \mathcal{A} \rceil$; this holds at each prime P_i on Y because D is integral and $\lceil \mathcal{A}_Y \rceil \ge 0$ by Klt. If P_i is exceptional on Y, each $\operatorname{mult}_{P_i}(\mathcal{A} - \overline{D}) > -1$ and so each $\operatorname{mult}_{P_i} \mathcal{A} > \operatorname{mult}_{P_i} \overline{D} - 1$. Taking $\lceil \cdot \rceil$, we get $\operatorname{mult}_{P_i} \lceil \mathcal{A} \rceil \ge \operatorname{mult}_{P_i} \overline{D}$ because D is Cartier on Y with only integral multiplicities. Thus, on any fixed sufficiently high model X_{hr}/Y, by the $\mathcal{A}$-saturation for $D := D'$ as a b-divisor, $D = D_{X_{\mathrm{hr}}} \ge \operatorname{Mov}\lceil D + \mathcal{A} \rceil_{X_{\mathrm{hr}}} \ge \overline{D}_{X_{\mathrm{hr}}}$, which contradicts the fact that D is fixed. (Compare the proof of Proposition 6.26.)

Hence, $D_i^2 > 0$ and the divisors D_i intersect each other. By Addendum 6.26.1, any of these D_i intersects any other $D_j \cong D_i$ at a finite number of points, bounded independent of D: namely, in $D_i \cap (\operatorname{Supp} B \cup \mathrm{CS})$, where CS denotes non-Gorenstein or noncanonical points. Therefore, an unbounded number of prime components D_i pass through the same point Q. This point depends on D. We can make a bounded resolution at such points Q, namely, the minimal resolution if Q is singular or the usual blowup if Q is nonsingular on X. Then we replace (X, B) by its crepant transform. The set of models depending on D can now be infinite.

The $\mathcal{A}$-saturation is birational, and the intersection points of D_i lie only over the log transform of $D_i \cap (\operatorname{Supp} B \cup \mathrm{CS})$. We consider the proper birational transform of D and of the prime components D_i. We again have an unbounded set of algebraically equivalent D_i, and again $D_i^2 > 0$. Again, we have some (new) point Q through which pass an unbounded number of components D_i. Indeed, the number of points in the intersection of D_i with the log transform of the above intersection is bounded and the bound is independent of D and the model. Again, we make a bounded resolution, and so on.

But this process must terminate since any two components are disconnected by a bounded resolution. Otherwise, we can see that the new self-intersection $D_i^2 \le$ the old self-intersection $D_i^2 - \mathrm{const}$, where const $= 1$ for the usual blowup of nonsingular points and some positive numbers at other points. $\square$

Example 6.38. Let $(X/\mathrm{pt.}, B = 0)$ be a (terminal) nonsingular del Pezzo surface. Then, in Proposition 6.26, every $D = \mathcal{D}_X$ is either free or is a -1-curve, except for del Pezzo surfaces of degree 1 with $D \in |-K|$ (see [16, Proposition 3.2.4]). The -1-curves D give fixed D of degree 1: $-K \cdot D = 1$. The standard sets are the disjoint sums of these -1-curves.

Then (FGA)$(X/\mathrm{pt.}, 0)$ gives the models $X_{\mathrm{st}} = X$, except for $= \mathbb{P}^1$ in the case of a pencil, and $=$ pt. (cf. Example 9.11 below).

Pencils and irreducibility in (RIR) of Proposition 6.26. There is no pencil, $\not\subset |C|$, of elliptic curves every element of which is tangent to the 1-complement boundary C, because the restriction of the pencil on C is an isomorphism for a general divisor. Thus, we have only a single element in the pencil with a single intersection (and tangent) point. However, for a higher genus, these are possible.

Example 6.39. For (CCS) on X/Z or on a triple Y/Z in the 3-fold or higher dimensional case (even the local case), we need the numerical condition that $\mathcal{M}_Y/Z$ is *nef* (cf. (NEF) in Remark 6.15(8)). In the one-dimensional case, it is implied because $\mathfrak{D}(X/Z,B)$ is effective in Conjecture 6.14; and in the two-dimensional case, because $\mathcal{M}$ is b-free or $|\mathcal{M}|_Y$ is free in codimension 1. But nef is important in dimension ≥ 3. Indeed, let $f\colon X \to Z$ be a small extremal contraction with Klt X that is negative with respect to D. Then any infinite set of b-divisors $\mathcal{D}$ of the form $\mathcal{D}_X = nD$ for a natural number n is not canonically confined, or equivalently, not log canonically confined. Otherwise, we have a very negative curve C contracted by f for $K+cD_X$ with C as a LCS (center of log canonical singularities), and log threshold c along C for all D_X. This is impossible by anticanonical boundedness [44, Theorem]. However, we can take another model Y/Z (for example, the D-flip) of X/Z, where $\{\mathcal{D}_Y\}$ is confined (if the flip conjecture holds).

Corollary 6.40. *Conjecture* 6.14 *holds in dimension* 2; *that is, for any weak log Fano contraction* $(X/Z,B)$ *with* $\dim X = 2$, *there exists a bounded family of desirable triples* $(Y/T/Z, B_Y, \mathfrak{F})$ *with induced standard* $\mathfrak{F} = \mathfrak{S}_Y$*—or with any other induced bounded and confined set of reduced divisors—such that, for each b-free* $\mathcal{M} \in \mathfrak{M}(X/Z,B)$, *the singularities of general* $\mathcal{M}_Y$ *are bounded* for some desirable Y/T. *In particular, they are canonically confined with respect to* B_Y.

Addendum 6.40.1. *In general, the only estimate is* $0 < c \leq \mathrm{mld}(X,B)$.

Addendum 6.40.2. $(\mathrm{CCS})_2^*$ *and* $(\mathrm{MOD})_2^*$ *hold. Moreover,*

- *it is enough to assume that the singularities of* $\mathcal{M} \in \mathfrak{M}(X/Z,B)$ *on the* $\mathrm{LCS}(X/Z,B)$ *are bounded*; *and*
- *a neighborhood of the* LCS *is preserved isomorphic on a desirable triple for* $\mathcal{M}$ *if* $\mathcal{M}$ *is Cartier over it, that is,* $\mathcal{M} = \overline{M}$ *for Cartier* M *in a neighborhood of the* LCS.

Then, in particular, (JAS) *and* (RST)$'$ *hold on the latter* (*desirable*) *triples for the characteristic system* $\mathcal{D}_\bullet$ *of any* $(\mathrm{FGA})_2^*(X/Z,B)$ *algebra.*

Remark 6.41. In applications, we can extend the induced standard set $\mathfrak{S}_Y$ to some $\mathfrak{F}$ by adding some fixed reduced divisors, e.g., $\operatorname{Supp} B_Y$, or even bounded sets (cf. the proof of Proposition 9.15 below). However, we then lose (SA$'$F), for example, on triples other than $(X/X,B,\mathfrak{S})$. Otherwise, to preserve (SA$'$F), we need to take $\mathfrak{S}_Y$ as an integral log transform: with $\sum E_i$, where E_i lie over integral components of $\mathcal{A}_Y$. Thus, we need (SA$'$F) only as a condition to establish that $\mathfrak{F}$ is bounded.

The integral condition on F in (SA$'$F) of Proposition 6.34 can be replaced by $\operatorname{Supp} F \cap \operatorname{Supp} B_Y = \varnothing$ when B_Y is a boundary and (X,B) is Klt.

Proof. We assume that the surface X is complete. This is a nontrivial assumption that is important in the applications below (cf. Theorem 9.9).

We associate desirable triples with respect to the bigness of $\mathcal{M}$, which proves (MOD). As in (CRP), each $(Y/T, B_Y)$ is a crepant model over (X,B) and satisfies (QFC). The corresponding set $\mathfrak{F}$ on Y is the log birational transform of the standard set $\mathfrak{S} = \mathfrak{S}(X,B)$. This is bounded by Corollary 6.37. We need to satisfy (TER) and on T (RPC). Choose a desirable triple *for each* $\mathcal{M} \in \mathfrak{M}(X,B)$ (see Definition 6.9).

Thus, for big $\mathcal{M}$, we take a terminal resolution $(Y/T, B_Y)$ as a desirable triple with the identity $Y/Y = T$. In particular, it is a weak log Fano contraction satisfying (TER) and (RPC). In addition,

$\mathcal{M}_Y = \mathrm{id}^*M$, where $M = \mathcal{M}_Y$ is big. Since $\dim X = 2$ and $\mathcal{M}$ is b-free, M is nef. On any model, all such sufficiently general $\mathcal{M}_Y$ are reduced and irreducible. Hence, by Proposition 6.26, all such sufficiently general divisors $\mathcal{M}_Y$ have bounded and confined singularities everywhere because (Y, B_Y) is terminal at points. We can take just one such triple.

The next case is when $\mathcal{M}$ gives a 1-dimensional image, that is, there exists a rational contraction $X \dashrightarrow C$ onto a curve C with $\mathcal{M}_X$ in the fibres. In the classical terminology, $|\mathcal{M}|_X$ is *composed of a pencil*. In this case, we define Y/T as the regular contraction defined by this pencil. In particular, $T = C$, and $\mathcal{M}_Y = g^*M$ for some divisor $M > 0$. Hence, M is nef and big, and, by construction, all such sufficiently general $\mathcal{M}_Y$ have bounded and confined singularities everywhere. This family of triples is bounded because the intersections of the elements of the pencil are bounded by Proposition 6.26. More precisely, by Addendum 6.26.2, every such pencil is elliptic and has a regularization in one blowup of a terminal resolution. In this case, $C = T = \mathbb{P}^1$ by the rational connectedness of X [46, следствие 6], satisfies (RPC), and (Y, B_Y) satisfies (TER).

Finally, suppose that $\dim |\mathcal{M}|_X = 0$, that is, the mobile divisor $\mathcal{M} = 0$. Take Y/T to be $X \to \mathrm{pt.}$; then $M = 0$. This triple is unique.

To prove Addendum 6.40.2, we can use the same arguments outside the LCS. Under the assumption that $\mathcal{M}$ is Cartier, $|\mathcal{M}|_X$ will be free in a neighborhood of the LCS, and we do not need additional blowups over the LCS to construct a desirable triple for $\mathcal{M}$. For example, it works for the mobile system of any algebra of $(\mathrm{FGA})_2^*$ type that gives the last statement under the choice of $\mathcal{C}$ and J of Example 5.25. $\square$

Proof of (FGA) in Main Theorem 1.7. Immediate by Theorem 6.19(3) and Corollary 6.40. The birational case was done in Example 6.25. $\square$

Corollary 6.42. *Each algebra of type* $(\mathrm{FGA})_2^*$ *is also f.g. Moreover, in Conjecture* 5.26, *we can assume that H is Cartier in a neighborhood of the* LCS.

Proof. Immediate by Addendum 6.19.1 and Addendum 6.40.2. $\square$

Corollary 6.43. *Any* $(\mathrm{FGA})_n(\mathrm{bir})$ *algebra* $\mathcal{L} = \mathcal{R}_{X/T}D$ *is f.g. up to codimension* 2 *over* T, that is, over points of codimension ≤ 2.

Proof. Immediate by $(\mathrm{FGA})_n$ with $n \leq 2$ in Main Theorem 1.7. $\square$

Corollary 6.44. $(\mathrm{FGA})_3^{\mathrm{pl}}$ *of Example* 4.52 *holds.*

Models of such algebras may give not a flip but a log quasiflip [47] by Remark 6.23. However, we still cannot drop the condition that $\mathcal{L}$ is bounded by D and satisfies Main Lemma 3.43 (cf. Remark 11.8).

Proof. Immediate by $(\mathrm{FGA})_d(\mathrm{bir})$ with $d \leq 2$, which was essentially done in Examples 6.25 and 4.52. $\square$

Of course, by Corollary 4.53, (CCS) also implies (RFA), but we can prove more.

Theorem 6.45. *Suppose LMMP and* (BP) *in dimensions* $d \leq n-1$, $(\mathrm{CCS})_{d-1}^*$, *and* $(\mathrm{SSB})_{d-1}(\mathrm{gl})$, *where* $*$ *means a modification in dimensions* $\leq d-1$ *including log singularities. Moreover, we can drop* $(\mathrm{CCS})_{d-1}^*$ *and* $(\mathrm{SSB})_{d-1}(\mathrm{gl})$ *with* $d \leq 3$, *that is, for* $n \leq 4$; *we can drop* (BP) *for* $n \leq 5$.

Then $(\mathrm{RFA})_{n,m}(\mathrm{bir})$ *holds with* $m \leq n-1$.

In fact, as one sees from the proof, we can replace condition $(\mathrm{CCS})_{d-1}^*$ by $(\mathrm{CCS})_{d-1}(\mathrm{gl})$, $(\mathrm{FGA})_{n-2}(\mathrm{bir})$, and $(\mathrm{FGA})_{d-1}^*(\mathrm{gl})$ (compare the proof of Corollary 10.14 below).

For 4-fold log flips, we need and essentially prove the theorem for $n = 4$, that is, $(\mathrm{RFA})_{4,m}(\mathrm{bir})$ proved in Section 11; the * modification of (CCS) is explained in Conjecture 6.14. The proof of the theorem is based on the stabilization of Theorem 9.9 in Section 9 and destabilization results in Section 10. However, before, we need to clarify restrictions of b-divisors in Section 7 and develop approximations in Section 8.

7. RESTRICTIONS OF b-DIVISORS

The new tools we introduce here are different flavors of b-restriction from X to a b-divisor E. The proof of Theorem 9.9 systematically uses restrictions of b-divisors (see also Proposition 4.50). There are essentially the two following types, or some mixture of them.

7.1. Mobile restriction. Let E be a prime b-divisor on X and $\mathcal{D}$ be a b-free divisor$/Z$. The *mobile restriction* of $\mathcal{D}$ to E is the b-free divisor $\mathcal{M} = \mathcal{D}_{\shortmid E}$ on E defined by the formula

$$\mathcal{M}_Y = \mathcal{D}_{Y|E},$$

where Y/X is a model such that E is a normal divisorial subvariety of Y and $\mathcal{D}$ is free over Y/Z. Thus, $\mathcal{M}_Y$ is also free over E/Z, and $\mathcal{M} = \overline{\mathcal{M}_Y}$. To cut out the restriction as a Weil divisor, it is enough to take a normal model Y/X on which E appears as a divisor and such that Y is nonsingular at every prime divisor of E. Similarly, we can define a Cartier b-divisor $\mathcal{D}_{\shortmid E}$ for any Cartier b-divisor $\mathcal{D}$ on X (compare (CAR) in the additional assumptions of 5.5). Thus, mobile restriction is well defined because the divisor has only 0 multiplicities up to linear equivalence. Mobile restriction is *defined up to linear equivalence.*

7.2. Fixed restriction. When we define the fixed restriction $D_{|E}$, we usually require D to be Cartier and $E \not\subset \operatorname{Supp} D$. Moreover, by Lemma 6.29, we can assume that D is K-Cartier (where $K = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$); then $D_{|E}$ is also K-Cartier and restriction preserves linear equivalence since it extends it. (Locally, each K-Cartier divisor is K-principal; cf. Definition 3.26. The restriction of any K-principal divisor is also K-principal.) In the same way, we can define the *fixed restriction* $\mathcal{D}_{\shortmid E}$ of a K-Cartier b-divisor $\mathcal{D}$ whenever $\operatorname{mult}_E \mathcal{D} = 0$. Again, we take

$$\mathcal{M}_Y = \mathcal{D}_{Y|E},$$

where Y/X is a model such that E is a normal divisorial subvariety of Y and $\mathcal{D}$ is K-Cartier$/Y$, that is, $\mathcal{D} = \overline{\mathcal{D}_Y}$. The b-divisor $\mathcal{M} = \overline{\mathcal{M}_Y}$ is well defined as a K-Cartier divisor$/E$. In this case, some multiplicities grow on subsequent blowups. This restriction also preserves *linear equivalence*. The same applies to restrictions to a subvariety $E \subset X$ such that E is not blown up on Y and $\mathcal{D} = 0$ over the generic points of E; e.g., under the latter assumption, such Y exists if $\mathcal{D}$ is b-semiample or even b-nef but X/Z is projective. We can also restrict any K-linear combination of such b-divisors.

In general, fixed restriction is well defined if it is independent of a sufficiently high resolution. For example, this holds for $\mathcal{A}'$ in the proof of Proposition 4.50, p. 130, or, equivalently, for the restriction of $(\mathcal{K} + E)_{\shortmid E}$ up to linear equivalence. This is the restriction in adjunction of a log canonical divisor to its log canonical center. Perhaps, up to multiplication, this is the only non-Cartier restriction of b-divisors.

7.3. Mixed restriction. If $\mathcal{D}$ is a K-Cartier b-divisor having $\operatorname{mult}_E \mathcal{D} \in K' \subset K$, we can define the *mixed restriction* $\mathcal{M} = \mathcal{D}_{\shortmid E}$ by setting

$$\mathcal{M}_Y = (\mathcal{D}_Y - (\operatorname{mult}_E \mathcal{D})E + (\operatorname{mult}_E \mathcal{D})E')_{|E},$$

where Y/X is a model such that E is a normal divisorial subvariety of Y, $\mathcal{D}$ is K-Cartier$/Y$, and $E' \sim E$ on Y, but $E \not\subset \operatorname{Supp} E'$. This is again a K-Cartier b-divisor.

In other words, if E is $\mathbb{Q}$-Cartier on Y and K has characteristic 0, we obtain

$$\mathcal{M} = (\mathcal{D} - (\operatorname{mult}_E \mathcal{D})\overline{E})_{\shortmid E} + (\operatorname{mult}_E \mathcal{D})\overline{E'}_{\shortmid E}.$$

Of course, this divisor depends on the choice of model Y/X and on the equivalence $E' \sim E$. Nonetheless, this restriction is compatible with *K'-linear equivalence*, in particular, with $\sim$ for $K' = \mathbb{Z}$, by Lemma 6.29 and because $E' \sim E$ implies $\overline{E'} \sim \overline{E}$.

Caution 7.4. In general, E' is not effective or prime. If Y is $\mathbb{Q}$-factorial, then E is automatically $\mathbb{Q}$-Cartier.

Where is the support of $\mathcal{D}\rceil_E$? For saturations, we need to take integral parts of fractional divisors and to estimate their fractional parts, in particular, for restrictions. For this, we now describe the support of restrictions on a case-by-case basis. In addition, for a birational contraction $f\colon X \to T$, we consider case (df) that behaves like a discrepancy: if $\mathcal{D}_1 = \mathcal{K}$ and $\mathcal{D}_2 = \overline{\mathcal{K}_T}$, then $(\mathcal{D}_1)_T = (\mathcal{D}_2)_T$.

Definition 7.5. A b-divisor $\mathcal{D}$ on X is K-Cartier at $p \in X$ if $\mathcal{D} = \overline{D'}$ locally near p, for some K-Cartier divisor D'. Note that in this case $D' = \mathcal{D}_X$ near p.

Example 7.6. Suppose that $\mathcal{D}$ is an *effective* b-free divisor/Z or a K-linear combination $\sum k_i\mathcal{D}_i$ of such (for example, a difference). We assume that the $\mathcal{D}_i$ are in general position on X, and in particular, $\operatorname{Supp}(\sum k_i\mathcal{D}_i)_X = \bigcup \operatorname{Supp}(\mathcal{D}_i)_X$. This condition is equivalent to requiring that $\mathcal{D}$ is a K-Cartier b-divisor up to $\sim$. Then $\mathcal{D}$ is *Cartier* outside $\operatorname{Supp}(\mathcal{D}_X)$ (a divisorial subvariety on X; cf. Lemma 10.9 below). Indeed, $\mathcal{D} \sim 0$ outside the base locus of the linear system of $|\mathcal{D}|_X$. The same holds for K-Cartier divisors $\mathcal{D}$ on X. They are K-Cartier and even *Cartier* outside $\operatorname{Supp}(\mathcal{D}_Y)$ on any model Y/Z of X/Z whenever D_Y has a presentation as above with $\mathcal{D}_i$ in general position on Y. Moreover, for the K-Cartier property, we can replace $\operatorname{Supp}\mathcal{D}_Y$ by the intersection

$$\bigcap \operatorname{Supp}(\mathcal{D}'_Y),$$

where $\mathcal{D}'$ runs through the b-divisors $\mathcal{D}' \sim_K \mathcal{D}$, at least locally/$Y$, and D' is a K-linear combination as above, also general on Y, although not necessarily effective.

For example, suppose that $X \to Y = \mathbb{P}^2$ is the blowup of $P \in \mathbb{P}^2$, with E the -1-curve over P, and let L and L_P be distinct lines, with $P \in L_P$. Then $\overline{E} = \mathcal{L} - \mathcal{L}_P$, where $\mathcal{L} = \overline{L}$ and $|\mathcal{L}_P| = |\mathcal{L} - P|$ are b-free. Thus, $\overline{E}$ is Cartier outside $\operatorname{Supp}((\mathcal{L} - \mathcal{L}_P)_Y) = L \cup L_P$. Note also that $\mathcal{L}_P \sim \mathcal{L} - \overline{E} \not\geq 0$ if $P \notin L$. However, $\mathcal{L}_P$ is not Cartier at P outside $\operatorname{Supp}((\mathcal{L} - \overline{E})_Y) = L$.

Proposition 7.7. *Let X be a model where E is a* (*normal*) *divisorial subvariety, and let $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_2$ be K-Cartier b-divisors that are K-Cartier outside* $\operatorname{Supp}((\mathcal{D}_*)_X)$. *Then there are two possible cases*[7] *for* $\operatorname{Supp}(\mathcal{D}\rceil_E)_E$:

(mv) *common* $\operatorname{Supp}(\mathcal{D}\rceil_E)_E = \varnothing$ *for $\mathcal{D}$ up to linear equivalence near each point of E if $\mathcal{D}$ is either Cartier or b-free on X*; *in other words, a Cartier b-divisor on X restricts to a Cartier b-divisor on E*;

(fx) $\operatorname{Supp}(\mathcal{D}\rceil_E)_E \subset E \cap \operatorname{Supp}\mathcal{D}_X$ *if* $\operatorname{mult}_E \mathcal{D} = 0$.

Suppose, in addition, that E is $\mathbb{Q}$-Cartier. Write $\mathcal{D}_*$ for any of $\mathcal{D}$ or $\mathcal{D}_i$, and set $\mathcal{D}'_* \sim \mathcal{D}_* - (\operatorname{mult}_E \mathcal{D}_*)\overline{E}$, so that $\operatorname{mult}_E \mathcal{D}'_* = 0$. Suppose that each $\mathcal{D}_* - (\operatorname{mult}_E \mathcal{D}_*)\overline{E}$ is K-Cartier outside $\operatorname{Supp}(\mathcal{D}'_*)_X$. *Then, up to linear equivalence, there are two possible cases for* $\operatorname{Supp}(\mathcal{D}\rceil_E)_E$:

(mx) $\operatorname{Supp}(\mathcal{D}\rceil_E)_E$ *is contained in the union of*

$$E \cap \operatorname{Supp}(\mathcal{D}_X - (\operatorname{mult}_E \mathcal{D})E) = E \cap \operatorname{Supp}\mathcal{D}'_X$$

and $E \cap \operatorname{Supp} E'$; *or*

(df) $\operatorname{Supp}(\mathcal{D}\rceil_E)_E$ *is contained in the union of*

$$E \cap \operatorname{Supp}((\mathcal{D}_1)_X - (\operatorname{mult}_E \mathcal{D}_1)E) = E \cap \operatorname{Supp}(\mathcal{D}'_1)_X$$

and

$$E \cap \operatorname{Supp}((\mathcal{D}_2)_X - (\operatorname{mult}_E \mathcal{D}_2)E) = E \cap \operatorname{Supp}(\mathcal{D}'_2)_X$$

for $\mathcal{D} := \mathcal{D}_1 - \mathcal{D}_2 - \operatorname{mult}_E(\mathcal{D}_1 - \mathcal{D}_2)\overline{E}$.

[7] Supp commutes with the trace on E.

Remark 7.8. If E is $\mathbb{Q}$-Cartier, by Example 7.6, the condition that the final b-divisor $\mathcal{D}$ is K-Cartier outside $\operatorname{Supp}\mathcal{D}$ holds if $\mathcal{D}$ is b-free or a K-linear combination of such b-divisors in general position on X.

Proof. (mv) and (fx) hold by definition (cf. Example 7.6). (fx) implies (mx) and (df). Indeed, $\mathcal{D} - (\operatorname{mult}_E \mathcal{D})\overline{E}$ is nontrivial and is not K-Cartier/E only over

$$E \cap \operatorname{Supp}(\mathcal{D} - (\operatorname{mult}_E \mathcal{D})\overline{E})_X = E \cap \operatorname{Supp}(\mathcal{D}_X - (\operatorname{mult}_E \mathcal{D})E),$$

that is, up to linear equivalence, only over $E \cap \operatorname{Supp}\mathcal{D}'_X$. In (df), we take $\mathcal{D}' = \mathcal{D}'_1 - \mathcal{D}'_2$. Note that E' is $\mathbb{Q}$-Cartier everywhere on X together with E. $\square$

Corollary 7.9. *Suppose that*

- K *is a field*;
- $a_2\mathcal{D}_2$ *is b-free/*Z *for some* $0 \neq a_2 \in K$; *and*
- X/T *is a birational contraction/*Z, *and* $(\mathcal{D}_1)_T = (\mathcal{D}_2)_T$ *on* T.

Then, up to linear equivalence, we can replace the union in (df) *by*

$$\Big(E \cap \operatorname{Supp}(\mathcal{D}_1)_X\Big) \cup \Big(E \cap \operatorname{Bs}|a_2\mathcal{D}_2|_X\Big) \cup \Big(E \cap \Big(\begin{matrix}\textit{exceptional divisors}\\ \textit{of } X/T \textit{ other than } E\end{matrix}\Big)\Big).$$

If, in addition,

- $a_1\mathcal{D}_1$ *is a sufficiently general effective divisor that is b-free/*Z *for some* $0 \neq a_1 \in K$,

then the first term in the union can be replaced by $E \cap \operatorname{Bs}|a_1\mathcal{D}_1|_X$. *In particular, for* $a_1 = 1$, *the* non-Cartier points *on* E *and the* fractional part *of* $(\mathcal{D}_{\rceil E})_E$ *are all contained in* $E \cap \operatorname{Bs}|\mathcal{D}_1|_X$, $E \cap \operatorname{Bs}|a_2\mathcal{D}_2|_X$, *and* $E \cap$ (*exceptional divisors of* X/T *other than* E).

In the local case over $P \in Z$ (*which is more general*), *modify the last assumptions to*

- $(\mathcal{D}_1)_T = (\mathcal{D}_2)_T$ *outside* $f^{-1}P$.

Then, up to linear equivalence, the last term in the union should be

$$E \cap (\textit{divisors of } X/P \textit{ other than } E).$$

Note that, in the proof below, we can assume that each $a_i\mathcal{D}_i$ is just b-free up to K-linear equivalence with the corresponding linear system instead of $|a_i\mathcal{D}_i|_X$.

Proof. $\mathcal{D}_1 - \mathcal{D}_2$ is exceptional/T (and in the local case, in divisors/P). The same holds over X up to exceptional divisors of X/T. More precisely, by (df) (with $\mathcal{D}'_1 = \mathcal{D}_1$ when $\mathcal{D}_1$ is a sufficiently general effective b-free divisor) and because $(\mathcal{D}_2)_X = (\mathcal{D}_1)_X$ up to exceptional divisors (and respectively, divisors/P), $\mathcal{D}_{\rceil E}$ is nontrivial up to $\sim$ only over $E \cap$the exceptional divisors ($\neq E$) of X/T (or respectively, $E \cap$ divisors/P), $E \cap \operatorname{Supp}(\mathcal{D}_1)_X$, or $E \cap \operatorname{Supp}(\mathcal{D}_2^{\mathrm{g}})_X$, where $\mathcal{D}_2^{\mathrm{g}} \sim a_2\mathcal{D}_2$ is sufficiently general effective. The case with $\operatorname{mult}_E \mathcal{D}_1 \neq 0$ is trivial: $E \subset \operatorname{Supp}(\mathcal{D}_1)_X$.

Indeed, $\mathcal{D}_X = 0$ near any point $Q \in E$ outside $E \cap$ exceptional divisors ($\neq E$) of X/T (and respectively, in $E \cap$ divisors of X/P) and $E \cap \operatorname{Supp}(\mathcal{D}_1)_X$. Hence, $(\mathcal{D}_2)_X = eE$ near Q with $e = \operatorname{mult}_E \mathcal{D}_2$. Since $a_2\mathcal{D}_2$ is b-free/X (near Q, in particular) and $a_2 \neq 0$, it follows that $\mathcal{D}_{\rceil E}$ is nontrivial over Q only if $\mathcal{D} \neq e\overline{E}$ or $a_2\mathcal{D} \neq a_2e\overline{E}$ near Q and, therefore, by Example 7.6, only if all $(D_2^{\mathrm{g}})_X$ pass through Q.

Taking the intersection of $E \cap \operatorname{Supp}(\mathcal{D}_2^{\mathrm{g}})_X$ for all effective $\mathcal{D}_2^{\mathrm{g}} \sim a_2\mathcal{D}_2$, we obtain that $Q \in E \cap \operatorname{Bs}(a_2\mathcal{D}_2)_X$ whenever $\mathcal{D}_{\rceil E}$ is nontrivial/Q. $\square$

8. APPROXIMATIONS

This section upgrades Section 5, especially Lemma 5.15 and Corollary 5.19, for application to 4-fold flips; the main addition to Section 5 is that approximations are not directed, and we are working with sets of b-divisors.

Definition 8.1. Let $F \subset X$ be a reduced divisor. A *set of* (*effectively*) *bounded semiample*/Z *divisors* (or a *bnd set*/Z) is defined to be a set $\mathfrak{N}_{\mathrm{bnd}}(F) = \mathfrak{N}_{\mathrm{bnd}}(X/Z, F)$ of $\mathbb{R}$-divisors of X that are semiample/Z and satisfy the following conditions:

(BFP) the *fractional part* of D is *bounded* by F: $D = D_{\mathrm{int}} + D_{\mathrm{fr}}$, where D_{int} is integral and $\operatorname{Supp} D_{\mathrm{fr}} \subset F$;

(CFG) $\mathfrak{N}_{\mathrm{bnd}}(F)$ has a set of generators that is *compact plus finite*: there is a compact rational polyhedron $\mathfrak{N}_{\mathrm{c}}$ and a (finite) set of integral divisors D_i in $\mathfrak{N}_{\mathrm{bnd}}(F)$ such that each $D \in \mathfrak{N}_{\mathrm{bnd}}(F)$ has a decomposition $D = D_{\mathrm{c}} + \sum n_i D_i$, or even the same conditions *up to linear equivalence*, where $D_{\mathrm{c}} \in \mathfrak{N}_{\mathrm{c}}$ and every $n_i \in \mathbb{N}$; and

(BND) as in Definition 5.17 for $\mathfrak{N}_{\mathrm{bnd}}(F)$, or equivalently, the same holds for $\mathfrak{N}_{\mathrm{c}}$.

Taking a bigger F, we can assume that the polyhedron of (CFG) is contained in the $\mathbb{R}$-vector space $\mathfrak{D}_F$ (for the notation, see Section 5, p. 141). (CFG) implies (BND) if $\{D_i\}$ is finite.

The *uniform neighborhood* of $\mathfrak{N}_{\mathrm{bnd}}(F)$ with *diameter* $\delta \in \mathbb{R}$ is the set of Weil $\mathbb{R}$-divisors

$$\mathfrak{U}_{\mathrm{bnd},\delta}(X/Z, F) = \mathfrak{U}_{\mathrm{bnd},\delta}(F) = \Big\{ D = \sum d_i P_i \;\Big|\; d_i \in \mathbb{R} \text{ and } \|D - \mathfrak{N}_{\mathrm{bnd}}(F)\| < \delta \Big\},$$

assuming that $\|\cdot\|$ is taken over $\operatorname{Supp}(D - \mathfrak{N}_{\mathrm{bnd}}(F)) \subset F$. In other words, the following two equivalent conditions hold:

(i) $D = D_{\mathrm{int}} + D_{\mathrm{fr}}$ such that D_{fr} is supported in F and there exists $D' = D_{\mathrm{int}} + D'_{\mathrm{fr}} \in \mathfrak{N}_{\mathrm{bnd}}(F)$ with $\|D - D'\| = \|D_{\mathrm{fr}} - D'_{\mathrm{fr}}\| < \delta$;

(ii) there exists $D' \in \mathfrak{N}_{\mathrm{bnd}}(F)$ such that $D - D'$ is supported in F and $\|D - D'\| < \delta$.

In particular, in either case $\|D - \mathfrak{N}_{\mathrm{bnd}}(F)\| \le \|D - D'\| < \delta$. (i) implies (ii) because $D - D' = D_{\mathrm{fr}} - D'_{\mathrm{fr}}$ is supported in F. (ii) implies (i) for $D_{\mathrm{fr}} = D - D' + D'_{\mathrm{fr}}$ because then $D = D_{\mathrm{fr}} + D' - D'_{\mathrm{fr}} = D_{\mathrm{fr}} + D_{\mathrm{int}}$ and $D - D' = D_{\mathrm{fr}} - D'_{\mathrm{fr}}$.

Examples 8.2. (1) Any rational polyhedral cone generated by a finite set of divisors D_i that are semiample/Z is a bnd cone/Z (cf. Example 5.18). We can assume that each D_i is integral. Then we can take

- F as the common support of all divisors D_i; and
- $\mathfrak{N}_{\mathrm{c}} = \{\sum r_i D_i \mid r_i \in [0,1]\}$.

(2) Let $X \to T/Z$ be a contraction. The most important example of a bnd set is the Abelian semigroup $\mathfrak{N}(F) = \mathfrak{N}(X/T/Z, F)$ of divisors D that are nef/Z and $\sim_{\mathbb{R}} 0/T$, and have fractional part bounded by F. (In particular, D is assumed to be an $\mathbb{R}$-Cartier divisor.) In this case, a *uniform* neighborhood $\mathfrak{U}_\delta(F) = \mathfrak{U}_\delta(X/T/Z, F)$ of $\mathfrak{N}(F)$ with *diameter* δ is the same as in Definition 8.1.

In fact, $\mathfrak{N}(F)$ is a bnd set, and is even semiample in several cases that, though special, are crucial for our purposes:

(WLF) $(X/Z, B)$ is a weak log Fano contraction for some boundary B and $T = X$ (see Proposition 4.42);

(0LP) $(X/Z, B)$ is a 0-log pair for some boundary B (cf. Remark 3.30(2)) and $T = Z$; and

(TRP) triples $(X/T/Z, B, \mathfrak{F})$ with $F \in \mathfrak{F}$ (see Definition 6.9).

In (0LP), the bound (BND) does not hold in general: torsion points of Abelian varieties contradict (BND). To satisfy (CFG) and (BND), we need to impose irregularity $= 0$, so that the Picard group is finitely generated. Thus, integral points of $\mathfrak{N}(X/Z/Z, F)$ are finitely generated up to linear equivalence, and $\mathfrak{N}_{\mathrm{c}}$ is a cube with bounded multiplicities as in (1) above. In particular, this is true if X/Z is a weak log Fano contraction. Moreover, the bnd property for (WLF) follows from (TRP) in the case of desirable triples (see Lemma 8.12(5) below).

Definition 8.3 (freedom with tolerance $\tau > 0$)**.** We fix a reduced divisor F; then $D \sim D_{\mathrm{bf}} + D_{\mathrm{fr}}$, where $\mathrm{Bs}\,|D_{\mathrm{bf}}| = \varnothing$. In particular, D_{bf} is Cartier and D_{fr} is supported in F with $\|D_{\mathrm{fr}}\| < \tau$ (cf. Lemma 5.20). We pronounce the linear system to be *free with tolerance* τ and write $\mathrm{Bs}\,|D| = \varnothing \mod \tau$.

(Integral) *nonvanishing with tolerance* $\tau > 0$ is similar: $D \sim D_{\mathrm{int}} + D_{\mathrm{fr}}$, where D_{int} is effective integral and D_{fr} is supported in F with $\|D_{\mathrm{fr}}\| < \tau$. We pronounce the linear system to be *nonempty with tolerance* τ and write $|D| \neq \varnothing \mod \tau$.

The main result of the section is

Theorem 8.4. *Suppose that we have*

- *a bounded family $(X_u/Z_u, F_u)$ for $u \in U$ of pairs with reduced divisor F_u on X_u;*
- *a* family $\mathfrak{N}_{\mathrm{bnd}}(X_u/Z_u, F_u)$ *of bnd sets/U of divisors* (see (7) and (7′) in the proof below); *and*
- *a tolerance $\tau > 0$.*

Then there exists $M > 0$ (depending on the family $(X_u/Z_u, F_u)$ and on τ) such that, for any $u \in U$ and any $D \in \mathfrak{N}_{\mathrm{bnd}}(X_u/Z_u, F_u)$, there exists an integer $m \in [1, M]$ (depending on D) with $\mathrm{Bs}\,|mD| = \varnothing \mod \tau$; *in particular,* $|mD| \neq \varnothing \mod \tau$.

Addendum 8.4.1. *Moreover, there is a real number $\delta > 0$ (depending on the family $(X_u/Z_u, F)$ and τ) such that the same freedom and nonvanishing hold for any $u \in U$ and any $D \in \mathfrak{U}_{\mathrm{bnd},\delta}(X_u/Z_u, F_u)$.*

Remark 8.5. Both choices of M and δ are very ineffective, as Dirichlet's theorem on simultaneous Diophantine approximation [6, теорема 1.1].

We start with properties of freedom with tolerance.

Proposition 8.6. (1) *Freedom with tolerance τ is invariant under linear equivalence*:

$$\textit{if } D \sim D', \textit{ then} \quad \mathrm{Bs}\,|D| = \varnothing \mod \tau \ \Rightarrow\ \mathrm{Bs}\,|D'| = \varnothing \mod \tau.$$

(2) *Freedom with tolerance is an* open property: *if* $\mathrm{Bs}\,|D| = \varnothing \mod \tau$, *then there is an $\varepsilon > 0$ such that, for any D' with $D - D'$ supported in F and $\|D - D'\| < \varepsilon$, also* $\mathrm{Bs}\,|D'| = \varnothing \mod \tau$.

(3) *If $D = D_{\mathrm{bf}} + D'$, where* $\mathrm{Bs}\,|D_{\mathrm{bf}}| = \varnothing$ *and* $\mathrm{Bs}\,|D'| = \varnothing \mod \tau$, *then* $\mathrm{Bs}\,|D| = \varnothing \mod \tau$.

The same holds for nonvanishing with tolerance τ.

Proof. (1) holds by definition. In (2), we can take any $0 < \varepsilon \le \tau - \|D_{\mathrm{fr}}\|$, where $D \sim D_{\mathrm{bf}} + D_{\mathrm{fr}}$ as in Definition 8.3 and $D' \sim D_{\mathrm{bf}} + D'_{\mathrm{fr}}$. (3) holds because the sum of two free linear systems is also free. $\square$

Proof of Theorem 8.4. Let $(X/Z, F)$ be a member of the family, possibly nonclosed (that is, the generic member of an irreducible subfamily). Let $F = \sum P_i$, and write $\mathfrak{D}_F = \{\sum d_i P_i \mid d_i \in \mathbb{R}\}$ for the $\mathbb{R}$-vector space of divisors generated over F.

For $\mathfrak{N}_{\mathrm{bnd}}(F)$, suppose that we have a compact subspace $\mathfrak{N}_{\mathrm{c}} \subset \mathfrak{N}_{\mathrm{bnd}}(F)$, a (finite) set of Cartier divisors $T_i \in \mathfrak{N}_{\mathrm{bnd}}(F)$, and a finite set of $\mathbb{Q}$-Cartier divisors $N_i \in \mathfrak{N}_{\mathrm{c}}$ with open neighborhoods U_i of $N_i \in \mathfrak{D}_F$, such that

(1) each T_i is free/Z;

(2) each N_i has a *freedom index* $n_i > 0$ for which $\mathrm{Bs}\,|n_i N_i| = \varnothing$;

(3) in addition, for each $D \in U_i$, $\operatorname{Bs}|n_iD| = \varnothing \mod \tau$; more precisely, $n_iD = n_iN_i + D_{\mathrm{fr}}$ with D_{fr} supported in F with $\|D_{\mathrm{fr}}\| < \tau$;
(4) $\mathfrak{N}_{\mathrm{c}}$ is a compact rational polyhedron with generators N_i;
(5) each $D \in \mathfrak{N}_{\mathrm{bnd}}(F)$ decomposes as $D \sim D_{\mathrm{c}} + \sum t_iT_i$ with natural numbers t_i and $D_{\mathrm{c}} \in \mathfrak{N}_{\mathrm{c}}$; and, in turn,
(6) $\mathfrak{N}_{\mathrm{c}} \subset \bigcup U_i$; and
(7) each T_i is defined and is free/Z_u for each specialization of $(X/Z, F)$ and for any general point of $(X/Z, F)$. That is, for every u in some nonempty Zariski open subset U, $\mathfrak{N}_{\mathrm{bnd}}(F)$ specializes isomorphically to $\mathfrak{N}_{\mathrm{bnd}}(F_u)$ preserving the structures T_i, N_i, and $\mathfrak{N}_{\mathrm{c}}$ in (1)–(4) above.

Then Theorem 8.4 holds for $\mathfrak{N}_{\mathrm{bnd}}(F)$, and there exists $\delta > 0$ such that Addendum 8.4.1 holds for $\mathfrak{U}_{\mathrm{bnd},\delta}(F)$. Moreover, the theorem and addendum both hold for every u in a nonempty $V \subset U$. In particular, we can use Noetherian induction in the proof of the theorem and addendum. That is, it is enough to check both on a single member $(X/Z, F)$ of the family. Thus, if $M_V = M_F$ and $\delta_V = \delta_F$ give the theorem and addendum at $(X/Z, F)$, and $M_{(X/Z,F)\setminus V}$ and $\delta_{(X/Z,F)\setminus V}$ over the closed proper subfamily $(X/Z, F)\setminus V$, then both hold over all points (= specializations) of $(X/Z, F)$ with $M = \max\{M_V, M_{(X/Z,F)\setminus V}\}$ and $\delta = \min\{\delta_V, \delta_{(X/Z,F)\setminus V}\}$.

Now (1)–(6) hold over some V. (1) holds by our assumption in (7). (2) and (5) define a nontrivial affine open subset V over which all n_iN_i are free, and restriction (= specialization) of linear equivalence on X_u/Z_u is well defined (= in general position; cf. (GNP) of Proposition 4.50). For specialization of linear equivalence to be well defined, it is enough to consider X_u/Z_u that are smooth at the generic point of X_u (= multiplicity 1), or just on the moduli space by Lemma 6.29. Thus, (2) and (5) hold in each $u \in V$. The inequality in (3), the generators (4), and the inclusion in (6) concern multiplicities of divisors P_i and are uniform over the connected component of F in U because we assume that F_U is reduced everywhere over U by (7′) below. This establishes the Noetherian induction by (7).

Now, for a single fixed $(X/Z, F)$, we derive the theorem and addendum from (1)–(6) and then check these together with (7). We take $M_F = \max\{n_i\}$ for the indexes in (2); δ_F is chosen later. Indeed, Proposition 8.6(1),(3) and the above (1) and (5) imply that it is enough to check the theorem and addendum, respectively, for $\mathfrak{N}_{\mathrm{c}}$ and for

$$\mathfrak{U}_{\mathrm{c},\delta} = \{D \mid D - D' \in \mathfrak{D}_F,\ D' \in \mathfrak{N}_{\mathrm{c}},\ \text{and}\ \|D - \mathfrak{N}_{\mathrm{c}}\| \le \|D - D'\| < \delta\}.$$

In turn, both the theorem and addendum follow from (2), (3), and (6). For the addendum, we need to choose $\delta = \delta_F$ such that $\mathfrak{U}_{\mathrm{c},\delta_F} \subset \bigcup U_i$; such a δ_F exists since $\bigcup U_i$ is open.

Finally, we check (1)–(7). Note for this that the boundedness of the family means the following:

(7′) on the total space of the family, there are a reduced divisor F, integral semiample divisors D_i, and semiample $\mathbb{Q}$-divisors V_i such that, for each u,
- $F_u = F_{|X_u}$ is reduced;
- $V_{i|X_u}$ are vertices of $\mathfrak{N}_{\mathrm{c}}(F_u)$ (= a simplicial structure with the vertexes);
- $D_{i|X_u}$ and $\mathfrak{N}_{\mathrm{c}}(F_u)$ generate $\mathfrak{N}_{\mathrm{bnd}}(F_u)$ as in (CFG); and, finally,
- (BND) holds *uniformly* for the family.

(BND) means that there is natural number $I > 0$ such that, for any semiample integral divisor D, $\operatorname{Bs}|ID| = \varnothing$. To satisfy (1) and (7), we take $T_i = ID_i$. We set $\mathfrak{N}_{\mathrm{c}} = I\mathfrak{N}_{\mathrm{c}}(F)$ and $N_i = IV_{i|X}$. This implies (2) with some n_i for each N_i, also by (BND). More precisely, $n_i \le Id_i$, where d_i is the minimal natural number such that d_iN_i is integral. Thus, (BFP) and Proposition 8.6(2),(3) imply (3) for some U_i. By definition, we can take U_i as an open disc with center N_i in a rational translate of $\mathfrak{D}_F$ and radius $r_i = \tau/n_i \ge \tau/Id_i$.

(4) and (7) hold by (7′), with U as the normal points of the parameter space for the subfamily given by the member $(X/Z, F)$. (5) holds by (CFG) for $(X/Z, F)$ in (7′).

To satisfy (6), we need to add $\mathbb{Q}$-divisors $N_i \in \mathfrak{N}_{\mathrm{c}}$ that do not affect (4). Since each such N_i is semiample, as above we can find n_i and U_i for (2) and (3). If it is an open covering, then, by the compactness of $\mathfrak{N}_{\mathrm{c}}$, we can take a finite subset of N_i that satisfies (6) and still (2)–(4). Actually, the discs form a covering. To verify this, we can assume that $\mathfrak{N}_{\mathrm{c}}$ is a simplex, and all its faces are covered by induction. Indeed, the $\mathbb{Q}$-points of the rational simplex $\mathfrak{N}_{\mathrm{c}}$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors, so these are covered.

On the other hand, by simultaneous approximation, Cassels [7, Ch. I, Theorem VII], for any $\varepsilon > 0$, each point of the affine span $\mathfrak{L}$ of $\mathfrak{N}_{\mathrm{c}}$, in particular, each point $D \in \mathfrak{N}_{\mathrm{c}}$, has a rational approximation N_i in $\mathfrak{L}$ such that $\|D - N_i\| < \varepsilon/d_i$. Hence, $N_i \in \mathfrak{N}_{\mathrm{c}}$ and $D \in U_i$, respectively, whenever $\varepsilon/d_i \le \varepsilon <$ the distance from $D \in \mathfrak{N}_{\mathrm{c}}$ to the complement $\mathfrak{L} \setminus \mathfrak{N}_{\mathrm{c}}$ and $\varepsilon < \tau/I$. Thus, for $\varepsilon \ll 1$, the internal points of $\mathfrak{N}_{\mathrm{c}}$ that are outside the covering of the faces are covered by U_i. This completes the induction and the proof of the theorem. □

Freedom with tolerance sometimes implies usual nonvanishing, but this is typically still a far cry from freedom.

Corollary 8.7. *Let C_u be a divisor on X_u with $\operatorname{mult}_{P_i} C_u > -1 + \tau$ for all prime divisors P_i of X. Then, for any $D \in \mathfrak{U}_{\mathrm{bnd},\delta}(F_u)$, there exists some integer $m \in [1, M]$ such that $|\lceil mD + C_u \rceil| \ne \varnothing$.*

Addendum 8.7.1. *If $\operatorname{mult}_{P_i} F_u = 0$, we can assume just $\operatorname{mult}_{P_i} C_u > -1$.*

Proof (compare the proof of Lemma 5.20)**.** In the proof below, "all ..." means for all prime divisors P_i of X. We drop the subscripts u.

Take m in Addendum 8.4.1 such that $|mD| \ne \varnothing \mod \tau$, that is, $mD \sim D_{\mathrm{int}} + D_{\mathrm{fr}}$, where the integral part $D_{\mathrm{int}} \ge 0$ and the fractional part $D_{\mathrm{fr}} = \sum d_{\mathrm{fr},i} P_i$ with all $|d_{\mathrm{fr},i}| < \tau$, in particular, $d_{\mathrm{fr},i} > -\tau$. Hence,

$$\operatorname{mult}_{P_i}(D_{\mathrm{fr}} + C) = d_{\mathrm{fr},i} + \operatorname{mult}_{P_i} C > \operatorname{mult}_{P_i} C - \tau > -1$$

and all $\operatorname{mult}_{P_i} \lceil D_{\mathrm{fr}} + C \rceil = \lceil \operatorname{mult}_{P_i}(D_{\mathrm{fr}} + C) \rceil \ge 0$. Thus, $\lceil D_{\mathrm{fr}} + C \rceil \ge 0$ and $\lceil D_{\mathrm{int}} + D_{\mathrm{fr}} + C \rceil = D_{\mathrm{int}} + \lceil D_{\mathrm{fr}} + C \rceil \ge 0$. This proves the required nonvanishing: $\lceil mD + C \rceil \sim \lceil D_{\mathrm{int}} + D_{\mathrm{fr}} + C \rceil \ge 0$ by Lemma 6.30 since $mD + C \sim D_{\mathrm{int}} + D_{\mathrm{fr}} + C$.

Since $d_{\mathrm{fr},i} = 0$ when $\operatorname{mult}_{P_i} F = 0$, Addendum 8.7.1 follows because then again

$$\operatorname{mult}_{P_i}(D_{\mathrm{fr}} + C) = d_{\mathrm{fr},i} + \operatorname{mult}_{D_i} C > 0 - 1 > -1. \qquad \square$$

Example 8.8. We can use Corollary 8.7 as a restriction on the values of τ for which the above nonvanishing holds:

(1) if $C \ge 0$, for any $\tau < 1$;

(2) if $C = \mathcal{A}_E$ is the discrepancy divisor, then for all

$$\tau < \min\{1 - b_i\}, \qquad \operatorname{mld}(X, B) \le \min\{1 - b_i\}.$$

In particular, there exists $\tau > 0$ provided that all $b_i < 1$, that is, (X, B) is Klt in divisors.

Corollary 8.9. *Suppose that each F_u is $\mathbb{Q}$-Cartier on X_u. Let $\alpha < 1$ and τ be positive real numbers, and $\mathcal{C}_u$ be an $\mathbb{R}$-b-divisor of X_u with $\mathcal{C}_u \ge 0/X_u$ (see* (EEF) *in* 5.9) *and*

$$\operatorname{mult}_{P_i} \alpha\mathcal{C}_u > -\alpha + \tau \operatorname{mult}_{P_i} F_u \qquad \textit{for all } P_i. \tag{8.9.1}$$

Then there exist positive real numbers M and β, δ giving the following nonvanishing/Z_u on any Y_u/X_u uniformly in u: for any u and for all b-divisors $\mathcal{D}$ such that the descent data $\mathcal{E}$ of $\mathcal{D}$ over X_u is confined by $\beta\mathcal{C}_u$, that is, $\mathcal{E} \le \beta\mathcal{C}_u/X_u$, and $\mathcal{D}_{X_u} \in \mathfrak{U}_{\mathrm{bnd},\delta}(F_u)$, we have

$$|\lceil m\mathcal{D}_{Y_u} + (\mathcal{C}_u)_{Y_u} \rceil| \ne \varnothing \qquad \textit{for some integer } m \in [1, M].$$

In particular, we assume that $\mathcal{D}_{X_u}$ is $\mathbb{R}$-Cartier if the descent data of $\mathcal{D}$ is confined (see Definition 5.7).

Proof. Again, we drop the subscripts u. By Addendum 8.4.1, we can choose M and $\delta > 0$, depending on the family $(X/Z, F)$ and τ. Now take any $\beta \in \mathbb{R}$ with

$$0 < \beta \leq \frac{1-\alpha}{M}.$$

Then take any b-divisor $\mathcal{D}$ that satisfies the nonvanishing conditions, and let $m \in [1, M]$ be an integer for which $\operatorname{Bs}|m\mathcal{D}_X| = \varnothing \mod \tau$. We check that $|\lceil m\mathcal{D}_Y + \mathcal{C}_Y \rceil| \neq \varnothing$ for arbitrary Y/X; this follows from Lemma 5.20 with $\mathcal{D} := m\mathcal{D}$. (5.20.1) follows from (8.9.1). Assumption (5.20.2) follows from confinement of the descent data for the original $\mathcal{D}$. Since $\mathcal{C} \geq 0/X$ and by (HOM) of Proposition 5.4, for any integer $m \in [1, M]$ (or even for any nonnegative real number), the descent data $\mathcal{E}$ of $\mathcal{D} := m\mathcal{D}$ is confined by $m\beta\mathcal{C} \leq M\beta\mathcal{C} \leq (1-\alpha)\mathcal{C}/X$, that is, $\mathcal{E} \leq (1-\alpha)\mathcal{C}/X$. □

Remark 8.10. For a given family $(X_u/Z_u, F_u)$ with $\mathcal{C}_u$, we find constants α, β, δ, τ, and M in the following order. First, we take any positive $\alpha < 1$. Next, we choose an appropriate positive τ (cf. Example 8.21 below). Next, by Theorem 8.4 and Addendum 8.4.1, we can find δ and M. Finally, we take β as in the preceding proof. Note that, to find τ, we need to assume that each F_u is $\mathbb{Q}$-Cartier. For this, we usually assume that X is $\mathbb{Q}$-factorial. In applications, we only need β, δ, and $M > 0$, and so can drop α and τ (cf. Corollary 8.26 below).

Remark 8.11. If $\mathcal{D}_X$ is $\mathbb{R}$-Cartier, the descent data $\mathcal{E}$ exists by (EXI) in Proposition 5.4; hence, it exists for all $\mathcal{D}$ if X is $\mathbb{Q}$-factorial.

Now we apply the results obtained above to triples $(X/T/Z, B, \mathfrak{F})$ (see Definition 6.9 and cf. Corollary 6.40). A *bounded* family of triples is a family with bounded moduli; for the details, see the proof of Theorem 8.15 below.

The next preliminary result explains also the role of conditions (CRP) and (RPC) in Definition 6.9 (cf. Example 8.18 below). We first fix notation: write

$$\operatorname{CDiv}^0_{\mathbb{R}}(X/T/Z) = \{D \in \operatorname{CDiv}_{\mathbb{R}}(X/Z) \mid D \sim_{\mathbb{R}} 0/T\}.$$

Equivalently, every such divisor $D \sim_{\mathbb{R}} g^*M/Z$ for some $\mathbb{R}$-Cartier divisor $M \in \operatorname{CDiv}_{\mathbb{R}}(T/Z)$ is defined up to $\sim_{\mathbb{R}}/Z$ (see Lemma 3.28); sometimes, $\mathbb{R}$-linear equivalence $\sim_{\mathbb{R}}/Z$ and numerical equivalence $\equiv/Z$ are the same on T/Z (cf. Lemma 8.12(1) below). Thus, we have an $\mathbb{R}$-linear projection

$$g_*\colon \operatorname{CDiv}^0_{\mathbb{R}}(X/T/Z) \to \operatorname{CDiv}_{\mathbb{R}}(T/Z)/\equiv.$$

This map is defined over $\mathbb{Q}$, but not over $\mathbb{Z}$ in general, since there may be multiple fibres. Thus, on each $\mathbb{R}$-linear subspace $L \subset \operatorname{CDiv}^0_{\mathbb{R}}(X/T/Z)$, we have an *induced* splitting

$$L = L^0 \oplus L^1,$$

where $L^0 = \ker g_{*|_L}$ and L^1 is isomorphic to g_*L. Note that L^0 is a subspace of $\ker g_* = \operatorname{CDiv}^0_{\mathbb{R}}(X/Z/Z)$. Although L^1 is not uniquely determined, we use it to identify the second summand and write g_*L for it. If L is defined/$\mathbb{Q}$, the splitting can also be defined over $\mathbb{Q}$. For example, if

$$L = \mathfrak{C} = \mathfrak{C}(X/T/Z, F) = \Big\{D = \sum d_i P_i \;\Big|\; d_i \in \mathbb{R} \text{ and } D \sim_{\mathbb{R}} 0/T\Big\} \subset \mathfrak{D}_F$$

with $F = \sum P_i$, we obtain a splitting $\mathfrak{C} = \mathfrak{C}^0 \oplus g_*\mathfrak{C}$, where

$$\mathfrak{C}^0 = \mathfrak{C}^0(X/T/Z, F) = \Big\{D = \sum d_i P_i \;\Big|\; d_i \in \mathbb{R} \text{ and } D \sim_{\mathbb{R}} 0/Z\Big\}.$$

Lemma 8.12. *Let $(X/T/Z, B, \mathfrak{F})$ be a desirable triple for a weak log Fano contraction and $F = \sum P_i \in \mathfrak{F}$ be a reduced divisor. Then*

(1) *g_* induces a splitting $\mathfrak{C} = \mathfrak{C}^0 \oplus g_*\mathfrak{C}$ over $\mathbb{Q}$, where now*

$$\mathfrak{C}^0 = \mathfrak{C}^0(X/T/Z, F) = \Big\{ D = \sum d_i P_i \;\Big|\; d_i \in \mathbb{R} \text{ and } D \equiv 0/Z \Big\};$$

(2) *Cone theorem: the nef cone or even the* semiample cone

$$\mathfrak{N}_{\mathrm{nef}} = \mathfrak{N}_{\mathrm{nef}}(X/T/Z, F) = \Big\{ D = \sum d_i P_i \;\Big|\; d_i \in \mathbb{R},\ D \text{ is nef}/Z \text{ and } \sim_{\mathbb{R}} 0/T \Big\}$$

is a rational polyhedral cone in $\mathfrak{C}$;

(3) *$g_*\mathfrak{C}$ is isomorphic/$\mathbb{Q}$ to $\mathrm{CDiv}_{\mathbb{R}}(T/Z)/\equiv$ whenever the components P_i generate the group of Weil divisors of X up to linear equivalence;*

(4) *there exists a uniform constant I, the* freedom index, *a natural number such that the multiple ID is free on X/Z for every integral $\mathbb{Q}$-Cartier $D \in \mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z)$ that is nef/Z;*

(5) *$\mathfrak{N}(X/T/Z, F) = \mathfrak{N}_{\mathrm{nef}}(X/T/Z, F)$ is a bnd set (cf.* (TRP) *in Example* 8.2(2)).

Proof. (1) Since the triple is a desirable triple for a weak log Fano contraction $(X'/Z, B')$, we have a diagram of contractions

$$\begin{tikzcd}
 & W \arrow[dl, "i"'] \arrow[dr, "h"] & & \\
X' & & X \arrow[r, "g"] & T/Z
\end{tikzcd}$$

where $i \circ h^{-1} \colon X \to X'$ is a birational map to the weak log Fano contraction X'/Z. We need to prove that each $\mathbb{R}$-Cartier divisor $D \equiv 0$ on X/Z is $\sim_{\mathbb{R}} 0/Z$. It is enough to check this for h^*D on W/Z. Since X' has rational singularities, the problem descends to X'/Z up to $\sim_{\mathbb{R}}$ (see [33, Corollary 1.3]). However, each $\mathbb{R}$-Cartier divisor $\equiv 0/Z$ is $\sim_{\mathbb{R}} 0/Z$ on any weak log Fano contraction/Z.

In addition, if D is a Cartier divisor and $D \equiv 0$, then $D \sim 0$ essentially by Contraction Theorem 2.1 in [40].

Remark 8.13. Apart from the final statement, the above holds if we assume only that X' has rational singularities and irregularity $0/Z$.

(2) follows from the numerical description of the splitting in (1):

$$\mathfrak{N}_{\mathrm{nef}} = \mathfrak{C}^0 \oplus g_*\mathfrak{N}_{\mathrm{nef}}.$$

By (1) and (RPC), this means that each nef/Z divisor in $\mathrm{CDiv}^0_{\mathbb{R}}(X/T/Z)$ is also semiample/Z, that is, the cone is also semiample/$\sim_{\mathbb{R}}$. Since

$$g_*\mathfrak{N}_{\mathrm{nef}} = g_*\mathfrak{C} \cap \text{nef cone in } \mathrm{CDiv}_{\mathbb{R}}(T/Z)/\equiv,$$

the nef cone $g_*\mathfrak{N}_{\mathrm{nef}}$ is rational polyhedral by (RPC) again. Hence, $\mathfrak{N}_{\mathrm{nef}}$ is also rational polyhedral.

(3) follows from the surjectivity of the projection g_* since the integral Cartier divisors are also generated by the components P_i.

To prove (4), as in (3), we can assume that an integral Weil divisor D/Z that is nef and $\mathbb{Q}$-Cartier is contained in $\mathfrak{C}$ up to linear equivalence. By (2), we can decompose D into $D_0 + D_1$, where $D_0 \in \mathfrak{C}^0$ and $D_1 \in g_*\mathfrak{N}_{\mathrm{nef}}$. This is a decomposition/$\mathbb{Q}$ but with bounded denominators, which we include in I; thus, we can assume that both D_0 and D_1 are integral. Moreover, (4) for D_1 follows from a similar decomposition by the cone property (RPC). Now D_0 is free provided it is Cartier, as remarked in the proof of (1). Hence, we can take I as the Cartier index for D_0, so that ID_0 is Cartier. This exists because $\mathfrak{C}^0$ is finite-dimensional.

(5) Each divisor in $\mathfrak{N}(X/T/Z,F)$ is semiample by (2). (BND) follows from (4). If F is big as in (3), (CFG) follows from (2) and (3) as in Example 8.2(1). For any subset of F, we need to cut $\mathfrak{N}_{\mathrm{c}}$ by equations $\operatorname{mult}_{P_i} D = j$ for $j \in \mathbb{Z}$, to drop P_i and keep integral generators D_i. In each compact rational polyhedron $\mathfrak{N}_{\mathrm{c}}$, this also cuts out a rational compact polyhedron. $\square$

Notation 8.14. For a triple $(X/Z, B, \mathfrak{F})$ with a bounded family $\mathfrak{F}$, we set

$$\mathfrak{N} = \mathfrak{N}(X/T/Z, \mathfrak{F}) = \bigcup_{F \in \mathfrak{F}} \mathfrak{N}(F)$$

(cf. Example 8.2(2)), and $\mathfrak{N} = \mathfrak{N}(X/Z, \mathfrak{F}) = \mathfrak{N}(X/X/Z, \mathfrak{F})$ if $T = X$ (cf. Theorem 8.23). In other words, each $D \in \mathfrak{N}$ is nef$/Z$, $\sim_{\mathbb{R}} 0/T$, and $D = D_{\mathrm{int}} + D_{\mathrm{fr}}$, where D_{int} is integral and $\operatorname{Supp} D_{\mathrm{fr}}$ is in $\mathfrak{F}$. We set

$$\mathfrak{U}_\delta = \mathfrak{U}_\delta(X/T/Z, \mathfrak{F}) = \bigcup_{F \in \mathfrak{F}} \mathfrak{U}_\delta(F),$$

and $\mathfrak{U}_\delta = \mathfrak{U}_\delta(X/Z, \mathfrak{F}) = \mathfrak{U}_\delta(X/X/Z, \mathfrak{F})$.

Theorem 8.15. *Let $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$ be a bounded family of triples desirable for some weak log Fano contractions* (see the proof), *and let $\tau > 0$ be a tolerance.*

Then there exists $M > 0$ (depending on the family and τ) such that, for any $(X_u/T_u, B_u, \mathfrak{F}_u)$ in the family and each divisor $D \in \mathfrak{N} = \mathfrak{N}(X_u/T_u/Z_u, \mathfrak{F}_u)$, we have $\operatorname{Bs}|mD| = \varnothing \mod \tau$ for some integer $m \in [1, M]$ (depending on D) and, in particular, $|mD| \neq \varnothing \mod \tau$.

Moreover, there exists some $\delta > 0$ such that the same nonvanishing and freedom results hold uniformly in the neighborhood $\mathfrak{U}_\delta(X_u/T_u/Z_u, \mathfrak{F}_u)$.

Proof. We reduce to the case of bnd sets in Theorem 8.4 and Addendum 8.4.1 using Lemma 8.12(5). To apply this to a bounded family of triples, we need to check (7′) in the proof of Theorem 8.4.

Note that a bounded family of triples $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$ is equivalent to a bounded family of simple triples $(X_u/T_u/Z_u, B_u, F_u)$ for $u \in U$, where the reduced divisor $F_u = \sum P_{u,i}$ is considered as a 1-element set $\mathfrak{F}_u = \{F_u\}$ (cf. spreading in Remark 6.10(3)). This means that we have a bounded family $X/T/Z/U$ of projective morphisms$/U$ and horizontal divisors $F = \sum P_i$ and B on X, such that each triple $(X_u/T_u, B_u, F_u)$ in our family arises as a specialization of $(X/T, B, F)$ for some $u \in U$.

In particular, the pairs $(X_u/Z_u, F_u)$ belong to a bounded family $(X/Z, F)$. To satisfy (7′), we need to transform the family: e.g., change the base U, split it into an open subset and a closed complement, etc. By Noetherian induction, this finally gives a family satisfying (7′).

Each $F_u = F_{|X_u}$ is reduced by definition. After a base change, we can assume that each prime component P_i of F specializes to a prime $P_{u,i}$. Thus, $\mathfrak{D}_F$, the finite-dimensional $\mathbb{R}$-vector space generated by F, is locally constant over U (with respect to the basis P_i), that is, it specializes isomorphically to $\mathfrak{D}_{F,u}$ from its generic point. By Lemma 8.12(1), the equivalence $\sim_{\mathbb{R}} 0/T$ is geometrically (= numerically) determined by the contraction $g\colon X \to T/Z$; namely, $D \sim_{\mathbb{R}} 0/T$ if and only if $D \equiv g^*M/Z$ for some $\mathbb{R}$-Cartier divisor M on T/Z up to $\equiv/Z$. Thus, for some nonempty open subset V of U, $\mathfrak{C} = \mathfrak{C}(X/T/Z, F)$ is a subspace of $\mathfrak{D}_F$ that is constant$/V$, and specializes isomorphically to $\mathfrak{C}_u = \mathfrak{C}(X_u/T_u/Z_u, F_u)$ for each $u \in V$.

Similarly, by Lemma 8.12(2), we can assume that over V the rational polyhedral cone $\mathfrak{N}_{\mathrm{nef}} = \mathfrak{N}_{\mathrm{nef}}(X/T/Z, F)$ specializes isomorphically to the rational polyhedral cone $\mathfrak{N}_{\mathrm{nef},u} = \mathfrak{N}_{\mathrm{nef}}(X_u/T_u/Z_u, F_u)$ for each $u \in V$. If the former is given by inequalities $D \cdot C_i \geq 0$ for a finite set of (bounded) curves C_i/Z, then the latter is given by inequalities $D \cdot C_{u,i} \geq 0$ for their specializations $C_{u,i}$. (Here it is better to assume that $k = \mathbb{C}$. Since each T_u/Z_u satisfies (RPC), it also satisfies (RPC) with generating curves of bounded degree. Otherwise, the complement to the union stratified by the cones that are generated by curves of bounded degree gives a very generic point where (RPC) does not hold. Indeed, each subfamily (*a priori* nonalgebraic) that corresponds to cones $\mathfrak{N}_{\mathrm{nef},u}$ generated by curves of bounded degree is *constructible*; see Remark 8.16.)

To find the decomposition, in (CFG) we can add some prime components P_i to F such that together they generate the group of Weil divisors of $X/\sim$, and the same after specialization. In a *big* case such as this with extended F, we get (CFG) and then deduce it for the original F as in the proof of Lemma 8.12(5) (cf. Example 8.2(1)).

Thus, the compact rational polyhedron $\mathfrak{N}_{\mathrm{c}}$ specializes isomorphically to $\mathfrak{N}_{\mathrm{c},u}$ for $(X_u/Z_u, F_u)$, and D_i specializes to $D_{u,i} = D_{i|X_u}$. Since we have a finite set of generators D_i, (CFG) implies (BND) uniformly for the family; as before, we do this over some open V of U, then repeat the same for generic points of the closed complement $U \setminus V$, and so on. □

Remark 8.16. As explained in the proof, it is crucial that $\overline{\mathrm{NE}}(T_u/Z_u)$ has bounded generators. For weak log Fano contractions $(X_u/Z_u, B_u)$, this follows from anticanonical boundedness [44]. This is not enough for nonalgebraic families: nonalgebraic subsets in algebraic families (cf. Example 8.18). However, for triples, any subfamily that corresponds to the bounded generators C_i is constructible. Indeed, we can assume that U is irreducible and the subfamily is *maximal*, that is, is not in a proper Zariski subset. Then we need to verify that $\mathfrak{N}_{\mathrm{nef}}$ is given by inequalities $D \cdot C_i \geq 0$. By maximality, each $D \in \mathfrak{N}_{\mathrm{nef}}$ is nef$/Z_u$ and semiample$/Z_u$ for very general $u \in U$. Exclude u with $D_{|X_u} \cdot C' < 0$ for some curve C'/Z. To prove that $\mathfrak{N}_{\mathrm{nef}}$ is given by the above inequalities over the generic point, it is enough to check that each $D \in \mathfrak{N}_{\mathrm{nef}}$ is semiample over the generic point. We can assume that D is $\mathbb{Q}$-Cartier and even Cartier. Then it is very semiample (free) over the very general point $u \in U$ by the standard properties of $f_*\mathcal{O}_X(D)$. This implies that it is also very semiample over the generic point of U.

As above, we derive results similar to Corollaries 8.7, 8.9 and new Corollary 8.20.

Corollary 8.17. *For some triple $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$, let C_u be a divisor on X_u such that all $\mathrm{mult}_{P_i} C_u > -1 + \tau$. Then, for any $D \in \mathfrak{U}_\delta(X_u/T_u/Z_u, \mathfrak{F}_u)$, there exists an integer $m \in [1, M]$ such that $|\lceil mD + C_u \rceil| \neq \varnothing$.*

Addendum 8.17.1. *For $D \in \mathfrak{U}_\delta(X_u/T_u/Z_u, F_u)$, if $\mathrm{mult}_{P_i} F_u = 0$, we can assume just $\mathrm{mult}_{P_i} C_u > -1$.*

Proof. The proof of Corollary 8.7, with Theorem 8.15 in place of Addendum 8.4.1. □

Example 8.18. Let $(X/T = \mathrm{pt.}/Z = \mathrm{pt.}, B = 0, \mathfrak{F})$ be a triple such that X is a complete elliptic curve, and $\mathfrak{F} = \{p + q \mid p \neq q \in E\}$. Then Theorem 8.15 does not hold for this triple. (Quiz time: why does not the theorem apply?)

Otherwise, by Corollary 8.17 with $C = 0$, there exists $M > 0$ such that, for any $D \in \mathfrak{N} = \mathfrak{N}(X/\mathrm{pt.}/\mathrm{pt.}, \mathfrak{F})$, there exists an integer $m \in [1, M]$ for which $\lceil mD \rceil \neq \varnothing$. Thus, if $D = p - q$ is a torsion point of the elliptic curve, then $D \sim_{\mathbb{R}} 0$ and $D \in \mathfrak{N}$. But then $m(p-q) = mD = \lceil mD \rceil \sim 0$, that is, the order of all torsion points is bounded; this is a contradiction.

Corollary 8.19. *Let $\alpha < 1$ and τ be positive real numbers and $\mathcal{C}_u$ be an $\mathbb{R}$-b-divisor of X_u such that $\mathcal{C}_u \geq 0/X_u$ and*

$$\text{all} \quad \mathrm{mult}_{P_i} \alpha \mathcal{C}_u > -\alpha + \tau \,\mathrm{mult}_{P_i} F_u \qquad \text{for any } F_u \in \mathfrak{F}_u. \tag{8.19.1}$$

Then there exist positive real numbers M and β, δ that give the following nonvanishing$/Z_u$ on any Y_u/X_u (uniformly in u). For all b-divisors $\mathcal{D}$ such that the descent data $\mathcal{E}$ of $\mathcal{D}$ over X_u is confined by $\beta\mathcal{C}_u$, that is, $\mathcal{E} \leq \beta\mathcal{C}_u/X_u$, and $\mathcal{D}_{X_u} \in \mathfrak{U}_\delta(X_u/T_u/Z_u, \mathfrak{F}_u)$, we have $|\lceil m\mathcal{D}_{Y_u} + (\mathcal{C}_u)_{Y_u} \rceil| \neq \varnothing$ for some integer $m \in [1, M]$.

Proof. The proof of Corollary 8.9 with Theorem 8.15 instead of Addendum 8.4.1. Note that each $F_u \in \mathfrak{F}_u$ is $\mathbb{Q}$-Cartier on X_u by (QFC) of desirable triples. □

Corollary 8.20. *There are real numbers β, δ, and $M > 0$ giving the nonvanishing as in Corollary* 8.19 *for the discrepancy b-divisors $\mathcal{C}_u = \mathcal{A}_u = \mathcal{A}(X_u, B_u)$.*

Example 8.21. Let (X, B) be a Klt pair with a subboundary B and $\mathbb{Q}$-factorial X, and $\mathfrak{F}$ be a *bounded* family of reduced divisors on X. Then there exists a real number $\gamma > 0$ such that $(X, B + \gamma F)$ is Klt for any $F \in \mathfrak{F}$ (compare stability (1.3.4) in [41]). Therefore, all

$$\operatorname{mult}_{P_i} \mathcal{A} - \gamma \operatorname{mult}_{P_i} F = \operatorname{mult}_{P_i}(\mathcal{A} - \gamma \overline{F}) = a(X, B + \gamma F, P_i) > -1$$

(cf. the proof of Lemma 6.1 and Example 5.27). Thus, for any real $\alpha > 0$, we have $\alpha \operatorname{mult}_{P_i} \mathcal{A} > -\alpha + \tau \operatorname{mult}_{P_i} F$ for $\tau = \alpha\gamma$. This gives (8.19.1) and the required τ in Corollary 8.19 for $\mathcal{C}_u = \mathcal{A}_u$.

If, in addition, (X, B) is canonical in codimension ≥ 2 and $\alpha < 1$ is a positive real number (for example, $\alpha = 1/2$), then there exists $\tau > 0$ that satisfies the inequalities, including (8.19.1) for $\mathcal{C}_u = \mathcal{A}_u$. Note that $\mathcal{A}_u \geq 0/X$ because (X, B) is canonical in codimension ≥ 2.

Lemma 8.22. *Let $\mathcal{C}_u = \mathcal{A}_u$ be discrepancy b-divisors on a bounded family of desirable triples for weak log Fano contractions* (*for example, as in Corollary* 8.20), *and let $\alpha \in (0, 1)$. Then there exists τ such that all $\mathcal{C}_u$ satisfy the conditions of Corollary* 8.19, *in particular,* (8.19.1).

Proof. Example 8.21 proves the lemma for each triple. Then we can use Noetherian induction over U as in the proof of Theorem 8.15. Note that Klt is open in families (e.g., by Inverse of adjunction 3.3 and deductions on p. 127 in [41]). For any desirable triple $(X_u/T_u/Z_u, B_u, \mathfrak{F}_u)$, each X_u is $\mathbb{Q}$-factorial by (QFC) of Definition 6.9, and each (X_u, B_u) is terminal in codimension ≥ 2 by (CRP) and (TER). □

Proof of Corollary 8.20. We can find $\beta, \delta > 0$ and M by Lemma 8.22 and then use Corollary 8.19 (cf. Remarks 8.10 and 8.11). □

An explanation on bounded nonvanishing: in fact, in applications, we can weaken the nef and big condition on M to just nef in Definition 6.9 of desirable triples. Thus, we are interested in two types of desirable triples corresponding to cases when $\mathcal{M}$ is big or a pencil in the proof of Corollary 6.40: (1) (weak) log del Pezzo $(X_t/\text{pt.}, B_t)$ with only terminal closed points (= terminal resolution) corresponding to triples $(X_t/X_t/\text{pt.}, B_t, \mathfrak{F})$ with crepant B_t, and (2) elliptic fibrations$/T_u = \mathbb{P}^1$ corresponding to triples $(X_u/T_u/\text{pt.}, B_u, \mathfrak{F})$ with an elliptic fibration $X_u \to T_u = \mathbb{P}^1$ and crepant B_u. In either case, $\mathfrak{F}$ is obtained from the standard family on $(X/\text{pt.}, B)$ by log birational transform (see Remark 6.13(2)).

As usual, the simplest form of freedom is on log Fano contractions.

Theorem 8.23. *Let $(X/Z, B)$ be a weak log Fano contraction*; *let $\mathfrak{F}$ be a bounded family of reduced divisors*; *and let $\tau > 0$ be a tolerance.*

Then there exists $M > 0$ (*depending on $(X/Z, B)$, $\mathfrak{F}$, and τ*) *such that, for each divisor $D \in \mathfrak{N} = \mathfrak{N}(X/Z, \mathfrak{F})$, we have $\operatorname{Bs}|mD| = \varnothing \mod \tau$, in particular, $|mD| \neq \varnothing \mod \tau$, for some integer $m \in [1, M]$* (*depending on D*).

Addendum 8.23.1. *Moreover, there exists a real number $\delta > 0$* (*depending on $(X/Z, B)$, $\mathfrak{F}$, and τ*) *such that the same nonvanishing holds for all $D \in \mathfrak{U}_\delta = \mathfrak{U}_\delta(X/Z, \mathfrak{F})$.*

Proof. Immediate by Theorem 8.15. We consider the weak log Fano contraction $(X/Z, B)$ with bounded $\mathfrak{F}$ as a triple $(X/X/Z, B, \mathfrak{F})$. □

Corollary 8.24. *Let C be a divisor such that all $\operatorname{mult}_{P_i} C > -1 + \tau$. Then, for any $D \in \mathfrak{U}_\delta$, there exists an integer $m \in [1, M]$ such that $|\lceil mD + C \rceil| \neq \varnothing$.*

Addendum 8.24.1. *If $\operatorname{mult}_{P_i} F = 0$, we can assume just $\operatorname{mult}_{P_i} C > -1$.*

Proof. Immediate by Corollary 8.17 and its addendum. □

Corollary 8.25. *Suppose that each $F \in \mathfrak{F}$ is $\mathbb{Q}$-Cartier. Let $\alpha < 1$ and τ be positive real numbers and $\mathcal{C}$ be a b-divisor such that $\mathcal{C} \geq 0/X$ and*

$$\text{all} \quad \operatorname{mult}_{D_i} \alpha\mathcal{C} > -\alpha + \tau \operatorname{mult}_{D_i} F \qquad \text{for any } F \in \mathfrak{F}.$$

Then there exist positive real numbers M and β, δ giving the following nonvanishing/Z on any Y/X (uniformly in F). For all b-divisors $\mathcal{D}$ such that the descent data $\mathcal{E}$ of $\mathcal{D}$ over X is confined by $\beta\mathcal{C}$, that is, $\mathcal{E} \leq \beta\mathcal{C}/X$, and $\mathcal{D}_X \in \mathfrak{U}_\delta$, $|\lceil m\mathcal{D}_Y + \mathcal{C}_Y \rceil| \neq \varnothing$ for some integer $m \in [1, M]$.

Proof. Immediate by Corollary 8.19. □

Corollary 8.26. *Let $(X/Z, B)$ be a weak log Fano contraction, with X $\mathbb{Q}$-factorial and with (X, B) canonical in codimension ≥ 2. Then, for any given bounded family $\mathfrak{F}$ of reduced divisors, there are real numbers β, δ, and $M > 0$ giving the nonvanishing/Z as in Corollary 8.25 for $\mathcal{C} = \mathcal{A}$.*

Proof. Immediate by Corollary 8.20. □

9. STABILIZATION AT A CENTRAL DIVISOR

We are now ready to establish the stabilization of a central multiplicity. We first describe what we mean by an inductive model and its central divisor E_c. Then, in Theorem 9.9, we state conditions under which we expect a limit $\mathcal{D} = \varinjlim \mathcal{D}_i$ of b-divisors to stabilize *at* E_c; this means that the multiplicity of E_c in $\mathcal{D}_i$ is eventually constant, for i sufficiently large and divisible. The next section (Theorem 10.13) treats stabilization *in a neighborhood* of E_c; this result includes, in particular, a nonvanishing for the limiting divisor on E_c. Finally, we prove that both hold for 3-folds.

After this, we extend this stabilization to the freedom result of Corollary 10.16 under Klt, in the realistic situation (CCS)(rfa) (cf. (CCS)(fga) in Conjecture 6.14). This is a freedom result on a Klt model that is terminal in codimension ≥ 2, and this leads to the proof of Theorem 6.45 in Section 11.

9.1. Conventions. In this section, "under LMMP" means that we assume LMMP in dimension $n = \dim X$; for $n \leq 3$, LMMP is proved, so this is not an extra assumption. In the same way, we can drop assumptions (CCS), (MOD), and (SSB) for $n \leq 2$ by Corollary 6.40.

Definition 9.2. We say that a limit $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ of $\mathbb{R}$-b-divisors *stabilizes near a set of prime b-divisors E_i* if there exists a model Y/X on which the limit stabilizes in a neighborhood of the centers of these b-divisors (viewed as closed subvarieties). That is, $\mathcal{D}_i = \mathcal{D}$ for some $i \gg 0$ over a neighborhood in Y of these centers.

Definition 9.3 (cf. Prokhorov–Shokurov [35, предложение 3.6])**.** An *inductive model* of a log pair $(X/Z, B)$ is a log model $(Y/Z, B_Y)$ of $(X/Z, B)$ such that

(1) $(X, B_X)/(Y, B_Y)$ is *log proper*, that is,

$$\mathcal{B}^Y = \mathcal{B}(Y, B_Y) \geq \mathcal{B}^X = \mathcal{B} = \mathcal{B}(X, B),$$

or, equivalently,

$$\mathcal{A}^Y = \mathcal{A}(Y, B_Y) \leq \mathcal{A}^X = \mathcal{A} = \mathcal{A}(X, B);$$

(2) (Y, B_Y) with boundary B_Y is *exceptional* in the sense that there is a single prime divisor E_c of Y with $\operatorname{mult}_{E_c} B_Y = 1$ and (Y, B_Y) is purely log terminal;

(3) $-(K_Y + B_Y)$ is nef and big/Z (big is automatic if X/Z is birational);

(4) $-(K_Y + B_Y)$ is nef and big on E_c/Z.

We say that E_c is the *central b-divisor* of the inductive model. For a b-divisor $\mathcal{D}$, the multiplicity $d_c = \operatorname{mult}_{E_c} \mathcal{D}$ is referred to as the *central multiplicity.* Note that (2) and (3) mean (WLF) of Proposition 4.42 except that Klt is replaced by purely log terminal (cf. (PFN) in Lemma 9.7 below); but it is still not a generalized log Fano contraction as in Conjecture 5.26.

In addition, we say that $(X/Z, B)$ is a *strict inductive model* if Y is $\mathbb{Q}$-factorial in (2) (so that (Y, B_Y) is strictly log terminal [41, c. 110], see also [27, Definition 2.13]), $-(K_Y + B_Y)$ is ample$/Z$ in (3) and ample$/E_{\rm c}/Z$ in (4).

Caution 9.4. In this definition, we do not assume that Y is $/X$.

Definition 9.5. A *local weak log Fano contraction* $(X/T, B)$ means (WLF) of Proposition 4.42 in the local situation, when

- X/T is a *local* contraction, that is, $\dim T \geq 1$ and $f\colon X \to T$ is onto T.

Lemma 9.6 (cf. [35, предложение 3.6]). *Under LMMP, let $(X/T, B)$ be a local weak log Fano contraction. Then, locally$/T$, there exists an inductive model. Moreover, we can assume that this model is such that*

- *Y is $\mathbb{Q}$-factorial and projective$/T$;*
- *B_Y is a $\mathbb{Q}$-divisor; and*
- *$E_{\rm c}$ is complete* (that is, $E_{\rm c}$ is $/P$).

Addendum 9.6.1. *Thus, we have a weak log Fano contraction $(E_{\rm c}, B_{\rm c})$ such that $K_{E_{\rm c}} + B_{\rm c} = (K_Y + B_Y)_{|E_{\rm c}}$. We call it a* central model. *We can assume that $B_{\rm c}$ is a $\mathbb{Q}$-divisor.*

Addendum 9.6.2. *Let X be a $\mathbb{Q}$-factorial inductive model, and assume that X is projective$/T$. Then we can upgrade X/T to a strict inductive model Y/T.*

Proof. We only consider the case when X/T is birational, which is all that we need. Prokhorov and Shokurov [35, предложение 3.6] treat a case that is not birational. After adding a complement, we assume that $K + B \equiv 0/T$. We can suppose, in addition, that B is a $\mathbb{Q}$-divisor. This increases $\mathcal{B}$, which is important for (1) in Definition 9.3. Hence, after a contraction, we can assume that $X = T$.

First, we can build a log canonical singularity by adding εH for an ample divisor through $P \in X$. Moreover, after perturbing H, we can assume that P is exceptionally log canonical for $B + H$ with some effective $\mathbb{Q}$-Cartier divisor H. For this, we again increase $\mathcal{B}$, strictly over P. By the exceptional property, we have the required central divisor $E_{\rm c}/P$ as a unique b-divisor with log discrepancy 0.

Now take a strict log terminal resolution $(Y/X, B_Y)$ resolving $E_{\rm c}$, where $E_{\rm c}$ is the only exceptional divisor on Y/X. The boundary B_Y is given by a crepant modification of $(X, B + H)$. In addition, we can assume that $g^{-1}H$ is nef$/X$; otherwise, we can apply LMMP to $B_Y + \varepsilon g^{-1}H$.

Finally, this gives the required model on replacing B_Y by $B_Y - \varepsilon g^{-1}H$. Properties (1)–(3) follow by construction. For (4), we need to check that $g^{-1}H$ is big on $E_{\rm c}$. Otherwise, since $g^{-1}H$ is nef$/X$ and strictly effective near $E_{\rm c}$, it defines a contraction $Y \to Z/X$, where Z/X is small nontrivial and H is positive on Z/X. Since H is a $\mathbb{Q}$-Cartier divisor, this is impossible (by the projection formula).

Addendum 9.6.1 follows from the adjunction formula [41, 3.1 и (3.2.3)] and the fact that $E_{\rm c}$ is normal [41, лемма 3.6]. Addendum 9.6.2 follows directly from the following result (cf. the proof of Lemma 3.3 in [35]). $\square$

Lemma 9.7. (*Under LMMP*; *cf. remark at the end of the proof*) *suppose that*

(PFN) *$(X/T, B)$ is a* purely log terminal *weak Fano contraction,* that is, (GLF) of Proposition 4.50 with purely log terminal (X, B) and with a boundary B,

such that X is projective$/T$, and suppose that $-(K+B)$ is nef and big$/T$ on each reduced component of B (by [41, теорема 6.9], *locally$/T$, there is* at most *one).*

Then we can find another boundary B^+ such that

- *$(X/T, B^+)$ still satisfies* (PFN) *with ample $-(K + B^+)/T$;*
- *$B^+ \geq B$; and*
- *B and B^+ have the same reduced components.*

Proof. We can contract all 0-curves C for $K+B$ (that is, curves with $C\cdot(K+B)=0$) by a contraction $X\to Y/T$. The contraction is given by $-(K+B)$ (see [40, следствие (2.7)]). (PFN) is preserved because the reduced divisors of B are not contracted. There exists an effective mobile divisor D on X that is negative$/Y$. In particular, for a general such D, the reduced divisors of B are not in $\operatorname{Supp} D$. Thus, we can replace B by the boundary $B^+=B+\varepsilon D$ for some $0<\varepsilon\ll 1$. Then $-(K+B^+)/T$ is ample, and $(X/T,B^+)$ satisfies all the required properties. For ampleness, we use the fact that the Kleiman–Mori cone of $(X/T,B)$ is polyhedral. We need projectivity of X/T for the existence of D and the polyhedral property. $\square$

Remark 9.8. We do not use the full strength of LMMP for Lemma 9.7; for our purposes, the cone and contraction theorems are enough, and they work under our current assumptions.

Now we are ready to state the stabilization of the limit $\mathcal{D}=\lim_{i\to\infty}\mathcal{D}_i$ *at* some central divisor E_{c}:

$$d_{\mathrm{c}}=d_{\mathrm{c},j}$$

for some $j\gg 0$, where $d_{\mathrm{c},j}=\operatorname{mult}_{E_{\mathrm{c}}}\mathcal{D}_j$. This holds in the following situation.

Theorem 9.9. *We assume LMMP,* $(\mathrm{CCS})_{n-1}(\mathrm{gl})$, *and* $(\mathrm{SSB})_{n-1}(\mathrm{gl})$. *Suppose that* $(X/T,B)$ *is a log pair, and let* $\mathcal{D}=\lim_{i\to\infty}\mathcal{D}_i$ *be a limit of b-divisors such that*

(LWF) $(X/T,B)$ *is a local weak log Fano contraction*;

(LBF) linear b-freedom: $i\mathcal{D}_i\sim\mathcal{M}_i/T$ *for all* i, *where* $\mathcal{M}_i$ *is b-free*;

(LCA) $\mathcal{D}_\bullet$ *is lca saturated over* (X,B);

(MXD) maximality of the limit: *each* $\mathcal{D}_i\le\mathcal{D}$; *and*

(BED) *each* $\mathcal{D}_i=\mathcal{D}$ *outside* $f^{-1}P$ *over* T (*cf. Proposition* 4.54).

Then there exists an inductive model with central divisor E_{c} *such that the limit* $d_{\mathrm{c}}=\lim_{i\to\infty}d_{\mathrm{c},i}$ *is a rational number, and the limit stabilizes*; *that is,* $d_{\mathrm{c}}=d_{\mathrm{c},j}$ *for infinitely many* j.

Corollary 9.21 below improves this result slightly. A more serious statement about local stabilization is contained in Theorem 10.13 of the next section.

Remark 9.10. Under (LWF), *linear b-free* (LBF) is equivalent to *numerically free* $i\mathcal{D}_i\equiv\mathcal{M}_i/T$, provided that $i\mathcal{D}_i$ is Cartier, or merely integral. It is enough to establish this for $\overline{i(\mathcal{D}_i)_Y}=i\mathcal{D}_i\equiv 0/T$. The required linear equivalence then follows from LMMP, or from descent of Cartier divisors for their contractions on rational singularities, and from stable freedom on $Y=X$ as in (WLF) [40, следствие (2.7)] (cf. Remark 8.13).

The main steps in the proof of Theorem 9.9 are as follows. We first establish in Proposition 9.13 the slightly more general rationality result $d_{\mathrm{c}}\in\mathbb{Q}$, together with a stabilization result. Then, using Proposition 9.15 on bounded presentation of the fixed part of linear systems on the central b-divisor, we reduce Theorem 9.9 to Proposition 9.13. Proposition 9.13 itself and its proof are similar to the proof of the rationality theorem in the Kleiman–Mori cone and the nonvanishing theorem (see [25, 4-1-1; 40, (0.2)]). However, this time, we need a birational version of nonvanishing, namely, Corollaries 8.20 and 8.26. We interpret this nonvanishing as stabilization of $\mathcal{D}=\lim_{i\to\infty}\mathcal{D}_i$ at E_{c}. We need some preliminary results, Lemmas 9.6 and 9.7 above, Lemma 9.18, and Corollary 9.19 to construct an appropriate inductive model.

The following example illustrates the use of an inductive model and is a paradigm for Theorem 9.9 and for other results in this section and in the paper (cf. Corollary 9.21 and Addendum 9.21.2).

Example 9.11 (projective space *et al.*)**.** Let $(X/Z,B)=(\mathbb{P}^n/\mathrm{pt.},0)$ be a projective space. Then (FGA) together with (SSB), (PRM), (PFC), (CCS), and (MOD) of Conjecture 6.14 hold on it. Moreover, $\mathfrak{S}=\mathfrak{P}=\varnothing$, and in (CCS), $\operatorname{Bs}|\mathcal{M}|_{\mathbb{P}^n}=\varnothing$ for each $\mathcal{M}\in\mathfrak{M}$; thus, $c=1$! Moreover, for any $\mathbb{R}$-b-divisor *under saturation* (SAT), $\mathrm{Bs}\,|\mathcal{D}|_{\mathbb{P}^n} = \varnothing$. Nontrivial (FGA) gives as stable models only $\mathbb{P}^n$ itself or pt.

We can prove this using restriction to $D = \mathcal{D}_Y$, as in the proof of Proposition 6.26. But this needs Fujita's bound for freedom, which is not yet established. Another *realistic* approach is to use inductive models. The first one is $(Y/\mathrm{pt.}, E = \mathbb{P}^{n-1})$, where $Y \to \mathbb{P}^n$ is the blowup in a closed point $P \in \mathcal{D}_{\mathbb{P}^n} \subset \mathbb{P}^n$ and E is the exceptional locus over P. If $\mathcal{D} = \mathcal{D}_Y \not\supset E$ is fixed (prime), we use saturation (STD) for $\mathcal{D}_{|E}$ for generic $P \in \mathcal{D}_{\mathbb{P}^n}$; saturation is preserved (consider $\varepsilon\mathcal{D}_{\mathbb{P}^n}$; cf. Lemma 9.16 and its proof below). Thus, by induction on n, $\mathcal{D}$ is mobile.

If $\mathcal{D}$ is mobile, for example, $\mathcal{D} = \mathcal{M} \in \mathfrak{M}$, we can also use the models $(\mathbb{P}^n/\mathrm{pt.}, H = \mathbb{P}^{n-1})$ with (general) hyperplane sections H. Again by induction, $|\mathcal{D}|_{\mathbb{P}^n}$ only has *closed* base points P (cf. Proposition 4.54). For such $P \in \mathrm{Bs}\,|\mathcal{D}|_{\mathbb{P}^n}$, we use induction. Namely, $|\mathcal{M}|_{Y|E}$ is free, where $\mathcal{M} = \mathrm{Mov}\,\mathcal{D}$ is *general* in the linear system on Y with $\mathcal{M}_{\mathbb{P}^n} = \mathcal{D}_{\mathbb{P}^n}$ (cf. Proposition 9.15). Hence,

$$\mathrm{Mov}\lceil \mathcal{D} + \mathcal{A} \rceil \geq \mathrm{Mov}(\mathcal{M} + \overline{E}) = \overline{(\mathcal{M})_Y} + \overline{E},$$

which contradicts saturation of $\mathcal{D}$ with respect to $\mathcal{A} = \mathcal{A}(\mathbb{P}^n, 0)$.

The same works for any other nonsingular Fano variety such that

- each blowup in a closed point is again a log Fano variety $(Y/\mathrm{pt.}, E)$; or
- there are (base point free) ladders of smooth log Fano varieties (down to surfaces),

either of which gives inductive models. For example, this holds for nonsingular quadrics. Finally, similar facts hold for nonsingular points $(X/X, B = 0)$ and *simple* singularities having such a Fano variety as exceptional locus of a minimal resolution. In addition to obvious models, the minimal resolutions appear as stable models when the singularity is nonterminal. All of this is aesthetically appealing, but unfortunately not deep.

The following result generalizes the Rationality Theorem for Cones.

Notation 9.12. In the following proposition, $\mathcal{A}' = \mathcal{A}'(X, B) = \mathcal{A} + E_{\mathrm{c}}$ is an *adjusted* (*truncated*) discrepancy, where (X, B) is *exceptionally* log terminal with a single E_{c} having $\mathrm{mult}_{E_{\mathrm{c}}} \mathcal{A} = -1$.

Proposition 9.13. *Let $(X/T, B)$ be a strict inductive model with central divisor E_{c}, $E' \sim E_{\mathrm{c}}$ be a linear equivalence with $E_{\mathrm{c}} \not\subset \mathrm{Supp}\, E'$, and $\mathcal{D}_\bullet$ be a system of b-divisors $\mathcal{D}_i$ such that $d_{\mathrm{c}} = \lim_{i\to\infty} d_{\mathrm{c},i}$. Then $d_{\mathrm{c}} \in \mathbb{Q}$ under the following conditions*:

(BNF) *each $\mathcal{D}_i$ is* b-nef$/T$ *in the sense of Lemma* 4.23;

(εA$'$S) *for some $\varepsilon > 0$,* integral weak asymptotic saturation over E_{c} (*cf. Definition* 4.43) *holds for $\mathcal{D}_\bullet$ of index* 1 *with respect to $\mathcal{C} = \mathcal{A}' + d\overline{E_{\mathrm{c}}}$, for any $0 \leq d \leq \varepsilon$;*

(MXC) maximality of the central limit: *each $d_{\mathrm{c},i} \leq d_{\mathrm{c}}$;*

(BWQ) *there exists a bounded family $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ of desirable triples as required for $(E_{\mathrm{c}}/\mathrm{pt.}, B_{\mathrm{c}})$ (cf. Theorem* 8.15) *such that each $(\mathcal{D}_{i\,|E_{\mathrm{c}}}^{\prime})_{E_i} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, \mathfrak{F}_i)$ (the mixed restriction for $K = \mathbb{R}$, cf.* (mx) *in Proposition* 7.7) *and the descent data of the restrictions $\mathcal{D}_{i\,|E_{\mathrm{c}}}^{\prime}$ is asymptotically confined by $\mathcal{A}_{\mathrm{c}} = \mathcal{A}(E_{\mathrm{c}}, B_{\mathrm{c}})$ over a* sequence of models E_i corresponding to $\mathcal{D}_{i\,|E_{\mathrm{c}}}^{\prime}$ (*see Remark* 5.8(6));

(BRE$'$) $\mathrm{Supp}(\overline{E'_{|E_{\mathrm{c}}}})_{E_u} \leq F \in \mathfrak{F}_u$ *for each $F \in \mathfrak{F}_u$ on each model E_u of triples in* (BWQ).

Moreover, the limit stabilizes: $d_{\mathrm{c}} = d_{\mathrm{c},j}$ *for infinitely many j.*

Remark 9.14. *$E' \sim E_{\mathrm{c}}$ induces $E'' \sim E = g^{-1}E_{\mathrm{c}}$* on any other model $g\colon Y \to X$ with $E \not\subset \mathrm{Supp}\, E''$. Take $E'' = g^*E' + E - g^*E_{\mathrm{c}}$. Then the mixed restriction of 7.3 defined by E'' on Y

is independent of Y/X: in fact, since $\operatorname{mult}_{E_c}\mathcal{D} = \operatorname{mult}_E\mathcal{D} = d$, we have

$$\begin{aligned}\mathcal{D}_{\restriction E_c} &= (\mathcal{D} - d\overline{E_c})_{\restriction E_c} + d\overline{E'}_{\restriction E_c} = (\mathcal{D} - d\overline{E_c})_{\restriction E_c} + d\overline{E'_{|E_c}}\\ &= (\mathcal{D} - d\overline{E_c})_{\restriction E} + d\overline{g^*E'_{|E}} = (\mathcal{D} - d\overline{E_c})_{\restriction E} + d\overline{(E'' + g^*E_c - E)}_{\restriction E}\\ &= (\mathcal{D} - d(\overline{E_c} - \overline{g^*E_c} + \overline{E}))_{\restriction E} + d\overline{E''}_{\restriction E} = (\mathcal{D} - d\overline{E})_{\restriction E} + d\overline{E''}_{\restriction E}\end{aligned}$$

because $g^*E' = E'' + g^*E_c - E$ and $\overline{E_c} = \overline{g^*E_c}$.

Proof of Proposition 9.13. Since $(K+B)/T$ is negative, we can replace ε by a smaller positive value so that the saturation (εA$'$S) still holds and $-(K+B) + \varepsilon E_c$ is still nef and big$/T$. Moreover, we can assume that the same holds for any smaller $\varepsilon \geq 0$.

In particular, for any $0 \leq d \leq \varepsilon$, for any real number $j \geq 0$, and for any natural number i, on a sufficiently high log resolution $g\colon Y \to X/T$ of $(X/T, B)$, we have the following vanishing:

$$R^1(f\circ g)_*\mathcal{O}(\lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil) = 0.$$

Note that the resolution Y depends only on i. Indeed, we take a log resolution *over* which $(\mathcal{D}_i)_Y$ is nef$/T$ and $\mathbb{R}$-Cartier. Then

$$\begin{aligned}\lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil &= K_Y + \lceil -g^*(K+B) + dg^*E_c + j(\mathcal{D}_i)_Y\rceil\\ &= K_Y + \lceil g^*(-(K+B) + dE_c) + j(\mathcal{D}_i)_Y\rceil,\end{aligned}$$

where $-(K+B) + dE_c$ is nef and big$/T$ and $(\mathcal{D}_i)_Y$ is nef$/T$. It follows that $g^*(-(K+B)+dE_c) + j(\mathcal{D}_i)_Y$ is nef and big$/T$, and Kawamata–Viehweg gives the required vanishing.

However, to apply this vanishing to a restriction to the birational image E of E_c in Y, we need to assume that $d\overline{E_c} + j\mathcal{D}_i$ has integral multiplicity in E_c, for integral weak saturation in (εA$'$S), or equivalently, that $d + jd_{c,i}$ is an integer. More precisely, then we have a surjective map of linear systems

$$|E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil| \dashrightarrow |(E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil)_{|E}|.$$

In addition, because $d + jd_{c,i}$ are integers, by normal crossing, Lemmas 6.29, 6.30, and Adjunction Formula 3.1 in [41], for a sufficiently high resolution Y (over which $\mathcal{D}_i$ is $\mathbb{R}$-Cartier and nef$/X/T$), the linear system

$$\begin{aligned}|(E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil)_{|E}| &= |\lceil (E + \mathcal{A}_Y)_{|E} + (d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y)_{|E}\rceil|\\ &= |\lceil (\mathcal{A}_c)_E + ((d\overline{E_c} + j\mathcal{D}_i)_{\restriction E_c})_E\rceil|\\ &= |\lceil (\mathcal{A}_c r)_E + (((d + jd_{c,i})\overline{E'} + j\mathcal{D}'_i)_{\restriction E_c})_E\rceil|\\ &= |\lceil (\mathcal{A}_c)_E + (((d + jd_{c,i})\overline{E''} + d(\overline{E_c} - \overline{E}) + j\mathcal{D}''_i)_{\restriction E})_E\rceil|\end{aligned}$$

is defined on E by the birational restriction $(d\overline{E_c} + j\mathcal{D}_i)_{\restriction E_c}$, that is, by

$$((d + jd_{c,i})\overline{E'} + j\mathcal{D}'_i)_{\restriction E_c} = ((d + jd_{c,i})\overline{E''} + d(\overline{E_c} - \overline{E}) + j\mathcal{D}''_i)_{\restriction E_c}$$

(as in Remark 9.14), where E is a sufficiently high resolution of E_c and $E' \sim E_c$ on X/T with $E_c \not\subset \operatorname{Supp} E'$, and where $E'' \sim E$ is induced on Y/T with $E \not\subset \operatorname{Supp} E''$, $\mathcal{D}'_i = \mathcal{D}_i - d_{c,i}\overline{E_c}$ and $\mathcal{D}''_i = \mathcal{D}_i - d_{c,i}\overline{E}$. This last resolution only depends on i. Hence, if we have nonvanishing for the linear system restricted to E, the corresponding linear system

$$|E + \lceil \mathcal{A}_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil| = |\lceil \mathcal{A}'_Y + d(\overline{E_c})_Y + j(\mathcal{D}_i)_Y\rceil|$$

is free on E. For saturation below, note that the system is independent of further resolutions$/Y$ by stabilization in Proposition 4.46 and Example 4.47.

In the present situation, this is impossible because it contradicts (MXC), the maximality of d_{c}. Namely,

(NBF) $|\lceil \mathcal{A}'_Y + d(\overline{E_{\mathrm{c}}})_Y + j(\mathcal{D}_i)_Y \rceil|$ cannot be free on E for a positive integer j and integers $d + jd_{\mathrm{c},i} > jd_{\mathrm{c}}$.

Indeed, then the multiplicity of the mobile part for the linear system is

$$\mathrm{mult}_E \lceil \mathcal{A}'_Y + d(\overline{E_{\mathrm{c}}})_Y + j(\mathcal{D}_i)_Y \rceil = \mathrm{mult}_{E_{\mathrm{c}}}(d\overline{E_{\mathrm{c}}} + j\mathcal{D}_i) = d + jd_{\mathrm{c},i}.$$

On the other hand, for any natural number i, by asymptotic saturation (see Definition 4.33), the multiplicity is $\le \mathrm{mult}_{E_{\mathrm{c}}} j\mathcal{D}_j = jd_{\mathrm{c},j}$, that is, $d + jd_{\mathrm{c},i} \le jd_{\mathrm{c},j}$. Hence, $d_{\mathrm{c},j} \ge d_{\mathrm{c},i} + d/j > d_{\mathrm{c}}$ by the inequality in (NBF). This contradicts (MXC).

By the above surjectivity, (NBF) is equivalent to the following vanishing:

(VN) on a sufficiently high resolution E/E_{c},

$$|\lceil (\mathcal{A}_{\mathrm{c}})_E + ((d + jd_{\mathrm{c},i})\overline{E'|_{E_{\mathrm{c}}}} + j\mathcal{D}'_{i|E_{\mathrm{c}}})_E \rceil| = \varnothing$$

whenever j is a positive integer and integer $d + jd_{\mathrm{c},i} > jd_{\mathrm{c}}$.

However, if $d_{\mathrm{c}} \notin \mathbb{Q}$, we can disprove this for some $i \gg 0$ using bounded nonvanishing.

For triples $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ in (BWQ), by Corollary 8.20, there exist real numbers β, δ, and $M > 0$ such that, for any b-divisors $\mathcal{D}$ with the descent data confined by $\beta\mathcal{A}_{\mathrm{c}}$ over E_u and $\mathcal{D}_{E_u} \in \mathfrak{U}_\delta$, we have the bounded nonvanishing

$$|\lceil m\mathcal{D}_{E_{\mathrm{hr}}} + (\mathcal{A}_{\mathrm{c}})_{E_{\mathrm{hr}}} \rceil| \ne \varnothing \qquad \text{for some integer } m \in [1, M]$$

on any E_{hr}/E_u, in particular, for some rather high $E_{\mathrm{hr}}/E_{\mathrm{c}}$.

We apply this to the b-divisor

$$\mathcal{D} = q(d/j + d_{\mathrm{c},i})\overline{E'|_{E_{\mathrm{c}}}} + q\mathcal{D}'_{i|E_{\mathrm{c}}}$$

for some natural number $q > 0$ and satisfying the nonvanishing conditions. Equivalently, $\mathcal{D} = p\overline{E'|_{E_{\mathrm{c}}}} + q\mathcal{D}'_{i|E_{\mathrm{c}}}$, where now $j = mq$ and $p = d/m + qd_{\mathrm{c},i}$ under the following conditions:

(1) $m > 0$ and p are integers, and

(2) $p/q > d_{\mathrm{c}}$.

These imply that $mq = j > 0$ and $mp = md/m + mqd_{\mathrm{c},i} = d + jd_{\mathrm{c},i}$ are integers and $mp = d + jd_{\mathrm{c},i} > jd_{\mathrm{c}}$ as in (VN).

Now we rewrite the nonvanishing conditions for $\mathcal{D}$:

(3) $\mathcal{D}_{E_i} \in \mathfrak{U}_\delta(E_i/T_i/\mathrm{pt.}, \mathfrak{F}_i)$, and

(4) the descent data of $\mathcal{D}$ is confined by $\beta\mathcal{A}_{\mathrm{c}}/E_i$.

Since

$$\mathcal{D} = (p - qd_{\mathrm{c},i})\overline{E'|_{E_{\mathrm{c}}}} + q(d_{\mathrm{c},i}\overline{E'|_{E_{\mathrm{c}}}} + \mathcal{D}'_{i|E_{\mathrm{c}}}),$$

we have $q(d_{\mathrm{c},i}\overline{E'|_{E_{\mathrm{c}}}} + \mathcal{D}'_{i|E_{\mathrm{c}}}) = q\mathcal{D}_{i|E_{\mathrm{c}}}$ (this mixed restriction is defined up to $\sim_{\mathbb{R}}$; see Mixed restriction 7.3), $(\mathcal{D}_{i|E_{\mathrm{c}}})_{E_i} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, F_i)$ for some $F_i \in \mathfrak{F}_i$, and $\mathrm{Supp}(E'|E_{\mathrm{c}})_{E_i} \le F_i$ by (BWQ) and (BRE$'$), it follows that, for a *natural number* q, (3) holds for $\mathcal{D} = p\overline{E'|_{E_{\mathrm{c}}}} + q\mathcal{D}'_{i|E_{\mathrm{c}}}$, provided that $\|(p - qd_{\mathrm{c},i})(\overline{E'|_{E_{\mathrm{c}}}})_{E_i}\| < \delta$. (For a natural number q, note that $\mathcal{D} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, F_i)$ implies $q\mathcal{D} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, F_i)$ because $\mathfrak{N}(E_i/T_i/\mathrm{pt.}, F_i)$ is an Abelian semigroup, although not a convex body.) This last inequality holds provided that

(3$'$) $0 \le p - qd_{\mathrm{c},i} < \delta/N$, where N is $\ge$ the *maximal* absolute multiplicity of $(\overline{E'|_{E_{\mathrm{c}}}})_{E_u}$ over all $F \in \mathfrak{F}_u$, because $\operatorname{Supp}(E'|E_{\mathrm{c}})_{E_i} \le F_i \in \mathfrak{F}_i$ by (BRE$'$). N is bounded under (BWQ) (by a Noetherian induction as in the proof of Theorem 8.15; cf. Lemma 8.22). The strict form $0 < p - qd_{\mathrm{c},i}$ of the left inequality in (3$'$) follows from (2) and (MXC).

If E_i/E_{c} (e.g., for surface E_{c} by Proposition 9.15(2)), the descent data for $(\overline{E'|_{E_{\mathrm{c}}}})$ is $0/E_i$, and the descent data for $\mathcal{D}$ is the same as for $q\mathcal{D}_{i}{}_{|E_{\mathrm{c}}}$ by (ADD) in Proposition 5.4. But for fixed q, the descent data for $q\mathcal{D}_{i}{}_{|E_{\mathrm{c}}}$ is asymptotically confined by $\mathcal{A}_{\mathrm{c}}/E_i$ (cf. Remarks 5.8(5),(6)). Hence, for some $i \gg 0$, the descent data of $\mathcal{D}$ is confined by $\beta\mathcal{A}_{\mathrm{c}}$ because, by definition, $\mathcal{A}_{\mathrm{c}} \ge 0/E_i$. This gives (4) for some $i \gg 0$.

If E_i is not$/E_{\mathrm{c}}$, the descent data for $(p - qd_{\mathrm{c},i})\overline{E'|_{E_{\mathrm{c}}}}$ is confined by $\gamma\mathcal{A}_{\mathrm{c}}/E_i$ with arbitrarily small $\gamma > 0$ for all $i \gg 0$, for instance, with $\gamma = \beta/2$. The rest can be confined by $(\beta/2)\mathcal{A}_{\mathrm{c}}$ for some $i \gg 0$ as above; this gives (4). Since E_i belongs to a bounded family and $E'|_{E_{\mathrm{c}}}$ is fixed, the descent data for $\overline{E'|_{E_{\mathrm{c}}}}$ is confined/each E_i (even E_u) with some $\gamma > 0$ by Example 6.7(1), Lemma 6.1, and the Noetherian induction. Then we can make $(p - qd_{\mathrm{c},i})$ arbitrarily small for all $i \gg 0$ as in (3$'$) (cf. the proof below).

Thus, we need to find integers $q > 0$ and p satisfying (2) and (3$'$). This gives

$$m\mathcal{D} = mp\overline{E'|_{E_{\mathrm{c}}}} + mq\mathcal{D}'_{i}{}_{|E_{\mathrm{c}}} = (d + jd_{\mathrm{c},i})\overline{E'|_{E_{\mathrm{c}}}} + j\mathcal{D}'_{i}{}_{|E_{\mathrm{c}}},$$

which contradicts (VN) for $j = mq$ and $d = m(p - qd_{\mathrm{c},i})$ with some natural number $1 \le m \le M$ by Corollary 8.20 and some $i \gg 0$.

Suppose that $d_{\mathrm{c}} \notin \mathbb{Q}$; then, for any real numbers N and $\varepsilon > 0$, there exists a rational approximation $r = p/q$ with integers $q > 0$ and p such that

- $r > d_{\mathrm{c}}$, and
- $r - d_{\mathrm{c}} < \varepsilon/Nq$.

See continued fractions in [7, pp. 2–5].

We apply this result for certain real numbers: $0 < \varepsilon \le \delta$, where ε is $\le$ that at the start of the proof, and $N \ge M$ as in (3$'$). Thus, we get the required integers $q > 0$ and p satisfying (2) and (3$'$) for all $i \gg 0$. Indeed, $p/q = r > d_{\mathrm{c}}$ implies (2) and the left inequality in (3$'$) by (MXC). On the other hand, since $d_{\mathrm{c}} = \lim_{i\to\infty} d_{\mathrm{c},i}$, for all $i \gg 0$ we have

$$p/q - d_{\mathrm{c},i} = r - d_{\mathrm{c},i} < \varepsilon/Nq \le \delta/Nq \qquad \text{or} \qquad p - qd_{\mathrm{c},i} < \delta/N.$$

We also need to verify that $0 \le d \le \varepsilon$: since $d = m(p - qd_{\mathrm{c},i})$ and the above argument gives more than (3$'$), $0 < p - qd_{\mathrm{c},i} < \varepsilon/N$; in fact, $0 < d < m\varepsilon/N \le M\varepsilon/N \le \varepsilon$ because $M \le N$. This leads to the promised contradiction. Therefore, $d_{\mathrm{c}} \in \mathbb{Q}$.

Finally, we verify the stabilization of the limit $d_{\mathrm{c}} = \lim_{i\to\infty} d_{\mathrm{c},i}$. We use the same arguments, but now the contradiction turns into an honest nonvanishing and stabilization. In particular, we replace (2) by the equation

(2$'$) $p/q = d_{\mathrm{c}}$.

Nonetheless, we get (3$'$) and the nonvanishing of the linear system in (VN) for some $i \gg 0$ by (3) and (4). As explained above, this implies that $|\lceil \mathcal{A}'_Y + d(\overline{E_{\mathrm{c}}})_Y + j(\mathcal{D}_i)_Y \rceil|$ is free on E. This time, by asymptotic saturation, we get the inequalities $d_{\mathrm{c},j} \ge d_{\mathrm{c},i} + d/j = d_{\mathrm{c}}$. Thus, by (MXC), $d_{\mathrm{c}} = d_{\mathrm{c},j}$ for $j = mq$. We can find infinitely many j if we replace p and q by lp and lq, respectively, for any natural number $l > 0$. (They may not be proportional to the first j since m depends on $\mathcal{D}$.) $\square$

To apply Proposition 9.13 in the proof of Theorem 9.9, we need to obtain bounded presentations on the central divisor E_{c} of the restrictions $(\mathcal{D}_{i}{}_{|E_{\mathrm{c}}})_{E_i}$ in (BWQ) for an appropriate inductive model of $(X/T, B)$. For this, we use Corollary 7.9 in conjunction with the following result, for a certain set of b-divisors $\mathcal{M}$, on the boundedness of their fixed components on E_{c}, that is, the boundedness of the divisorial components of $\operatorname{Bs}|\mathcal{M}|_X \cap E_{\mathrm{c}}$.

Proposition 9.15. *Under* $(\mathrm{CCS})_{n-1}(\mathrm{gl})$ *and* $(\mathrm{SSB})_{n-1}(\mathrm{gl})$, *let* $(X/T,B)$ *be a strict inductive model with central divisor* E, $\mathrm{Bs}\subset X$ *be a proper reduced subvariety, and* $\{\mathcal{M}\}$ *be a set of* (*sufficiently general*) *b-free b-divisors* $\mathcal{M}$ *such that*

(BSP) $\mathrm{Bs}\,|\mathcal{M}|_X\subset\mathrm{Bs}$; *and*

(SA′) $\mathcal{M}$ *is* $\mathcal{A}'$*-saturated on sufficiently high models* X_{hr}/X with $'$ over P and Bs, that is, $\mathcal{A}'=\mathcal{A}+\sum E_i$ with $\sum E_i$ *exactly* over the integral components of $\mathcal{A}$ over P and Bs, that is, with $f(\mathrm{center}_T\,E_i)=P$ or with $\mathrm{center}_X\,E_i\subset\mathrm{Bs}$;

then the set of b-divisors $\mathcal{M}$ *satisfies*

(1) boundedness of the fixed component *of each* $|\mathcal{M}|_X$ *on* E, *that is, the support of a* whole *divisorial component of the intersection* $\mathrm{Bs}\,|\mathcal{M}|_X\cap E$ *belongs to a bounded set* $\mathfrak{F}$ *of reduced divisors on* E;

(2) *in each linear system* $|\mathcal{M}_{|E}|$, *a* general element *has confined canonical singularities on a triple* $(E_u/T_u/\mathrm{pt.},B_u,\mathfrak{F}_u)$ (with E_u/E, desirable for $(E/\mathrm{pt.},B)$, and also just bounded if E is a surface); *and*

(3) *the desirable triples in* (2) *belong to a bounded algebraic family.*

Note that $\mathcal{M}$ is free on X outside $f^{-1}P$ if $\mathrm{Bs}=\varnothing$ *there.*

Lemma 9.16 (cf. Proposition 4.50 and its proof)**.** *Under the assumptions of Proposition* 9.15, (SA′) *for* $\mathcal{M}$ *implies* (SA′F) *of Proposition* 6.34(4) *with* $\mathcal{M}:=\mathcal{M}_{|E}$ *and* $F=\mathrm{Supp}\,\mathrm{Fix}(|\mathcal{M}|_{X|E})$, except over $\mathrm{Supp}\,B_E$.

Proof. Suppose that $\mathcal{M}_X$ is sufficiently general in its linear system $|\mathcal{M}|_X$ and set $M=\mathcal{M}_X$.

Then, for any $0<\varepsilon\ll 1$, on a sufficiently high log resolution $g\colon Y\to X/T$ of $(X/T,B)$, we have the following vanishing:

$$R^1g_*\mathcal{O}(\lceil\mathcal{A}_Y+\varepsilon\overline{M}_Y\rceil)=0;$$

$\overline{M}$ is well defined because X is $\mathbb{Q}$-factorial (by the strict property in Definition 9.3(2)). This resolution Y depends only on $\mathcal{M}$. We take a log resolution such that $\mathrm{Bs}\,|\mathcal{M}|_Y=\varnothing$. For general M, $g^{-1}M=\mathcal{M}_Y$ is also general in $|\mathcal{M}|_Y$. Then the resolution is also a log resolution for $B+\varepsilon M$, and

$$\lceil\mathcal{A}_Y+\varepsilon\overline{M}_Y\rceil=K_Y+\lceil -g^*(K+B)+\varepsilon g^*M\rceil=K_Y+\lceil g^*(-(K+B)+\varepsilon M)\rceil,$$

where $-(K+B)+\varepsilon M$ is ample$/T$ for all $0<\varepsilon\ll 1$ since $-(K+B)/T$ is ample$/T$ (by the strict form of Definition 9.3(3)). Thus, $g^*(-(K+B)+\varepsilon M)$ is nef and big$/T$, and Kawamata–Viehweg gives the required vanishing.

However, to apply this vanishing downstairs to the restriction to the birational image E_Y of E in Y, we need to assume that εM has rather small multiplicities e_i in the exceptional divisors E_i of Y/X. More precisely, each $e_i<1-\{a_i\}$ for each discrepancy $a_i=\mathrm{mult}_{E_i}\mathcal{A}$, where, as usual, $\{a_i\}$ is the fractional part; in particular, $e_i<1$ for integral a_i. Equivalently, $\lceil a_i+e_i\rceil=\lceil a_i^*\rceil$, where $a_i^*=\mathrm{mult}_{E_i}\mathcal{A}^*=a_i+1$ for integral a_i, when E_i is over M and even exactly over $\mathrm{Bs}\,|\mathcal{M}|_X$ for the *general* M, and $a_i^*=a_i$ otherwise. Thus, as compared to $'$, the operation * increases *exactly* by 1 those integral values a_i of components that are exceptional on X and lie over $\mathrm{Bs}\,|\mathcal{M}|_X$; in particular, * does not hold for E itself. (When $\mathcal{M}_X$ is free outside $f^{-1}P$, we can assume that * *only* increases values over P.)

Hence,

$$\lceil\mathcal{A}_Y+\varepsilon\overline{M}_Y\rceil=\left\lceil\mathcal{A}_Y+\varepsilon\mathcal{M}_Y+\sum e_iE_i\right\rceil=\lceil\mathcal{A}_Y^*\rceil+\mathcal{M}_Y,$$

and we get the vanishing $R^1g_*\mathcal{O}(\lceil\mathcal{A}_Y^*\rceil+\mathcal{M}_Y)=0$. This gives the surjective map of linear systems

$$|E_Y+\lceil\mathcal{A}_Y^*\rceil+\mathcal{M}_Y|\dashrightarrow|(\lceil\mathcal{A}_Y^*+E_Y\rceil+\mathcal{M}_Y)_{|E_Y}|=|\lceil(\mathcal{A}_E^*)_{E_Y}\rceil+(\mathcal{M}_{|E})_{E_Y}|.$$

Indeed, by definition and our choice of Y, $\mathcal{M}_{Y|E_Y} = (\mathcal{M}_{\restriction E})_{E_Y}$. The same holds for *any* sufficiently high resolution X_{hr}/X and *some* sufficiently high resolution E_{hr}/E. In addition, $(\mathcal{A}_Y + E_Y)_{|E_Y} = (\mathcal{A}_E)_{E_Y}$ by Adjunction Formula 3.1 in [41], where $\mathcal{A}_E = \mathcal{A}(E, B_E)$. Hence, $(\mathcal{A}_Y^* + E_Y)_{|E_Y} = (\mathcal{A}_E^*)_{E_Y}$, where $(\mathcal{A}_E^*)_{E_Y} = (\mathcal{A}_E)_{E_Y} + \mathcal{F}_{E_Y}$ for $\mathcal{F}_{E_Y} = \sum F_i$ with $F_i = E_i \cap E_Y$ and E_i under *. This assumes that we can extend $\mathcal{F}_{E_Y}$ birationally to $\mathcal{F}$, or on (any) sufficiently high E_{hr}/E. Since $\overline{M}_{\restriction E}$ has the same support on E as $\mathrm{Mov}(|\mathcal{M}|_{X|E})$, $F = \mathcal{F}_E$ is exactly the fixed divisorial component of

$$E \cap \mathrm{Bs}\,|\mathcal{M}|_X \supset \mathrm{Supp}\,\mathrm{Fix}(|\mathcal{M}|_{X|E})$$

(in general, $\neq \mathrm{Fix}\,|\mathcal{M}_{\restriction E}|_E = (\mathrm{Fix}\,\mathcal{M}_{\restriction E})_E = 0$ for b-free $\mathcal{M}$!) over the locus where B_E is integral, that is, *except over* $\mathrm{Supp}\,B_E$, because (E, B_E) is Klt. Since the resolution is divisorial, the whole $\mathcal{F}$ is exactly over the integral components of $\mathcal{A}_E$ and over $E \cap \mathrm{Bs}\,|\mathcal{M}|_X = \mathrm{Bs}\,|\mathcal{M}|_{X|E} \supset \mathrm{Bs}\,|\mathcal{M}_{\restriction E}|_E$, in particular, over F (cf. (Sa$'$F) in Proposition 6.34(4)). Finally, by normal crossings, $\lceil \mathcal{A}_Y^* + E_Y \rceil_{|E_Y} = \lceil (\mathcal{A}_E^*)_{E_Y} \rceil$.

Now, by Lemma 4.44, (SA$'$), and Remark 4.34(1) (again because (E, B_E) is Klt), we get saturation on Y/T:

$$|E_Y + \lceil \mathcal{A}_Y^* \rceil + \mathcal{M}_Y| = |\mathcal{M}_Y| + E_Y + \lceil \mathcal{A}_Y^* \rceil,$$

where $|\mathcal{M}_Y|$ is free on Y. Indeed, $E_Y + \lceil \mathcal{A}_Y^* \rceil \geq 0$ and $\mathcal{M}_Y$ is integral; $\mathrm{mult}_{E_Y}(E_Y + \lceil \mathcal{A}_Y^* \rceil) = 0$. Hence, by the above surjectivity, on E_Y

$$\left|\lceil (\mathcal{A}_E^*)_{E_Y} + (\mathcal{M}_{\restriction E})_{E_Y} \rceil\right| = \left|\lceil (\mathcal{A}_E^*)_{E_Y} \rceil + (\mathcal{M}_{\restriction E})_{E_Y}\right| = \left|(\mathcal{M}_{\restriction E})_{E_Y}\right| + \lceil (\mathcal{A}_E^*)_{E_Y} \rceil,$$

where $|(\mathcal{M}_{\restriction E})_{E_Y}|$ is free on E_Y. Moreover, we can add to $\mathcal{F}$ exceptional/X F_i over the integral components of $\mathcal{A}_E$, but not over $E \cap \mathrm{Bs}\,|\mathcal{M}|_X$, preserving the mobile part $(\mathcal{M}_{\restriction E})_{E_Y}$. This gives $\mathcal{A}_E^* = \mathcal{A}_E + \mathcal{F} = \mathcal{A}' + F$ with $F, \mathcal{A}'$ and with $\mathcal{M} := (\mathcal{M}_{\restriction E})$ under the saturation (SA$'$F), where F is considered as a b-divisor. Then

$$\left|\lceil (\mathcal{M}_{\restriction E})_{E_Y} + F + (\mathcal{A}_E')_{E_Y} \rceil\right| = \left|\lceil (\mathcal{A}_E^*)_{E_Y} + (\mathcal{M}_{\restriction E})_{E_Y} \rceil\right| = \left|(\mathcal{M}_{\restriction E})_{E_Y}\right| + F + \lceil (\mathcal{A}_E')_{E_Y} \rceil,$$

which means the saturation (SA$'$F). $\square$

The proof of the lemma used the invariance under linear equivalence of saturations (SA$'$) and (SA$'$F).

Proposition 9.17 (invariance of saturations)**.** *If $\mathcal{D} \sim \mathcal{D}'/Z$, then the $\mathcal{C}$-saturation of $\mathcal{D}$ is linearly equivalent to that of $\mathcal{D}'$. The same holds for $\mathbb{R}$-divisors.*

For asymptotic saturation, we can replace each $\mathcal{D}_i$ by $\mathcal{D}_i \sim \underline{\mathcal{D}_i'}/Z$ uniformly (*that is, there exists a rational function $a \neq 0$ on X/T such that each $\mathcal{D}_i = \mathcal{D}_i' + \overline{(a)}$; cf. linear equivalence of systems on p.* 111)*. The same holds for similarity of* characteristic type *of $\mathcal{D}_\bullet$* (*see Remark* 4.34(7)); *on truncation by I', the index I of asymptotic saturation equals $I/\gcd(I, I')$.*

Proof. It is enough to verify the divisorial version, when $D = \mathcal{D}_X \sim D' = \mathcal{D}_X'/Z$ and $C = \mathcal{C}_X$. Thus, $D' = D + (a)$.

Indeed,

$$\mathrm{Mov}\lceil D' + C \rceil = \mathrm{Mov}(\lceil D + C \rceil + (a)) = \mathrm{Mov}\lceil D + C \rceil + (a) \leq D + (a) = D'$$

because the fixed part is invariant under linear equivalence and the mobile part changes by the principal divisor (a).

For asymptotic saturation, on a sufficiently high model $Y/\underline{X}$, we replace $j(\mathcal{D}_i)_{X_{\mathrm{hr}}}$ by $j(\mathcal{D}_i')_{X_{\mathrm{hr}}} = j(\mathcal{D}_i)_{X_{\mathrm{hr}}} + j\overline{(a)}_{X_{\mathrm{hr}}}$ and $j(\mathcal{D}_j)_{X_{\mathrm{hr}}}$ by $j(\mathcal{D}_j')_{X_{\mathrm{hr}}} = j(\mathcal{D}_j)_{X_{\mathrm{hr}}} + j\overline{(a)}_{X_{\mathrm{hr}}}$.

By the above, to prove invariance under similarity, it is enough to consider a truncation $\mathcal{D}_\bullet^{[I']} = I'\mathcal{D}_{iI'}$. Then, by asymptotic $\mathcal{C}$-saturation of $\mathcal{D}_\bullet$ with index I, we have

$$\operatorname{Mov}\lceil j\mathcal{D}_i^{[I']} + \mathcal{C}\rceil = \operatorname{Mov}\lceil jI'\mathcal{D}_{iI'} + \mathcal{C}\rceil \le jI'\mathcal{D}_{jI'} = j\mathcal{D}_j^{[I']}$$

for any i, j divisible by I. This means that $\mathcal{D}_\bullet^{[I']}$ is asymptotically $\mathcal{C}$-saturated with the same index I. In fact, we can replace I by $I/\gcd(I, I')$. $\square$

Proof of Proposition 9.15. By Lemma 9.16, we obtain (SA$'$F), and this then applies, in particular, to the fixed component of $|\mathcal{M}|_X$ on E because $\operatorname{Supp} B_E$ is fixed (cf. Remark 6.41). This gives (1) by (SSB)$_{n-1}$(gl) and Proposition 6.34(4). It also gives (2) and (3) by (CCS)$_{n-1}$(gl) (cf. Corollary 6.40 for $n = 3$) because (SA$'$F) implies (SAT) for $\mathcal{D} = \mathcal{M}$ by Lemmas 4.44 and 6.36.

Finally, by the construction in the proof of Corollary 6.40, we can assume that each E_u is$/E$ if E is a surface. $\square$

To secure the saturation (εA$'$S), we need to strengthen the inductive model.

Lemma 9.18. *Let $(X/T, B)$ be a weak log Fano contraction and D be any $\mathbb{R}$-Cartier divisor on X. Then there exists a boundary $B^+ \ge B$ such that*

- $(X/T, B^+)$ *is again a weak log Fano contraction*; *and*
- $\mathcal{A}(X, B) \ge \mathcal{A}(X, B^+) + \varepsilon\overline{D}$ *for any real number* $0 < \varepsilon \ll 1$.

Proof. Since $(X/T, B)$ is a weak log Fano contraction, we can find an effective $\mathbb{R}$-Cartier divisor $D' \sim_{\mathbb{R}} -(K + B)$. Since this is big, we can assume also that there is an *effective* $D^+ = ND' \ge D$ for some real number $N > 0$. Then, for any $0 < \delta \ll 1$, $(X, B^+ = B + \delta D^+)$ is again a weak log Fano contraction with the required properties. Indeed, $\mathcal{A}^+ = \mathcal{A}(X, B^+) = \mathcal{A} - \delta\overline{D^+}$ by definition. Thus, for any $0 < \varepsilon \le \delta$, we have

$$\mathcal{A} = \mathcal{A}^+ + \delta\overline{D^+} \ge \mathcal{A}^+ + \varepsilon\overline{ND'} \ge \mathcal{A}^+ + \varepsilon\overline{D}. \quad \square$$

Corollary 9.19. *Under the assumptions of Lemma* 9.18, *saturation* (LCA) *implies*

(εAS) *asymptotic saturation for $\mathcal{D}_\bullet$ with respect to $\mathcal{C} = \mathcal{A}^+ + \varepsilon\overline{D}$.*

Proof. By Lemma 4.44, we need the inequality $\mathcal{A} \ge \mathcal{A}^+ + \varepsilon\overline{D}$, which we know by Lemma 9.18. $\square$

Proof of Theorem 9.9. To prove stabilization, we use Proposition 9.13. But before this application, we need to construct an appropriate inductive model of $(X/T, B)$. Note that assumptions (LBF), (MXD), (BED), and (LCA) are birational$/T$. Thus, they hold on any birational model of X/T. However, (LCA) is sensitive to changes of the boundary B. This allows us to improve (LCA) on an inductive model of $(X/T, B)$ as follows.

Let D be an effective Cartier divisor whose support contains the special fibre $f^{-1}P$ of X/T. We can find such a D in our local case. By Corollary 9.19, we can increase B so that $(X/T, B)$ is still a weak log Fano contraction (in particular, is still Klt) and (εAS) holds:

(εAS) asymptotic saturation holds for $D_\bullet$ with respect to $\mathcal{C} = \mathcal{A} + \varepsilon\overline{D}$ for some $\varepsilon > 0$.

Now we modify our weak log Fano contraction $(X/T, B)$ to an inductive model $(X/T, B)$ that is projective$/T$. Such a model exists by Lemma 9.6. All our assumptions are preserved, except for (LWF), but including (εAS); this follows from monotonicity (1) in Definition 9.3 of inductive models and Lemma 4.44. After a $\mathbb{Q}$-factorialization, we can assume that X is $\mathbb{Q}$-factorial. This time (εAS) is preserved since this modification is crepant.

Note that this final modification affects not only the boundary B, but also the contraction X/T itself. Thus, $\overline{D}$ in (εAS) is replaced by an effective Cartier b-divisor $\mathcal{D}$ that may not be Cartier on the new X itself. However, it contains the special fibre $f^{-1}P$, that is, $\mathcal{D} \geq \overline{f^*H}$, where H is a hypersurface through P. (The latter is a birational invariant of modifications of X/T.) In particular, $\mathcal{D} \geq \overline{F}$ for any prime $\mathbb{Q}$-Cartier divisor F on X over P. By Lemma 4.44, we can replace $\mathcal{D}$ by $\overline{F}$ in (εAS). Taking $F = E_{\mathrm{c}}$, we get an inductive model that satisfies (εAS) with $D = E_{\mathrm{c}}$. By Proposition 9.17, after a truncation, saturation holds with index $I = 1$.

Moreover, by Addendum 4.44.1, (εA$'$S) holds for asymptotic saturations with respect to $\mathcal{A}'+d\overline{E_{\mathrm{c}}}$ which are integral *over* E_{c} (cf. Example 4.45), for any $d < \varepsilon$. Take $\mathcal{C}_1 = \mathcal{A} + \varepsilon\overline{E_{\mathrm{c}}} > \mathcal{C}_2 = \mathcal{A} + d\overline{E_{\mathrm{c}}}$. Since $\operatorname{mult}_{E_{\mathrm{c}}} \mathcal{A} = -1$ is integral for the inductive model $(X/Z, B)$, the integral weak property means that $j\mathcal{D}_i + d\overline{E_{\mathrm{c}}}$ also has integral multiplicity in E_{c}. For smaller $\varepsilon > 0$, this holds for all $d \leq \varepsilon$.

Now, by Lemma 9.7, after increasing the boundary B, we can assume that $(X/T, B)$ is a purely log terminal Fano contraction. This gives the required strict inductive model of Proposition 9.13 satisfying (BNF), (εA$'$S), and (MXC). Now (εA$'$S) was proved above; also (εAS), without the integral condition. (BNF) follows from (LBF). (MXC) follows from (MXD).

Fix a linear equivalence $E' \sim E_{\mathrm{c}}$ with $E_{\mathrm{c}} \not\subset \operatorname{Supp} E'$. Since the sets $\mathfrak{F}$ in (BWQ) can be defined up to bounded addition of a fixed divisor (cf. Remark 6.35(1)) and E' is fixed, we get (BRE$'$) whenever (BWQ) is known. Indeed, for a bounded family of models E of E_{c}, the divisors $(\overline{E'_{|E_{\mathrm{c}}}})_E$ have bounded support, that is, they *add* a bounded set (really a single element!) to each element of $\mathfrak{F}$. The sum is included in the log transform of that from E_{c} whenever E/E_{c}.

Hence, to use Proposition 9.13, we want to verify (BWQ) after increasing B. This property is birational, and it results from the following conditions:

(1) Boundedness of the fixed component of each $|\mathcal{M}_i|_X$ on E_{c}; that is, the divisorial components of $\operatorname{Bs}|\mathcal{M}_i|_X \cap E_{\mathrm{c}}$ form a bounded family $\mathfrak{F}$.

(2) In each $|\mathcal{M}_{i|E_{\mathrm{c}}}|$, a general element has canonically confined singularities on a desirable triple of $(E_{\mathrm{c}}/\mathrm{pt.}, B_{\mathrm{c}})$ for the element (also just bounded if E_{c} is a surface); and

(3) desirable triples in (2) form a bounded algebraic family.

Thus, by (3), there exists a bounded family of triples $(E_u/T_u/\mathrm{pt.}, B_u, \mathfrak{F}_u)$ of the required form for $(E_{\mathrm{c}}/\mathrm{pt.}, B_{\mathrm{c}})$ (cf. Corollary 6.40). On the other hand, by Proposition 9.15, we obtain (1)–(3) because $(X/T, B)$ is now a strict inductive model with central divisor E_{c}, and (εAS) for b-divisors $i\mathcal{D}_i$ implies (SA$'$) for the b-divisors $\mathcal{M}_i \sim i\mathcal{D}_i$. Indeed, by definition, asymptotic saturation (εAS) for $\mathcal{D}_i$ and $j = i$ means the saturation of $i\mathcal{D}_i$ with respect to $\mathcal{A} + \varepsilon\overline{E_{\mathrm{c}}}$ (see Remark 4.34(5)). Increasing B again, we can extend E_{c} to $E_{\mathrm{c}} + M$ for any $0 < \varepsilon \ll 1$, where $M \geq M_1 = (\mathcal{M}_1)_X$ for a sufficiently general $\mathcal{M}_1$, and where $E_{\mathrm{c}} + M$ contains the fibre $f^{-1}P$. Then, by the invariance of saturation in Proposition 9.17 and by (LBF), the saturation of $i\mathcal{D}_i$ with respect to $\mathcal{A} + \varepsilon\overline{E_{\mathrm{c}} + M}$ implies the same saturation of $\mathcal{M}_i$. The latter implies (SA$'$) by Addendum 4.44.1 because each $\mathcal{M}_i$ is integral with $\operatorname{Bs} = \operatorname{Supp}(E_{\mathrm{c}} + M)$ in (BSP) (cf. Example 4.45). Indeed, then, by (BED), for all $\mathcal{M}_i$, $\operatorname{Bs}(\mathcal{M}_i)_X \subset \operatorname{Supp} M_1 \cup f^{-1}P \subset \operatorname{Bs}$.

In fact, we proved (1)–(3) in a slightly more general setting: we can consider an arbitrary family of b-free b-divisors $\mathcal{M} = \mathcal{M}_i$ that satisfy (BSP) and the saturation (SA$'$).

Before applying Corollary 7.9, note that we can assume that $\mathcal{D}_1 \sim \mathcal{M}_1$ is sufficiently general. Otherwise, we can replace $\mathcal{D}_1$ by general $\mathcal{D}_1^{\mathrm{g}} \sim \mathcal{D}_1$, that is, by $\mathcal{D}_1^{\mathrm{g}} = \mathcal{D}_1 + \overline{(a)}$, where $a \neq 0$ is a rational function on X/T. Then we replace each $\mathcal{D}_i$ uniformly by $\mathcal{D}_i^{\mathrm{g}} = \mathcal{D}_i + \overline{(a)} \sim \mathcal{D}_i$ (similarity!). Thus,

$$\mathcal{D}^{\mathrm{g}} = \lim_{i\to\infty} \mathcal{D}_i^{\mathrm{g}} = \lim_{i\to\infty} \mathcal{D}_i + \overline{(a)} = \mathcal{D} + \overline{(a)}.$$

These changes preserve the assumptions of the theorem. (LWF) is not affected. (LBF) continues to hold because $i\mathcal{D}_i^{\mathrm{g}} = i\mathcal{D}_i + i\overline{(a)} \sim i\mathcal{D}_i$; (LCA) by Proposition 9.17; (MXD) because $\mathcal{D}^{\mathrm{g}} = \mathcal{D} + \overline{(a)} \geq \mathcal{D}_i + \overline{(a)} = \mathcal{D}_i^{\mathrm{g}}$; (BED) because, outside $f^{-1}P$, each $\mathcal{D}_i^{\mathrm{g}} = \mathcal{D}_i + \overline{(a)} = \mathcal{D} + \overline{(a)} = \mathcal{D}^{\mathrm{g}}$.

Finally, we derive (BWQ) from (1)–(3). By Corollary 7.9 with $K = \mathbb{Q}$, $a_1 = 1$, $a_2 = i$, and $\mathcal{D}_2 = \mathcal{D}_i$, by (BED) which gives a similar condition in Corollary 7.9, and by (1), we obtain that each restriction $(\mathcal{D}_1 - \mathcal{D}_i)_{|E_c}$ up to linear equivalence, and in particular, its fractional components, has support bounded by some $F \in \mathfrak{F}$, where $\mathfrak{F}$ is a bounded set of reduced divisors. Since $\mathcal{D}_{1|E_c}$ is fixed, this last bound holds for each $\mathcal{D}_{i|E_c}$, and even on each E_u/T_u of our bounded family of triples. To be complete, we include also $\overline{E'|_{E_c}}$ into consideration by definition of mixed restrictions. Hence, each $(\mathcal{D}_{i|E_c})_{E_i} \in \mathfrak{N}(E_i/T_i/\mathrm{pt.}, \mathfrak{F}_i)$ for a desirable triple $(X_i/T_i/\mathrm{pt.}, B_i, \mathfrak{F}_i)$ because, by definition of such triple, $(\mathcal{M}_{i|E_c})_{E_i} = g_i^* M_i$, where $g_i \colon E_i \to T_i/\mathrm{pt.}$ and M_i is nef and big on $T_i/\mathrm{pt.}$ In particular,

$$(\mathcal{D}_{i|E_c})_{E_i} \sim_{\mathbb{Q}} ((\mathcal{M}_i/i)_{|E_c})_{E_i} = g_i^* M_i/i \sim_{\mathbb{Q}} 0/T_i$$

and is nef on $E_i/\mathrm{pt.}$

By (LBF), $i\mathcal{D}_{i|E_c} \sim \mathcal{M}_{i|E_c}$. So, by (2) and Addendum 6.8.2, the descent data of the restrictions $\mathcal{D}_{i|E_c}$ is asymptotically confined with respect to $\mathcal{A}_c$ over a sequence of desirable models E_i/T_i for $\mathcal{M}_i$ because B_i is the crepant boundary for B_c as in the definition of desirable triple (Definition 6.9), that is, $\mathcal{A}(E_i, B_i) = \mathcal{A}_c$, and $\lim_{i\to\infty} i = \infty$. □

Now we can prove stabilization of $\lim_{i\to\infty} \mathcal{D}_{i|E_c}$ on E_c. It is enough to prove this for a similar system $\mathcal{D}_\bullet$ (such as for characteristic systems, cf. Remark 4.34(7)).

Corollary 9.20. *Under the assumptions of Theorem* 9.9, *up to similarity of* characteristic type, *the system $\mathcal{D}_\bullet$ satisfies the theorem with $d_{c,1} = d_c = 0$.*

In addition, we can assume that the new $\mathcal{D}_1 = \mathcal{M}_1$ *is sufficiently general*; *in particular, $\mathcal{D}_1 \geq 0$.*

Proof. By the theorem, there exists a natural number $j > 0$ such that $d_{c,j} = d_c$. Take sufficiently general $\mathcal{D}_1^{[j]} = \mathcal{M}_j \sim jD_j$. This data defines the required similarity.

Indeed, as in the proof of Theorem 9.9, we can verify that any similarity with $j = 1$ preserves the assumptions of the theorem, and also its conclusions. The same holds for any truncation, that is, for new $\mathcal{D}_i := \mathcal{D}_i^{[j]} = j\mathcal{D}_{ij}$. (LWF) is not affected by the change. (LBF) continues to hold because $i\mathcal{D}_i^{[j]} = ij\mathcal{D}_{ij} \sim \mathcal{M}_i^{[j]} = \mathcal{M}_{ij}$; (LCA) by Proposition 9.17; (MXD) because

$$\mathcal{D}^{[j]} = \lim_{i\to\infty} \mathcal{D}_i^{[j]} = \lim_{i\to\infty} j\mathcal{D}_{ij} = j\mathcal{D} \geq j\mathcal{D}_{ij} = \mathcal{D}_i^{[j]}$$

for any natural number i; (BED) because each $\mathcal{D}_i^{[j]} = j\mathcal{D}_{ij} = j\mathcal{D} = \mathcal{D}^{[j]}$ outside $f^{-1}P$.

Finally, for general $\mathcal{M}_j$, $\mathrm{mult}_{E_c} \mathcal{M}_j = 0$ by (LBF) in Theorem 9.9. Hence, by our choice of j, $d_c = d_{c,1} = \mathrm{mult}_{E_c} \mathcal{M}_j = 0$. Note that stabilization takes place for exactly the same divisors (but their indexes may be different and the values of multiplicities in E_c are j times the old ones).

The final assertion of Corollary 9.20 holds by our choice of *new* $\mathcal{D}_1$. □

Corollary 9.21. *Under the conditions of Theorem* 9.9, *assume in addition*

(AMN) arithmetic monotonicity: $\mathcal{D}_i \geq \mathcal{D}_j$ *for any* $j \mid i$; *and*

(RRF) $d_{c,1} = d_c = 0$ *as in Corollary* 9.20.

Then $\lim_{i\to\infty} \mathcal{D}_{i|E_c}$ *stabilizes* (*that is,* $= \mathcal{D}_{j|E_c}$ *for some* $j \gg 0$; *compare Remark* 9.22(4) *below*).

More precisely, there exist a natural number $N > 0$ and a b-free divisor $\mathcal{M}$ of E_c such that $\mathcal{M}$ satisfies (*birational*) *saturation with respect to* $\mathcal{A}_c = \mathcal{A}(E_c, B_c)$ (*cf.* (SAT) *and Addendum* 6.26.2), *and if* $N \mid j$, *then*

- $d_{c,j} = d_c = 0$; *and*
- $j\mathcal{D}_{j|E_c} = (j/N)\mathcal{M}$.

Thus, each $\mathcal{M}_{j|E_c} \sim j\mathcal{D}_{j|E_c} = (j/N)\mathcal{M}$.

We also use this corollary in the following generalized form.

Addendum 9.21.1. *Under the assumptions of Theorem* 9.9, *we can weaken condition* (LWF) *to an arbitrary local log pair with $K+B$ $\mathbb{R}$-Cartier, weakening inductive model in the conclusion to a* projective generalized inductive model $(Y/T, B_Y)$; *that is, purely log terminal in Definition* 9.3(2) *is replaced by*

(ELT) E_{c} *is an* exceptionally log terminal LCS center *for* (Y, B_Y), *that is,* (Y, B_Y) *is exceptionally log terminal near the general point of* E_{c}, *but not Klt* (E_{c} need not be a divisor on this model);

(GIM) $B_Y \geq 0$, $K_Y + B_Y$ *is* $\mathbb{R}$*-Cartier with*

$$-1 = \mathrm{mld}(Y, B_Y, E_{\mathrm{c}}) < a(X, B, E_{\mathrm{c}}),$$

and is ample/$Z = T$ in a neighborhood U of $\mathrm{LCS}(Y, B_Y)$ *minus the open subset of Klt points in* $(E_{\mathrm{c}}, B_{\mathrm{c}})$ (*see the proof below and compare Conjecture* 5.26); *and stabilization of $\mathcal{D}_\bullet$ with the ample property of the limit holds in the neighborhood U.*

We also need to replace $(\mathrm{CCS})_{n-1}(\mathrm{gl})$ *by* $(\mathrm{CCS})^*_d$ *or by* $(\mathrm{FGA})^*_d$ *with* $d = \dim E_{\mathrm{c}}$ (*and only the global case if* E_{c}/P); *we can omit LMMP and* (SSB) *together. In addition, we need to assume that* $\operatorname{Supp} B_Y \supset E_{\mathrm{c}}$ *and* (BP) *holds.*

Of course, we assume that $\mathcal{D}_{\bullet\restriction E_{\mathrm{c}}^{\nu}}$ *is well defined*: *e.g., each* $\mathcal{D}_i = 0$ *over the generic point of* E_{c} (*a generalization of* (RRF); *see also Fixed restriction* 7.2); *it is automatic up to similarity if U intersects E_{c} or $(E_{\mathrm{c}}, B_{\mathrm{c}})$ is not Klt.*

Addendum 9.21.2. *In dimension $n \leq 3$, LMMP,* $(\mathrm{CCS})^*$ *and* (SSB), (BP) *for $n \leq 4$ are proved, so they are no longer assumptions.*

Remarks 9.22. (1) (AMN) is preserved for any similarity, in the same way as (MXD) of Theorem 9.9.

(2) Under the assumptions of Theorem 9.9, (AMN) implies (MXD). For this, we need that $\mathcal{D}_\bullet$ is bounded by a Cartier divisor (possibly nonzero at E_{c}). Then, to prove (MXD), we use (LBF) or (BNF) and Lemma 4.23 (cf. Proposition 4.22).

(3) We can take a truncation of $\mathcal{D}_\bullet$ such that the corollary holds with $N = 1$.

(4) We *only* expect $\mathcal{D}_{j\restriction E_{\mathrm{c}}} = \mathcal{D}_{\restriction E_{\mathrm{c}}}$ and $\mathcal{M}_{j\restriction E_{\mathrm{c}}} = j\mathcal{D}_{\restriction E_{\mathrm{c}}}$ if stabilization holds in a neighborhood of E_{c} on some model (cf. Theorem 10.13 below).

(5) In Addendum 9.21.1, (RRF) and, moreover, the assumption that each $\mathcal{D}_i = 0$ over the generic point of E_{c} do not follow from the other assumptions even up to similarity for $\mathcal{D}_\bullet$ (cf. Corollary 9.20).

(6) If $d_{\mathrm{c},j} = 0$, $\mathcal{D}_{j\restriction E_{\mathrm{c}}}$ is the *fixed* restriction as in (fx) of Proposition 7.7.

(7) (AMN) and (LBF) imply the stabilization equation $\mathcal{D}_j = \mathcal{D}_i = \mathcal{D}$ for certain multiplicities, namely, for the multiplicities in E_i with centers on X outside $(\mathcal{D}_j)_X \neq \mathcal{D}_X$ and $\mathrm{Bs}\,|\mathcal{D}_j|_X$, and the bound

$$\mathcal{D}_j \leq \mathcal{D}_i \leq \mathcal{D} \leq \overline{(\mathcal{D}_j)_X}$$

outside $(\mathcal{D}_j)_X \neq \mathcal{D}_X$ and the non-$\mathbb{Q}$-Cartier points of $(\mathcal{D}_j)_X$, if $j \mid i$, under the lca saturation for such j, i, and over $\lceil \mathcal{A} \rceil \geq 0$ (by Lemmas 10.9 and 4.23, respectively); this holds, in particular, over E_{c} outside $\operatorname{Supp} \mathcal{D}_X$ on the inductive model of our corollary. Under (AMN) and (LBF), the converse on saturation also holds outside $\mathrm{Bs}\,|\mathcal{D}|_X$ over $\mathcal{A}_X \leq 0$ (cf. Example 4.35).

Lemma 9.23. *Let* $(X/T, B)$ *be an inductive model* (*possibly generalized as in Addendum* 9.21.1 *under* (BP)) *with central divisor* E_{c} (*respectively, center* $\mathrm{center}_Y\, W = E_{\mathrm{c}}$ *for a prime b-divisor* W *with* $a(Y, B_Y, W) = -1$), *and* $\mathcal{D}_\bullet$ *be a system of* $\mathbb{R}$*-b-divisors such that*

(BSA) *each* $\mathcal{D}_i$ *is* b-semiample *as in Proposition* 4.50 (*respectively, in Addendum* 4.50.3), *or just satisfies* (BNF);

(ASA$'$) *the system* $\mathcal{D}_\bullet$ *is asymptotically saturated with respect to* $\mathcal{C} = \mathcal{A}' = \mathcal{A}(X,B) + E_{\mathrm{c}}$ (*respectively, with* W *instead of* E_{c}); *and*

(FXR) *each* $d_{\mathrm{c},i} = 0$ (*see the notation before Theorem* 9.9; *respectively, each* $\mathcal{D}_i = 0$ *over the generic point of* E_{c}); *in particular, each of the fixed restrictions* $\mathcal{D}_{i}{}_{\restriction E_{\mathrm{c}}}$ *is well defined.*

Then the restricted system $\mathcal{D}_\bullet{}_{\restriction E_{\mathrm{c}}}$ *satisfies lca saturation over* $(E_{\mathrm{c}}^\nu, B_{\mathrm{c}})$ (*cf.* (LCA) *in Conjecture* 4.39), *where* E_{c}^ν *is the normalization of* E_{c}.

For usual inductive models, $E_{\mathrm{c}} = E_{\mathrm{c}}^\nu$ is normal.

Proof. Immediate by Proposition 4.50 with $Y = E_{\mathrm{c}}$. (GLF) and (LCC) follow from properties of inductive models. (FXR) ensures general position (GNP). Moreover, (BSA) can be replaced by (BNF) because b-nef is enough for Kawamata–Viehweg vanishing and saturation in the proof of Proposition 4.50. (Respectively, use Addendum 4.50.3. See the proof of Addendum 9.21.1.) $\square$

Lemma 9.24. *Let* $\mathcal{D}_\bullet$ *be a system of b-divisors on a log pair* $(X/Z, B)$ *such that*

- *either* (X,B) *is Klt or* $\mathcal{D}_\bullet$ *stabilizes over* $\mathrm{LCS}(X,B)$;
- $\mathcal{D}_\bullet$ *is bounded by a b-divisor and satisfies* (AMN), (LBF), *and lca saturation over* (X,B).

Then, up to a truncation, it converges to a b-divisor.

Thus, (AMN) works in the same way as convexity in Lemma 4.24.

Proof (compare the proof of Lemma 4.24)**.** Up to a similarity preserving all the assumptions (cf. the end of the proof of Theorem 9.9 and Remark 4.34(7)), we can assume that $\mathcal{D}_1 \ge 0$ by (LBF), moreover, $\mathrm{Mov}\lceil \mathcal{D}_1 + \mathcal{A}\rceil \ge 0$ over Klt points not over $c(\mathrm{Supp}\, J)$, where c is given by $\mathcal{D}_1$ with the maximal numerical dimension (over $c(\mathrm{Supp}\, J)$, $\mathcal{D}_\bullet$ stabilizes; see (RST) in Corollary 5.21 and the proof of Addendum 5.21.1), and that $\mathcal{D}_\bullet$ satisfies lca saturation with index $I = 1$. Hence, each $\mathcal{D}_i \ge 0$ by (AMN). Since the system is bounded, (AMN) implies that it has a convergent subsequence. In addition, for any natural numbers i, j, q, and r satisfying $i = jq + r$, lca saturation and stabilization imply the estimate

$$\frac{jq}{jq+r}\mathcal{D}_j \le \mathcal{D}_i.$$

This implies that the sequence has a limit because it is bounded and has a convergent subsequence $\mathcal{D}_j$. Indeed, by lca saturation for $\mathcal{D}_j$ and since $\lceil r\mathcal{D}_j + \mathcal{A}\rceil \ge 0$ over above Klt points, we get the required inequality there:

$$jq\mathcal{D}_j \le qj\mathcal{D}_j + \mathrm{Mov}\lceil r\mathcal{D}_j + \mathcal{A}\rceil \le \mathrm{Mov}\lceil (jq+r)\mathcal{D}_j + \mathcal{A}\rceil \le (jq+r)\mathcal{D}_i.$$

Over $c(\mathrm{Supp}\, J)$, the above estimate holds by stabilization $\mathcal{D}_i = \mathcal{D}_j \ge 0$. $\square$

Proof of Corollary 9.21. First, by (AMN), (MXD), and (RRF), we have $d_{\mathrm{c},i} = d_{\mathrm{c}} = 0$ for each natural number i. Thus, the fixed restriction $\mathcal{D}_{i}{}_{\restriction E_{\mathrm{c}}}$ is well defined for each i.

Secondly, restriction preserves properties (LBF), (LCA), (MXD), and (AMN) of Theorem 9.9 and the corollary. Now we define the new objects

$$(X/T, B) := (E_{\mathrm{c}}/\mathrm{pt.}, B_{\mathrm{c}}), \qquad T := \mathrm{pt.}, \qquad \mathcal{D} := \mathcal{D}_{\restriction E_{\mathrm{c}}} \text{ (if exists)},$$

$$\mathcal{D}_i := \mathcal{D}_{i}{}_{\restriction E_{\mathrm{c}}}, \qquad \mathcal{M}_i := \mathcal{M}_{i}{}_{\restriction E_{\mathrm{c}}}, \qquad \text{and} \qquad \mathcal{A} := \mathcal{A}_{\mathrm{c}} = \mathcal{A}(E_{\mathrm{c}}, B_{\mathrm{c}}).$$

Indeed, we get (LBF), (MXD), and (AMN) by the definition of restriction; (AMN) implies (MXD) by Remark 9.22(2). For (AMN), note that a birational fixed restriction of an effective divisor is effective.

By Lemma 9.23, (LCA) for $\mathcal{D}_\bullet{}_{\restriction E_{\mathrm{c}}}$ on $(E_{\mathrm{c}}/\mathrm{pt.}, B_{\mathrm{c}})$ follows from (ASA$'$) of Lemma 9.23 for the inductive model in the proof of Theorem 9.9. In turn, (ASA$'$) follows from (εA$'$S) of Proposition 9.13 with $d = 0$ by (FXR), and (εA$'$S) itself follows from (LWF) and (LCA) of Theorem 9.9 by Addendum 4.44.1 and properties (1), (2) in Definition 9.3 of inductive models (cf. Example 4.52).

By Lemma 9.24 and the bound on $\mathcal{D}_{\bullet|E_c}$ (cf. Remark 9.22(2)), up to a truncation, the limit $\lim_{i\to\infty}\mathcal{D}_{i|E_c}$ exists and is a *candidate* for $\mathcal{D}_{|E_c}$ (cf. Remark 9.22(4)).

Now the corollary follows from the solution of the asymptotic descent problem in Addenda 5.12.1 and 5.13.1 for the limit $\mathcal{D} := \lim_{i\to\infty}\mathcal{D}_i$ with $\mathcal{D}_i := \mathcal{D}_{i|E_c}$. (FDS) follows from (MXD) for effective $\mathcal{D}_\bullet$. Asymptotic saturation holds for $\mathcal{C} = \mathcal{A}(E_c, B_c)$ by (LCA). (LBF) implies (BNF). A prediction model exists by (CCS)(E_c/pt., B_c), or (CCS)$_{n-1}$(gl). This can be done exactly as in the proof of Theorem 6.19(3). The only difference between our current situation and (FGA) is that we do not have the convexity of Lemma 4.24. (However, in our applications, this is satisfied.) But this is not at all required for Theorem 6.19(3) and Addendum 5.12.1. We only need (AMN) under (LCA)! □

Proof of Addendum 9.21.1. This follows by modifying the above arguments as follows. First, after increasing B_Y (as in the proof of Lemma 9.7; the required contraction exist due to [2, Theorem 5.6]), we can assume that $(Y/T, B_Y)$ is a generalized Fano contraction. Second, we replace E_c by its normalization E_c^ν (in our application, E_c can be nonnormal) and set $\mathcal{A} := \mathcal{A}(E_c^\nu, B_c)$, where (LCA) means lca saturation over (E_c^ν, B_c). (In general, $E_c^\nu/T := T_{E_c^\nu}$, where $E_c^\nu \to T_{E_c^\nu} \to f(E_c)^\nu$ is the Stein factorization of the normalization $E_c^\nu \to f(E_c)^\nu$. If the last map is not a contraction, we also use Remark 4.40(1).)

Here the main difficulty is adjunction, as we discuss presently. Then we use Corollary 5.23 with Example 5.25 (cf. also Addendum 5.21.1) instead of Addenda 5.12.1 and 5.13.1 and the proof of Addendum 6.19.1 instead of the proof of Theorem 6.19(3); stabilization of $\mathcal{D}_\bullet$ on a neighborhood of $\mathrm{LCS}(E_c^\nu, B_c)$ follows from stabilization on U of (GIM) (see p. 199). A prediction model now exists by (CCS)*(E_c^ν/pt., B_c), or (CCS)$^*_{n-1}$(gl). Indeed, $\mathcal{M}_i$ and $K_{E_c^\nu} + B_c$ ample in a neighborhood of the LCS again follow from ampleness on U of (GIM). If E_c is divisorial, we can use Adjunction Formula 3.1 in [41] and Addendum 4.50.1. If it is not divisorial, we use Addendum 4.50.3 instead (cf. also Lemma 9.23).

Unfortunately, for nondivisorial adjunction, the divisorial part of the boundary $B_{\mathrm{div}} = (\mathcal{B}_{\mathrm{div}})_{E_c^\nu}$ does not give a good boundary of $(E_c^\nu, B_{E_c^\nu})$; even lca saturation holds with respect to its associated discrepancy $\mathcal{A}_{\mathrm{div}}$. Here we have two problems in general:

- $K_{E_c^\nu} + B_{\mathrm{div}}$ can be non-$\mathbb{R}$-Cartier; and
- adjunction does not hold, even numerically: $(K+B)_{|E_c^\nu} \not\equiv K_{E_c^\nu} + B_{\mathrm{div}}/T$ (in the case under consideration $T = \mathrm{pt.}$).

As indicated by Kawamata [23, 24], both problems can be solved at one go by introducing a moduli contribution B_{mod} to adjunction $B_{E_c^\nu} = B_{\mathrm{div}} + B_{\mathrm{mod}}$. For example, if $B \ge 0$ and (Y, B_Y) is exceptionally log terminal near E_c, then Kawamata proved that, on some good model of E_c^ν, B_{mod} is nef/T under the technical assumption that $\operatorname{Supp} B_Y \supset E_c$; this holds in our case. If the nef holds on E_c^ν, since $(E_c^\nu/T, B_{\mathrm{div}})$ is a log Fano map, B_{mod} is semiample/T with $\sim_{\mathbb{R}}$ in the adjunction. Taking general effective B_{mod}, we get a log Fano map $(E_c^\nu/T, B_{E_c^\nu})$ with $B_{E_c^\nu} \ge B_{\mathrm{div}}$ under (BP) over E_c^ν. Hence, lca saturation also holds over $(E_c^\nu, B_{E_c^\nu})$. If we cannot use the log Fano property and (BP) over E_c^ν, again on some good model, over which (BP) holds, we can assume that $B_{\mathrm{div}} \ge H$, where H is ample/T $\mathbb{R}$-divisor. This holds after increasing B_Y. Then we can do a perturbation: $B_{E_c^\nu} := B_{\mathrm{div}} - \varepsilon H + B'_{\mathrm{mod}}$, where B'_{mod} is a general effective $\mathbb{R}$-divisor $\sim_{\mathbb{R}} B_{\mathrm{mod}} + \varepsilon H$ that is ample/T, and push down all this to E_c^ν (see [2, Theorem 4.9 and Remark 4.10.2]). However, in this case, to apply Lemma 4.44, we again need to increase B_Y so that lca saturation also holds over $(E_c^\nu, B_{\mathrm{div}} - \varepsilon H)$.

In our case, $B_Y \ge 0$, but it may have log singularities near E_c other than E_c itself. In this case, we expect that all the above works, and again it is better to use b-concepts, e.g., that B_{mod} is b-semiample. (Actually, it is better to consider $\mathcal{B}_{\mathrm{div}}$ as a b-divisor of boundary type, but $\mathcal{B}_{\mathrm{mod}}$ should be of b-Cartier type, compare [14].) Then, taking the image on E_c^ν of a general divisor of this type, we get an $\mathbb{R}$-divisor that is effective up to $\sim_{\mathbb{R}}$, and only has "fixed locus up to $\sim_{\mathbb{R}}$" but $\mathrm{LCS}(E_{\mathrm{c}}^{\nu}, B_{E_{\mathrm{c}}^{\nu}})$ is in the closure of $\mathrm{LCS}(X,B) \setminus E_{\mathrm{c}}$. This is still conjectural.

However, by Kawamata [24], over a log resolution and under our assumption, $\mathcal{B}_{\mathrm{mod}}$ is b-nef$/T$. Combined with the above perturbation, this is enough under (BP). Here log resolution means a model where $B_Y = D + B$, with $D \geq 0$, (X,B) exceptionally log terminal, and the same LCS center E_{c}. To construct such a model as a log canonical model over Y, we apply LMMP to a log resolution with exceptional boundary coefficients

$$1 > b > 1 - (\text{mld among} > 0)$$

(such exists), except for E below, the same boundary multiplicity for nonexceptional divisors with boundary multiplicities ≥ 1, and a single multiplicity $b = 1$ for E with $\mathrm{center}_Y E = E_{\mathrm{c}}$ and $a(Y, B_Y, E) = -1$. It preserves Y exactly out of the above closure of $\mathrm{LCS}(X,B) \setminus E_{\mathrm{c}}$. (We can use the generalized Fano contraction given by the resolution if (FGA)* and (CCS)* hold with the Cartier semiample assumption on the LCS instead of ampleness; cf. Conjectures 5.26, 6.14 and Remark 6.15(9).)

At the end of the proof, we give a more extended explanation when $n = \dim X = \dim Y = 3$; this is enough for 4-fold flips. (In this case, Kawamata proved that the modular part is b-semiample.) Thus, $\dim E_{\mathrm{c}} \leq 2$. In the current situation, $(E_{\mathrm{c}}^{\nu}/T, B_{\mathrm{c}})$ is a generalized Fano contraction, usually with worse than log canonical singularities. As above, we apply Corollary 5.23 instead of Addendum 5.12.1. In addition, we sketch below a proof of the former in this low-dimensional circumstances. Suppose that $E = E_{\mathrm{c}}^{\nu}$ is a surface; then, by Proposition 4.50, together with the original $\mathcal{D}_{\bullet}$, the new $\mathcal{D}_{\bullet} := \mathcal{D}_{\bullet}|_E$ satisfies lca saturation (LCA). The only novelty is that we now have essentially negative components of $\mathcal{A} := \mathcal{A}(E, B := B_{\mathrm{c}})$ (that is, components with discrepancies ≤ -1). The negative components defined the ideal $J = J_E$ (see Example 5.25). Since each $j\mathcal{D}_i$ is Cartier on $\mathrm{LCS}(E,B)$, saturation is equivalent to the following inclusion (see Example 5.25 and cf. (JAS) in Corollary 5.21):

(JLC) On any sufficiently high model E_{hr}/T of E/T that is the *identity* or *contractible* in the neighborhood $U := U \cap E$ of the LCS, we have

$$f_* J(\lceil j\mathcal{D}_i + \mathcal{A} \rceil_{E_{\mathrm{hr}}}) \subset h_* \mathcal{O}_{E_{\mathrm{hr}}}((j\mathcal{D}_j)_{E_{\mathrm{hr}}}),$$

where $f\colon E \to T$ and $h\colon E_{\mathrm{hr}} \to T$.

We construct a prediction model $(E/T, \mathcal{C} = \mathcal{A}, F, \gamma)$ for the descent problem $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ on a modification of $(E/T, B)$ to a triple (we assume that E is normal). The modification is *contractible* if it does not blow up exceptional divisors over U. However, it can contract divisors: increase U and contract a complete subvariety in the resulting bigger open set (see below for an example). This gives (EEF) on any terminal resolution over $E \setminus U$. The system $\mathcal{D}_{\bullet}$ is bounded by Example 4.21. Up to similarity, each $\mathcal{D}_i$ is effective and $(\mathcal{D}_i)_U = \mathcal{D}_U$ is stable on U. The unstable part is over an effective divisor F supported in the complement $E \setminus U$. This gives (UAD) and (LGD) for some real number $\gamma > 0$ (cf. Example 8.21).

In the two-dimensional case, $\mathcal{D}_E$ is nef $=$ nef$/T$ in codimension 2 (that is, nef over general complete curves$/T$ in any prime divisor; it is $\mathbb{R}$-Cartier in codimension 2 by stabilization in U) because this holds for each $(\mathcal{D}_i)_E/T$. The latter follows from (LBF): each $\mathcal{M}_i := i\mathcal{D}_i = \mathcal{L}_i|_E$ is b-free$/T$. Then (SAM) follows from generalized semiampleness (see Ambro [2, Theorem 5.6]): if D is an $\mathbb{R}$-Cartier divisor on E/T that is Cartier, ample on U/T, and nef$/T$, then it is semiample$/T$. By the rationality of the nef cone for such divisors (variations only over F; LMMP* contracts all exceptional curves in F), it is enough to verify this for $\mathbb{Q}$-divisors. This is essentially a consequence of the generalized Fano property. For more details, see [2, Theorem 6.2]. To check (SAC), we can use Proposition 6.26. However, to apply this to approximations, we need more from the triple and the prediction model: namely, they should be strictly desirable (WAM) for $\mathcal{D}$, as explained below. If the Kodaira dimension of $\mathcal{D}$ is 0, then each $\mathcal{D}_i = 0$ and we get the stabilization of $\mathcal{D}_\bullet$. If the Kodaira dimension is 1, then, after a truncation, $|\mathcal{M}_i|_E$ is a pencil for each i. Its base points are only at good points, outside U. Resolving these gives a prediction model in a form of a triple $(E/W/T, B, F)$ with crepant B, single F, and $g\colon E \to W/T$ given by the pencil. Moreover, each $(\mathcal{D}_i)_E = g^*M$ for some ample M on W/T. In general, this allows us to improve desirable triples for $\mathcal{M} = \mathcal{D}$ (possibly, not b-free) and $\mathcal{M} = \mathcal{M}_i$:

(WAM) a desirable triple for $\mathcal{M}$ is *strict* if M is ample on W/T (cf. Definition 6.9); or at least $C \cdot M > 0$ on each complete curve in W/T that does not intersect $g(\operatorname{Supp} J)$ (cf. Remark 6.10(2)).

If U dominates W/P, we get the stabilization from W, induced in turn from U (cf. (RST) in Corollary 5.21). Otherwise, U and $\operatorname{Supp} J$ do not dominate W; thus, in the approximation methods in the proof of Theorem 5.12, for $j \gg 0$, we can disregard J in (JLC). This means that

(BFF) for any ample divisor H on W/T, or H that is Cartier in a certain cone of semiample/T divisors under the weak assumption in (WAM), and any proper subscheme of W given by an ideal J_W, $|J_W(NH)|$ is free for $N \gg 0$ outside $\operatorname{Supp} J_W$ and the union of contractible curves/T, that is, with $C \cdot H = 0$ and $C \cap \operatorname{Supp} J_W \neq \varnothing$.

We can also use this in general (cf. the proof of Addendum 4.50.2). In our case, $J_W = g(J)$, and the system $\mathcal{D}_\bullet$ stabilizes in an open set that is bigger than U and includes the above contractible curves. Note that, in both previous cases, we get (SAC) without (CCS) (cf. Example 6.24). But we need Example 4.41 with bad singularities. Finally, suppose that $\mathcal{D}$ is big; then, up to a truncation, the same holds for each $\mathcal{D}_i$ (by the above considerations, (AMN), and Lemma 6.22(3)).

By saturation, say (LCA) with $i = j$, and Proposition 6.26, each $|\mathcal{M}_i|_E$ is free on a terminal resolution (again up to a truncation). This gives (CCS) and (SAC). The triple is the resolution $(E/E/T, B, F)$. However, to apply approximations as in the proof of Theorem 5.12, we need (WAM) and (BFF) for $\mathcal{D}$; that is, $D = \mathcal{D}_E$ should be ample/T. If this fails, there is a complete curve C/T with $C \cdot D = 0$. If C does not intersect $\operatorname{Supp} J$, we can contract it because C is exceptional (as in LMMP* with good divisors on the bad singularities [2, Theorem 5.6]). This gives a birational contraction $g\colon E \to W/T$. The triple $(E/W/T, B_T, F_W = g(F))$ is a generalized log Fano contraction with the same properties as $(E/E/T, B, F)$, except possibly for *nonterminal* points near F but still Klt.

After a number of contractions of this type, we get (WAM) in the weak form, that is, each C with $C \cdot D = 0$ intersects $g(\operatorname{Supp} J)$, where $D := \mathcal{D}_W$. Since $D \geq 0$, if we take $\mathcal{D}$ general on U, then C is disjoint from $\operatorname{Supp} D$. Then stabilization holds over C (see Lemma 10.9). We can increase U to include a neighborhood of $g^{-1}C$ and then contract C (together with all the curves in the same numerical class, to preserve the algebraic category, but maybe not the generalized Fano property). In the *big* case, we now consider E/W that are not the identity (cf. Example 5.27 and the proof of Corollary 6.40). After that, the methods of the proof of Theorem 5.12 apply to the prediction model $(E/T, \mathcal{A}, F, \gamma)$ with approximations from W/T (cf. Example 5.18 and see the proof of Corollary 5.21). If E_{c} is a curve, the same arguments lead to a (rational) pencil of rational curves on E/E_{c} that do not intersect the LCS on the surface E with log discrepancy 0 over the curve. Such a pencil is free, and we can treat it as the cases of Kodaira dimensions 0 and 1 above. Or we can use Addendum 4.50.3 as above with adjunction on E_{c}^ν [23]. □

Stabilization *near* E_{c} needs an opposite technique that we discuss in the next section.

10. DESTABILIZATION

We find a neighborhood of E_c on a 0-log pair as in Remark 3.30(2), with a *certain* boundary, over which the system $\mathcal{D}_\bullet$ stabilizes. We also work with generalizations of such pairs (cf. (WLF) versus (GLF) and (PFN)).

Definition 10.1. Let $(X/T, B)$ be a log pair such that

- X/T is a projective contraction;
- $K + B \equiv 0/T$;
- $(X/T, B)$ is *exceptionally* log terminal; and, moreover,
- there exists *at most one* prime b-divisor E_c with log discrepancy 0 for $K + B$, and E_c is/P.

If such an E_c exists, we say that $(X/T, B)$ is a 0-*exceptional log terminal pair*, or 0-*elt* pair, and that the b-divisor E_c is *central*. Otherwise, $(X/T, B)$ is Klt and is just a 0-*log pair*.

The role of 0-log pairs is based on the following property that we use throughout this section:

- any weakly log canonical model [45, Definition 2.1] of a 0-log pair is a weak log Fano contraction.

Moreover, then X/T is birational (or just generically finite/T), it can be perturbed into a Fano contraction, and each weakly log canonical model is crepant [45, Proposition 2.4].

Example 10.2. Usually a 0-elt pair arises as a *complement* as follows. Suppose that $B = B_\eta + \eta C$ such that $(X/T, B_\eta)$ is a 0-log pair and $C \geq 0$ with $\eta = \operatorname{mult}_{E_c} C > 0$. Note that in this situation $C \sim_{\mathbb{R}} 0/T$, and it defines canonically a b-divisor $\mathcal{C}$ such that $C = \mathcal{C}_X$ and $\mathcal{C}$ is possibly b-free over T whenever B_η and η are rational.

A typical example is building a log singularity. Let $(X/T, B)$ be a *birational* weak log Fano contraction, that is, such that X/T is birational. Then, after taking a complement as in [41, предложение 5.5], we can assume that $K + B_\eta \equiv 0/T$. Thus, after a contraction, X/T is the identity. Now we can find the required $C \geq 0$ passing through P such that $K + B_\eta + \eta C$ has only one log discrepancy 0 in E_c for some threshold $\eta > 0$, and E_c is/P. This is how we constructed our inductive model in Lemma 9.6.

To be more precise, we introduce (birational) neighborhoods of E_c on certain special and quite nice (log) canonical models. They give an induction on the *rank* of non-$\mathbb{Q}$-factoriality and the number of seminegative discrepancies (an analogue of the notion of *difficulty* [40, определение 2.15]; cf. Lemma 10.12 below).

Key 10.3. For given b-free $\mathcal{L}$, we need to construct a birational contraction Y/T and an open set U in it (in applications, a neighborhood of some subvariety; see Definition 10.11 below) with an effective $\mathbb{R}$-divisor $D = \sum d_i E_i$ such that

- $d_i > 0$ in U precisely on the prime divisors E_i that are exceptional on T;
- $L + \sum d_i E_i$ is nef and even semiample/T in U, where $L = \mathcal{L}_Y$; and
- $\mathcal{L}$ is free over Y/T.

We call such a model a *destabilizing model* (destab model) for $\mathcal{L}$, because we can add to it (or to its multiple) a divisor D on Y that violates (destabilizes) D-saturation (for example, the exceptional saturation of Example 4.35; see Step 2 of the proof of Theorem 11.1) when $D \neq 0$ or Y/T is nonsmall (see Caution 10.4). The divisor D is *destabilizing*, and it defines the b-divisor $\mathcal{D} = \overline{D}$.

Caution 10.4. The contraction Y/T is not necessarily divisorial.

Example 10.5. (1) If $L = \mathcal{L}_T = 0$, we can take $Y = T$. Slightly more generally, if $L = \mathcal{L}_Y$ is ample/T and free on Y, we can take any Cartier D on Y that is effective in each exceptional divisorial component of Y/T (replacing $\mathcal{L}$ by a rather high multiple).

(2) If $\dim X = 2$ and L is a curve passing through P on T, then, for any $\mathbb{Q}$-factorial model Y/T over which $\mathcal{L}$ is free, we can find a $\mathbb{Q}$-divisor $D > 0$ that converts Y/X to a destab model. Indeed, we can find a semiample divisor $(D + L)/T$ with $\mathbb{Q}$-Cartier $D > 0$ (cf. Lemma 11.2). The same holds over the open set $Y \setminus \operatorname{Bs} |\mathcal{L}|_Y$ if this linear system has a nontrivial mobile component on Y/P.

(3) In some cases in dimension ≥ 3, we cannot construct a *divisorial* contraction Y/T: e.g., if X/T is a flopping 0-elt pair and $L = \mathcal{L}_X$ is its hyperplane section. This means that, in this case, by Lemma 10.9, we cannot destab $\mathcal{L}$ on any divisorial Y/T (compare the proof of Theorem 10.13).

Probably, a destab model does not exists for every $\mathcal{L}$ in dimensions ≥ 3. However, in certain important situations, it exists by LMMP.

Proposition 10.6. *Let $(X/T, B)$ be a local weak log Fano contraction with birational contraction X/T and $\mathcal{L}$ be a b-free divisor. Then LMMP in dimension $n = \dim X$ implies the existence of a destab model for $\mathcal{L}$.*

The same holds for a purely log terminal model whose reduced part is not exceptional/T; or if (X, B) has only log terminal singularities with reduced centers not over P and $\mathcal{L}$ is free/T over $X \setminus f^{-1}P$. Or we can assume log terminal singularities and that $\mathcal{L}$ is in rather general position, *that is, not passing through the log canonical centers of (X, B) and under* LSEPD (*see* [41, 4.5] *or* [27, 2.30]).

We only need flips as noninductive new objects in LMMP *under the* LSEPD trick [41, 0-сдутия 4.5].

However, sometimes *very* special flips are enough (cf. Example 10.10 and Theorem 10.13 below).

Addendum 10.6.1. *If we can construct such a destab model over a neighborhood U on Z/T, where Z is the model defined by $\mathcal{L}/T$, then destab holds over the neighborhood. The construction in the proof is log canonical; thus, it can be done locally/Z.*

The proof of the proposition is completely effective.

10.7. Proof of Proposition 10.6. Construction of a destab (destabilizing) model. (1) Increasing B, we can convert our model into a 0-elt pair as in Example 10.2. Then we can identify $(X = T/T, B = B_T)$.

(2) We take a log resolution $(Y/T, B_Y)$ of $(T/T, B_T)$ such that

- $B_Y = f^{-1}B_X + \sum E_i$ (a noncrepant boundary!), where $f\colon Y \to X = T$ and the divisors E_i are exceptional on T; and
- $L = \mathcal{L}_Y$ is free over Y, in particular, $\mathcal{L} = \overline{L}$, where $L = \mathcal{L}_Y$.

In plt and log terminal cases, we can do this without resolving the LCS centers. Then we proceed as follows:

(2′) We suppose that $L > 0$ is rather powerful: $L \equiv NH/T$, where H is Cartier (semiample/T): for example, $N \geq 2\dim X + 2$ is enough. Otherwise, we replace L by a multiple up to $\equiv/T$. Finally, we get the same multiple for the destab divisor D.

(3) We apply LMMP to $(Y/T, B_Y + L)$. At each step, we contract an extremal ray R with $R \cdot (K_Y + B_Y + L) < 0$ by a birational contraction $g\colon Y \to Z/T$. Note that $R \cdot L = 0$ by (2′) and boundedness of negative contractions [44, Theorem]. Thus, modifications only contracting or blowing up curves C with $C \cdot L = 0$ preserve the freedom of L and (2′). Finally, we obtain a model where $K_Y + B_Y + L$ is nef; it is always big because Y/T is birational, or by (2′) again.

(4) Then we make a semiample contraction for $K_Y + B_Y + L$. (By [25, Theorem 3-1-1], for a $\mathbb{Q}$-boundary B_Y, this is enough for our application; for the general case, cf. [45, Conjecture 2.6, Theorem 2.7, and Remark 6.23.5].) We get this because the divisor is always big, because Y/T is birational, and because, after a perturbation, B_Y is a $\mathbb{Q}$-divisor (the supporting face for $K + B_Y + L$ in the cone $\overline{\mathrm{NE}}(Y/T)$ is rational polyhedral and acute since, after another perturbation, we can

put it into the negative part of the cone with respect to $K + B_Y + L$ and use the Cone Theorem [25, 4-2-1]). We also need the LSEPD trick [41], and can do it if we secure it. This also preserves the freedom of L because, by ($2'$) again, $C \cdot (K_Y + B_Y + L) = 0$ implies $C \cdot L = 0$ for a curve C. This is a destab model.

(5) Finally, we take a destab divisor $D = l\mathcal{A}_Y$ (so that $\mathcal{D} \le l\mathcal{A}(T, B_T)$), where l means that we take log discrepancies. By the definition of log discrepancies,

$$D = \sum d_i D_i = K_Y + B_Y - g^*(K + B) \equiv K_Y + B_Y/T$$

for $g\colon Y \to T$ because $K + B \equiv 0/T$ for a 0-elt model. Since it is Klt, all the $d_i > 0$. The same holds for a purely log terminal model if its reduced part is not exceptional. For a terminal model under (2).

$D + L \equiv K_Y + B_Y + L$ is ample$/T$ by construction. In particular, it is nef and semiample$/T$; in addition, Addendum 10.6.1 holds over U by ampleness of $\mathcal{L}/T$ whenever the model exists$/U$; over $Z \setminus U$, the model can be enlarged arbitrarily.

Finally, from LMMP in dimension n, we only need flips. Indeed, the termination is special because we have pl flips (cf. Example 10.10). Thus, we get it by induction by Special Termination 2.3. However, at present, the main problem is with flips. The model Y is really over Z/T; so, if we have flips for Y/Z over some neighborhood $U \subset Z$, then we can construct a model Y/U and get the destab divisor over U/T, where $\equiv/U$ and now $K_Y + B_Y$ is ample$/U$. Since the model is log canonical$/U$, it is *unique$/U$ for a given choice* of B_Y by the local uniqueness of the log canonical model [41, (1.5.1)], and it can be constructed locally$/U$. □

Lemma 10.8. *Let D be an effective $\mathbb{R}$-Cartier divisor$/X$. For any map $f: Y \to X$, any point of $Y/\operatorname{Supp} D$ and, therefore, any curve $C/\operatorname{Supp} D$ are contained in the divisorial subvariety* $\operatorname{Supp} f^*D$.

Proof. This is well known if D is Cartier. This implies also the $\mathbb{Q}$-Cartier case because any positive multiple rD with $r > 0 \in \mathbb{R}$ preserves all the supports. Finally, any $\mathbb{R}$-Cartier D has a $\mathbb{Q}$-perturbation to the $\mathbb{Q}$-Cartier case preserving the conditions. □

Lemma 10.9. *Let $\mathcal{D}$ be a b-divisor$/X$ such that*

- $\mathcal{D}_X = 0$;
- effective: $\mathcal{D} \ge 0$; *and*
- *nef on general curves$/X$: on (some) sufficiently high models Y/X, the restriction $\mathcal{D}_Y$ is nef$/X$ on curves covering the exceptional divisors of Y/X; in particular, $\mathcal{D}_Y$ is nef$/X$ on such models.*

Then $\mathcal{D} = 0$.

Proof. Immediate from the negativity of Lemma 3.22. □

Example–Construction 10.10. If we work in dimension $n = \dim X$ and do not yet know the existence of all flips, it would be very helpful to know in advance what types of flips are needed. Suppose that we have a 0-lt pair $(X/T, B)$ (that is, replace elt by log terminal in Definition 10.1) and that B only has reduced exceptional components on T. For example, $(X/T, B)$ may have come from a purely log terminal Fano contraction by taking a complement. Suppose that $\mathcal{L} \sim_{\mathbb{R}} \mathcal{S} = S + \sum e_i E_i/T$ is b-free and that Z/T is a model for $\mathcal{L}$ over T such that

- LSEPD is secured for B_Z (the log transform of B_T or B) on a model Z/T such that Z is nonsingular outside B_Z;
- $S = \sum s_i S_i$ is supported in the reduced part of B, considered as a b-divisor; and
- $\mathcal{S} \ge 0$ with the E_i exceptional on T.

Then, *over some neighborhood* U of $\operatorname{Supp}\mathcal{S}_Z$, strictly log terminal and extremal $(S{+}{-})$ flips are sufficient: with $B_Y = S^-{+}S^+{+}B'$, where $(Y/U, B_Y)$ is log terminal, B' has no reduced components, $S^-, S^+ \neq 0$, and their intersections with the extremal ray R satisfy $R\cdot S^- < 0$, $R\cdot S^+ \geq 0$.

Indeed, in the construction,

$$B_Y = g^{-1}B_T + \sum E_i \qquad \text{and} \qquad R\cdot(K_Y + B_Y + L) < 0 \quad \text{with } L = \mathcal{L}_Y.$$

Since R/Z, $R\cdot L = 0$ and $R\cdot(K_Y + B_Y) < 0$. Thus, the log discrepancy is supported in the exceptional locus$/T$:

$$D = l\mathcal{A}(X,B)_Y = K_Y + B_Y - g^*(K+B) \equiv K_Y + B_Y,$$

and there exists E_i with $R\cdot E_i \neq 0$. Moreover, $R\cdot E_i < 0$ since $D \geq 0$.

On the other hand,

$$R\cdot(S + \sum e_iE_i) = R\cdot L = 0, \qquad \text{where } \mathcal{S}_Y = S + \sum e_iE_i \text{ on } Y.$$

Hence, if $R\cdot E_i \neq 0$ with $e_i \neq 0$, then there exists another divisor E_j or a component of S having opposite intersection with R; both components are in the reduced part of B_Y because, by our conditions, $\mathcal{S}_Y$ is supported in the reduced part of B_Y. (In general, some e_i may be 0.) Since R has a curve over $\operatorname{Supp}\mathcal{S}_Z$, also R has a curve on $\operatorname{Supp}\mathcal{S}_Y$ by Lemma 10.8. Indeed, $\mathcal{S}_Y = (\overline{\mathcal{S}_Z})_Y = h^*\mathcal{S}_Z$, where $h\colon Y \to Z/T$, because $\mathcal{S} \sim_{\mathbb{R}} \mathcal{L} \sim 0/Z$ is $\mathbb{R}$-Cartier over Z by the Cartier property of $\mathcal{L}/Z$, by construction and Lemma 10.9 (cf. Example 7.6). Thus, either some component E_j with $e_j \neq 0$ intersects the curve, or a component of S. If this component has nonzero numerical intersection with R, we get an $(S{+}{-})$ flip by the above. Otherwise, the intersection is $= 0$, and we again get an $(S{+}{-})$ flip with $S^- \geq E_i$.

Definition 10.11. Let $(X/T,B)$ be a 0-log pair and E be a prime b-divisor. A (birational) *Klt neighborhood* $(Y/T, B_Y)$ of $\operatorname{center}_Y E$ is a *crepant projective* model $(Y/T,B)$ that is also a 0-log pair and is a Zariski neighborhood of $\operatorname{center}_Y E$ in it.

Lemma 10.12. *Under LMMP in dimension $n = \dim X$ and for birational X/T, Klt neighborhoods terminate for inclusions.*

Proof. On each model, this is a Noetherian property. But there only exist a finite number of 0-log pair models (the finiteness of minimal models in the big case) [45, Corollary 6.22 and Remark 6.23.5]. □

We are now ready to establish stabilization in a birational neighborhood of $E = E_{\mathrm{c}}$.

Theorem 10.13. *Under LMMP in dimension $\leq n-1$, let $(X/T,B)$ be a log pair, E be a prime b-divisor over a neighborhood of P, and $\mathcal{L}_\bullet$ be a system of b-divisors such that*

- (0LP) $(X/T,B)$ *is a birational* 0*-log pair with* $\dim X = d \leq n-1$;
- (LBF) *each $\mathcal{L}_i$ is b-free*; *moreover,*
- (STB) *the free restrictions $\mathcal{L}_{i}{}_{\shortmid E}$ stabilize over E, and each $\mathcal{L}_{i}{}_{\shortmid E} = i\mathcal{M}$, where $\mathcal{M} = \mathcal{L}_{1}{}_{\shortmid E}$ is b-free*; *and*
- (RFA) *the situation is a restriction of a pl contraction in dimension n*, that is, $\mathcal{L}_i$ is a restriction of the mobile system for some $(\mathrm{RFA})_{n,d}$ algebra (see Definition 3.47 and the proof below).

Suppose also that $(S{+}{-})$ flips as in Example 10.10 *exist in dimension n. Then there exists a Klt neighborhood $(Y/T,B_Y)$ of* $\operatorname{center}_Y E$ *on which all $\mathcal{L}_i$ with $i \gg 0$ are free, ample, and $\mathcal{L}_i/i$ stabilizes* (over this neighborhood, all $\mathcal{L}_i/i = \mathcal{D}$, the limit). *Thus, after a truncation, this holds for each $i > 0$, and the contraction defined by each $\mathcal{L}_i$ gives this quasiprojective neighborhood.*

Proof. First, we recall and explain (RFA). There exists an extension

$$(X/T, B) \subset (X'/T', B'),$$

with $(X'/T', B'')$ a pl contraction of dimension n, X the intersection of the reduced divisor in B'', and $B' \geq B''$. Such a complement always exists preserving (X', B') log terminal, with B' having the same reduced part, and we fix it. Then we define B by successive adjunction [41, 3.1]:

$$(K_{X'} + B')_{|X} = K + B.$$

We assume also that the system $\mathcal{L}_\bullet$ is the mobile restriction of a b-free system $\mathcal{L}'_\bullet$ (see 7.1), that is, $\mathcal{L}_i \sim \mathcal{L}'_{i|X}$, where $\mathcal{L}'_\bullet$ is the mobile system for the $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ algebra given by the pl contraction. Equivalently, the mobile system of the restriction for the algebra, where in either case we assume general position.

Fix some $i \gg 0$ and consider the model Z'/T' defined by $\mathcal{L}'_i$. By the construction of the (RFA) algebra (cf. Example 4.35) and, in particular, by (AMN) for its characteristic system, we can assume that

$$\mathcal{L}'_i \sim_{\mathbb{Q}} \mathcal{S}' = S' + \sum e_i E_i \geq 0,$$

as in Example 10.10. Note also that, since X/T is birational, the model Z/T of X/T for $\mathcal{L}_i$ is in $\operatorname{Supp} \mathcal{S}'_{Z'}$ (and Z is normal by Addendum 4.50.1 and Example 4.52). It is completely contained in any neighborhood $U' \subset Z'$ of $\operatorname{Supp} \mathcal{S}'_{Z'}$. Thus, the destab model$/T$ over Z/T constructed below is projective$/T$ (complete$/T$).

Now, by Addendum 10.6.1 and Example 10.10, for each $\mathcal{L}'_i$, there exists a destab$/T'$ model $(Y'/U', B'_{Y'})$ over some neighborhood U' of $\operatorname{Supp} \mathcal{S}'_{Z'}$ in Z' (possibly not complete$/T'$). Then, by exceptional saturation, we can destabilize multiples of $\mathcal{L}'_i/U'$ over the destab divisor whose support includes the exceptional divisors$/T$. By (STB), $\operatorname{center}_{Y'} E$ does not intersect the destab divisor, that is, the exceptional divisorial locus of Y'/T' for the log canonical model for $(Y'/U', B' + NL')$ with $L' = (\mathcal{L}'_i)_{Y'}$. Note: $\operatorname{center}_{Y'} E$ is well defined for a sufficiently high resolution ($g''\colon Y'' \to X'$ that is also regularly dominated over Y' by a Hironaka hut and E is a divisor on $(g'')^{-1}X$) for a blowup of E. In particular, $\operatorname{center}_{Y'} E \subset X_0$, where X_0 is the birational transform of X on Y'; and $\operatorname{center}_{Y'} E = \operatorname{center}_{X_0} E$.

After a normalization, this gives the required neighborhood. Indeed, if we apply adjunction of the subboundary $(B')^{Y'} = \mathcal{B}(X', B')_{Y'}$ on the normalization of X_0, which we also denote by X_0 (in fact, X_0 is normal at least in required U; see the end of the proof), we get the different (divisor) that is *effective* outside the exceptional divisors of Y'/T'. In particular, the latter holds in a neighborhood of $\operatorname{center}_{Y'} E = \operatorname{center}_{X_0} E$. On the other hand, if we identify X' with T' as we did for 0-log pair, we get the adjunction $(K_{X'} + B')_{|X} = K + B$ with effective B. In addition, by the commutative diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ T & \subset & T' \end{array}$$

the above adjunction gives the subboundary $B^{X_0} = \mathcal{B}(X, B)_{X_0}$ as the adjunction for the subboundary that is effective on a neighborhood $U \subset U' \cap Z$ of $\operatorname{center}_{X_0} E$. The log canonical divisor $K_{X_0} + B^{X_0}$ is numerically trivial$/T$. In particular, X_0/U is the identity. The discrepancies$/U$ for (X_0, B^{X_0}) are the same as for (X, B). Thus, if we replace the subboundary B^{X_0} by a boundary $B_0 \geq B^{X_0}$ that is the same where the subboundary is effective, and 0 (or even 1) where it is negative, we obtain $K_{X_0} + B_0$ that is nef on U, that is, on the closure of each curve in U. If

$K_{X_0}+B_0$ is not $\mathbb{R}$-Cartier and not log canonical outside U, we can make it so after some projective modification/X_0 outside U, by uniqueness [41, (1.5.1)], as in Addendum 10.6.1. The discrepancies over U for (X_0, B_0) are the same as for (X, B). Now we can apply LMMP to $(X_0/T, B_0)$. The modification does not affect U because we increase a discrepancy [40, 2.13.3] that is the same as for $(X/T, B)$ [45, 2.4.2]. However, at the end we obtain a projective 0-log pair of $(X/T, B)$ with the same discrepancies, that is, a crepant model.

Finally, we can give $\mathcal{L}_i \sim \mathcal{L}'_{i|X}$ that is free and ample/T on U for $i \gg 0$. Indeed, by Proposition 10.6, it gives a multiple of $N\mathcal{L}'_i + \mathcal{D}$ since, in the proposition, the destab $D = \mathcal{D}_{Y'}$ is ample/Z' in U', and $N\mathcal{L}'_i$ is b-free/T' with ample descent on Z'/T'. Since the neighborhood on Y'/T' is not divisorial near $\operatorname{center}_{X_0} E$, it stabilizes (we cannot add exceptional divisors by Lemma 10.9; cf. Example 10.5(3)). See also Addendum 4.50.1 and Example 4.52. $\square$

Now we can apply Theorem 10.13 to the inductive model in the proof of Theorem 9.9 to get a complete stabilization. This is the main result of this section.

Corollary 10.14. *Assume LMMP and* (BP) *in dimensions* $d' \le d = \dim X$, *together with* $(\mathrm{CCS})^*_{d'-1}(\mathrm{gl})$ (*or* $(\mathrm{CCS})_{d'-1}(\mathrm{gl})$ *and* $(\mathrm{FGA})^*_{d'-1}(\mathrm{gl})$) *and* $(\mathrm{SSB})_{d'-1}(\mathrm{gl})$. *Moreover,* $(\mathrm{CCS})^*_{d'-1}$ *and* $(\mathrm{SSB})_{d'-1}(\mathrm{gl})$ *are known for* $d' \le 3$, *so we can omit them as assumptions*; (BP) *for* $d' \le 4$.

Assume also that $(S{+}{-})$ *flips exist in dimension* n. *Let* $\mathcal{L}_\bullet$ *be a mobile system of type* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *on* X/T. *Then there exists a* 0*-log pair* $(Y/T, B_Y)$ *over which each* $\mathcal{L}_i$ *with* $i \gg 0$ *is free and ample, up to a truncation.*

This gives a solution for $(\mathrm{CCS})_n(\mathrm{rfa}), (\mathrm{bir})$ *under the assumptions.*

Lemma 10.15. *Under LMMP in dimension* $d = \dim X$,

- (0LP) *let* $(X/T, B)$ *be a birational* 0*-log pair that is projective*/T;
- (NBU) *let* U *be a nonempty open subset in* X *that is incomplete*/P; *and*
- (CMP) *let* $(X/T, B + C)$ *be a complement* nonlog canonical (*or just non-Klt*) 0*-log pair with* $C = \mathcal{C}_X$ *such that* $\mathrm{LCS}(X, B{+}C)$ *is completely contained inside* U (*cf. Example* 10.2).

Then there exists a different complement with a new $\mathcal{C}$ *such that the* new $\mathrm{LCS}(X, B + C)$ *is not completely contained in* U, *but all its* LCS *centers are in* U, *and* $(X, B + C)$ *is log canonical outside* U. *Thus, the nonlog canonical part* $X_{-\infty}$ *of the* LCS (*see Example* 5.25) *is inside* U, that is, does not intersect the complement of U. *In addition, there is a* transient LCS center E *such that* $(X, B + C)$ *is log canonical in its generic point and* E *is not completely contained in* U. *After a perturbation of* C, *this* E *is unique, and is purely log terminal in its generic point and outside* U.

Moreover, we can find C^+ *such that* $(X/Z, B{+}C^+)$ *is a generalized log Fano contraction* (*as in Conjecture* 5.26) *with the same support of* LCS (*and* LCS *centers*) *and the same purely log terminal center* E. *Thus, adjunction on* E *gives a* generalized (possibly nonnormal) *log Fano contraction* $(E/T_E, \omega)$ (*cf. Example* 5.25), *where*

- $E/T_E = f(E)$ *and* E^ν/T_{E^ν} *are proper contractions, where* $E^\nu \to T_{E^\nu} \to (T_E)^\nu$ *is the Stein factorization of the normalization map* $E^\nu \to (T_E)^\nu$;
- $(E^\nu/T_{E^\nu}, B_{E^\nu})$ *is a normal generalized log Fano contraction* (*and the map is*/$(T_E)^\nu$) *with* $-(K_{E^\nu} + B_{E^\nu}) \equiv -(K + B + C^+)_{|E}$ *ample*/$(T_E)^\nu$; *and*
- J_E *is the fractional ideal sheaf of the* $\mathrm{LCS}(E, \omega)$ *on* E, the direct image of that on the normalization; *it is supported in the intersection of* E *with the closure of* $\mathrm{LCS}(X, B{+}C^+)\backslash E$. *In particular,* (E, B_E) *is normal and Klt outside* $U_E = U \cap E$ (*and* $(E/T_E) = E^\nu/T_{E^\nu} = (T_E)^\nu$ *is a contraction where* $E \setminus U_E$ *is complete*/T_E).

Addendum 10.15.1. *We can also suppose that the transient* E *is*/P.

We can omit LMMP in the lemma (see [2, Theorem 6.6]).

Proof. We can find a maximal $c > 1$ such that $K + B + cC$ is log canonical (near) outside U; this gives the required new value of $\mathcal{C} := c\mathcal{C}$. It exists by (NBU) and Lemma 10.8 because $\mathcal{C}_X = (\overline{\mathcal{C}_T})_X = f^*\mathcal{C}_T$ and $P \in \operatorname{Supp}\mathcal{C}_T$. Set $X_0 = X \setminus X_{-\infty}$; that is, this is the open subset of X where (X, B) is log canonical. Note that $X_{-\infty}$ is completely contained in U.

Indeed, we verify that the generic points of the log canonical centers are in U. Suppose that there exists such a center E_1 outside U, in particular, E_1 is completely contained in X_0. On the other hand, by (CMP), $c > 1$, there is a center E_n in $X_{-\infty}$. By LMMP, the LCS is connected (see the proof of Theorem 6.9 in [41]). Hence, we have a chain of LCS centers $E_1, E_2, \dots, E_n$, with E_1 completely contained in X_0, E_2 in general in X_0, and E_n completely outside, that is, contained in $X_{-\infty}$. Note that $K + B + cC \equiv 0/P$ on this chain. This is impossible. Now we consider only $d = 3$, that is, X is a 3-fold. Then E_i are at most of dimension 2. We check that each $E_i \subset X_0$, which gives a contradiction. If E_2 is completely in X_0, we can use induction on n. Thus, suppose that E_2 intersects $X_{-\infty}$. If E_2 is a curve, then $E_2/f(E_2) = P$ is complete$/P$. Otherwise, we delete E_2 from our chain. On the other hand, by adjunction [23] (see also the proof of Addendum 9.21.1), on the normalization E_2^ν, there is a boundary $B_2 \geq 0$ such that $K_{E_2^\nu} + B_2 \equiv 0$. In addition, E_2 has two points $\nu(P_1) \in E_1 \cap E_2$ and $\nu(P_2) \in E_2 \cap X_{-\infty}$ with $\operatorname{mult}_{P_1} B_2 \geq 1$ and $\operatorname{mult}_{P_2} B_2 > 1$. This is impossible because $\deg K_{E_2} \geq -2$. Thus, E_2 is a surface that is in general in X_0. Now $E_2/T_{E_2} = f(E_2)$ is a complete surface or a curve fibration. Again, adjunction on the normalization gives a boundary $B_2 \geq 0$ that has a reduced divisor or log canonical center P_1 in $E_2^0 = E_2 \cap X_0 \subset E_2^\nu$ and the nonlog canonical center P_2 outside. On the other hand, $K_{E_2} + B_2 \equiv 0/T_{E_2}$, so we have a chain of LCS centers for $(E_2^\nu/T_E, B_2)$ as above (if it is not a contraction$/T_{E_2}$, make a base change or argue in the connected component of the fibre E_2^ν/T_{E_2} using inverse adjunction). This gives a contradiction by induction on the dimension of E_2.

For $d \geq 4$, we can do similarly, by LMMP. We can again use adjunction (see the proof of Addendum 9.21.1), but it is easier to do this after a reduction (using a log resolution) to the situation when all E_i are divisorial and the adjunction is divisorial [41, 3.1]. In either case, the required connectedness is a higher dimensional version of [41, теорема 6.9] and can also be derived from LMMP. Without LMMP, see [2, Theorem 6.6].

Thus, the new $\mathcal{C} := c\mathcal{C}$. After a perturbation of $C_T = \mathcal{C}_T$, we can get a single transient center E. Perturbation: there is $H \geq 0$ a $\mathbb{Q}$-Cartier divisor on T with $P \in \operatorname{Supp} H$ such that, on a (sufficiently high) log resolution $g\colon Y \to T$ of $(T, B_T + C_T + H)$, $g^*H \geq H'$, where H' is ample$/T$. Then we can perturb C_T and so $\mathcal{C} = \overline{C_T}$: replace by $(1-\varepsilon)C_T + (1-\delta)H''$, where $g^*H'' \geq H - H' + \sum e_i E_i$ with $0 < \varepsilon, \delta, e_i \ll 1$ and each E_i is exceptional$/T$ or is in the support of B_T and C_T (under the assumption that C_T is rather powerful).

Finally, before taking the complement (that is, assuming $C \geq C' \geq 0$, where C' is ample$/T$), we can improve $(X/T, B + C)$ to a Fano contraction$/T$ as in Lemma 9.7 (in the non-$\mathbb{Q}$-factorial case). Again, as in the perturbation, $\|B^+ - B\|$ should be $\ll \|C\|$.

The properties of $(E/T_E, \omega)$ are standard. Addendum 10.15.1 can be obtained by adding $c\overline{H}_X$ instead of $(c-1)C$, where H is quite generic effective Cartier divisor through P on T (cf. the proof of Lemma 9.6). $\square$

Proof of Corollary 10.14. By definition, the mobile system $\mathcal{L}_\bullet$ is defined on a log Fano contraction $(X/T, B)$. Its characteristic system is $\mathcal{D}_i = \mathcal{L}_i/i$. Using complements, we can convert the pair into a 0-log pair $(X/T, B)$ with a new fixed B [41, предложение 5.5]. It is also a weak log Fano contraction because X/T is birational. Now applying Corollaries 9.20 and 9.21 to $\mathcal{D}_\bullet$ gives the stabilization of $\mathcal{D}_\bullet$ on some prime b-divisor $E = E_{\mathrm{c}}/P$, that is, (STB) of Theorem 10.13 for $\mathcal{L}_\bullet$; by Example 10.2, this also gives the first $\mathcal{C}$ and $C = \mathcal{C}_X$. Indeed, (LWF) holds by construction. (LBF), (LCA), and (MXD) hold by Corollary 4.53 as for a (FGA) system. In addition, (AMN) also holds. Note that any complement preserves (LCA) by Lemma 4.44. (BED) for $\mathcal{D}_\bullet$ after a truncation follows from Proposition 4.54 and Theorem 6.19(3),(4) under the assumption $(\mathrm{CCS})_{d-1}$. We can replace this last assumption by $(\mathrm{CCS})^*_{d'-1}(\mathrm{gl})$ (or $(\mathrm{CCS})_{d'-1}(\mathrm{gl})$ and $(\mathrm{FGA})^*_{d'-1}(\mathrm{gl})$) and $(\mathrm{SSB})_{d'-1}(\mathrm{gl})$ with $d' \le d-1$ by induction on $\dim X = d$.

Therefore, by Theorem 10.13, there is a Klt neighborhood $(Y/T, B_Y)$ of E on which after a truncation each $\mathcal{L}_i$ is ample and free, and $\mathcal{L}_i/i$ stabilizes. If U is complete$/P$, we are done (compare the next corollary). If not, we enlarge the neighborhood. This is eventually complete by Lemma 10.12 on Noetherian induction. (The final destab D is disjoint from Y.)

Suppose that U is still not complete$/P$. Thus, it satisfies (NBU). By Noetherian induction, $U \neq \varnothing$ (because it includes old centers) and satisfies (CMP). Hence, Lemma 10.15 applied to $(X/T, B) := (Y/T, B_Y)$ with the previous $\mathcal{C}$ gives a new transient center E and new C and C^+. Therefore, if $\mathcal{D}_{\bullet}\rceil_E$ stabilizes$/T_E$ or (STB) holds for $\mathcal{L}_{\bullet}\rceil_E$, then again by Theorem 10.13, up to a truncation, we get that each $\mathcal{L}_i$ is free, ample$/T$, and $\mathcal{D}_\bullet$ stabilizes in a Klt neighborhood that enlarges the old one by a complete center $\mathrm{center}_X E$ over T! (Compare the proof of the theorem.)

Thus, we need to verify (STB) on E. This time we use Addendum 9.21.1 on $(X/T, B + C^+)$. Indeed, all the conditions are satisfied: (LWF) is generalized by (ELT), and (GIM) holds by properties of the transient center. Note that this also implies needed generalization of (RRF): each $\mathcal{D}_i = 0$ over the generic point of E, and the ampleness and stabilization of $\mathcal{D}_\bullet$ on the LCS inside U. By Lemma 4.44 and since $\mathcal{B}^Y$ increases at each step of our construction (Lemma 10.15), (LCA) holds over $(Y/T, B_Y)$ again, and we can assume also that $\operatorname{Supp} B + C^+ \supset \operatorname{Supp} C^+ \supset E$. For required stabilization (STB), $(\mathrm{CCS})^*_{d'-1}(\mathrm{gl})$ is enough by Addendum 10.15.1.

Above we verified $(\mathrm{SSB})_{d'-1}$ and $(\mathrm{CCS})^*_{d'-1}$ for $d' \le 3$, so we can drop it as an assumption. The next case $d' = 4$ is unknown: that is, $(\mathrm{SSB})_3$ and $(\mathrm{CCS})^*_3$ for a 3-fold E in a 4-fold X are unknown even without *. (BP) see in the proof of Addendum 4.50.3. $\square$

Corollary 10.16. *Under the assumptions of Corollary* 10.14, *the limit* $\mathcal{D} = \lim_{i\to\infty} \mathcal{D}_i$ *of the characteristic system of any* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *algebra stabilizes. Thus, f.g. for* $(\mathrm{RFA})_{n,d}(\mathrm{bir})$ *holds under the assumptions.*

Proof. Immediate from the proof of Corollary 10.14 or by Corollary 10.14 itself, Theorem 6.19(3),(4), and Limiting Criterion 4.28. $\square$

Corollary 10.17. *Under the assumptions of Corollary* 10.14, *pl flips exist in dimension* n.

Proof. Immediate by Main Lemma 3.43, Corollary 10.16, and Corollary 3.32. $\square$

The next section gets rid of the assumption on the existence of $(S{+}{-})$ flips under $(\mathrm{CCS})^*$.

11. THE MAIN RESULT

Theorem 11.1. *Under LMMP and* (BP) *in dimension* $\le n-1$, *and* $(\mathrm{CCS})^*_d$ *with* $d \le n-2$, $(S{+}{-})$ *flips exist in dimension* n. *Moreover, when* $n \le 4$, *we can omit assumption* $(\mathrm{CCS})^*_d$; (BP) *when* $n \le 5$.

We can also replace $(\mathrm{CCS})^*_d$ *by* $(\mathrm{FGA})_{n-2}(\mathrm{bir})$ *and* $(\mathrm{FGA})^*_d(\mathrm{gl})$.

Lemma 11.2. *Let* $\mathcal{L} = \overline{L}$ *be an ample b-divisor on* X/T, *and* E *and* E' *be two prime divisors on* X *such that* E *intersects* E' *divisorially. Then there is an effective Cartier b-divisor* $\mathcal{E}'$ *supported over* E' *that* destabilizes $\mathcal{L}\rceil_E$, that is, $(\operatorname{Mov}(\mathcal{E}' + N\mathcal{L}))\rceil_E > N\mathcal{L}\rceil_E$.

Proof. Taking hyperplane sections of X/T by L reduces the lemma to the two-dimensional case, with X a surface. Then we can use Example 10.5(2). The main problem is that E' may not be $\mathbb{Q}$-Cartier. This can be resolved by a $\mathbb{Q}$-factorialization and perturbation in exceptional components to preserve ampleness$/T$.

In higher dimensions, we take the closure of a family of $\mathcal{E}'$ constructed in dimension 2. $\square$

Lemma 11.3. *Under LMMP in dimension n, let B be a subboundary on X with* $\dim X = n \geq 3$ *and S be a prime divisor in X such that*

- *$K + S + B$ is $\mathbb{R}$-Cartier*;
- *B has no reduced components*;
- *the adjunction* [41, 3.1] $(K + S + B)_{|S^\nu}$ *is Klt, where S^ν is the normalization of S*; *and*
- *the negative components of B intersect S only in points.*

Then there exists a weak $\mathbb{Q}$-factorialization, *that is, a model, projective/X, on which all the divisors that are exceptional/X and the divisors that intersect S in a point, including the negative components of B, do not intersect S.*

Remarks 11.4. (1) S is normal outside its intersection points with negative components of B [41, лемма 3.6] because (X, B) is purely log terminal there (cf. the conditional inverse adjunction [41, 3.4]).

(2) We expect a more perfect form of the lemma in any dimension n, with the intersections in codimension ≥ 3 rather than in points (cf. Remark 11.7 at the end of Step 1 in the proof of Theorem 11.1).

Proof of Lemma 11.3. Take a strictly log terminal resolution of $(X, S + B)$ that we do with *reduced* exceptional divisors, intersecting S in a point, and negative components, that is, we set or change their multiplicities to 1. The reduced components of the resolution do not intersect new S because the construction gives a log terminal resolution of (S, B_S), under the adjunction of original (X, B), and the resolution of (S, B_S) does not have reduced components. $\square$

Proof of Theorem 11.1. Since the flip is strictly log terminal, after slightly decreasing the other reduced components, we can assume that both S^+ and S^- are irreducible (cf. Example 10.10). Thus, the boundary is $B + S^+ + S^-$ with $\lfloor B \rfloor = 0$.

We can suppose that this is a *small* flip with the exceptional locus of dimension $\leq n - 2$. Moreover, the contraction is birational on S^+, S^- and on their intersection $E = S^+ \cap S^-$. Indeed, E contractible contradicts the normal crossing of S^+ and S^- ([41, следствие 3.8] and the irreducibility of Example 3.40) because then $S^+{}_{|S^-} = E$ is nef/T and exceptional on S^-. This last conclusion is impossible by Negativity 1.1 in [41].

Let $\mathcal{L}_\bullet$ be the mobile system of a divisorial algebra $\mathcal{R}_{X/T}D$ with $D \sim S^-$, numerically negative/T. Since the flip is extremal, the algebra defines it if it is f.g. We have a translation t by Addendum 3.43.1 that corresponds to the inductive sequence with $s = t = 2$ and single $S_2 = S^-$ (cf. Example 3.40 with $S_1 = S^+$).

By Proposition 4.42, the system $\mathcal{L}_\bullet$ is of type (FGA)$_n$(bir) over the pair

$$(X/T, B + (1 - \varepsilon)(S^+ + S^-))$$

for $0 < \varepsilon \ll 1$ because

(EXS) $\mathcal{L}_\bullet$ is exceptionally asymptotically saturated/T.

Unfortunately, we cannot preserve condition (EXS) when restricting, but the (FGA) is preserved for the birational restrictions $|$ of $\mathcal{L}_\bullet$ on S^+, S^-, and E by Example 4.45 and Proposition 4.50 (cf. Example 4.52).

In particular, $\mathcal{L}_{i|E}$ is lca saturated. Thus, by our assumption, f.g. in (FGA)$_{n-2}$(bir) follows from (CCS)$_{n-2}$(bir) and Theorem 6.19(3),(4). Thus, the characteristic system of $\mathcal{L}_{i|E}$ stabilizes. Since the translation t is preserved under the restriction, we get f.g. in (FGA) for $\mathcal{L}_{i|S^+}$ on S^+ by Main Lemma 3.43 with a single element inductive sequence $S^+ \cap S^-$ (with possibly nonflipping model; cf. Corollary 6.44).

For the same reasons (but without S^+) or LMMP, the characteristic system of $\mathcal{L}_\bullet$ stabilizes over points (in general) of dimension 1. Thus, Proposition 4.54 applied just to $\mathcal{L}_\bullet$ gives a stabilization for the characteristic system of $\mathcal{L}_\bullet$ over $T \setminus P$, with closed P, with (BED) after a truncation.

11.5. Preamble. We also establish below f.g. in (FGA) for $\mathcal{L}_{i|S^-}$. Thus, by Main Lemma 3.43 again, with a single element inductive sequence S^- on X we get f.g. in (FGA) for $\mathcal{L}_i$ itself and get the flip on X.

To prove f.g. on S^-, we construct a 0-log pair $(S^-_m/T^- = f(S^-), B^-_{S^-_m})$ over which each $\mathcal{L}_{i|S^-}$ with $i \gg 0$ is free and ample, up to a truncation (cf. Corollary 10.14). This gives (CCS) and f.g. by Theorem 6.19(3),(4) because we can secure lca saturation for the system $\mathcal{L}_{\bullet|S^-}$ as above.

First, in Step 1 below, we construct a Klt neighborhood U_m of E (as E_c but now no longer/P) on which all $\mathcal{L}_{i|S^-}$ with $i \gg 0$ are free, ample, and $\mathcal{L}_i/i_{|S^-}$ stabilize (*over this neighborhood, all* $\mathcal{L}_i/i = \mathcal{D}$, *the limit*). Thus, after a truncation, this holds for each $i > 0$, and each $\mathcal{L}_i$ gives this quasiprojective neighborhood on a contraction given by $\mathcal{L}_i$. To compare with Theorem 10.13, now we cannot use the (S+−) flips.

By Lemma 10.12 and LMMP assumption, neighborhoods U_m of this form considered under inclusions terminate. The maximal U_m is complete/T^- and is the required 0-log pair.

Finally, in Step 2 we extend each incomplete U_m.

Step 1. Take $\mathcal{L} = \mathcal{L}_m$ for $m \gg 0$ for which the restricted linear systems $\mathcal{L}_{m|E}$ and $\mathcal{L}_{m|S^+_m}$ are very semiample and have compatible restrictions

$$\mathcal{L}_m/m_{|E} = \mathcal{D}_{|E} \qquad \text{and} \qquad \mathcal{L}_m/m_{|S^+_m} = \mathcal{D},$$

that is, they stabilize. This follows from the normality of restricted algebras in Addendum 4.50.1. Then the linear system of $\mathcal{L}_m$ defines a flag of normal varieties $E_m \subset S^+_m \subset X_m$, whereas $\mathcal{L} = \overline{L}$ for *just* ample but free $L = (\mathcal{L})_{X_m}/T$. This can be done by a normalization of the model given by $\mathcal{R}_{X/T}L$.

However, the birational transform of S^- in X_m may still be nonnormal. We denote its normalization by $\nu\colon S^-_m \to X_m$. We claim that U_m is a neighborhood of E_m in S^-_m. Note that E_m is also embedded in S^-_m by the above normality Addendum 4.50.1.

By (BED) and (EXS), or since $\mathcal{L}$ gives the flip over $T \setminus P$, the only exceptional divisors E_i of X_m/T are/P (cf. Lemma 3.19).

In addition, no exceptional E_i/T intersects S^+_m *divisorially*. For otherwise, an effective exceptional/T b-divisor $\mathcal{E}'/E' = E_i$ destabilizes $\mathcal{L}_{|S^+_m}$ by Lemma 11.2, where $E = S^+_m$. This is impossible by (EXS) on X and (MXD) on S^+_m since $\mathcal{L}/m_{|S^+_m} = \mathcal{L}_m/m_{|S^+_m} = \mathcal{D}$ stabilizes.

In particular, this implies that adjunction for the crepant subboundary B_{X_m} on S^+_m has an *effective different* (as we expect by the stabilization; cf. Addendum 5.12.2). More precisely, by a perturbation, we convert $(X/T, B)$ into a (fixed) 0-elt pair $(X/T, B^+)$ (actually, it is purely log terminal) with only reduced S^+. Thus, S^+_m is a 0-log pair for the adjunction $(K_T + B^+_T)_{|f(S^+_m)}$ on $T^+ = T^+_m = f(S^+_m)$ (which is independent of m and normal). Hence, by adjunction and the commutative diagram

$$\begin{array}{ccc} S^+_m & \subset & X_m \\ \downarrow & & \downarrow \\ T^+ & \subset & T \end{array}$$

up to codimension 2, that is, for general surface sections of X_m/T (cf. the different in [41, §3]), and near S_m^+, $K_{X_m}+B_{X_m}^+$ and $K_{S_m^+}+B_{S_m^+}^+$ are log canonical with *crepant* B^+, in particular, *birationally* transformed B^+ on the nonexceptional/T part. The adjunction can be extended on the whole of S_m^+ to a 0-log pair $(S_m^+/T^+, B_{S_m^+}^+)$ with

$$K_{S_m^+}+B_{S_m^+}^+ = (K_{X_m}+B_{X_m}^+)_{|S_m^+};$$

in particular, the pair is Klt everywhere on S_m^+ with $B_{S_m^+}^+ \geq 0$ (crepant for $(S^+, B_{S^+}^+)$) by [41, (3.2.2)].

Moreover, every exceptional E_i intersects S_m^+ in at most one point. Indeed, we can assume that the subvarieties in S_m^+ of dimension ≥ 1 in general are *weakly* $\mathbb{Q}$-factorial *for the exceptional divisors* E_i/T; that is, if the dimension of intersection is ≥ 1, these divisors intersect S_m^+ divisorially (in codimension 2 in X_m). Indeed, taking a hyperplane section up to the intersection with E_i by a point, by Lemma 11.3, we get weakly $\mathbb{Q}$-factorial for divisors E_i on a weak $\mathbb{Q}$-factorialization. Note that LMMP in the lemma holds under our assumptions of the theorem. On the other hand, by the numerical and geometric properties of S^+ and S^-, $af(S^+)+b(\mathcal{L}_m)_T$ is an effective Cartier divisor on T for some $a>0$ and $b\geq 0$; $b>0$ if $R\cdot S^+>0$. Then

(PRS) on any model over T, $aS_m^+ + bL + \sum e_iE_i \equiv 0/T$ and, in particular, *locally* on X_m, Cartier with multiplicities $e_i \geq 0$ (depending on m as L) in exceptional E_i; and by Lemma 10.8, $e_i>0$ on E_i/P.

The latter is preserved on restricting to any general hyperplane section. Again by Lemma 10.8, this is impossible for some curve/X_m (over a point in X_m) on a weak $\mathbb{Q}$-factorialization and, if the latter is nontrivial, its exceptional curves are on the modification of S_m^+, by the connectedness of the fibre/X_m. Thus, as in the $\mathbb{Q}$-factorial case, the divisorial part always intersects in a divisor if it intersect in a curve. (In the same way, we could eliminate intersections with S_m^+ in points if we knew the existence and termination of n-fold flips! Thus, we expect this near S_m^+, and that S_m^- is normal by the argument below. But for now we just disregard such intersections in points.)

Therefore, the *exceptional divisors* E_i *intersect* S_m^+ *at most in points*. In particular, the negative components in $B_{X_m}^+$ intersect S_m^+ at most in points. The same holds for $E_m\subset S_m^+$ with the adjoint subboundary B_E^+ given by the adjunction of a log terminal 0-log pair $(T/T, B_T^{+-})$. The pair has two reduced divisors S^+ and S^-. The third needed 0-elt pair $(X/T, B^-)$ with reduced S^- induces the 0-log pair $(T^-/T^-, B_{T^-}^-)$ which corresponds to a Klt U_m under construction. We can also assume that B^{+-} has a *substantial* nonreduced part, that is, $B^- = B^{+-}-C$ for some $C>0$.

Now, if we consider $E_m\subset S_m^-$, by the above, the negative components of the crepant subboundary $B_{X_m}^{+-}$ intersect E_m in points. Thus, we have an adjunction of (T, B_T^{+-}) on $(S_m^-, B_{S_m^-}^{+-})$ with the negative components in $B_{S_m^-}^{+-}$ intersecting E_m at most in points, and in turn an effective adjunction of $(S_m^-, B_{S_m^-}^{+-})$ on $(E_m, B_{E_m}^{+-})$ with $B_{E_m}^{+-}\geq 0$. Hence, again by Lemma 11.3 and the presentation (PRS), the intersections even in points on S_m^- are impossible, and $B_{S_m^-}^{+-}$ is a *boundary* near E_m. The same holds for $B_{S_m^-}^-$ on S_m^- because, for the same reason, the support of each restriction $E_{i|S_m^-}$ in the sense of Mumford (restricted to the normalization S_m^- and defined only divisorially) *does not intersect* E_m *as closed subvarieties*. This is a crucial point that *gives* a neighborhood U_m as the complement to the supports. Indeed, in the neighborhood, $B_{S_m^-}^-$ is a boundary and $\mathcal{L}_{jm}/j_{|U_m}$ stabilizes for all $j\geq 1$! This is the boundary by [41, (3.2.2)]. Thus, as at the end in the proof of Theorem 10.13 (based on the uniqueness in Addendum 10.6.1), we can convert (= embed) U_m into a Klt neighborhood (into a crepant log pair for $(T^-/T^-, B_{T^-}^-)$). Stabilization holds by the following:

Trick 11.6. By (AMN) and (BED) and since the exceptional E_i in X_m are$/P$, $\mathcal{L}_{jm} = j\mathcal{L}_m + \mathcal{E}_{m,j}$, where each $\mathcal{E}_{m,j} \geq 0$ is b-free (in particular, b-Cartier)$/X_m$ with $\operatorname{Supp}(\mathcal{E}_{m,j})_{X_m}$ in a union of exceptional E_i. Hence, by Proposition 7.7, (fx), we have $\operatorname{Supp}(\mathcal{E}_{m,j}{}_{|U_m})_{U_m} = 0(/X_m)$ *divisorially*, or $(\mathcal{E}_{m,j}{}_{|U_m})_{U_m} = 0$, and by Lemma 10.9 $\mathcal{E}_{m,j}{}_{|U_m} = 0$. Then $\mathcal{L}_{jm}/j_{|U_m} = \mathcal{L}_{m|U_m}$. This implies the required stabilization on U_m.

Remark 11.7. The proof is essentially higher dimensional: that is, it works better if we assume $n \geq 4$. For $n \leq 2$, there are no flips under the small condition. For $n = 3$, under LMMP in dimension 3, Lemma 11.3 and (PRS) imply that each E_i is disjoint from E_m (cf. Remark 11.4(2)) and we have the same stabilization as in Step 1. However, the existence of 3-fold log flips follows from $(\mathrm{FGA})_2(\mathrm{bir})$ (see the proof of (FGA) in Main Theorem 1.7 at the end of Section 6) or from [41].

Step 2. Using Lemma 10.15, we can enlarge the neighborhood U_m, whenever it is still incomplete, by a transient center E_t in a crepant model of $(T^-/T^-, B^{+-}_{T^-})$, and establish a stabilization of the restricted system $\mathcal{L}_{jm}/j_{|E_t}$ as in the proof of Corollary 10.14 by $(\mathrm{FGA})^*_d$ or $(\mathrm{CCS})^*_d$ with $d = \dim E_t \leq n-2$ and the $(\mathrm{FGA})^*$ property (with singularities) of the restricted system (see Conjecture 5.26). By Addendum 10.15.1, we need only global $(\mathrm{FGA})^*_d(\mathrm{gl})$ or $(\mathrm{CCS})^*_d(\mathrm{gl})$. To apply Addendum 9.21.1 and Lemma 10.15 with U_m in a 0-elt (actually 0-plt) model of $(T^-/T^-, B^{+-}_{T^-})$, we need that $(X_m, B^{+-}_{X_m})$ is log terminal and that $B^{+-}_{X_m}$ is a boundary at the generic point of $E_m = S^+_m \cap S^-_m$, and $B^{+-}_{S^-_m}$ is a boundary in U_m by [41, следствие 3.11] and the *first time* by Step 1, because in general $E = S^+ \cap S^-$ goes birationally to E_m and the exceptional$/T$ prime b-divisors with log discrepancy 0 are only/generic point of E_m (cf. the proof of Theorem 10.13). In the *induction*, we use (BP) to secure $(\mathrm{FGA})^*$ property, and the disjoint property of each $E_{i|S^-_{jm}}$ from U_{jm} below.

Finally, we extend U_m to U_{jm} for $j \gg 0$ such that the restricted $\mathcal{L}_{jm}/j_{|E_t}$ reaches stabilization. Indeed, if $E_t \not\subset U_{jm}$ on S^-_{jm}, then we can make a destabilization as in the proof of Corollary 10.14. But now we replace Proposition 10.6 by Example 10.5(1) because $\sum e_i E_i$ is Cartier and the e_i are positive for E_i/P. This follows from ampleness of L/T (which is, in particular, Cartier) and because the $\sum e_i E_{i|S^-_{jm}}$ are disjoint from $E_{jm} = S^+_{jm|S^-_{jm}}$ by Step 1 with $m := jm$. Indeed, by (PRS) with $m := jm$, $aS^+_{jm} + bL + \sum e_i E_i$ is Cartier with $e_i > 0$, and $\equiv 0/T$. This implies also that each $E_{i|S^-_{jm}}$ is disjoint from some neighborhood U_{jm} of U_m and E_t.

Thus, if we set new $m := jm$, we can return to Step 2 again. The termination mentioned above completes the proof. □

Proof of Theorem 6.45. Immediate by Corollary 10.17 and Theorem 11.1. □

Remark 11.8. During the proof of Theorem 6.45, we essentially established that each $(\mathrm{FGA})_3(\mathrm{bir})$ algebra restricted from any *exceptionally* saturated algebra on a contraction of Example 3.40 with $\dim X \leq 4$ is f.g. In general, this does not imply f.g. of the algebra under restriction. Perhaps, this affects all $(\mathrm{FGA})_3(\mathrm{bir})$ algebras (cf. Remark 4.40(6)).

Similar arguments apply to $(\mathrm{FGA})_3(\mathrm{bir})$ if there exists a boundary divisor $B^+ \geq B$ such that $(X/T, B^+)$ is a weak plt Fano, and an effective Cartier divisor $aS^+ + bL \equiv 0/T$ for some $a > 0$, $b \geq 0$, the reduced part $S^+ \neq 0/P$ of B^+, and $L = (\mathcal{D}_i)_X$ with $i \gg 0$ up to truncation, since then (PRS) holds; instead of Lemma 11.2, we can use the more precise Examples 6.25, 10.5(2) and Addendum 5.12.2. For example, required a and b exist if, as in Theorem 11.1, X/T is extremal, S^+ is nef$/T$, and L is numerically negative$/T$.

Proof of (RFA) in Main Theorem 1.7. Immediate by Theorem 6.45 because for $n \leq 4$ we can drop LMMP, (BP), $(\mathrm{CCS})^*$, and (SSB)(gl). □

ACRONYMS AND OTHER ABBREVIATIONS

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