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THE EXISTENCE OF A STRAIGHT LINE ON FANO 3-FOLDS

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ABSTRACT. In this paper is shown the existence of a straight line on a Fano 3-fold of the principal series (under the anticanonical embedding) if the 3-fold has index 1 and is not isomorphic to the product $\mathbf{P}^1 \times \mathbf{P}^2$.

Bibliography: 13 titles.

In the papers [6] and [7] about Fano 3-folds, as well as in the classical papers [4] and [11], a considerable role in studying the geometry of these varieties is played by the question of existence of straight lines on them under the anticanonical embedding. In [4] Fano states a proposition about the existence of a straight line on an algebraic 3-fold V, whose Picard group is generated by the very ample anticanonical class $-K_V$. Such varieties V are called in [7] Fano 3-folds of the first species. However, his considerations [3], to which Roth refers in [11], are based mainly on counting parameters, which does not give a precise proof. The importance of the question about the existence of a straight line was pointed out by Iskovskih [7]. In the present paper a complete answer is obtained to this question (see, for example, Theorem 1.2).

§1. Statement of the main result

1.1. We shall assume that the ground field k is algebraically closed and has zero characteristic.

As in [6], [7] and [12], by a *Fano 3-fold* we mean a complete, nonsingular, irreducible variety V of dimension 3 over the field k with ample anticanonical class $-K_V$. The integer $g = g(V) = -K_V^3/2 + 1$ is called the *genus* of V. The largest integer $r \geq 1$ such that $\mathcal{H}^r \approx \mathcal{O}_V(-K_V)$ for some invertible sheaf $\mathcal{H} \in \operatorname{Pic} V$ is called the *index* of V. An effective one-dimensional cycle $l \subset V$ with $-K_V l = 1$ will be called a *line*.

Following [7], a Fano 3-fold with a very ample anticanonical class $-K_V$ will be called a Fano 3-fold *of the principal series*. For every such 3-fold V the anticanonical linear system gives an embedding $\varphi_{|-K_V|}\colon V \to V_{2g-2} \subset \mathbf{P}^{g+1}$, where V_{2g-2} is the subvariety of $\mathbf{P}^{g+1}$ of degree $2g-2$.

V_{2g-2} is called the *anticanonical model* of the 3-fold V. In the case of a Fano 3-fold of the principal series the straight line l has the usual geometric sense. It is a straight line on the anticanonical model V_{2g-2}.

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1.2. THEOREM. *Let V be a Fano 3-fold of the principal series. Then precisely one of the following alternatives is true.*

(1.2.1) *On V there is a line.*

(1.2.2) *V has index $r \geqslant 2$.*

(1.2.3) *$V \approx \mathbf{P}^1 \times \mathbf{P}^2$.*

From this theorem it is easy to deduce the following criterion for the existence of a straight line.

1.3. CRITERION FOR THE EXISTENCE OF A STRAIGHT LINE. *Let V be a Fano 3-fold of the principal series. On V there exists a straight line if and only if the anticanonical class $-K_V$ cannot be represented as a sum of two ample divisor classes.* ■

1.4. REMARKS.

(1.4.1) Criterion 1.3 is a weaker proposition than Theorem 1.2. Nevertheless from it one can deduce the following moral: the obstruction to the existence of a straight line has a topological nature at least in the case of a ground field k of characteristic zero.

(1.4.2) Apparently, Theorem 1.2 and Criterion 1.3 remain valid in the case of an arbitrary Fano 3-fold (compare Corollary 1.5), i.e. without assuming the very ampleness of the anticanonical divisor.

As a second simple corollary of Theorem 1.2 we obtain the truth of the following proposition, which is called in [6] Hypothesis 1.14. We recall that a Fano 3-fold V with $\operatorname{Pic} V = \mathbf{Z}$ is called a Fano 3-fold *of the first species.*

1.5. COROLLARY. *On a Fano 3-fold of the first species and of index one there exists a straight line.*

PROOF. By Theorem 1.2 of [12], Proposition 4.4 of [6], and Theorem 1.2 the above corollary remains unproven only in the following two cases:

(a) V is a hyperelliptic Fano 3-fold (see Definition 7.1 of [6]);

(b) the linear system $|-K_V|$ has a nonempty base set.

In case (b) the fiber of the elliptic pencil $|Y|$ from Proposition 3.1 (b) of [6] gives the needed straight line. In case (a) the anticanonical linear system gives a morphism $\varphi_{|-K_V|}$: $V \to W \subset \mathbf{P}^{g+1}$ of degree 2. Then by Theorem 7.2 of [6] the variety W is nonsingular and V is uniquely determined by the pair (W, D), where $D \subset W$ is the ramification divisor of $\varphi_{|-K_V|}$. Since $\operatorname{Pic} V \approx \mathbf{Z}$, by Corollary 7.6 of [6] either $W = \mathbf{P}^3$ and D is a smooth hypersurface of degree 6, or $W = V_2 \subset \mathbf{P}^4$ is a smooth quadric and $D = V_2 \cap V_4$ a smooth intersection of a quadric with a quartic. If b is a bitangent line in $W \subset \mathbf{P}^i$ ($i = 3$ or 4) to D, then $\varphi^{-1}_{|-K_V|}(b)$ splits into two straight lines in V. The existence of a bitangent straight line is an elementary geometric fact. ■

According to Theorem 6.1 in [7], from the above corollary we obtain

1.6. COROLLARY. *For a Fano 3-fold of the first species and of index* 1 *we have* $-K_V^3 \leqslant 22$. ■

1.7. NOTATION AND CONVENTIONS. The basic object of study in this paper is a Fano 3-fold V of the principal series. V is always identified with its anticanonical model V_{2g-2}. By the *degree* of an algebraic cycle c on V we mean the degree of the cycle with respect to the anticanonical linear system which coincides with the usual degree of c on $V_{2g-2} \subset \mathbf{P}^{g+1}$. Let X be an algebraic subvariety of $\mathbf{P}^n$. By $\langle X \rangle$ we denote the linear hull of X in $\mathbf{P}^n$, i.e. the smallest projective subspace of $\mathbf{P}^n$ which contains X. In particular, $\langle X \rangle \subset \mathbf{P}^{g+1}$ is defined for every algebraic subvariety $X \subset V$. By T_x we denote the embedded tangent space to V at the point x.

1.8. Let D be a divisor on V. The rational mapping defined by the linear system $|D|$ will be denoted by $\varphi_{|D|}\colon V \dashrightarrow \mathbf{P}^{\dim|D|}$.

1.9. Let D be a divisor on a smooth irreducible surface X, $x_1, \ldots, x_n$ a set of n distinct points on X and $k_1, \ldots, k_n$ a set of natural numbers. By $|D - \sum_1^n k_i x_i|$ we denote the linear subsystem of the complete linear system $|D|$ which consists of all the divisors $D' \in |D|$ which have multiplicity $\geqslant k_i$ at x_i, $i = 1, \ldots n$. By $\operatorname{codim}|D - \sum_1^n k_i x_i|$ we mean the codimension of the projective space $|D - \sum k_i x_i|$ in $|D|$. We say that the linear system $|D - \sum k_i x_i|$ is *nondegenerate at* x_i if the monoidal transformation with center x_i removes the indeterminacy at that point. Correspondingly the divisor D' has a nondegenerate singularity of degree k at x if this singularity is resolved by a single monoidal transformation with center x.

1.10. In this paper a *curve* (respectively a *surface*) is a one-dimensional (two-dimensional) complete irreducible and reduced algebraic variety.

1.11. Replacing in 1.9 the surface X by a smooth 3-fold and the points x_i by curves q_i, we can define the linear system $|D - \sum k_i q_i|$ and $\operatorname{codim}|D - \sum k_i q_i|$. Also the notion of a *nondegenerate linear system* $|D - \sum_1^n k_i q_i|$ at the generic point of q_i makes sense, as well as the nondegeneracy of a surface singularity at the generic point of q, where q is a curve.

1.12. Let H be a smooth divisor on a smooth variety V and let D be a divisor on V. We denote by $\mathcal{O}_H(H, D)$ the restriction of the invertible sheaf $\mathcal{O}_V(D)$ to H, by $|(H, D)|$ the corresponding complete linear system and by $(H, |D|)$ the linear subsystem of it obtained by restricting the linear system $|D|$ to H.

§2. Plan of the proof of Theorem 1.2

2.1. DEFINITION. A linear system $|D|$ on an irreducible nonsingular variety V is called a *linear system with splittings* if there exist two divisors $D_1, D_2 > 0$ such that $D_1 + D_2 \in |D|$.

If V is a Fano 3-fold of the first species with index 1, then $|-K_V|$ is a linear system without splitting. The author does not know if the converse is true.

The role played by splittings of the anticanonical system $|-K_V|$ of a Fano 3-fold is explained in the following four propositions.

2.2. PROPOSITION. *For a Fano 3-fold V of the principal series one of the following conditions is fulfilled.*

(2.2.1) *There is a line on V.*

(2.2.2) *V has index $r \geqslant 2$.*

(2.2.3) $V \approx \mathbf{P}^1 \times \mathbf{P}^2$.

(2.2.4) *On V there is a surface $S \approx \mathbf{P}^2$ of degree* 4 *with* $\dim\langle S \rangle = 5$.

(2.2.5) *$|-K_V|$ is without splitting.*

For the proof, see §8.

2.3. PROPOSITION. *For a Fano 3-fold V of the principal series one of the following conditions is fulfilled.*

(2.3.1) *If $\sigma\colon \tilde{V} \to V$ is a monoidal transformation with center at any point $x \in V$, then $\tilde{V}$*

is a Fano 3-fold of the principal series, and also $\varphi_{|-K_{\tilde V}|}$ *maps* $S = \sigma^{-1}(x)$ *onto a Veronese surface* $S \approx \mathbf{P}^2$ *of degree* 4.

(2.3.2) $|-K_V|$ *has splittings.*

(2.3.3) V *contains a conic* (*i.e. a smooth curve of degree* 2).

(2.3.4) V *contains a straight line.*

For the proof, see §7.

2.4. PROPOSITION. *Let* V *be a Fano* 3-*fold of the principal series which contains a conic. Then one of the following statements is true.*

(2.4.1) *If* $\sigma\colon \tilde V \to V$ *is a monoidal transformation with center in a sufficiently general conic* $q \subset V$, *then* $\tilde V$ *is a Fano* 3-*fold of the principal series, and also the exceptional surface* $S = \sigma^{-1}(q) \subset \tilde V$ *has degree* 4, $\dim\langle S\rangle = 5$, *and* $S \approx \mathbf{P}^1 \times \mathbf{P}^1$.

(2.4.2) $|-K_V|$ *has splittings.*

(2.4.3) V *contains a straight line.*

For the proof, see §6.

2.5. PROPOSITION. *Let* V *be a Fano* 3-*fold of the principal series which contains a surface* S *of degree* 4 *and is such that* (i) $\dim\langle S\rangle = 5$ *and* (ii) $S \approx \mathbf{P}^2$ *or* $S \approx \mathbf{P}^1 \times \mathbf{P}^1$. *Then either*

(2.5.1) *there exists a straight line not meeting* S, *or*

(2.5.2) *there exist two effective divisors* D, $D' > 0$ *and a positive integer* n *such that* $S \notin \mathrm{Ass}(D) \cup \mathrm{Ass}(D')$ *and* $nS + D + D' \in |-K_V|$.

For the proof, see §5.

The most fundamental role in this paper is played by Propositions 2.3–2.5. From them it is already easy to deduce the Fano Hypothesis 1.15 of [7]. Also they allow us to strengthen Proposition 2.2 substantially; namely, the following proposition holds:

2.6. COROLLARY. *Proposition* 2.2 *remains true even if one omits its last statement* (2.2.5).

PROOF. Let V be a Fano 3-fold of the principal series on which $|-K_V|$ does not split. It is enough to establish the existence of a straight line on V. Let us assume that V contains neither straight lines nor conics. Then the 3-fold $\tilde V$ obtained by the monoidal transform $\sigma\colon \tilde V \to V$ with center at a general point $x \in V$ is a Fano 3-fold of the principal series by Proposition 2.3. Also the surface $S = \sigma^{-1}(x) \subset \tilde V$ satisfies the requirements of Proposition 2.5. Because of the absence of straight lines on V, also $\tilde V$ contains no straight lines which do not intersect S, since $K_{\tilde V} \sim \sigma^*(K_V) + 2S$. From this last relation and from the absence of splittings in $|-K_V|$ we obtain that for $\tilde V$ (2.5.2) is impossible. This leads to a contradiction with Proposition 2.5. Therefore on V there exist either straight lines or conics.

Let us assume that V contains conics but no straight lines. Then the monoidal transform $\sigma\colon \tilde V \to V$ with center at a general conic $q \subset V$ leads to a Fano 3-fold $\tilde V$ of the principal series (see Proposition 2.4). Proceeding as above for the monoidal transform with center at a point, we obtain a contradiction with Proposition 2.5. (We recall that in this case $K_{\tilde V} \sim \sigma^*(K_V) + S$.) This last contradiction establishes the existence of a straight line on V. ∎

The next step in proving Theorem 1.2 is to exclude (2.2.4) from Proposition 2.2. An important role in doing this is played by the following lemma.

2.7. LEMMA. *Let V be a Fano 3-fold of the principal series, and let $S \subset V$ be a nonsingular surface of degree $\geqslant 3$ such that* $\deg S + 1 = \dim\langle S \rangle$. *Then S is an exceptional surface whose contraction $\sigma: V \to V'$ gives a Fano 3-fold V' of the principal series.*

PROOF. By the classification of surfaces of degree $n - 1$ in $\mathbf{P}^n$ (see [7]), either

(i) $S \approx F_n$, a rational ruled surface embedded in $\langle S \rangle$ by means of the complete linear system

$$\left| b_n + \frac{\deg S + n}{2} s_n \right|,$$

where s_n and b_n are the standard generators of the Picard group of F_n, or

(ii) $S \approx \mathbf{P}^2$, a plane embedded in $\langle S \rangle$ by means of the linear system of quadrics.

In case (i) $s_n S = -1$, and in case (ii) we have $(s, S) \sim -l$, where l is a straight line on $\mathbf{P}^2$. Therefore by the numerical criterion we have a contraction $\sigma: V \to V'$ of the surface S. In case (i), $\sigma(S)$ is a smooth rational curve, and in case (ii) it is a point. Also $\sigma^*(-K_{V'}) \sim -K_V + S$ in case (i), and $\sim -K_V + 2S$ in case (ii), from which, using the numerical criterion, it is easy to prove the ampleness of $-K_{V'}$. Also in case (i) it is necessary to use the equivalence $(S, S - K_V) \sim (\deg S - 2)s_n$ on S, which follows from the adjunction formula for K_S. Consequently, V' is a Fano 3-fold of genus $g + \deg S - 1$ in case (i) and of genus $g + 4$ in case (ii). Since V is a Fano 3-fold of the principal series, it follows that $g \geqslant 3$. Hence V' has genus $g' \geqslant 5$, and $-K_{V'}^3 = 2g' - 2 \geqslant 8$. Therefore in the linear system $|-K_V|$ there are no base points, by Theorem 1.2 of [12] and Proposition 3.1 in [6].(¹) Also $V' - \sigma(S)$ is mapped biregularly under the anticanonical morphism $\varphi_{|-K_{V'}|}$, since $\sigma^*(-K_{V'}) \sim -K_V + nS$, $n \geqslant 1$. Consequently V' is a Fano 3-fold of the principal series, by Proposition 4.4 in [6]. ■

PROOF OF THEOREM 1.2. Let V contain no straight lines and let the index of V be equal to 1. Then it is enough to prove that $V \approx \mathbf{P}^1 \times \mathbf{P}^2$. Assume this is not so. Then, because of Corollary 2.6, V contains a Veronese surface $S \approx \mathbf{P}^2$ of degree 4, which is exceptional by Lemma 2.7. The 3-fold V' obtained by contracting S to a point x is a Fano 3-fold of the principal series. V' does not contain straight lines; and $V' \not\approx \mathbf{P}^1 \times \mathbf{P}^2$ since on blowing up x one obtains a Fano 3-fold and therefore no conics or straight lines pass through x. Also under blowing up the point x of a Fano 3-fold the parity of the index is preserved. Therefore the index of V' is either one or three. As is known, on a Fano 3-fold of index 3 through every point there passes a one-dimensional family of smooth rational curves of degree 3. This is the family of straight lines on the quadric $V^1 \approx Q_2 \subset \mathbf{P}^4$ through x (see Theorem 4.2 in [6]). Under blowing up with center x the proper transform of such a curve of degree 3 is a straight line on V, which is impossible by the proposition. Consequently the index of V' is equal to 1. In such a case by Corollary 2.6 V' contains a Veronese surface $S \approx \mathbf{P}^2$ of degree 4. Further we pass from V' to V'', etc. Because of the finiteness of rk Pic V this is impossible, which completes the proof of Theorem 1.2. ■

(¹) In point (b) of Theorem 3.1 in [6] the possibility that $\mathcal{H}_H \approx \mathcal{O}_H(Z + 4Y)$ and $V \approx F \times \mathbf{P}^1$, where F is a del Pezzo surface of degree 1, is omitted. (This correction was communicated to the author by V. A. Iskovskih, and was also noted on p. 471 of the English translation of [7].)

§3. Lemmas about linear systems on surfaces

In this section, by a surface we mean a complete, irreducible and nonsingular variety of dimension two.

3.1. LEMMA. *On the surface X let there be given a divisor D and a set of distinct points $x_1, \ldots, x_n$ with positive integral multiplicities $k_1, \ldots, k_n$ such that*

(i) $\dim|D - \Sigma_1^n k_i x_i| > \max\{\dim|K_X|, 0\}$, *and*

(ii) *the general element of $|D - \Sigma_1^n k_i x_i|$ is irreducible and reduced.*

Then

$$\operatorname{codim}|D - \sum_{i=1}^n k_i x_i| \geqslant \left(\sum_{i=1}^n \frac{k_i^2}{2}\right) + \chi(X) - 2 + \min\left\{\frac{-K_X D}{2}, 1\right\},$$

where

$$\chi(X) = \sum_{i=0}^2 (-1)^i h^i(X, \mathcal{O}_X)$$

is the Euler characteristic of X. If X is a $K3$ surface, then

$$\operatorname{codim}|D - \sum_{i=1}^n k_i x_i| \geqslant \sum_{i=1}^n \frac{k_i^2}{2},$$

where in the case of equality all the fixed points of the linear system $|D - \Sigma_1^n k_i x_i|$ are nondegenerate; they coincide with one of the points x_i $(1 \leqslant i \leqslant n)$ and have multiplicity k_i.

PROOF. By the Riemann-Roch theorem, for the divisor $D_0 \in |D - \Sigma_1^n k_i x_i|$ we have

$$\dim|D_0| \geqslant \frac{D_0(D_0 - K_X)}{2} + \chi(X) - 1, \tag{3.2}$$

since

$$h^2(X, \mathcal{O}_X(D_0)) = h^0(X, \mathcal{O}_X(K_X - D_0)) = 0.$$

Indeed, in the opposite case

$$\dim|K_X| \geqslant \dim|D_0| \geqslant \dim|D - \sum_{i=1}^n k_i x_i|,$$

which contradicts (i).

For D_0 we choose a general element of $|D - \Sigma_1^n k_i x_i|$. Then we can assume that D_0 satisfies the following conditions. Let $\sigma = \sigma_1 \circ \cdots \circ \sigma_m \colon \tilde{X} \to X$ be a sequence of monoidal transformations $\sigma_1, \ldots, \sigma_m$ which removes the points of indeterminacy of the linear system $|D - \Sigma_1^n k_i x_i|$. Then the proper transform $\tilde{D}_0$ of the divisor D_0 is a smooth (because of (ii)) curve on X such that

$$\dim|\tilde{D}_0| = \dim|D - \sum_{i=1}^n k_i x_i|. \tag{3.3}$$

Also,

$$g(\tilde{D}_0) = p_a(D_0) - \sum_{i=1}^{m} \frac{l_i(l_i-1)}{2}, \tag{3.4}$$

$$0 \leqslant \tilde{D}_0^2 = D_0^2 - \sum_{i=1}^{m} l_i^2, \tag{3.5}$$

where l_i is the multiplicity of the point of the monoidal transformation σ_i on the proper transform of D_0 under the sequence of monoidal transformations $\sigma_1, \ldots, \sigma_{i-1}$. We may assume that $m \geqslant n$ and $l_i \geqslant k_i$ for $i = 1, \ldots, n$. From the exact cohomology sequence of the triple

$$0 \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}(\tilde{D}_0) \to \mathcal{O}_{\tilde{D}_0}(\tilde{D}_0, \tilde{D}_0) \to 0$$

we obtain the inequality

$$\dim|\tilde{D}_0| \leqslant \dim|(\tilde{D}_0, \tilde{D}_0)| + 1. \tag{3.6}$$

Consequently the linear system $|(\tilde{D}_0, \tilde{D}_0)| \neq \varnothing$. If $h^1(\tilde{D}_0, \mathcal{O}_{\tilde{D}_0}(\tilde{D}_0, \tilde{D}_0)) = 0$, then by (3.4), (3.5) and the Riemann-Roch theorem

$$\dim|(\tilde{D}_0, \tilde{D}_0)| = \tilde{D}_0^2 - g(\tilde{D}_0) = \frac{D_0(D_0 - K_X)}{2} - 1 - \sum_{i=1}^{m} \frac{l_i(l_i+1)}{2}$$

$$\leqslant \frac{D_0(D_0 - K_X)}{2} - 1 - \sum_{i=1}^{n} \frac{k_i^2}{2},$$

since

$$p_a(D_0) = \frac{D_0(D_0 + K_X)}{2} + 1.$$

Therefore from (3.6), (3.2) and (3.3) we obtain in this case

$$\operatorname{codim}|D - \sum_{i=1}^{n} k_i x_i| \geqslant \sum_{i=1}^{n} \frac{k_i^2}{2} + \chi(X) - 1. \tag{3.7}$$

Otherwise $h^1(\tilde{D}_0, \mathcal{O}_{\tilde{D}_0}(\tilde{D}_0, \tilde{D}_0)) > 0$, i.e. $|(\tilde{D}_0, \tilde{D}_0)|$ is a special linear system. Then by Clifford's theorem (see Theorem 5.4 in [5]) $\dim|(\tilde{D}_0, \tilde{D}_0)| \leqslant \tilde{D}_0^2/2$.

From (3.6), (3.2), (3.3) and (3.5) we get

$$\operatorname{codim}|D - \sum_{i=1}^{n} k_i x_i| \geqslant \sum_{i=1}^{m} \frac{l_i^2}{2} - \frac{K_X D_0}{2} + \chi(X) - 2. \tag{3.8}$$

This completes the proof of the lemma. The proof of the last assertion is easily deduced from (3.7) and (3.8). ■

Now we will give some corollaries of the lemma.

3.9. Corollary. *Under the assumptions of Lemma* 3.1, *if X is a $K3$ surface and* $\operatorname{codim}|D - \sum_1^n h_i x_i| \leqslant 1$, *then the following assertions are true*:

(3.9.1) *The general divisor D_0 of the linear system $|D - \sum_1^n k_i x_i|$ is a smooth curve.*

(3.9.2) *The linear system $|D - \sum_1^n k_i x_i|$ has at most two fixed points, and it has two only if D_0 is hyperelliptic.*

PROOF. By Lemma 3.1 we have the inequality $1 \geqslant \Sigma_1^n k_i^2/2$, from which we obtain (3.9.1) and the first half of (3.9.2).

Let us assume that $x_i \neq x_2 \in X$ are two distinct points of $|D - \Sigma_1^n k_i x_i|$. In this situation (3.7) is not satisfied and (3.8) is an equality. Therefore, as is clear from the proof of Lemma 3.1, we also have the equality $\dim|(\tilde{D}_0, \tilde{D}_0)| = \tilde{D}_0^2/2$, and $m = 2$, $l_1 = l_2 = 1$. On the other hand, on a $K3$ surface one always has $2g(D_0) - 2 = D_0^2 = \tilde{D}_0^2 + 2$ (the latter by (3.5)). Hence by Clifford's theorem the second half of (3.9.2) follows in the case $\tilde{D}_0^2 > 0$. If $\tilde{D}_0^2 = 0$, then D_0 has genus 2. ∎

3.10. COROLLARY. *Under the assumptions of Lemma* 3.1, *if X is a rational surface, with* $\operatorname{codim}|D - \Sigma_1^n k_i x_i| \leqslant 1$ *and* $-K_X D \geqslant 3$, *then the following assertions are true*:

(3.10.1) *The general divisor of the linear system $|D - \Sigma_1^n k_i x_i|$ is a smooth curve.*

(3.10.2) *The linear system $|D - \Sigma_1^n k_i x_i|$ has at most one nondegenerate fixed point of multiplicity one.*

PROOF. By Lemma 3.1 we have $1 \geqslant \Sigma_1^n k_i^2/2$; hence we obtain (3.10.1).

In the given situation (3.8) implies (3.10.2). Hence difficulty may only arise in the case of (3.7); but, as one sees from its proof, it is obtained by weakening the inequality

$$\operatorname{codim}|D - \sum_{i=1}^{n} k_i x_i| \geqslant \sum_{i=1}^{m} \frac{l_i(l_i+1)}{2} + \chi(X) - 1.$$

Therefore if (3.8) is satisfied we have $1 \geqslant \Sigma_1^m l_i(l_i + 1)/2$; hence we also obtain (3.10.2). ∎

3.11. COROLLARY. *Let X be a $K3$ surface and D a curve on it. Then the following assertions are true*:

(3.11.1) *If* $\dim|D| = 0$, *then D is a smooth rational curve with* $D^2 = -2$.

(3.11.2) *If* $\dim|D| > 0$, *then the linear system $|D|$ has no fixed points.*

(3.11.3) *The general member of the linear system $|D|$ is a smooth curve.*

PROOF. By duality and the Ramanujan vanishing theorem for a regular surface (see the remark on page 180 of [1]), $h^1(X, \mathcal{O}_X(D)) = 0$. Also $h^2(X, \mathcal{O}_X(D)) = 0$. Hence by the Riemann-Roch theorem $\dim|D| = D^2/2 + 1$. On the other hand, $2p_a(D) - 2 = D^2$, whence we obtain (3.11.1). (3.11.2) is a direct consequence of Lemma 3.1; and (3.11.1), (3.11.2) and Bertini's theorem imply (3.11.3). ∎

3.12. COROLLARY. *Let X be a rational surface and D a curve on it with* $-K_X D \geqslant 1$. *Then the following assertions are true*:

(3.12.1) *If* $\dim|D| = 0$, *then D is a smooth rational curve and* $K_X D = D^2 = -1$.

(3.12.2) *The general element of $|D|$ is smooth.*

(3.12.3) *The linear system $|D|$ has no base points for* $-K_X D \geqslant 2$.

PROOF. (3.12.1) is an immediate consequence of the Riemann-Roch theorem and the arithmetic genus formula for a curve. (3.12.3) and (3.12.2) follow from Bertini's theorem and Lemma 3.1. ∎

3.13. COROLLARY. *Let X be a smooth rational surface and D a curve on it such that the linear system $|D|$ is ample and* $-DK_X \geqslant 3$. *Then $|D|$ is very ample.*

PROOF. As was shown in [9], to prove the very ampleness of D it is sufficient to show the surjectivity of the natural homomorphism of graded algebras

$$S^*H^0(X, \mathcal{O}_X(D)) \to \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)), \tag{3.14}$$

where S^*E denotes the graded symmetric algebra of the vector space E. Let $D_0 \in |D|$ be a smooth curve (see (3.12.2)). Then from the cohomology sequence of the triple $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_{D_0}(D_0, D) \to 0$ we obtain the epimorphism $H^0(X, \mathcal{O}_X(D)) \to H^0(D_0, \mathcal{O}_{D_0}(D_0, D))$, so that $h^1(X, \mathcal{O}_X) = 0$ by assumption. Therefore for the proof of surjectivity in (3.14) it is enough to show it for the homomorphism

$$S^*H^0(D_0, \mathcal{O}_{D_0}(D_0, D)) \to \bigoplus_{n=0}^{\infty} H^0(D_0, \mathcal{O}_{D_0}(D_0, nD))$$

(see Lemma (2.9) in [6]). The latter map is surjective since $\deg(D_0, D) = D^2 \geqslant 2g(D_0) + 1$ [9]. ■

3.15. LEMMA. *Let X be a regular (i.e. $h^1(X, \mathcal{O}_X) = 0$) surface. Assume that on it two distinct curves D and D' are given for which* $\dim|D + D'| > \dim|D| + \dim|D'|$. *Then the general element of $|D + D'|$ is irreducible and reduced.*

PROOF. First of all, from the conditions of the lemma we obtain the absence of fixed components for divisors of the linear system $|D + D'|$. We may assume also the absence of base points in $|D + D'|$–in the opposite case one needs to remove the points of indeterminacy of $|D + D'|$. The general element $E \in |D + D'|$ is a smooth divisor. If $|D + D'|$ is not a pencil, the lemma follows from Bertini's theorem. If $|D + D'|$ is a pencil, then $\dim |D + D'|$ = the number of components of E (because of the regularity of X). On the other hand, from the inequality $\dim|D + D'| > \dim|D| + \dim|D'|$ it is easy to get that $\dim|D + D'| = 1$. Therefore E is irreducible. ■

3.16. LEMMA. *Let X be a regular surface on which are given effective divisors D and D' such that*

(i) Ass $D' \cap$ Ass $D = \varnothing$,

(ii) $h^1(X, \mathcal{O}_X(-D)) = 0$, *and*

(iii) *D' is reduced and $D' > 0$.*

Then $h^1(X, \mathcal{O}_X(-D - D')) =$ (the number of connected components of $D + D'$) $- 1$.

PROOF. By Ramanujan's theorem for a regular surface (see the remark on page 180 in [1])

$$h^1(X, \mathcal{O}_X(-D-D')) = h^0(D+D', \mathcal{O}_{D+D'}) - 1.$$

The case $D = 0$ is obvious by (iii). If $D > 0$, then one can limit oneself to the case of connected $D + D'$ by (i). Then $h^0(D + D', \mathcal{O}_{D+D'}) = 1$ by (iii), (i) and $h^0(D, \mathcal{O}_D) = 1$. The last equality follows from (ii) and Ramanujan's theorem. ■

3.17. LEMMA. *Let X be a rational surface with $|-K_X| \neq \varnothing$ on which there are given two distinct curves D and D' that are not fixed components of $|-K_X|$. If* (i) *$DD' \geqslant 2$, or* (ii) *DD', $-DK_X$, $-D'K_X \geqslant 1$, then the general element of $|D + D'|$ is irreducible and reduced.*

PROOF. Subtracting from the Riemann-Roch equality for the divisor $D + D'$ the analogous equalities for D and D', we obtain that

$$\dim|D+D'|-\dim|D|-\dim|D'| = h^1(X, \mathcal{O}_X(D+D')) - h^1(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D')) + D'D,$$

since

$$h^2(X, \mathcal{O}_X(D)) = h^2(X, \mathcal{O}_X(D')) = h^2(X, \mathcal{O}_X(D+D')) = 0.$$

Therefore by Lemma 3.15 it is enough to establish the inequality

$$DD' > h^1(X, \mathcal{O}_X(D')) + h^1(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D+D')).$$

The latter is a simple consequence of duality and of Lemma 3.16. Let us analyze for example the case (i) where D and D' do not intersect any anticanonical divisor $D'' \in |-K_X|$. Then

$$h^1(X, \mathcal{O}_X(D)) = h^1(X, \mathcal{O}_X(-D''-D)) = 1,$$

since

$$h^1(X, \mathcal{O}_X(-D'')) = h^1(X, \mathcal{O}_X(K_X)) = 0.$$

Analogously

$$h^1(X, \mathcal{O}_X(D')) = h^1(X, \mathcal{O}_X(D+D')) = 1.$$

This yields the needed inequality. ■

3.18. LEMMA. *Let X be a $K3$ surface and let D and D' be two distinct intersecting curves on X. Then in the linear system $|D + D'|$ one can find an irreducible and reduced divisor if and only if $D \cdot D' \geqslant 2$.*

PROOF. Subtracting from the Riemann-Roch equality for $D + D'$ the analogous equalities for D and D', we obtain that

$$\dim|D+D'|-\dim|D|-\dim|D'| = D\cdot D'-1,$$

because the second and first cohomology groups of the sheaves $\mathcal{O}_X(D)$, $\mathcal{O}_X(D')$, $\mathcal{O}_X(D + D')$ vanish. Then using Lemma 3.15 we obtain the assertion. ■

3.19. LEMMA. *Let X be a $K3$ surface and let D, D' and D'' be three distinct pairwise intersecting curves on it. Then in the linear system $|D + D' + D''|$ there exists an irreducible and reduced divisor.*

PROOF. If among the given curves there are two with intersection index $\geqslant 2$, then by applying Lemma 3.18 twice we get the conclusion. Therefore we may assume that $DD' = DD'' = D'D'' = 1$. If $\dim|D|$, $\dim|D'| > 0$, then by (3.11.2) the linear system $|D + D'|$ has no base points and $(D + D')^2 \geqslant 2$. Therefore its general element is irreducible and reduced. Then Lemma 3.18 for the general elements of $|D + D'|$ and $|D''|$ gives the conclusion. Consequently, we may assume that $\dim|D| \geqslant 0$ and $\dim|D'| = \dim|D''| = 0$ (after suitably renaming the curves). By Lemma 3.11, $(D')^2 = (D'')^2 = -2$. By the Riemann-Roch theorem,

$$\dim|D| = \frac{D^2}{2} + 1, \quad \dim|D+D'+D''| = \frac{(D+D'+D'')^2}{2};$$

hence $\dim|D + D' + D''| > \dim|D|$. If D'' is a fixed component of $|D + D' + D''|$, then $\dim|D + D'| > \dim|D|$. Then by Lemmas 3.15 and 3.18 we get the conclusion. Therefore we may assume that $|D + D' + D''|$ does not have fixed components. If $|D + D' + D''|$ is not a pencil, Bertini's theorem proves what we need. If $|D + D' + D''|$ is a pencil, then it is of the form $|nE|$, where $|E|$ is an elliptic pencil on X. The general element of $|D + D' + D''|$ is connected, since $h^1(X, \mathcal{O}_X(-D - D' - D'')) = 0$. Therefore $n = 1$, which completes the proof of the lemma. ■

3.20. Lemma. *Let X be a surface with $|-K_X| \neq 0$ and $K_X \neq 0$, and let D be a rational curve (perhaps singular) on X with $-K_X D \geqslant 2$. Then the family of effective divisors algebraically equivalent to the divisor D has dimension $\geqslant 1$.*

Proof. X has Kodaira dimension -1. If X is a regular surface then by Corollary (3.12.1) $\dim|D| \geqslant 1$. If X is an irregular surface, then by Theorem 4.1 in [2] there exists a canonical projection π: $X \to Y$, where Y is a smooth curve of genus $q(X) = h^1(X, \mathcal{O}_X)$ and the general fiber of π is a smooth rational curve. Also every curve in every fiber of π is a smooth rational curve. Therefore D is a smooth curve from some fiber of π. Then $D^2 \geqslant 0$ and $D = \pi^{-1}(d)$ for some point $d \in Y$, since the minimal model of X is a ruled surface. By virtue of the algebraic movability of the fiber of π, this completes the proof of the lemma. ■

3.21. Lemma. *Let $X \subset \mathbf{P}^g$ be a $K3$ surface embedded by a very ample linear system $|D|$ (hyperplane sections of X), and assume that $D_0 \in |D|$ is a curve which has nondegenerate quadratic singularities at the points $x_1, \dots, x_n$ and which is smooth outside of these points. Then the points $x_1, \dots, x_n$ are in general position, i.e.* $\dim\langle\sum_1^n x_i\rangle = n - 1$.

Proof. Let $r = \dim\langle\sum_1^n x_i\rangle$. Obviously $r \leqslant n - 1$. We will show that $r \geqslant n - 1$. Let σ: $\tilde{X} \to X$ be the sequence of monoidal transforms with center in x_i, $E_i = \sigma^{-1}(x_i)$, and $\tilde{D}_0$ the proper transform of D_0. $\tilde{D}_0$ is a smooth curve of genus $p_a(D_0) - n = g - n$, and

$$\dim|\tilde{D}_0 + \sum_{i=1}^{n} E_i| = \dim|D - \sum_{i=1}^{n} x_i| = g - r - 1,$$

since

$$\sigma^*(D_0) - \sum_{i=1}^{n} E_i \sim \tilde{D}_0 + \sum_{i=1}^{n} E_i.$$

Moreover, $K_{\tilde{X}} \sim \sum_1^n E_i$, and by the adjunction formula $K_{\tilde{D}_0} \sim (\tilde{D}_0, \tilde{D}_0 + \sum_1^n E_i)$. From the cohomology sequence of the triple

$$0 \to \mathcal{O}_{\tilde{X}}(K_{\tilde{X}}) \to \mathcal{O}_{\tilde{X}}\left(\tilde{D}_0 + \sum_{i=1}^{n} E_i\right) \to \mathcal{O}_{\tilde{D}_0}(K_{\tilde{D}_0}) \to 0$$

we have the inequality

$$\dim|\tilde{D}_0 + \sum_{i=1}^{n} E_i| \leqslant \dim|K_{\tilde{D}_0}| + \dim|K_{\tilde{X}}| + 1 = g - n.$$

Consequently, $g - r - 1 \leqslant g - n$ and $n - 1 \leqslant r$. ■

3.22. LEMMA. *Let $X \subset \mathbf{P}^g$ be a $K3$ surface embedded by the very ample linear system $|H|$, and assume that D is a curve and that D' is a reduced connected divisor on X such that $D + D' \in |H|$. Then D is fully linearly embedded in $\langle D \rangle$.*

PROOF. If we consider the cohomology sequence of the short exact sequence

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(H) \to \mathcal{O}_D(D, H) \to 0$$

we see that to show the surjectivity of

$$H^0(X, \mathcal{O}_X(H)) \to H^0(D, \mathcal{O}_D(D, H))$$

it suffices to show that $h^1(X, \mathcal{O}_X(D')) = 0$. The latter is obtained by duality and from Ramanujan's theorem for regular surfaces (see the remark on page 180 in [1]). ■

3.23. LEMMA. *Let C be a nonhyperelliptic smooth curve and let $D \neq 0$ be a special divisor on it (i.e. $|K_C - D| \neq \varnothing$ and $|D| \neq \varnothing$). Then* $\deg D_0' \leqslant \deg D$, *and equality holds only for $D_0' = D'$, where D_0' is the fixed part of the linear system $|D'| = |K_C - D|$.*

PROOF. By duality and by the definition of D_0' we have

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(D')) = h^0(C, \mathcal{O}_C(D' - D_0')) = h^1(C, \mathcal{O}_C(D + D_0'));$$

hence, subtracting the Riemann-Roch formula for the sheaf $\mathcal{O}_C(D)$ from the corresponding formula for $\mathcal{O}_C(D + D_0')$, we obtain

$$h^0(C, \mathcal{O}_C(D + D_0')) = h^0(C, \mathcal{O}_C(D)) + \deg D_0'.$$

Now from Clifford's theorem for the special divisor $D + D_0'$ there follows the inequality

$$h^0(C, \mathcal{O}_C(D)) + \deg D_0' \leqslant \frac{\deg(D + D_0')}{2} + 1.$$

Hence we obtain the inequality required in the lemma, because $h^0(C, \mathcal{O}_C(D)) \geqslant 1$, and, in the case of equality, by Clifford's theorem either $D + D_0' \sim 0$ or $D + D' \sim K_C$ (C is nonhyperelliptic). The former is impossible by assumption. ■

3.24. COROLLARY. *Let $X \subset \mathbf{P}^g$ be a $K3$ surface embedded by a very ample linear system $|H|$, and let D be a curve on it such that $|H - D| \neq \varnothing$. Then* $\deg D_0' \leqslant \deg D$ (*the degree under the inclusion $X \subset \mathbf{P}^g$*), *and equality holds only for $D_0' = D'$, where D_0' is a fixed component of the linear system $|D'| = |H - D|$.*

PROOF. The general hyperplane section $C \in |H|$ is a smooth canonical curve of genus g. The linear system $|D'|$ restricts to the linear system $|(C, D')|$ on C isomorphically. The latter follows from the exact cohomology sequence of the triple

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X(D') \to \mathcal{O}_C(C, D') \to 0$$

and from the vanishing of

$$h^0(X, \mathcal{O}_X(-D)) = h^1(X, \mathcal{O}_X(-D)) = 0.$$

Then we obtain the assertion from the previous lemma. ■

§4. Lemmas on linear systems on Fano 3-folds

4.1. LEMMA. *Let D be an effective divisor on a Fano 3-fold V. If $D > 0$, then*

$$h^0(V, \mathcal{O}_V(-D))=0,$$
$$h^1(V, \mathcal{O}_V(-D))=h^0(D, \mathcal{O}_D)-1,$$
$$h^2(V, \mathcal{O}_V(-D))=h^1(D, \mathcal{O}_D),$$
$$h^3(V, \mathcal{O}_V(-D))=h^2(D, \mathcal{O}_D);$$

if $D = 0$, then

$$h^0(V, \mathcal{O}_V)=1,$$
$$h^1(V, \mathcal{O}_V)=h^2(V, \mathcal{O}_V)=h^3(V, \mathcal{O}_V)=0.$$

PROOF. The statement for $D = 0$ is a direct consequence of the Kodaira vanishing theorem and Serre duality. The case $D > 0$ follows from the cohomology sequence of the short exact sequence $0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V \to \mathcal{O}_D \to 0$ and from the previous proposition. ■

4.2. COROLLARY. *If an element of the linear system $D_0 \in |D|$ on a Fano 3-fold V consists of n connected components, then $h^1(V, \mathcal{O}_v(-D)) \geqslant n - 1$. If each component of D_0 is reduced, then $h^1(V, \mathcal{O}_V(-D)) = n - 1$.* ■

4.3. Let H be an effective divisor. We say that D is a *divisor from the splitting of the linear system* $|H|$ if D is an effective divisor and if there exists an effective divisor D' for which $D + D' \in H$. If $D > 0$ and $D' > 0$, we say that D *gives a nontrivial splitting* (compare Definition 2.1). The divisor D' will be called *residual to D in H* (in the case $|H| = |-K_V|$ simply *residual*), and the corresponding linear system $|D'|$ will be called *residual to D in $|H|$* (respectively, just *residual*).

4.4. LEMMA. *Let H be a smooth surface on a Fano 3-fold V, and let D be a divisor from splitting of the linear system H for which the general residual divisor $D' \in |H - D|$ is reduced. Then the natural restriction homomorphism*

$$r: H^0(V, \mathcal{O}_V(D)) \to H^0(H, \mathcal{O}_H(H, D))$$

is surjective for $D' = 0$ and has a cokernel of dimension $\leqslant n - 1$, where n is the number of connected components in the general element $|D'|$ for $D' > 0$. The map r is injective if $D' > 0$, and it has a one-dimensional kernel if $D = 0$.

We prove this by applying Corollary 4.2 to the exact cohomology sequence of the short exact sequence

$$0 \to \mathcal{O}_V(-D') \to \mathcal{O}_V(D) \to \mathcal{O}_H(H, D) \to 0. \quad ■$$

4.5. LEMMA. *Let D and D' be two surfaces on a Fano 3-fold V which give a nontrivial splitting $D + D' \in |-K_V|$, and suppose D is also smooth. Then D is a smooth rational surface with $|-K_D| \neq \varnothing$.*

PROOF. If $D \neq D'$, then, using the adjunction formula $-K_D \sim (D, D') > 0$, we obtain that $|-K_D|$ is nonempty when $D \neq D'$. If $D = D'$, the nonemptiness follows by the same reasoning with D' replaced by a general divisor in $|D'|$ ($\dim|D'| = \dim|-K_V/2| > 0$; see Proposition 1.9(ii) in [6]). Therefore to show the rationality of D it is enough to establish the triviality of $h^1(D, \mathcal{O}_D) = 0$ and $h^2(D, \mathcal{O}_D) = 0$ (see [2]). The latter follows from Lemmas 4.1 and 4.2, and from Serre duality:

$$h^2(D, \mathcal{O}_D) = h^3(V, \mathcal{O}_V(-D)) = h^0(V, \mathcal{O}_V(-D')) = 0,$$
$$h^1(D, \mathcal{O}_D) = h^2(V, \mathcal{O}_V(-D)) = h^1(V, \mathcal{O}_V(-D')) = h^0(D', \mathcal{O}_{D'}) - 1 = 0. \blacksquare$$

4.6. LEMMA. *Let $q_1, \ldots, q_n$ be a set of distinct curves with positive integral multiplicities $k_1, \ldots, k_n$ on a Fano 3-fold of the principal series such that*

(i) $\dim\left|-K_V - \sum_{i=1}^{n} k_i q_i\right| > 0$;

(ii) *the general element of $|-K_V - \sum_1^n k_i q_i|$ is irreducible and reduced, and*

(iii) $n > 0$.

Then

$$\operatorname{codim}\left|-K_V - \sum_{i=1}^{n} k_i q_i\right| \geqslant 1 + \sum_{i=1}^{n} \frac{k_i^2}{2} \deg q_i,$$

and in the case of equality every base curve q for $|-K_V - \sum_1^n k_i q_i|$ coincides with one of the q_i $(1 \leqslant i \leqslant n)$ and has a nondegenerate multiplicity k_i.

PROOF. We consider a sufficiently general hyperplane section $H \in |-K_V|$. Also we may assume that H is a smooth $K3$ surface ([6], Corollary 1.5) on which are given $\sum_1^n \deg q_i$ distinct points x_i^j $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant \deg q_i)$, the points of intersection of $\bigcup_1^n q_i$ with H. Then by Lemma 4.4 and the nontriviality of $\sum_1^n k_i q_i$ (see (iii)) it is enough to show that

$$\operatorname{codim}\left|D - \sum_{i=1}^{n} \sum_{j=1}^{\deg q_i} k_i x_i^j\right| \geqslant \sum_{i=1}^{n} \frac{k_i^2}{2} \deg q_i, \tag{4.7}$$

where D is a smooth canonical curve-section of genus g on H. For a sufficiently general H the general element of

$$\left|D - \sum_{i=1}^{n} \sum_{j=1}^{\deg q_i} k_i x_i^j\right|$$

is irreducible and reduced (see (ii)), and the linear system $|-K_V - \sum_1^n k_i q_i|$ restricts injectively to the system

$$\left|D - \sum_{i=1}^{n} \sum_{j=1}^{\deg q_i} k_i x_i^j\right|.$$

Therefore

$$\dim\left|D - \sum_{i=1}^{n} \sum_{j=1}^{\deg q_i} k_i x_i^j\right| > 0$$

(see (i)). In this situation (4.7) follows from Lemma 3.1. The last assertion of the lemma being proved follows from the corresponding assertion in Lemma 3.1. ■

4.7. LEMMA. *Under the assumptions of Lemma* 4.6, *if* $k_1 = \cdots = k_n = 1$ *and*

$$2\dim\left\langle \bigcup_{i=1}^{n} q_i \right\rangle = \sum_{i=1}^{n} \deg q_i,$$

then $\sum_1^n \deg q_i = 2g - 2$, *and* $\bigcup_1^n q_i$ *is a curve-section of the Fano* 3-*fold* $V \subset P^{g+1}$.

PROOF. Indeed,

$$\operatorname{codim}\left| -K_V - \sum_{i=1}^{n} q_i \right| = \dim\left\langle \bigcup_{i=1}^{n} q_i \right\rangle + 1.$$

Therefore in Lemma 4.6 we have equality. Consequently, the general divisor $D \in |-K_V - \sum_1^n q_i|$ is a surface which admits only isolated singularities. The intersection of D with a general hyperplane section is a nonhyperelliptic canonical curve $C \subset \mathbf{P}^{g-1}$. On it lie $2\dim\langle \bigcup_1^n q_i\rangle$ distinct points which generate a subspace of dimension equal to $\dim\langle \bigcup_1^n q_i\rangle - 1 \leqslant g - 2$. The latter, as is known by Clifford's theorem, is only possible if $\dim\langle \bigcup_1^n q_i\rangle - 1 = g - 2$, which completes the proof. ■

4.8. LEMMA. *Let D be a surface from the splitting of the linear system $|-K_V|$ on a Fano 3-fold V of the principal series. Then* $\deg D_0' \leqslant \deg D$, *and equality holds only if* $D_0' = D'$, *where D_0' is a fixed component of the residual linear system* $|D'| = |-K_V - D|$.

By means of a general hyperplane section the proof reduces to Corollary 3.24 just as 3.24 reduces to Lemma 3.23. In addition, to prove that the linear system $|D'|$ isomorphically restricts to the linear system $|(H, D')|$ on the general hyperplane section $H \in |-K_V|$ one needs to use Lemma 4.4. ■

4.9. LEMMA. *Let D be a surface from the splitting of the linear system $|-K_V|$ of a Fano 3-fold V of the principal series such that the general element of* $|D'| = |-K_V - D|$ *is reduced and connected. Then for any point $x \in V$ the following assertions are true.*

(4.9.1) *The general element of $|D|$ is not a cone with vertex x if* $\deg D \geqslant 3$.

(4.9.2) *The general element of $|D - x|$ is not a cone with vertex x if* $\deg D \geqslant 4$ *and* $\dim|D| \geqslant 1$.

PROOF. Using 4.4, we reduce this lemma using a general hyperplane section through x to 3.19 ■

4.10. COROLLARY. *Under the assumptions of the previous lemma the following assertions hold.*

(4.10.1) $\dim|D| = 0$ *if and only if* $\dim\langle D\rangle = \deg D + 1$, *and if* $\deg D \geqslant 3$ *then D is a smooth surface.*

(4.10.2) *The linear system $|D|$ has no base points if* $\dim|D| \geqslant 1$ *and its general element is a smooth surface.*

PROOF. By Lemma 4.4 the first half of (4.10.1) is reduced by a general hyperplane section to Lemmas 3.22 and 3.11. Also we need to use the fact that $\dim\langle X\rangle = \deg X$ for a fully embedded smooth rational curve. The second half of (4.10.1) follows from the previous lemma.

Let x be a fixed point of $|D|$, i.e. $|D - x| = D$. By the previous lemma (see (4.9.1)) a sufficiently general element of $|D|$ and a hyperplane section $H \in |-K_V|$ through x give in the intersection a curve from $|(H, D)|$. Then, by Lemma 4.4 and (3.11.2), $\dim|D| = \dim|(H, D)| = 0$. This proves the first half of (4.10.2). The last assertion follows from Bertini's theorem. ■

4.11. LEMMA. *Under the conditions of Lemma* 4.9, *suppose that*
(i) $\dim|D| \geqslant 1$ (*then by* (4.10.2) $\varphi_{|D|}$ *is a morphism*),
(ii) $\dim \varphi_{|D|}(V) = 3$, *and*
(iii) $\varphi_{|D|}$ *does not map any surface* $S \subset V$ *to a point.*
Then either
(4.11.1) V *contains a straight line*, *or*
(4.12.2) *the linear system* $|D|$ *is ample.*

PROOF. D is ample by the numerical criterion if $\dim \varphi_{|D|}^{-1}(x) = 0$ for every point $x \in \varphi_{|D|}(V)$.

Suppose that for some point $x \in \varphi_{|D|}(V)$ this is not true. We may also assume that $D \not\sim 0$. Then it is enough to show that some component of $l \subset \varphi_{|D|}^{-1}(x)$ is a straight line (the inverse image is considered as a reduced subvariety of V).

Indeed, $\dim l = 1$ by (ii). The general element of $|D - l|$ is irreducible and reduced by Bertini's theorem, since it has no fixed components (see (iii)) and is not a pencil (ii). Also $\operatorname{codim}|D - l| = 1$ and $\dim|D - l| > 0$. Using a sufficiently general hyperplane section $H \in |-K_V|$, we obtain the inequality $\deg l \leqslant 2$ by 4.4 and 3.9. The curve l cannot be a conic, since otherwise by Proposition 4.3 of [7] the image of the morphism $\varphi_{|D|}$ would be two-dimensional and its general fiber would be a conic. This ends the proof. ■

4.12. LEMMA. *Under the assumptions of Lemmas* 4.9 *and* 4.11, *if* $D^2D' > 3$, *then the following assertions are true*:

(4.12.1) $V' = \varphi_{|D|}(V)$ *is a smooth* 3-*fold, and* $\varphi_{|D|}\colon V \to V'$ *is the contraction of nonintersecting ruled surfaces* $S_1, \ldots, S_n \subset V$ *whose fiber consist of lines* $l \subset V$ *contracted to a point by* $\varphi_{|D|}$.

(4.12.2) *If* V *contains no straight lines* l *with* $D'l = 1$, *then* $|D|$ *is very ample.*

PROOF. (4.12.2) is a special case of (4.12.1). First we show that $\deg \varphi_{|D|} = 1$. To do this we consider a sufficiently general divisor $D_0 \in |D|$. D_0 is a smooth rational surface (see Corollary 4.10 and the proof of Lemma 4.5). The linear system $|D|$ restricts epimorphically to the linear system $|L| = |(D_0, D)|$ on D_0. The general element $L_0 \in |L|$ is a smooth curve by Bertini's theorem. Let $x \in L_0$ be some point. Then the linear system $|L - x|$ has the unique base point x (see Corollary 3.10), since $-K_{D_0}L = D_0DD' \geqslant 3$. Hence $\varphi_{|D|}$ is a birational morphism. If $\dim \varphi_{|D|}^{-1}(x) = 1$, then $l = \varphi_{|D|}^{-1}(x)$ is a straight line on V. This is shown by the method used above in Lemma 4.11, only in doing this one has to note that the general hyperplane H is mapped birationally and therefore the general element of the linear system $|(H, D)|$ on H is not hyperelliptic. Let $x \in V$ be a point. We will show the smoothness of the generic element $D_0 \in |D - x|$. If x does not lie on a contractible straight line l, then the linear system $|D - x|$ has a finite number of base points $x_1 = x, \ldots, x_m$. It will suffice to prove the smoothness of D_0 at every one of the points x_i ($1 \leqslant i \leqslant m$). By Bertini's theorem D_0 is irreducible, and by Lemma 4.9 it is not a cone with vertex at x_i since

$$\deg D = DK_V^2 = -D^2K_V - DD'K_V \geqslant -D^2K_V = D^3 + D^2D' \geqslant 4.$$

Then by Lemma 3.9 the general element of $|(H, D) - x_i|$ on the general hyperplane section of H through x_i is smooth. Thus D_0 is also smooth at x_i ($1 \leqslant i \leqslant m$). Therefore by Bertini's theorem in this case the general element $D_0 \in |D - x|$ is smooth. Let us consider the linear system $|L| = |(D_0, D)|$ on D_0. By Lemma 4.4, $\varphi_{|D|}|_{D_0} = \varphi_{|L|}$ ($\varphi|_{D_0}$ denotes the restriction of the morphism φ to the subvariety D_0). If L is a contractible curve for $\varphi_{|L|}$, then it is a straight line on V. Also, L is an exceptional curve on D_0 because $-lK_{D_0} = lD' = 1$. Then from Lemma 3.13 it is easy to show that $\varphi_{|L|}$ is the contraction of these exceptional curves to the smooth surface D_0' (which is a hyperplane section of V'), since $-LK_{D_0} = D^2D' \geqslant 3$. Hence follows the biregularity of $\varphi_{|D|}$ outside the subvariety $S = \bigcup_1^n S_i$, which is swept by the contractible straight lines. The proof of the lemma will be complete if we show the smoothness of the general member $D_0 \in |D - x| = |D - l|$ for x lying on the contractible straight line l. Indeed, in this case a smooth hyperplane section D_0' obtained from D_0 by contracting exceptional curves will pass through $\varphi_{|D|}(x)$. Also the number of such curves is equal to $K_{D_0'}^2 - K_{D_0}^2$. In order to show the smoothness of D_0 it is enough to show that $\operatorname{codim}|D - 2x| \geqslant 3$ for the general point $x \in l$ (the inequality $\geqslant 2$ is proved in the same way as for the points outside of S_i). A sufficiently general hyperplane $H \in |-K_V|$ does not contain contractible straight lines, and the mapping $\varphi_{|D|}|_H = \varphi_{|M|}$ is birational; here $|M| = |(H, D)|$. Therefore $|M|$ is a very ample system on the $K3$ surface H. Consequently,

$$\operatorname{codim}|D - 2x| \geqslant \operatorname{codim}|M - 2x| \geqslant 3 \quad \text{for } x = H \cap l. \quad \blacksquare$$

§5. Proof of Proposition 2.5

5.1. Let V a Fano 3-fold of the principal series satisfying the assumptions but not the conclusion of Proposition 2.5. Then

(5.1.1) *every straight line on V intersects S; and*

(5.1.2) *every divisor $D \in |-K_V|$ can be represented in the form $D = nS + F$, where $n \geqslant 0$ is an integer and F is a surface on V.*

Using the method of multiple projection of V with vertex in $\langle S \rangle$, we will establish a contradiction (see 5.13).

5.2. S is an exceptional surface (see 2.7). Therefore $\operatorname{rk}\operatorname{Pic} V \geqslant 2$. If $g \leqslant 5$, then V is a complete intersection in $\mathbf{P}^{g+1}$ (see [7], Proposition 1.3), and $\operatorname{Pic} V \approx \mathbf{Z}$. Consequently $g \geqslant 6$ and $\dim|S'| = g - 5 \geqslant 1$, where $|S'| = |-K_V - S|$.

5.3. *The general element of the linear system $|S'| = |-K_V - S|$ is irreducible and reduced and* $\neq 0$. Because of 5.2 and (5.1.2) a fixed component of $|S'|$ has the form nS ($n \geqslant 0$), and $nS \not\sim S'$. Then $n = 0$ (see 4.8), and 5.3 follows from (5.1.2).

5.4. *The linear system $|S'|$ gives a morphism $\varphi_{|S'|}\colon V \to \mathbf{P}^{\dim|S'|}$ with a three-dimensional image.* By Corollary 4.10, $|S'|$ has no base points. On the other hand, $S(S')^2 = K_S^2 \geqslant 8$. Therefore the image of $\varphi_{|S'|}$ has dimension $\geqslant 2$. If $\dim \varphi_{|S'|}(V) = 2$, then, because of the linear normality of $\varphi_{|S|}$ and the rationality of the general smooth surface $S_0' \in |S'|$ (see 4.5), $\varphi_{|S'|}(V)$ is a surface of degree $\dim|S'| - 1$ in $\mathbf{P}^{\dim|S'|}$, $\dim|S'| = g - 5 = 5$, since $(S')^3 = 2g - 12 - K_S^2$ (see [6], Lemma 2.11), and $(S')^3 = 0$. On the surface $\varphi_{|S'|}(V)$ the linear system of hyperplane sections of $\varphi_{|S'|}(V)$ has a nontrivial splitting $D + D' \in |L|$ with $\dim|D| \geqslant 1$ and $\dim|D'| \geqslant 1$. This contradicts (5.1.2), which completes the proof of 5.4.

5.5. *The linear system* $|S'|$ *is very ample*. The linear system $|S'|$ restricts to an ample linear system $|(S, S')| = |-K_S|$ on S. Therefore $\varphi_{|S'|}$ does not map to a point surfaces that intersect S; hence by (5.1.2) and because of the connectedness of any divisor in $|-K_V|$ (see [12], Lemma 2.1) we obtain that $\varphi_{|S'|}$ does not map any surface to a point. Also $(S')^2S = K_S^2 \geqslant b$. Consequently, $\varphi_{|S'|}$ is a contraction of mutually nonintersecting ruled surfaces $S_1, \ldots, S_m$ whose fibers consist of straight lines $l \subset V$ which are contracted to a point by $\varphi_{|S'|}$ (see Lemma 4.13).

$V' = \varphi_{|S'|}(V) \subset \mathbf{P}^{g-5}$ has degree $(S')^3 = 2g - 12 - K_S^2 > 0$, i.e. $g - 5 > 5$. If $\varphi_{|S'|}(V)$ contained a movable family of planes, then as in 5.4 we would reach a contradiction with (5.1.2). This means that

$$\deg \varphi_{|S'|}(V) = 2g - 12 - K_S^2 > g - 7$$

(see [6], Lemma 2.8) and $g - 5 > K_S^2$. Therefore $\varphi_{|S'|}(S)$ is contained in a hyperplane in $\mathbf{P}^{g-5}$. Any contractible surface S_i lies outside of S, intersects S and is contracted to $\varphi_{|S'|}(S)$. Consequently, $\bigcup_1^m S_i$ lies in the splitting of $|-K_V - S|$. Then $m \leqslant 1$. If $m = 1$, then $nS + S_1 \in |-K_V|$ for some $n \geqslant 2$ (see (5.1.2)). Also $\varphi_{|S'|}(S)$ is a smooth hyperplane section, $n = 2$ and $-K_{V'} \sim 2(\varphi_{|S'|}(S))$, i.e. V' is a Fano 3-fold of index 2 with $(-K_V/2)^3 = K_S^2 \geqslant 8$. The latter is impossible because of Theorem 4.2 of [6]. This means that $m = 0$ and $|S'|$ is very ample.

5.6. *The linear system* $|S''| = |S' - S|$ *is nonempty, and its general element is representable in the form* $kS + G$, *where* $\dim|G| = \dim|S''| \geqslant 1$, *k is an integer* $\geqslant 0$ *and G is a surface in V.*

In proving 5.5 it was shown that $g - 5 > K_S^2$ and that $\varphi_{|S'|}(S)$ is contained in a hyperplane. Hence we obtain the nonemptiness of $|S' - S|$. If $g - 5 = K_S^2 + 1$, then $\deg V' = 2g - 12 - K_S^2 = K_S^2$, $S'' \sim 0$ and $\varphi_{|S'|}(S)$ is a hyperplane section of $V' \subset \mathbf{P}^{g-5}$. The latter contradicts the fact that S is exceptional. Consequently, $g - 5 > K_S^2 + 1$ and $\dim|S'| \geqslant 1$. This completes the proof of 5.6 (see (5.1.2)).

5.7. *The linear system* $|G|$ *has no base points*. Let $x \in V$; then by Lemmas 3.17 and 4.4 the general element $G_0 \in |G|$ is not a cone with vertex x under the inclusion $\varphi_{|S'|}$ (the case when G_0 is a plane is obviously impossible because of the exceptionality of S). Also the linear system $|S''|$ restricts isomorphically to the linear system $|(S_0', S'')|$ for the general hyperplane section $S_0' \in |S'|$ through x. The linear system $|(S_0', G)|$, and therefore also G, has no fixed points, since $-(S_0', G)K_{S_0'} = GS'S \geqslant 2$ (see 3.12).

5.8. *The general element of* $|S''|$ *is irreducible and reduced*, i.e. $k = 0$ in 5.6. By 5.7 the general element $G_0 \in |G|$ is a smooth surface. Also we may assume that G_0 and S intersect transversally and give a smooth curve B in the intersection. By the adjunction formula, $(k + 2)B \in |-K_{G_0}|$. Therefore $k + 2 \leqslant 3$, and for $k + 2 = 3$ we have $G_0 \approx \mathbf{P}^2$ and $B \approx \mathbf{P}^1$ (see [8], §2). This last case is impossible since a smooth rational curve does not lie in the linear system $|(S, S'' - S)|$ on S. Consequently, $k = 0$.

5.9. *The surface S is fully linearly embedded by the map* $\varphi_{|S''|}|_S$. From the cohomology sequence of the triple

$$0 \to \mathcal{O}_V(S'' - S) \to \mathcal{O}_V(S'') \to \mathcal{O}_S(S, S'') \to 0$$

it is enough to show that $h^1(V, \mathcal{O}_V(S'' - S)) = 0$. The latter follows from duality, the ampleness of $|S' + S''|$ (see 5.5–5.8) and Kodaira's vanishing theorem. Indeed,

$$h^1(V, \mathcal{O}_V(S'' - S)) = h^2(V, \mathcal{O}_V(-S' - S')) = 0.$$

5.10. $\dim \varphi_{|S''|}(V) \geqslant 3$. In the opposite case we would have $\varphi_{|S''|}(V) = \varphi_{|S''|}(S)$. The latter leads to a contradiction with (5.1.2) because of 5.9, since $S \approx \mathbf{P}^2$ is embedded by the complete linear system of curves of degree 4 and $S \approx \mathbf{P}^1 \times \mathbf{P}^1$ is embedded by the system $|3b_0 + 2s_0|$.

5.11. *The linear system* $|S''|$ *is ample.* The absence of surfaces contracted to a point is shown exactly as it is shown for $|S'|$ in 5.5. If q is a contracted one-dimensional subvariety, i.e. $q = \varphi_{|S''|}^{-1}(x)$, $x \in \varphi_{|S''|}(V)$, then by Bertini's theorem the general element of $|S'' - q|$ is irreducible and reduced (see 5.10). Consequently, $q \cdot S' = 1$ (see 4.4 and 3.10), since $\dim|S'' - q| > 0$ and $\operatorname{codim}|S'' - q| = 1$, while

$$K_{S_0'}(S_0', S'') = SS''S' = S(S')^2 - S^2S' > K_S^2 \geqslant 8$$

for the general hyperplane section $S_0' \in |S'|$. On the other hand, $qS'' = 0$ and $\dim(q \cap S) \leqslant 0$ (see 5.9).

Also, by 5.5 and (5.1.1) there are no lines on V which do not lie on S. Therefore q is a conic. Conics on V form at most a two-dimensional set (see [7], Proposition 4.3), contradicting 5.10. This means that $|S''|$ is ample.

5.12. The general element $S_0'' \in |S''|$ is a smooth surface intersecting S transversally along a smooth curve B (see 5.7 and 5.8). Also $g(B) > 0$ (see the end of 5.10), and $2B \in |-K_{S_0''}|$. Consequently S_0'' is a nonrational ruled surface, and $h^1(S'', \mathcal{O}_{S_0''}) > 0$ (see §2 of [8]).

5.13. By Kodaira's vanishing theorem and the ampleness of $|S''|$ we have $h^2(V, \mathcal{O}_V(-S'')) = 0$. Thus $h^1(S_0'', \mathcal{O}_{S_0''}) = 0$ according to 4.1. The latter contradicts 5.12. ∎

§6. Proof of Proposition 2.4

6.1. Let V be a Fano 3-fold of the principal series which contains a conic. Let us also assume that V does not satisfy (2.4.2) and (2.4.3), i.e. that the following conditions are satisfied:

(6.1.1) *All the divisors of the linear system* $|-K_V|$ *are irreducible and reduced.*

(6.1.2) V *does not contain a straight line.*

In order to prove Proposition 2.4 it is sufficient to establish the truth of (2.4.1), which will be done below (see 6.16).

6.2. Let $q \subset V$ be a conic and let $\sigma\colon \tilde{V} \to V$ be the monoidal transform with center q. Denote by π the rational map $\pi\colon \tilde{V} \dashrightarrow \mathbf{P}^{g-2}$ corresponding to the linear system $|-K_{\tilde{V}}|$. π has a simple geometric meaning. It is the lifting under σ of the projection of V from $\langle q \rangle$, since $|-K_{\tilde{V}}| = |\sigma^*(-K_V) - S|$, where $S = \sigma^{-1}(q)$ is the exceptional surface.

6.3. The existence of a straight line on trigonal Fano 3-folds and on Fano 3-folds which are complete intersections is well known. Therefore V is not a complete intersection, i.e. $g(V) \geqslant 6$, and also V is not trigonal; consequently V is an intersection of quadrics which contain it in $\mathbf{P}^{g+1}$ (see [7], Proposition 1.7). Hence $\langle q \rangle \cap V = q$ (see (6.1.1)) and the linear system $|-K_{\tilde{V}}|$ has no base points. Also the morphism π on $\tilde{V} - S \approx V - q$ coincides with the projection of $V - q$ from the plane $\langle q \rangle$. The latter assertion will be called in what follows the *geometric interpretation of* π.

6.4. Let b_n and s_n be the standard generators of the Picard group of $S \approx F_n$, s_n a fiber of the ruled surface S (over q) and b_n the base curve. Then

$$|(S, -S)| = \left| b_n + \frac{n}{2} s_n \right|,$$

$$|(S, -K_{\tilde V})| = \left| b_n + \left(2 + \frac{n}{2}\right) s_n \right|,$$

where $n \equiv 0 \bmod 2$ and $0 \leqslant n \leqslant 4$ (see 4.2 and 4.3 (iv) in [7], 2.11 in [6] and (6.1.1) of the present paper).

6.5. *The morphism π does not map any surface $F \subset \tilde V$ to a point, and* $\dim \pi(S) = 2$. If $F \neq S$ and $\pi(F)$ is a point, then by the geometric interpretation $\dim\langle\sigma(F)\rangle = 3$. The latter contradicts (6.1.1) since $g \geqslant 6$. If $F = S$, then $SK_{\tilde V}^2 = 4$ (see 6.4) and $\dim \pi(S) = 2$.

6.6. $\dim \pi(\tilde V) = 3$ *and* $-K_{\tilde V}^3 = 2g - 8$. It is enough to prove the latter, since $g \geqslant 6$. Indeed,

$$2g-2 = -K_V^3 = -K_{\tilde V}\sigma^*(K_V)^2 = -K_{\tilde V}^3 + 2K_{\tilde V}^2 S - K_{\tilde V} S^2 = -K_{\tilde V}^3 + 6$$

(see 6.4).

6.7. *If C is a curve on $\tilde V$ such that $\pi(C)$ is a point, then either*

(6.7.1) *$\sigma(C) \neq q$ and $\sigma(C)$ is a conic on V doubly intersecting q, i.e. q and $\sigma(C)$ are tangent at some point or intersect in two distinct points, or*

(6.7.2) *$\sigma(C) = q$, and in addition $S \approx F_4$ and $C = b_4$.*

Let $\sigma(C) \neq q$. By the geometric interpretation $\dim\langle q \cup \sigma(C)\rangle = 3$. Then by (6.1.1) $\deg \sigma(C) \leqslant 2$, since V is the intersection of quadrics (see 6.3). Therefore $\sigma(C)$ is a conic (see (6.1.2)). The second half of (6.7.1) follows from the fact that $S \cdot C = 2$ since $-CK_{\tilde V} = 0$. If $\sigma(C) = q$, then $C \subset S$ and $C \cdot K_{\tilde V} = 0$; hence it is easy to get (6.7.2) by 6.4.

6.8. Denote by Q the subvariety of $\pi(\tilde V)$ consisting of the points $x \in \pi(\tilde V)$ such that $\dim \pi^{-1}(x) \geqslant 1$. Obviously $\dim Q \leqslant 1$ (see 6.6). Also $\dim \pi^{-1}(x) = 1$ for $x \in Q$ and $Q \subset \pi(S)$ (see 6.5 and 6.7). We will show that *over each point $x \in Q$ there lies precisely one contractible curve*. Let us assume the contrary; then by 6.7 there exist two curves $C_1 \neq C_2$ with $\pi(C_1) = \pi(C_2) \in Q$. If both curves have type (6.7.1), then q, $q_1 = \sigma(C_1)$ and $q_2 = \sigma(C_2)$ are three distinct conics on V. By the geometric interpretation of the map π we have $\dim\langle q \cup q_1 \cup q_2\rangle = 3$; this contradicts Lemma 4.7. Consequently one of the curves C_1 or C_2 coincides with b_4. Let $C_2 = b_4$. Then on V we have two distinct conics $q = \sigma(C_2)$ and $q_1 = \sigma(C_1)$, and $\dim\langle q \cup q_1\rangle = 3$. By Lemma 4.6 we obtain that the general divisor $D \in |-K_V - q - q_1|$ can have only isolated singularities. Also in the general point of q all the divisors from $|-K_V - q - q_1|$ have a common tangent plane, i.e. $\dim(T_x \cap \langle q \cup q_1\rangle) \geqslant 2$ for points $x \in q$. The general hyperplane section D gives a nonhyperelliptic canonical curve $C \subset \mathbf{P}^{g-1}$ of genus g. On this curve there are 4 distinct points (the points of the hyperplane section of $q \cup q_1$) which span a plane, and two of these points (the sections of q) lie in this plane together with tangent lines to C. The latter contradicts Clifford's theorem. This proves the assertion.

6.9. *The general element $H \in |-K_{\tilde V}|$ is a $K3$ surface.* Indeed, H is a smooth surface by Bertini's theorem and by 6.3 and 6.5. By the adjunction formula, $K_H = 0$. On the other hand, for the 3-fold $\tilde V$ we have the vanishing

$$h^1(\tilde V, \mathcal{O}_{\tilde V}) = h^2(\tilde V, \mathcal{O}_{\tilde V}) = h^3(\tilde V, \mathcal{O}_{\tilde V}) = 0,$$

since it is true for V. By the method used in the proof of Lemma 4.1, we conclude from this that

$$h^1(H, \mathcal{O}_H) = h^2(\tilde{V}, \mathcal{O}_{\tilde{V}}(K_{\tilde{V}})) = 0.$$

Then by Serre duality and by the above-mentioned vanishing we have $h^1(H, \mathcal{O}_H) = 0$.

6.10. *The linear system $|-K_{\tilde{V}}|$ restricts surjectively to the linear system $|(H, -K_{\tilde{V}})|$ on the K3 surface $H \in |-K_{\tilde{V}}|$.* This is shown, using the vanishing of $h^1(\tilde{V}, \mathcal{O}_{\tilde{V}})$, by the methods of Lemma 4.4.

6.11. *π is a birational map.* From 6.6 it follows that

$$\deg \pi(\tilde{V}) = \frac{1}{\deg \pi}(-K_{\tilde{V}}^3) = (2g-8)/\deg \pi \geqslant g-4.$$

Therefore $\deg \pi \leqslant 2$. If π is not birational, then $\pi(\tilde{V}) \subset \mathbf{P}^{g-2}$ is a three-dimensional subvariety of degree $g-4$. From the inequality $g-2 \geqslant 4$ (see 6.3) and the requirement (6.1.1) we obtain the absence of a family of surfaces L with $\dim\langle L\rangle \leqslant g-4$ on $\pi(\tilde{V})$. Hence it follows that $g=6$ and $\pi(\tilde{V})$ is a smooth quadric in $\mathbf{P}^4$ (see [6], Lemma 2.8). We will show that the latter is also impossible. Indeed, a smooth quadric in $\mathbf{P}^4$ does not contain any planes. Then from 6.5 and 6.4 it is easy to deduce that $\pi(S)$ is a surface of degree 2 or 4.

Let us first assume that $\deg \pi(S) = 2$, i.e. $\pi(S)$ is a hyperplane section of $\pi(\tilde{V})$, and that $\deg \pi|_S = 2$. Then $\dim Q \geqslant 1$, since the inverse image of the general point of $\pi(S)$ coincides with S (see 6.10 and [13]) and $\dim|S| = 0$. The ramification divisor D of π is cut out on $\pi(\tilde{V})$ by a form of degree 4, and Q is a curve of singularities of D (see [13]). The fibers of the ruled surface S have degree 1 with respect to $-K_{\tilde{V}}$ and are mapped isomorphically onto straight lines which lie on the quadric $\pi(S)$. Therefore the ramification divisor of the morphism $\pi\colon S \to \pi(S)$ consists of two distinct straight lines l_1, $l_2 \subset \pi(S)$. Also by (6.1.1) we have that $l_1, l_2 \not\subset Q$ and Q is a curve with $\dim\langle Q\rangle \geqslant 3$. On the other hand, $l_1, l_2, Q \subset D \cap \pi(S)$; hence by the preceeding $\deg Q \leqslant 3$. In such a case Q is a smooth curve of degree 3. In the anticanonical system of the 3-fold $\tilde{V}$ we have the splitting $S + F \in |-K_{\tilde{V}}|$, where $F = \pi^{-1}(Q)$ is a surface by (6.1.1). By the adjunction formula, $(S, F) \in |-K_S| = |2b_n + (n+2)s_n|$ if $S \approx F_n$. The surface F is smooth, since according to 6.8 and 6.7 all of its fibers are smooth rational curves. (To prove the last fact it is first necessary to show that singularities of F must be singularities of fibers and then to use the existence of a section, which follows from Tsen's theorem.) Thus F is a rational ruled surface. Then, on the one hand, $FS(F+S) = K_S^2 + K_F^2 = 16$ by the adjunction formula; and, on the other hand,

$$FS(F+S) = -SK_{\tilde{V}}F = \left(b_n + \left(2+\frac{n}{2}\right)s_n\right)(2b_n + (n+2)s_n) = 6.$$

Therefore $\deg \pi(S) \neq 2$.

Thus $\deg \pi(S) = 4$. Because of the smoothness of $\pi(\tilde{V})$ the surface $\pi(S)$ is the complete intersection of two quadrics. Also $\pi(S)$ is not contained in a hyperplane section of $\pi(\tilde{V})$ and is not a cone with vertex at a point (i.e. $S \not\approx F_4$). The 3-fold $\tilde{V}$ contains a contractible curve C–otherwise $\tilde{V}$ would be a hyperelliptic Fano 3-fold and Pic $\tilde{V} = \mathbf{Z}$ by [6], Corollary 7.6, which leads to a contradiction. We will show that $\dim Q = 0$. In the opposite case all curves of type (6.7.1) are transformed to singular points of $\pi(S)$, since π is birational on S. Hence $Q \subset \operatorname{sing} \pi(S)$. On the other hand, it is easy to show that singularities of $\pi(S)$ can only lie along a straight line. This contradicts (6.1.1). Thus $Q \neq \varnothing$ and $\dim Q = 0$.

Before finishing the proof of assertion 6.11 we will show that $\tilde{V}$ *contains only finitely many contractible curves when* $g = 6$ (compare 6.13).

We have already considered the case when π is not birational from this point of view. If π is birational, so is $\pi|_S$ (see 6.10, 6.11 and [13]). Therefore $Q \subset \pi(S)$, and $\pi(S)$ is singular at the points of Q. Suppose $\dim Q \geqslant 1$. As we see from (6.1.1), $\dim\langle Q\rangle \geqslant 3$; hence $\deg Q \geqslant 3$. On the other hand, the general hyperplane section of $\pi(S)$ is a rational curve of degree 4 by 6.4. Consequently, $\dim\langle\pi(S)\rangle \leqslant 3$. Therefore Q is a smooth curve of degree 3 and $\pi(S)$ is a hyperplane section of $\pi(\tilde{V}) \subset \mathbf{P}^4$. The latter leads to a contradiction just as in the case $\deg \pi(S) = 2$, which we considered above.

Let us return to the case when $\deg \pi = 2$ and $\deg \pi(S) = 4$. Let C be a contractible curve. The curve C has type (6.7.1), since $S \not\approx F_4$. Let us consider the general hyperplane section H passing through $q \cup \sigma(C)$. The surface H can be chosen smooth (see the end of 6.8 and 6.15). We denote by W the cone swept by lines on $\pi(\tilde{V})$ through the point $x = \pi(C) \in \pi(\tilde{V})$. Let $\tilde{H}$ be the strict transform of H under σ. Then $\pi(\tilde{H})$ cuts out two straight lines l_1 and l_2 on W if H is sufficiently general. Also we may assume that l_1, $l_2 \not\subset \pi(S)$. Let $\tilde{l}_1$ and $\tilde{l}_2$ be the inverse images of these straight lines on $\tilde{V}$ under π with C excluded, and let $\tilde{W}$ be the inverse image of W. Then the hyperplane section of $\sigma(\tilde{W})$ cuts out on H a curve-section $(H, \sigma(\tilde{W}))$ whose irreducible components are only q, $\sigma(C)$, $\sigma(\tilde{l}_1)$ and $\sigma(\tilde{l}_2)$. For a general H we have $\deg \sigma(\tilde{l}_1) = \deg \sigma(\tilde{l}_2)$, and by (6.1.2) this quantity is $\geqslant 2$. From this it follows that $\tilde{l}_1$ and $\tilde{l}_2$ are irreducible curves, since the curve-section $(H, \sigma(\tilde{W}))$ has degree 10. Also one may assume that the components $\sigma(\tilde{l}_1)$ and $\sigma(\tilde{l}_2)$ are reduced in $(H, \sigma(\tilde{W}))$. Therefore

$$(H, \sigma(\tilde{W})) = nq + m\sigma(C) + \sigma(\tilde{l}_1) + \sigma(\tilde{l}_2),$$

where n and m are natural numbers. If $m = n = 1$, then $\deg \sigma(l_1) = \deg \sigma(l_2) = 3$. By (6.1.2), $\sigma(\tilde{l}_1)$ and $\sigma(\tilde{l}_2)$ are curves of degree 3. Let us consider one of them, denoted by R. The space $\langle q \cup \sigma(C) \cup R\rangle$ has dimension 4. Therefore $\operatorname{codim}| - K_V - q - \sigma(C) - R| = 5$. From (6.1.2) and 4.6 it follows that the general divisor $D \in | - K_V - q - \sigma(C) - R|$ is a surface with isolated singularities. The general hyperplane section of D gives a nonhyperelliptic canonical curve-section $X \subset \mathbf{P}^5$ of genus 6 on which a certain subspace of dimension 3 (the section of $\langle q \cup \sigma(C) \cup R\rangle$) cuts out 7 distinct points (the section $q \cup \sigma(C) \cup R$). Hence it is easy to see that X is trigonal, which means that so is V. The latter is impossible. Therefore either n or $m \geqslant 2$. Then $\sigma(\tilde{W})$ has singularities along q for $n \geqslant 2$, or along $\sigma(C)$ if $m \geqslant 2$. Indeed, in the opposite case the general H is tangent along q to either $\sigma(C)$ or $\sigma(\tilde{W})$. From this, arguing as at the end of §6.8, we obtain a contradiction. Therefore there exists $\sigma(\tilde{W}) \in |-K_V|$ which is singular along q or along $\sigma(C)$. Since $\pi(S)$ is not contained in a hyperplane section of $\pi(\tilde{V})$, it follows that $\sigma(\tilde{W})$ is smooth along q and singular along $\sigma(C)$. Let us denote by $\sigma'\colon V' \to V$ the monoidal transformation with center $\sigma(C)$, and by π' the corresponding anticanonical morphism. Then $\pi'(S')$ is contained in a hyperplane section $\pi'(V') \subset \mathbf{P}^4$, where $S' = \sigma'^{-1}(\sigma(C))$. This leads to a contradiction, since there are only finitely many contractible curves and $\dim|S'| = 0$. This completes the proof that π is birational.

6.12. *The map π is birational on S, and* $\deg \pi(s) = 4$. This is an immediate consequence of 6.10, 6.11 and the results of [13].

6.13. *$\tilde{V}$ contains only finitely many curves which π maps to a point.*

Let us assume that $\dim Q = 1$. By 6.8 $Q \subset \pi(S)$, and $\pi(S)$ is singular along Q because every contractible curve of type (6.7.1) is smooth and it intersects S twice. As is seen from (6.1.1), $\dim\langle Q\rangle \geqslant g - 3$. Therefore $\deg Q \geqslant 3$. On the other hand, the general hyperplane section of $\pi(S)$ is a rational curve of degree 4 by 6.12 and 6.4; hence $\dim\langle \pi(S)\rangle \leqslant 3$. Therefore $g = 6$, Q is a smooth curve of degree 3 and $\pi(S)$ is a hyperplane section of $\pi(\tilde{V}) \subset \mathbf{P}^4$. The latter leads to a contradiction (see 6.11).

6.14. *There are no contractible curves of type* (6.7.1).

Case 1 ($g \geqslant 8$). If C is a contractible curve on $\tilde{V}$ of type (6.7.1), then $\pi(S)$ is a smooth surface of degree 4, and it has at least two singular points when $S \approx F_4$ (see 6.8 and 6.7). Hence it is evident that $S \approx F_n$ is embedded in $\langle \pi(S)\rangle$ by the proper subsystem $|b_n + (2 + n/2)s_n|$. Therefore $\dim\langle\pi(S)\rangle \leqslant 4$. Let $q' = \sigma(C)$ be a conic on V. Then $|-K_V - 2q| = |-K_V - 2q - q'|$ and $\operatorname{codim}|-K_V - 2q - q'| \leqslant 8$. Therefore by Lemma 4.6 and the conditions (6.1.1) and (6.1.2) the general element $H \in |-K_V - 2q - q'|$ can only have a curve of quadratic singularities q. Let $\tilde{H}$ be the strict transform of H under σ. The surface $\tilde{H}$ gives a splitting $\tilde{H} + S \in |-K_{\tilde{V}}|$. Let us resolve the singularities of $\tilde{H}$ by means of monoidal transforms with centers in the singular sets. We denote this resolution by $\sigma'\colon V' \to \tilde{V}$ and the strict transform of $\tilde{H}$ by H'. At the general point of C the surface $\tilde{H}$ is smooth, and its strict transform will be a smooth rational curve C' on H'. From the canonical class formula for V' we have the splitting $\sigma'^*(S) + H' + H'' \in |-K_{V'}|$ and also $\sigma'(H'') \subset \operatorname{sing}(\tilde{H})$ (sing( ) denotes the set of singular points). Hence $C' \not\subset \sigma'^*(S) \cup H''$ and $H' \not\subset \sigma'^*(S) \cup H''$. By the adjunction formula and the connectedness of the elements of $|-K_{H'}|$ we have

$$(H',\ \sigma'^*(S) + H'') \in |-K_{H'}| \text{ and } h^2(H',\ \mathcal{O}_{H'}) = 0.$$

Also

$$-C'K_{H'} = C'(\sigma'^*(S) + H'') \geqslant C'\sigma'^*(S) = CS = 2.$$

Consequently C' is algebraically movable on H' by Lemma 3.20; hence by the projection formula the number of contractible curves is not finite. This contradicts 6.13. Hence for $g \geqslant 8$ there are no contractible curves of type (6.7.1).

Case 2. $g = 7$. Let us assume the existence of a contractible curve of type (6.7.1). Just as in the previous case we get that $\pi(S)$ must be a surface of degree 4, and $\dim\langle\pi(S)\rangle \leqslant 4$. In any hyperplane section of $\pi(\tilde{V})$ which passes through $\pi(S)$ there also lies a surface of degree 2, since $\deg \pi(\tilde{V}) = 6$. By the geometric interpretation of π the existence of such a surface of degree 2 contradicts (6.1.1). This completes the analysis of this case.

Before completing the proof of assertion 6.14 and the proof of the proposition, we establish the following result.

6.15. LEMMA. *Let V be a Fano 3-fold of the principal series of genus 6 which satisfies the condition of* §6.1. *Then for every one-dimensional reduced and connected subscheme $R \subset V$ of degree 4 we have* $\dim\langle R\rangle = 4$.

PROOF. Let us assume that $\dim\langle R\rangle \leqslant 3$. Then, because V is not trigonal (see 6.3), we have $\dim\langle R\rangle = 3$, and $R \subset \langle R\rangle$ is a complete intersection of two quadrics. Also

$\dim T_x \cap \langle R \rangle \leqslant 2$ for every point $x \in R$. This follows from (6.1.2) and from Proposition 1.7 (iii) in [7]. For the general point $x \in R$ we have $\dim(T_x \cap \langle R \rangle) = 1$. The latter is shown using the methods of the end of §6.8, where one also has to remember that the degree of every component of R is $\geqslant 2$ by (6.1.2). Hence one can pass through R a smooth hyperplane section $H \in |-K_V|$, because $R \cap \langle R \rangle = R$. Being the complete intersection of two quadrics, R has arithmetic genus one. Knowing the splitting types of R on H, by 3.11 and 3.18 we obtain the existence of a smooth elliptic curve of degree 4 on $H \subset V$. For the rest of this proof, R will denote such an elliptic curve.

Let $\sigma'\colon V' \to V$ be the monoidal transform with center R. We will show that V' is a hyperelliptic Fano 3-fold of genus 2. Following the ideas explained in 6.2–6.6, we establish the absence of base points in the linear system, the three-dimensionality of the image and the absence of surfaces contracted to a point. Let C be a contractable curve, i.e. $-C \cdot K_{V'} = 0$. We assume first that $C \not\subset \sigma'^{-1}(R)$. Then $\sigma'(C) \neq R$ is a curve, and $\dim\langle R \cup \sigma'(C)\rangle = 4$ (the latter follows from the fact that $\pi' = \varphi_{|-K_{V'}|}$ is a projection from $\langle R \rangle$ at the points $V' - \sigma'^{-1}(R) \approx V - R$). From Lemmas 4.6 and 4.7 we deduce that $\deg \sigma'(C) \leqslant 3$. If $\deg \sigma'(C) = 3$, then by 4.6 we can pass through $R \cup \sigma'(C)$ a hyperplane section which can have at most only isolated singularities and also is irreducible and reduced. Cutting this section by another general one, we obtain in the intersection a smooth canonical curve-section $X \subset \mathbf{P}^5$ of genus 6 on which there lie 7 distinct points (the intersection with $R \cup \sigma'(C)$) which span a subspace of dimension $\leqslant 3$. From this it is easy to deduce that X is trigonal. (Consider the residual linear system for the given seven points with respect to the canonical system $|K_X|$.) The latter is impossible. Hence $\sigma'(C) = q$ is a conic. Further we use the already-proved results for the monoidal transformation with center in the conic q.

Let $\tilde{R}$ be the strict transform of R under σ. Then, by the geometric description of π, π maps the elliptic curve $\tilde{R}$ onto a straight line. Consequently, $\deg(\pi|\tilde{C}) \geqslant 2$. In such a case, by 6.10, the results of [13] and the birationality of π we have the inclusion $\pi(\tilde{C}) \subset Q$. The latter contradicts 6.13. Therefore if C is a contractible curve for π', then $C \subset \sigma'^{-1}(R)$, i.e. $\sigma'(C) = R$ (since C is not a fiber of $\sigma'^{-1}(R)$). Let us consider the 4-dimensional projective space $T \subset \mathbf{P}^7$ which contains R and which projects to the point $\pi'(C)$. By Lemma 4.6 and the condition (6.1.2) the general hyperplane section through T cuts out on V a surface H which can at most have isolated singularities. Also such general surfaces are tangent along R, because their strict transforms on V' will be elements of $|-K_{V'} - C|$. Therefore $\dim(T_x \cap \langle R \rangle) \geqslant 2$ for the points $x \in R$. Intersecting H with a general hyperplane section, we obtain a monomial nonhyperelliptic curve-section X of genus 6 on which there are 4 distinct points (intersection with R) which are in a three-dimensional subspace together with tangents to X. By Clifford's theorem this is impossible. Consequently there are no contractible curves, V' is a hyperelliptic Fano 3-fold, and rk Pic $V' \geqslant 2$. This contradicts Corollary 7.6 of [6]. ■

End of the proof of Proposition 6.14. Case 3 ($g = 6$) is an immediate corollary of the previous lemma, since $\dim\langle q \cup \sigma(C)\rangle = 3$ and $\deg(q \cup \sigma(C)) = 4$ for a curve C of type (6.7.1).

6.16. The linear system $|-K_{\tilde{V}}|$ is very ample if $S \approx F_4$, i.e. $N_q|_V \approx \mathcal{O}_q(-2) \oplus \mathcal{O}_q(2)$ (see Proposition 4.4 in [6], and 6.14, 6.11 and 6.7 above). By Proposition 4.3 in [7], a sufficiently general conic on V satisfies the last condition, since V does not contain two-dimensional quadrics. Indeed, from (6.1.1) and the inequality $g + 1 \geqslant 7$ it follows that $N_{q|V} \simeq \mathcal{O}_q \oplus \mathcal{O}_q$ and $S \approx \mathbf{P}^1 \times \mathbf{P}^1$ for the general conic q. It remains to show that $\dim\langle\pi(S)\rangle = 5$. As in §5.2, we show that $g - 2 - 1 \geqslant 6$, i.e. $g \geqslant 9$. As in §5.3, we show that the general element of $|S'| = |-K_{\tilde{V}} - \pi(S)|$ is irreducible and reduced. By Lemma 3.22 a hyperplane section of $\pi(S)$ is a fully linearly embedded curve. On the other hand, $\deg \pi(S) = 4$. Therefore $\dim\langle\pi(S)\rangle = 5$, which completes the proof of Proposition 2.4. ■

§7. Proof of Proposition 2.3

7.1. Let V be a Fano 3-fold of the principal series which does not satisfy the requirements (2.3.2), (2.3.3) and (2.3.4), i.e. let the following conditions hold:

(7.1.1) *All the divisors in the linear system* $|-K_V|$ *are irreducible and reduced.*

(7.1.2) *V does not contain a straight line or a conic.*

Then in order to prove Proposition 2.4 it is enough to show the truth of (2.3.1), which will be shown below (see §7.17).

The methods of proof in this section are in many ways analogous to the methods of §6. Therefore some details will be omitted. For complete understanding it would be useful for the reader to recall them.

7.2. Let $x \in V$ be a point, and let $\sigma\colon \tilde{V} \to V$ be a monoidal transformation with center x. Let $\pi\colon \tilde{V} \to \mathbf{P}^{g-3}$ be the rational map corresponding to the linear system $|-K_{\tilde{V}}|$. Then $|-K_{\tilde{V}}| = |\sigma^*(-K_V) - 2S|$, where $S = \pi^{-1}(x)$ is the exceptional surface.

7.3. The 3-fold V is not trigonal, and $g \geqslant 6$ (compare 6.3). If $y \in V \cap T_x$ and $y \neq x$, then the straight line $\langle\{x\} \cup \{y\}\rangle$ lies in V, since V is given as the intersection of quadrics that contain it. This contradicts (7.1.2). Therefore $V \cap T_x = x$, and π is the projection from T_x for the points $\tilde{V} - S \approx V - x$. The last statement will be called the *geometric interpretation of* π. Let $t_v \in T_x$ be the tangent straight line corresponding to a point $v \in S$.

Let us consider two sufficiently general hyperplane sections $H_1, H_2 \in |-K_V|$ through t_v which cut out a smooth canonical curve section X. The latter curve is not trigonal. That means that there exists a hyperplane H through T_x for which the intersection index of H, H_1 and H_2 at the point x is equal to 2. The proper transform H' of H with respect to σ will be an element of $|-K_{\tilde{V}}|$ which does not pass through v. Therefore v is not a base point of $|-K_{\tilde{V}}|$. Then from the geometric interpretation it follows that *π is a morphism.*

7.4. Let S be the standard generator of the Picard group $S \approx \mathbf{P}^2$. Then

$$|(S, -S)| = |s|,$$

$$|(S, -K_{\tilde{V}})| = |2s|.$$

7.5. *The morphism π does not map any surface into a point, and* $\dim \pi(S) = 2$. The proof is analogous to 6.5.

7.6. $\dim \pi(\tilde{V}) = 3$ *and* $-K_{\tilde{V}}^3 = 2g - 10$. This is a direct consequence of 7.4 and the inequality $g \geqslant 6$.

7.7. *If C is a curve on $\tilde{V}$ such that $\pi(C)$ is a point, then either*

(7.7.1) *$\sigma(C)$ is a rational curve of degree 4 with a single singular point x of degree 2, i.e.* $SC = 2$, *or*

(7.7.2) *$\sigma(C)$ is a rational curve of degree 6 with a single singularity of degree 3, i.e.* $SC = 3$.

From the geometric interpretation of π we have $\dim\langle T_x \cup \sigma(C)\rangle = 4$, and $\sigma(C)$ is a curve since $C \not\subset S$ by 7.4 and 7.3. Then, by Lemmas 4.6 and 4.7, $\deg \sigma(C) < 8$ since $g \geqslant 6$. On the other hand, by the projection formula $\deg \sigma(C) \equiv 0 \bmod 2$. By assumption V does not contain conics; hence we obtain the assertion. The rationality and the assertion about singularities in case (7.7.1) follows from the fact that $\dim\langle\sigma(C)\rangle = 3$ (V is not trigonal), and in case (7.7.2) from Lemma 4.7.

7.8. We are now ready for the proof that $g \geqslant 7$. Indeed, let $g = 6$. Let us assume that V contains a curve C for which $\pi(C)$ is a point. Then by Lemma 6.15 C has the type (7.7.2). From assumption 7.1 it follows that any curve-section through $\langle\sigma(C)\rangle$ splits into two curves $\sigma(C)$ and R, where $\deg R = 4$. (We recall that in this case the curve section has degree $2g - 2 = 10$.) On the other hand, by Lemma 4.6 we may assume that through $\sigma(C) \cup R$ there passes a hyperplane section without singular curves. Intersecting this hyperplane section without singular curves. Intersecting this hyperplane section with another sufficiently general one, we obtain a canonical curve-section X of genus 6. The curve X contains 2 effective divisors P_1 (section with $\sigma(C)$) and P_2 (section with R), where P_1 consists of 6 distinct points, $\dim\langle P_1\rangle = 3$ and P_2 consists of four points. Also $P_1 + P_2 \in |K_X|$. By the Riemann-Roch theorem it is easy to deduce that $\dim\langle P_2\rangle = 2$. Consequently $\dim\langle R\rangle = 3$. The latter contradicts Lemma 6.15. Therefore π does not contract curves into a point, the linear system $|-K_{\tilde{V}}|$ is ample and $\tilde{V}$ is a Fano 3-fold. Of course $\tilde{V}$ is hyperelliptic, since the index of $\tilde{V}$ is equal to the index of V, i.e. equal to one. This too is impossible by Corollary 7.6 of [6]; hence $g \geqslant 7$.

7.9. *If there exists a curve C of type* (7.7.2), *then it is unique and there are no other curves contracted to a point.* Let C' be another contracted curve. Then from the geometric description of π we have

$$\dim\langle\sigma(C)\cup\sigma(C')\rangle\leqslant 5 \quad \text{and} \quad \deg \sigma(C)\cup\sigma(C')\geqslant 10.$$

The latter contradicts Lemmas 4.6 and 4.7.

7.10. Let us denote by Q the subvariety of $\pi(\tilde{V})$ consisting of the points $x \in \pi(\tilde{V})$ with $\dim \pi^{-1}(x) \geqslant 1$. Then, just as in §6.8, we have $\dim Q \leqslant 1$, $\dim \pi^{-1}(x) = 1$ for $x \in Q$, and $Q \subset \pi(S)$. From the geometric interpretation of π and Lemmas 4.6 and 4.7 it follows that *over every point $x \in Q$ there lies exactly one contracted curve.* Arguing as at the end of §6.8, we can show that there exists a smooth hyperplane section H through $\sigma(C)$ if C is a contracted curve of type (7.7.1). The curve $\sigma(C)$ is a complete intersection of two quadrics in $\langle\sigma(C)\rangle = \mathbf{P}^3$, and $p_a(\sigma(C)) = 1$. Therefore $\sigma(C) \subset H$ has at x a nondegenerate quadratic singularity. Thus *every curve of type* (7.7.1) *will be smooth.*

7.11. *The mapping π is birational.* By 7.8, $g \geqslant 7$. Furthermore, as at the beginning of §6.11, we conclude that π may fail to be birational only in the case $g = 7$. Then $\pi(\tilde{V})$ is a smooth quadric in $\mathbf{P}^4$. We will show that Lemma 6.15 is true for a Fano 3-fold of the principal series of genus 7 which satisfies the assumptions of 7.1. Let us assume the contrary. Then, just as in Lemma 6.15, we find a smooth elliptic curve R of degree 4. Further we show that the 3-fold V' obtained by a monoidal transformation with center R is a Fano 3-fold. In contrast to Lemma 6.15, in our case for every contracted curve $C \subset V'$ we have $\deg \sigma'(C) = 3$, $\sigma'(C) \neq R$, and $\dim\langle\sigma'(C) \cup R\rangle = 4$. The contradiction in this case follows from the fact that on a nontrigonal nonhyperelliptic curve X of genus 7 there does not exist a special divisor D of degree 7 with $h^0(X, \mathcal{O}_X(D)) \geqslant 4$ (a special divisor $D' \sim K_X - D$ has degree 5, and $\dim h^0(X, \mathcal{O}_X(D')) \geqslant 3$). This means that V' is a Fano 3-fold, $|-K_{V'}|$ has no fixed points, rk Pic $V' > 2$ and the index of V' is 1. It is easy to show that this is impossible. Therefore *for $g = 7$ there can exist no more than one contracted curve, and the contracted curve has type* (7.7.2). Further, just as at the end of 6.11 we show that deg $\pi(S) \neq 4$, i.e. that $\pi(S)$ is a hyperplane section of a quadric. This contradicts the exceptionality of S, since there are only finitely many contracted curves. Hence follows the birationality of π.

7.12. *The mapping π is birational on S and* deg $\pi(S) = 4$, as in 6.12.

7.13. *There are finitely many contracted curves.* Let us assume that dim $Q = 1$. The surface $\pi(S)$ is singular along Q, since every contracted curve is of type (7.7.1) (see §7.9), is smooth by 7.8 and intersects S with multiplicity 2. On the other hand, $\dim\langle Q\rangle \geqslant g - 4$ by (7.1.1), and deg $Q \geqslant 3$. Then, as in 6.13, $\dim\langle\pi(S)\rangle \leqslant 3$. Therefore $g = 7$, and in this case the finiteness has already been established in §7.11.

7.14. *If C is a contracted curve of type* (7.7.2), *then* $\dim\langle\pi(S)\rangle \leqslant 3$. We consider the linear system $|-K_{\tilde{V}} - C|$. It restricts to the subsystem $(S, |-K_{\tilde{V}} - C|) \subset |2S|$, which has no fixed components and is not a pencil. The latter follows from $\pi(S)$ being a fourth degree surface and therefore not a cone (an element of $|2S|$ does not contain more than two components!). Therefore the general element $q \in (S, |-K_V - C|)$ is a smooth conic on $S \approx \mathbf{P}^2$. Then the linear system $|-K_{\tilde{V}} - C|$ restricts on q to the linear system $(q, |-K_{\tilde{V}} - C|) \supset |4y|$, where y is a point of q, and it has *at most* a fixed divisor with support in $C \cap S \subset q$ of degree $\geqslant 3$. Consequently $\dim(q, |-K_{\tilde{V}} - C|) \leqslant 1$; hence

$$\dim(S, |-K_{\tilde{V}} - C|) \leqslant 2 \quad \text{and} \quad \dim(S, |-K_{\tilde{V}}|) \leqslant 3,$$

which was to be shown.

7.15. *There are no contracted curves of type* (7.7.2). Let C be a contracted curve of type (7.7.2).

Case 1 ($g \geqslant 8$). Since $\dim\langle\pi(S)\rangle \leqslant 3$, we have that

$$\operatorname{codim}|-K_V - 3x - \sigma(C)| \leqslant 8$$

($-3x$ means that all the elements of $|-K_V - 3x - \sigma(C)|$ have singularities of degree $\geqslant 3$ at x). Then by Lemma 4.6 on the general element $H \in |-K_V - 3x - \sigma(C)|$ the curve $\sigma(C)$ is not a curve of singularities. Let $\tilde{H}$ be the strict transform of H under σ. We have a splitting $kS + \tilde{H} \in |-K_{\tilde{V}}|$ with $k \geqslant 1$. Further, as in the analysis of Case 1 in 6.14 (with kS in place of S) we obtain that there are finitely many contracted curves. This contradiction completes the analysis of the present case.

Case 2 ($g = 7$). Then $\pi(S)$ is a hyperplane section of a quartic $\pi(\tilde{V}) \subset \mathbf{P}^4$. This together with 7.13 leads to a contradiction.

7.16. *There are no contracted curves.* If C is a curve contracted to a point, then it has type (7.7.1).

Case 1 ($g \geqslant 9$). Since C is a smooth curve and $SC = 2$, it follows that $\pi(S)$ is a singular surface of degree 4 and $\dim\langle\pi(S)\rangle \leqslant 4$. Hence $\operatorname{codim}|-K_V - 3x - \sigma(C)| \leqslant 9$. Then by Lemma 4.6 the general hypersurface $H \in |-K_V - 3x - \sigma(C)|$ has in the worst case a nondegenerate quadratic singularity along $\sigma(C)$ and perhaps also finitely many isolated singularities. The surface H cannot be singular along $\sigma(C)$, since $\dim\langle\sigma(C)\rangle = 3$. (For the proof consider the general hyperplane section $H' \in |-K_V|$ and apply to it Lemma 3.21 with $D_0 = (H', H)$.) Therefore the general H is smooth along

$\sigma(C)$. Now we show that $\dim Q \geqslant 1$, as in Case 1 in §§7.15 and 6.14. This gives a contradiction.

Case 2 ($g = 8$). This is proved using the method of Case 2 in §6.14.

Case 3 ($g = 7$). This is an immediate consequence of 7.11 and 7.15.

7.17. The previous subsection completes the proof of the ampleness of $|-K_{\tilde{V}}|$. Then by Proposition 4.4 of [6] and our assertion 7.11 $\tilde{V}$ is a Fano 3-fold of the principal series. It remains to show that $\dim\langle\pi(S)\rangle = 5$; this is done just as at the end of §6.16. ■

§8. Proof of Proposition 2.2

8.1. Let F be a Fano 3-fold of the principal series for which the requirements (2.2.1)–(2.2.4) are not fulfilled, i.e., suppose that the following conditions hold:

(8.1.1) *V does not contain a line.*

(8.1.2) *V has index one.*

(8.1.3) $V \approx \mathbf{P}^1 \times \mathbf{P}^2$.

(8.1.4) *V does not contain a Veronese surface $S \approx \mathbf{P}^2$ of degree* 4.

Then in order to prove Proposition 2.2 it is necessary to establish nonsplitting in the anticanonical linear system $|-K_V|$, which will be shown below (see (8.10)). In all the assertions of this section we consider a fixed 3-fold V which satisfies the above assumptions.

8.2. LEMMA. *In the splittings of the system $|-K_V|$ there are no irreducible and reduced divisors D of degree* $\dim\langle D\rangle - 1$.

PROOF. From the classification of surfaces D of degree $n-1$ in $\langle D\rangle = \mathbf{P}^n$ (see [10]) there follows the existence of a line on D (therefore also on V) except for the case when $D \approx \mathbf{P}^2$ is a Veronese surface of degree 4. The latter is impossible by (8.1.4). ■

8.3. LEMMA. *Suppose that the surface D gives a nontrivial splitting of $|-K_V|$ and that the general element of the residual system $|D'| = |-K_V - D|$ is reduced and connected. Then the linear system $|D|$ has no base points.*

This follows from Corollary 4.11 and Lemma 8.2. ■

8.4. LEMMA. *Suppose that the surface D gives a nontrivial splitting of $|-K_V|$ and that all the divisors of the residual linear system $|D'|$ are connected. Then the linear system $|D'|$ has no base points and consequently, by Bertini's theorem, its general element is irreducible, reduced and smooth.*

PROOF. Consider the general element of the residual linear system $D'_0 \in |D'|$. Let us intersect the divisor D'_0 with a sufficiently general hyperplane section H so that the number of components of the divisor (H, D'_0) on H and their multiplicities should be the same as for D'_0, i.e. so that every irreducible and reduced component of D'_0 should restrict to the same kind of component of (H, D'_0). If the divisor D'_0 has two distinct mutually nonintersecting irreducible and reduced components, then by (8.1.1) they intersect along a one-dimensional subvariety of degree $\geqslant 3$. Then the divisor (H, D'_0) on the $K3$ surface H contains two distinct curves which intersect at least in two distinct points. By Lemmas 4.4 and 3.18 and the generality of D'_0, the latter is impossible. Therefore, because of the connectedness of all the members of $|D'|$, the general divisor D'_0 is of the form $D'_0 = nS$, where S is a surface on V.

We will show that $n = 1$. Let us assume the contrary, i.e. $n \geqslant 2$. Then $\dim|nS| = 0$. By the considerations of the previous paragraph and by Bertini's theorem one can show that the general element of the linear system $|D + (n - 1)S|$ is of the form $\Sigma_1^m k_i F_i$, where $F_1, \ldots, F_m$ are nonintersecting surfaces, and $k_i F_i$ is a fixed component of $|D + (n - 1)S|$ for $k_i > 1$. The divisor $D + (n - 1)S$ is connected by Lemma 2.1 of [12]. Therefore the fixed part of the linear system $|D + (n - 1)S|$ is of the form wS, where $0 \leqslant w \leqslant n - 1$. In the case $w \geqslant 1$ the divisor $\Sigma_1^m k_i F_i$ as well as $|D + (n - 1)S|$ is connected, i.e. $|D + (n - 1)S| = |wS|$, which is impossible. Consequently, $|D + (n - 1)S|$ does not have fixed components. If the latter linear system is not a pencil, then by Bertini's theorem its general element is irreducible and reduced. In this case, by Lemma 8.3 we obtain a contradiction to the fact that S is fixed. Consequently, the system $|D + (n - 1)S|$ is a pencil. By Lemma 4.4, under restriction to the general hyperplane section we obtain a pencil $|(H, D + (n - 1)S)|$. This pencil has no fixed components on the $K3$ surface H. Therefore $|(H, D + (n - 1)S)| = |lE|$, where $|E|$ is an elliptic pencil on H. In the linear system $|lE|$ the divisor $(H, D + (n - 1)S)$ is present. The latter divisor is connected, as is $D + (n - 1)S$. Also, $(H, D) \neq (H, S)$, since $D \neq S$ by (8.1.2). Consequently $l = 1$, and the general element of the linear system $|D + (n - 1)S|$ is irreducible and reduced. This, as above, leads to a contradiction with Lemma 8.3. This means that $n = 1$. Then $|D'| = |S|$, and by Lemma 5.3 it has no fixed points. ■

8.5. Lemma. *In the splittings of the linear system $|-K_V|$ there are no divisors whose linear system is a pencil without fixed points.*

Proof. Let us assume the contrary. Since $\mathrm{Pic}^0 V = 0$, in the splittings of $|-K_V|$ there exists an irreducible and reduced divisor D whose linear system $|D|$ is a (projectively) one-dimensional pencil with no fixed points.

By the Bertini-Zariski theorem the general divisor $D_0 \in |D|$ is irreducible, reduced and smooth. Also $-K_{D_0} = (D_0, D') = (D, -K_V)$, where $D' \in |-K_V - D|$, and therefore D_0 is a del Pezzo surface. On a del Pezzo surface, as on a Fano 3-fold, every effective anticanonical divisor is connected. From the connectivity of the elements of $|-K_V|$ and the movability of D we obtain the connectedness of all divisors in $|D'|$. Then, by Lemma 8.4, $|D'|$ has no fixed points.

Let us show that any divisor in $|D|$ is irreducible and reduced. Indeed,

$$\deg D = (D, D + D', D + D') = (D, D'^2) = K_D^2 \leqslant 9.$$

Any surface of degree $\leqslant 3$ contains a straight line. Therefore by (8.1.1) only the following nontrivial splittings of $|D|$ are possible: $2E \in |D|$ and $E + E' \in |D|$, where E and E', $E \neq E'$, are irreducible and reduced. The general divisor $D_0' \in |D'|$ is irreducible, reduced and smooth by Lemma 8.4. From the connectedness of the members of $|-K_V|$ we can deduce that D_0' correctly and nontrivially intersects E (under a suitable naming of the divisors of the second splitting). Then by Lemma 4.2

$$h^1(V, \mathcal{O}_V(-D_0' - E)) = 0$$

and all the elements of the system $|D_0' + E|$ are connected. Therefore by Lemma 8.4 the linear system $|E|$ or $|E'|$ (corresponding to the cases of possible splitting) is movable. This leads to a contradiction, snce $\dim|D| = 1$ and $|D|$ has no fixed components.

Since $(D, D')^2 = K_D^2 \geqslant 1$, the linear system $|D'|$ cannot be a pencil. By Lemma 4.5 the general divisor $D_0' \in |D'|$ is a smooth rational surface with $|-K_{D_0'}| \neq \varnothing$.

We will show that $\dim \pi(V) = 3$ for the mapping $\pi = \varphi_{|D'|}: V \to \mathbf{P}^{\dim|D'|}$, given by the linear system $|D'|$. For this it is enough to show that the restriction of the linear system $|D'|$ to D_0' is not a pencil. Indeed, in the opposite case $|(D_0', D')| = |nE|$, where E is an irreducible and reduced curve whose linear system $|E|$ is a one-dimensional (projectively) pencil without fixed points. The latter assertion follows from the triviality $\mathrm{Pic}^0 D_0' = 0$ and from the absence of fixed points in $|(D_0', D')|$ (Lemmas 4.4 and 8.4). By the adjunction formula for the canonical class of $-K_{D_0'} \sim (D_0', D)$ and the fact that $D(D')^2 \geqslant 1$ we deduce that $-EK_{D_0'} \geqslant 1$. Therefore by the adjunction formula for the canonical class of E the curves of $|E|$ are a pencil of rational curves. These rational curves are conics on V, since $\deg E = -EK_V = ED = -EK_{D_0'} = 2$. Then by (8.1.1) the surface will be a ruled rational surface. Consequently $K_{D_0'}^2 = 8$. On the other hand, $K_{D_0'}^2 = D^2D' = 0$. The latter contradiction shows that the image $\pi(V)$ is three-dimensional.

The mapping π does not contract surfaces into points. Let S be an irreducible and reduced surface, and let $\pi(S)$ be a point. Since $DD'^2 > 0$ and any element of $|D|$ is irreducible and reduced, it follows that $S \notin |D|$ and the general member $D_0 \in |D|$ correctly and nontrivially intersects S. Hence by the ampleness of $-K_{D_0}$ on D_0 we obtain that the general element $D_0' \in |D'|$ correctly and nontrivially intersects S. Therefore $\pi(S)$ is not a point. Then by Lemma 4.11 we obtain the ampleness of $|D'|$. We will show that $|D'|$ is very ample. To do this, by Lemma 4.12 it is enough to establish the inequality $(D')^2D = K_{D_0}^2 \geqslant 8$, which follows from the absence of exceptional curves of the first kind on a general del Pezzo surface $D_0 \in |D|$, i.e. $D_0 \approx \mathbf{P}^2$ or $D_0 \approx \mathbf{P}^1 \times \mathbf{P}^1$. Indeed, every such exceptional curve l would be a straight line on the Fano 3-fold V: $\deg l = -lK_V = lD' = -lK_{D_0} = 1$.

We will show that D gives a splitting in the linear system $|D'|$ which is nontrivial by (8.1.2), i.e. there exists an effective divisor $D'' > 0$ such that $|D'| = |D + D''|$. Let us assume the contrary. Then by Lemma 4.1 and duality we have

$$h^2(2D', \mathcal{O}_{2D'}) = h^3(V, \mathcal{O}_V(-2D')) = h^0(V, \mathcal{O}_V(D' - D)) = 0.$$

By Lemma 4.1 and the Kodaira vanishing criterion, for an ample sheaf $\mathcal{O}_V(D')$ we have

$$h^1(2D', \mathcal{O}_{2D'}) = h^2(V, \mathcal{O}_V(-2D')) = 0.$$

Also, $h^0(2D', \mathcal{O}_{2D'}) = 1$ by the ampleness of $\mathcal{O}_V(D')$. Hence $1 = \chi(F) = 2\chi(D_0') - \chi(C) = 2 - (1 - g(C)) = 1 + g(C)$ and $g(C) = 0$, where C is a general curve-section under the embedding of V by the linear system $|D'|$ and F is a general divisor of the system $|2D'|$. From Lemma 4.4 and from the regularity of the surface D_0' it is easy to deduce that the curve C is fully linearly embedded. Consequently, $\deg \pi(V) = \dim|D'| - 2$. This means by [10] that $D_0' \approx \mathbf{P}^2$ or $D_0' \approx F_n$. The latter is impossible, since $K_{D_0'}^2 = 0$. Hence we obtain the splitting $|D'| = |D + D''|$, $D'' > 0$.

Let us show that the general member of the linear system $D_0'' \in |D''|$ is a smooth surface. Since

$$\begin{aligned} h^0(D'', \mathcal{O}_{D''}) &= h^1(V, \mathcal{O}_V(-D'')) = h^2(V, \mathcal{O}_V(-2D)) \\ &= h^1(2D, \mathcal{O}_{2D}) = 2h^1(D, \mathcal{O}_D) = 0, \end{aligned}$$

by Lemma 4.2 any divisor in $|D''|$ is connected. By Lemma 4.4 the linear system $|D''|$ restricts isomorphically to the linear system $|(D_0', D'')|$ on D_0'. For the general divisors

$D_0' \in |D'|$ and $D_0'' \in |D''|$ the number of irreducible and reduced components of divisors D_0'' and (D_0', D_0'') on D_0' is preserved under restriction, because $|D'|$ is very ample. Suppose D_0'' has two irreducible and reduced intersecting components $S_1 \neq S_2$. By Lemma 3.17 these cannot simultaneously intersect the general member of $|D|$ correctly and nontrivially. Consequently one of these divisors lies in $|D|$, because all the elements of $|D|$ are irreducible and reduced. Suppose, for example, that $S_1 \in |D|$. Then S_2 correctly and nontrivially intersects the general member of $|D|$; hence $S_1S_2D_0' = S_2D_0'D \geqslant 2$, since on $\mathbf{P}^2$ the anticanonical class intersects every curve at least triply and on $\mathbf{P}^1 \times \mathbf{P}^1$ at least doubly. In such a case, by Lemma 3.17 we obtain a contradiction with the generality of the divisor D_0''. This means that the general element $D_0'' \in |D''|$ is irreducible because of the connectedness of all the elements of $|D''|$. If D_0'' is irreducible, then $D_0'' = nS$, where $n \geqslant 2$, S is a surface in V and $\dim|S| = 0$. The curve (D_0', S) is also linearly fixed on the general D_0'. By (8.1.2), S correctly and nontrivially intersects the general divisor in $|D|$. Hence, as above, $SD_0'D \geqslant 2$ and $-K_{D_0'}(D_0', S) = SD_0'D > 2$. The latter by (3.12.1) contradicts the linear immovability of (D_0', S). Hence the general divisor in $|D''|$ is irreducible and reduced. Then from Lemma 3.12 we obtain the absence of fixed points $x \in V$ in $|D''|$, provided that for the general $|D_0''| \in |D''|$ this point x is not the vertex of the cone D_0'' under the inclusion $\varphi_{|D'|}$, since $-K_{D_0'}(D_0', D'') = D''D'D = K_{D_0}^2 \geqslant 8$. This is shown by introducing a general hyperplane $D_0' \in |D'|$ through x under the embedding π. The surface D_0'' nontrivially intersects the general element of $|D|$. Hence by Lemma 3.17 the general $D_0'' \in |D''|$ cannot be a cone with vertex at a point $x \in V$ except in the case when D_0'' is a plane under the inclusion π. The latter case is analyzed just as the nonconical case, since a general hyperplane section $D_0' \in |D'|$ through the fixed point x gives a curve on D_0'. Consequently the linear system $|D''|$ has no base points, and its general element D_0'' is irreducible, reduced and smooth by Bertini's theorem.

The general $D_0'' \approx E \times \mathbf{P}^1$, where E is an elliptic curve. Indeed by the adjunction formula for the canonical class of the surface D_0'' we obtain that the linear system $|-K_{D_0''}|$ has no fixed points, and its general element has at least two connected components. Hence from the classification of surfaces with $|-K_{D_0''}| \neq \varnothing$ and $K_{D_0} \neq 0$ (see, for example, [2] and [8]) we obtain the assertion. We will denote by L the class of the factor $\mathbf{P}^1$ in the Picard group Pic D_0''.

Let $\mu = \varphi_{|D''|}\colon V \to \mathbf{P}^{\dim|D''|}$. Obviously $|D''|$ is not a pencil, since $(D'')^2D = (D')^2D > 0$. Let us assume that the image $\mu(V)$ is two-dimensional. Then by Lemma 4.4 the restricted linear system $|(D_0'', D'')|$ on D_0'' is a pencil without base points. Then it is easy to show that $|(D_0'', D'')| = |nL|$ and $n = K_{D_0}^2$ (from the inequality $(D'')^2D = K_{D_0}^2 > 0$ there follows the rationality of the smooth components of this pencil). Let us now establish the surjectivity of the restriction of the linear system $|D''|$ to the system $|(D_0, D'')|$ on the general surface $D_0 \subset |D|$. To do this it is enough to show that $h^1(V, \mathcal{O}_V(D'' - D)) = 0$. By duality,

$$h^1(V, \mathcal{O}_V(D''-D)) = h^2(V, \mathcal{O}_V(-D'-D'')).$$

The pencil $\mathcal{O}_V(D' + D'')$ is ample, since $|D''|$ has no base points, and the pencil $\mathcal{O}_V(D')$ is ample. Hence by Kodaira's vanishing theorem we have $h^2(V, \mathcal{O}_V(-D' - D'')) = 0$. Consequently we have the surjectivity of restriction indicated above. The restricted linear system $|(D_0, D'')|$ is the anticanonical linear system on D_0. Hence under the assumption that $\mu(V)$ is two-dimensional the linear system $|D''|$ gives a mapping μ onto $\mu(V) \approx D_0$, i.e. $\mu(V) \approx \mathbf{P}^2$ or $\mu(V) = \mathbf{P}^1 \times \mathbf{P}^1$. If the surface S lies in a fiber of μ, then $\dim|S| = 0$ and therefore its intersection with a general element of $|D|$ is nontrivial. Also S must not intersect the general element of $|D''|$. The latter contradicts the ampleness of the anticanonical class on the general D_0. Consequently the fibers of the morphism μ are one-dimensional. The general fiber of μ is a curve L on D_0''. Therefore it will be a conic in V. Hence by (8.1.1) all the fibers of the morphism μ are conics; therefore $V \approx D_0 \times \mathbf{P}^1$ and the fiber product structure is given by the projections $\theta = \varphi_{|D|}\colon V \to \mathbf{P}^1$ and by μ. Consequently either $V \approx \mathbf{P}^2 \times \mathbf{P}^1$ or $V \approx \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Because of conditions (8.1.2) and (8.1.3) the index of V is equal to one and V is not isomorphic to $\mathbf{P}^1 \times \mathbf{P}^2$. This contradiction completes the proof that $\dim \mu(V) = 3$. In the course of the proof, without using the assumption that $\dim \mu(V) = 2$ we showed the surjectivity of the restriction of the linear system $|D''|$ to D_0 and the vanishing $h^1(V, \mathcal{O}_V(-D' - D'')) = 0$. Hence by Lemma 4.1 we have $h^1(D' + D'', \mathcal{O}_{D'+D''}) = 0$.

Now we will show that D gives a splitting in the linear system $|D''|$, which is nontrivial by (8.1.2), i.e. there exists an effective divisor $D''' > 0$ and $|D''| = |D + D'''|$. Indeed, in the opposite case, by duality and Lemma 4.1 we have

$$0 = h^0(V, \mathcal{O}_V(D'' - D)) = h^2(D' + D'', \mathcal{O}_{D'+D''}).$$

Then by the ampleness of $|D'|$ there exist irreducible and reduced smooth surfaces $D_0' \in |D'|$ and $D_0'' \in |D''|$ transversally intersecting each other along an irreducible, reduced and smooth curve C. Therefore $\chi(D_0') + \chi(D_0'') - \chi(C) = \chi(D_0' + D_0'') = 1$. In addition, $\chi(D_0') = 1$ because of the rationality of the surface D_0'; and $\chi(D_0'') = \chi(E \times \mathbf{P}^1) = 0$. This means that $\chi(C) = 0$ and $g(C) = 1$. By the adjunction formula for the canonical class of the surface D_0'' we have $|(D_0'', D)| = |E|$. On the other hand, since $|D''|$ does not restrict to a pencil on D_0'', it follows that

$$|(D_0'', D'')| = |mE + \sum_{i=1}^{n} L_i|,$$

where $m > 0$ and $n > 0$. Therefore

$$C \in |(D_0'', D')| = |(D_0'', D'' + D)| = |(m+1)E + \sum_{i=1}^{n} L_i|.$$

Consequently

$$0 = 2g(C) - 2 = \left((m+1)E + \sum_{i=1}^{n} L_i\right) \cdot \left((m+1)E + \sum_{i=1}^{n} L_i\right) = 2mn,$$

since $K_{D_0''} \sim -2E$. Hence $2mn = 0$, which is impossible for $m, n > 0$. Consequently there exists a divisor $D''' > 0$ which gives a splitting $|D''' + D| = |D''|$.

The linear system $|D'''|$ does not have fixed components. Since the morphism μ does not contract surfaces and the linear system $|D'''|$ by Lemma 4.4 restricts to the system $|(D_0'', D''')|$ on D_0'' isomorphically, it is sufficient to show the absence of fixed components in the system

$$|(D_0'', D''')| = \left|(m-1)E + \sum_{i=1}^{n} L_i\right|$$

on the surface D_0'', where $n = DD'''D_0'' = K_{D_0}^2 \geqslant 8$. Obviously in the latter system $|(m-1)E + \Sigma_1^n L_i|$ there are not even fixed points, since every divisor of degree $\geqslant 2$ on an elliptic curve has no fixed points.

The linear system D''' is a pencil. Indeed, assume the contrary. Then the general element $D_0''' \in |D'''|$ is by Bertini's theorem an irreducible and reduced surface. The restriction of $|D'''|$ to the system $(H, |D'''|)$ on the general hyperplane section $H \in |-K_V|$ has by Lemma 4.4 codimension $\leqslant 2$ in the complete linear system $|(H, D''')|$, i.e. by Lemma 3.1 the general surface D_0''' has on the hyperplane section H at most one nondegenerate quadratic singularity. Thus, for the general divisor D_0''' the curve of singularities can only be a straight line. From (8.1.1) it follows that D_0''' is a surface with finitely many isolated singular points. The latter easily leads to a contradiction. To see this, resolve the singularities of D_0''' and obtain the surface $\tilde{D}_0'''$, which by the adjunction formula has a divisor in the anticanonical system which consists of at least three connected components (corresponding to the intersection of D_0''' with $|3D|$).

Because of the triviality $\operatorname{Pic}^0 V = 0$ and the absence of fixed components in $|D'''|$ we have $|D'''| = |nS|$, where S is a surface giving the one-dimensional pencil $|S|$. We will show that $|S|$ is a linear system without base points. The general element of $|S|$ correctly and nontrivially intersects the general member of $|D|$; hence by Lemma 4.2 we have

$$h^1(V, \mathcal{O}_V(-3D - (n-1)S)) = 0$$

for $n \geqslant 2$, which implies the connectedness of all divisors of the linear system $|3D + (n-1)S|$. The case $n = 1$ is impossible (see the previous paragraph). Then, because the divisors in $|3D + (n-1)S|$ are connected (see Corollary 4.2 and Lemma 8.4), the system $|S|$ has no base points. Consequently $|D'''|$ is a pencil without base points. The latter contradicts the inequality $(D''')^2D = K_{D_0}^2 \geqslant 8$, which completes the proof. ■

8.6. Lemma. *Let D be a divisor from the splitting of the anticanonical linear system $|-K_V|$. Then the linear system $|D|$ has no base points.*

Proof. Let us choose among the divisors which give the splitting of $|-K_V|$ a nontrivial divisor with base points of minimal degree. We denote it by D. Obviously D is irreducible and reduced. By the choice of D and Lemma 4.8, the residual linear system $|D'|$ has a fixed component only if $\dim|D'| = 0$ and, for the surface D', $\deg D' = \deg D = g - 1$. By Lemma 8.3 this is impossible. Therefore the linear system $|D'|$ has no fixed components. By Bertini's theorem and Lemma 8.3 the linear system $|D'|$ is a pencil. Then by the triviality of $\operatorname{Pic}^0 V$ we obtain that $|D'| = |nS|$, where S is a surface which gives a one-dimensional pencil $|S|$, and $n \geqslant 2$ by Lemma 8.3. Using Lemmas 4.2 and 2.1 of [12], we show that

$$h^1(V, \mathcal{O}_V(-D-(n-1)S) = 0$$

and that all the divisors $|D + (n-1)S|$ are connected. Then by Lemma 8.4 the pencil $|S|$, and therefore also $|D'|$, will be pencils without base points. By Lemma 8.5 this is impossible. Consequently there are no divisors D with the properties noted in the beginning of the proof. ■

8.7. Lemma. *Let D be a nontrivial divisor from the splitting of the anticanonical system $|-K_V|$. Then the linear system $|D|$ is not a pencil and has no base points. Therefore by Bertini's theorem its general element is irreducible, reduced and smooth.*

This is an immediate consequence of Lemmas 8.5 and 8.6. ■

8.8. LEMMA. *On a Fano 3-fold V every nontrivial divisor D from the splitting of the anticanonical system is ample if the mapping $\varphi_{|D|}$ corresponding to it has a three-dimensional image.*

This is an immediate consequence of condition (8.1.1) and Lemmas 8.7 and 4.12. ■

8.9. LEMMA. *On a Fano 3-fold V an arbitrary nontrivial divisor D from the splitting of the anticanonical linear system $|-K_V|$ is ample.*

PROOF. Obviously we may assume that D gives a nontrivial splitting. If the linear system $|D|$ has nontrivial splittings, then by Lemma 8.6 it is sufficient to show the ampleness of some nontrivial divisor from the splitting of $|D|$. Therefore we may assume that all the elements of $|D|$ are irreducible and reduced. By Lemma 8.8 it is enough to establish the three-dimensionality of the image of V under the map $\varphi_{|D|}$, or, equivalently, that $|D|$ does not restrict to a pencil on its general member $D_0 \in |D|$. Let us assume the contrary. Then, by Lemma 4.4, on the surface D_0 the linear system $|(D_0, D)|$ is a pencil without base points. By Lemmas 8.7 and 4.5, D_0 is a smooth rational surface with $|-K_{D_0}| \neq \varnothing$, since by assumption $D' > 0$. Then $|(D_0, D)| = |nL|$, where L is a smooth curve with a one-dimensional pencil $|L|$ with no base points; $L^2 = 0$ and

$$2g(L)-2=LK_{D_0}=-\frac{1}{n}D^2D'=\frac{1}{n}D^2K_V<0.$$

Consequently, L is a smooth rational curve and $LK_{D_0} = -2$, so that L is a conic on V since $LD = 0$. From (8.1.1) it follows that the surface D_0 is rational and ruled. The ruled structure on D_0 is given by the pencil $|L|$. We will show now that $n = 1$, i.e. $|(D_0, D)| = |L|$. Indeed, by Lemma 4.4 the linear system $|D|$ gives a mapping into projective space of dimension $n + 1$ onto a surface of degree n. From the classification of such surfaces [10] we have splittings in the system $|D|$ in all cases except $n = 1$. This means that $n = 1$ and $D^2D' = 2$.

The residual linear system $|D'| = |-K_V - D|$ is ample. By Lemma 8.8 it is enough to prove that the image under the map $\varphi_{|D'|}$ is three-dimensional. If that is not the case, then, just as in the case of $|D|$, we obtain that the general element $D'_0 \in |D'|$ is a rational ruled surface and $D^2D' = K^2_{D'_0} = 8$. The latter contradicts the equality $D^2D' = 2$ obtained above.

We will show that D gives a splitting of the linear system $|D'|$. We assume the contrary. Then by Lemma 4.1 and by duality

$$h^2(2D', \mathcal{O}_{2D'}) = h^3(V, \mathcal{O}_V(-2D')) = h^0(V, \mathcal{O}_V(D'-D)) = 0.$$

By Lemma 4.1 and Kodaira's vanishing theorem we have

$$h^1(2D', \mathcal{O}_{2D'}) = h^2(V, \mathcal{O}_V(-2D')) = 0$$

and

$$h^0(2D', \mathcal{O}_{2D'}) = h^1(V, \mathcal{O}_V(-2D')) + 1 = 1.$$

Consequently $2\chi(D') - \chi(C) = \chi(2D') = 1$, where C is a smooth curve-section of $|D'|$. By Lemma 4.5, $\chi(D') = 1$; hence $\chi(C) = 1$ and $g(C) = 0$. The system $|D'|$ is very ample by Lemma 4.13, since $D(D')^2 = K^2_{D_0} = 8 \geqslant 3$. It is easy to deduce the full linear embedding of the rational curve-section C under the embedding $\varphi_{|D'|}$ from the fact that $h^1(V, \mathcal{O}_V) = h^1(D_0', \mathcal{O}_{D_0'}) = 0$. Consequently by [10] the general hyperplane section $D_0' \in |D'|$ will be either a ruled surface or isomorphic to $\mathbf{P}^2$. Both cases are impossible, since $K_{D_0'}^2 = D^2D' = 2$. Therefore we have a splitting $D + D'' \in |D'|$, and $D'' > 0$ by (8.1.2).

By Lemmas 8.7 and 4.5 and the classification of rational surfaces the general surface $D_0'' \in |D''|$ is a rational ruled surface, since $D^3 = 0$ and

$$K_{D_0''}^2 = (2D)^2D'' = 4D'(D'' + D) = 4D^2(D'' + D) = 4D^2D' = 8.$$

The linear system $|D''|$ has nontrivial splittings. To prove this, let us assume the contrary, i.e. that every divisor from $|D''|$ is irreducible and reduced. We first show that $|D''|$ is ample. Indeed, in the contrary case, arguing as in the case of the linear system $|D|$, we obtain the relation $2(D'')^2D = 2$ and the ampleness of $|2D| = |-K_V - D''|$. On the other hand, the system $|2D|$ is not ample since $D^3 = 0$. This contradiction proves the ampleness of the system $|D''|$. By duality and Kodaira's vanishing theorem we have

$$h^1(V, \mathcal{O}_V(D''-2D)) = h^2(V, \mathcal{O}_V(-2D'')) = 0.$$

Since $D'' \nsim D$ by (8.1.2), we have $|D'' - D| = \varnothing$. Consequently $h^0(V, \mathcal{O}_V(D'' - D)) = 0$. Then by the cohomology sequence of the short exact sequence

$$0 \to \mathcal{O}_V(D''-2D) \to \mathcal{O}_V(D''-D) \to \mathcal{O}_{D_0}(D_0, D''-D) \to 0$$

we obtain the vanishing

$$h^0(D_0, \mathcal{O}_{D_0}(D_0, D''-D)) = 0.$$

By the adjunction formula and by the canonical class formula for the surface $D_0 \approx F_n$ we have $|(D_0, D'' - D)| = |2b_n + ns_n| \neq \varnothing$, where s_n is a fiber and b_n is the base curve of F_n. This contradicts the vanishing just obtained. This means that $|D''|$ has a nontrivial splitting.

Splittings of the form $|D''| = |D''' + D|$ are impossible. Otherwise for the general smooth rational surface $D_0''' \in |D'''|$ we would have

$$K_{D_0}^2 = (3D)^2 D''' = 9D^2D''' = 9D^2D' = 18.$$

Consequently there is a splitting $|D''| = |D_1'' + D_2''|$, where $D_1'', D_2'' > 0$ and the divisor D does not give a splitting $|D_1''|, |D_2''|$. Therefore the restriction of the linear systems $|D_1''|$ and $|D_2''|$ to D_0 are injective. Therefore, by Lemma 8.7, $\dim|(D_0, D_1'')| \geqslant 2$ and $\dim|(D_0, D_2'')| \geqslant 2$. Using the adjunction formula and the canonical class formula for the ruled surface $D_0 \approx F_n$, we have

$$(D_0, D_1'') + (D_0, D_2'') \sim 2b_n + (n+1)s_n,$$

where $b_n^2 = -n$, $b_ns_n = 1$ and $s_n^2 = 0$. Then, because of the absence of base points in D'' and therefore also in $2b_n + (n+1)s_n$, we have $n \leqslant 1$. On the other hand, since $2b_0 + s_0$ cannot be decomposed into the sum of two two-dimensional systems, it follows that $n \geqslant 1$. Hence $D_0 \approx F_1$ and $(D_0, D_1'') \sim (D_0, D_2'') \sim b_1 + s_1$. The latter systems do not split further into two-dimensional ones. Therefore the linear systems $|D_1''|$ and $|D_2''|$ do not have nontrivial splittings. Also $D_1'' \nsim D_2''$, because V has index 1. If D'' is ample, then, since D does not appear in the splittings of $|D''|$, we may as above prove that $h^0(D_0, \mathcal{O}_{D_0}(D_0, D'' - D)) = 0$. The latter leads to a contradiction, since $|(D_0, D'' - D)| = |2b_1 + s_1|$. Consequently, $|D''|$ and together with it also the systems $|D_1''|$ and $|D_2''|$ are not ample. Then, as above, we show that the general divisor D_1'' is a ruled rational surface F_1. By the adjunction formula the anticanonical system $|-K_{D_1''}| = |2b_1 + 3s_1|$ contains a divisor $(D_1'', D) + (D_1'', D) + (D_1'', D_2'')$ whose every summand has dimension $\geqslant 2$ because of Lemma 8.7 and because of the injectivity of restriction. But the latter is impossible. Consequently the divisor $|D|$ is ample. ■

8.10. Lemma. *The linear system $|-K_V|$ has no splittings.*

Proof. Let there be a nontrivial splitting $D + D' \in |-K_V|$. We may assume that $|D|$ has no nontrivial splittings. Then by (8.1.2)

$$h^0(V, \mathcal{O}_V(D-D'))=0.$$

We will show that D gives a splitting of $|D'|$, which is nontrivial by (8.1.2). We will assume the contrary, i.e. that $h^0(V, \mathcal{O}_V(D - D')) = 0$. By Lemmas 4.4 and 8.7 we have an isomorphic restriction of the linear systems $|D|$ and $|D'|$ to a general hyperplane section $H \in |-K_V|$. We consider the exact sequence of sheaves

$$0\to\mathcal{O}_V(-2D)\to\mathcal{O}_V(D'-D)\to\mathcal{O}_H(H, D'-D)\to 0.$$

Because of ampleness (Lemma 8.9), in the corresponding cohomology sequence we have $h^1(V, \mathcal{O}_V(-2D)) = 0$. Therefore $h^0(H, \mathcal{O}_H(H, D' - D)) = 0$. Analogously

$$h^0(H, \mathcal{O}_H(D - D')) = 0.$$

Consequently, by duality and the Riemann-Roch theorem on the $K3$ surface H we have

$$-h^1(H, \mathcal{O}_H(H, D'-D)) = \frac{(D'-D)^2 H}{2} + 2.$$

Since $H \sim D + D'$, we get

$$-h^1(H, \mathcal{O}_H(H, D'-D)) = \frac{D^2(D-D')}{2} + \frac{(D')^2(D'-D)}{2} + 2.$$

By the adjunction formula and by Lemmas 8.7 and 8.9 the general divisor $D_0 \in |D|$ is a del Pezzo surface. The general element $D_1 \neq D_0$ of D cuts out on D_0 a smooth curve C by Lemma 8.7. Therefore $D^2(D - D') = D_0D_1(D_1 - D') = 2g(C) - 2 \geqslant -2$. Consequently in the relation obtained above we have

$$h^1(H, \mathcal{O}_H(H, D'-D))=0, \quad D^2(D-D')=(D')^2(D'-D)=-2;$$

hence $D^2D' \geqslant 3$ and $(D')^2D \geqslant 3$. Then by Lemmas 4.12 and 8.9 the sheaves $|D|$ and $|D'|$ are very ample. The curve-sections of the corresponding embeddings are rational and fully embedded. Then by [10] the general divisors from $|D|$ and $|D'|$ are either ruled surfaces or isomorphic to $\mathbf{P}^2$; hence $K_{D'}^2 = D'D^2 = 8$ or 9. Therefore $D^3 = 6$ or 7, because of the relation $D^2(D - D') = -2$ obtained above. This means that the very ample divisor $|D|$ embeds V in $\mathbf{P}^8$ or $\mathbf{P}^9$ as a smooth, irreducible and reduced 3-fold of degree 6 or 7, respectively. Hyperplane sections of such 3-folds split. This contradicts the choice of D. Therefore we have a splitting $2D + D'' \in |-K_V|$, and $D'' > 0$ by (8.1.2). Then the general element $D_0'' \in |D''|$ is a del Pezzo surface by Lemma 8.9. The del Pezzo surface D_0'' has index 2, i.e. its anticanonical divisor is effectively divisible by two. Consequently $D_0'' \approx \mathbf{P}^1 \times \mathbf{P}^1$. By the ampleness of $|D''|$ the restriction of $|D''|$ to D_0'' can be represented in the form $|(D_0'', D'')| = |nb_0 + ms_0|$, where $m, n \geqslant 1$ and b_0 and s_0 are general curves for the projection of the product $\mathbf{P}^1 \times \mathbf{P}^1 \approx D_0''$. By the adjunction formula and by the canonical class formula for $\mathbf{P}^1 \times \mathbf{P}^1$ we have $|(D_0'', D)| = |b_0 + s_0|$. We consider the exact sequence

$$0 \to \mathcal{O}_V(-D) \to \mathcal{O}_V(D'' - D) \to \mathcal{O}_{D_0''}(D_0'', D'' - D)$$
$$\approx \mathcal{O}_{D_0''}((n-1)b_0 + (m-1)s_0) \to 0.$$

By the ampleness of D we have

$$h^1(V, \mathcal{O}_V(-D)) = h^0(V, \mathcal{O}_V(-D)) = 0.$$

Therefore from the cohomology sequence we obtain that $|D'' - D| \neq \varnothing$, since

$$|(n-1)b_0 + (m-1)s_0| \neq \varnothing \quad \text{for } m, n \geqslant 1.$$

Consequently we have the splitting $3D + D''' \in |-K_V|$, and $D''' > 0$ by (8.1.2). Then as above the general element $D_0''' \in |D'''|$ is a del Pezzo surface of degree $K^2_{D_0'''} = (3D)^2 D_0''' \geqslant 9$. This means that $D_0''' \approx \mathbf{P}^2$. Also, $|D_0''', D'''| = |nl|$, where l is a straight line on $\mathbf{P}^2 \approx D_0'''$ and $n \geqslant 1$, by the ampleness of D'''. Using the same exact sequence as above but replacing D_0'', D'' and $(n-1)b_0 + (m-1)s_0$ by D_0''', D''' and $(n-1)l$, we obtain $|D''' - D| \neq \varnothing$. Hence we have the splitting $4D + D^{\mathrm{IV}} \in |-K_V|$, where $D^{\mathrm{IV}} > 0$ by (8.1.2). Then the general element of $D_0^{\mathrm{IV}} \in |D^{\mathrm{IV}}|$ is a del Pezzo surface of degree $K^2_{D_0^{\mathrm{IV}}} = (4D)^2 D_0^{\mathrm{IV}} \geqslant 16$. Such surfaces do not exist. This completes the proof. ■

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