LETTERS OF A BI-RATIONALIST
IV. Geometry of log flips

V.V. Shokurov *

October 13th, 2000/January 2nd, 2001

Flops and flips first appeared in mathematics as geometrical constructions:

(1) during Fano's modification of a 3-fold cubic into a Fano 3-fold $X_{14} \subset \mathbb{P}^9$ [10, Theorem 4.6.6];

(2) Atiyah's flop: one of his first papers [3] in 1958 treated the simultaneous resolution of the surface ODP, and was the initial stimulus for Brieskorn's simultaneous resolution of Du Val singularities[1];

(3) Kulikov's perestroikas [15, Modifications in 4.2-3];

(4) Francia's flip (see Example 3 below);

(5) Reid's pagodas [18];

(6) semistable flips [28] [12] [22];

(7) Kawamata's nonsingular 4-fold flip [13];

*Partially supported by NSF grants DMS-9800807 and DMS-0100991.

[1]Thanks to Miles Reid for this historical remark. He added also "Possibly a little later, Moishezon (and Hironaka) were using the same kind of thing to construct algebraic spaces (minischemes) that were not varieties.

However, as I said in my Old Person's View, one can trace the idea back through Zariski and Kantor and Cremona, even as far back as papers of Beltrami in 1863 and Magnus in 1837 referred to in Hilda Hudson's bibliography – these papers study the standard monoidal involution $\mathbb{P}^3 - \to \mathbb{P}^3$ given by $(x, y, z, t) \mapsto (1/x, 1/y, 1/z, 1/t)$, which flops the 6 edges of the coordinate tetrahedron".

(8) geometrical 4-fold flips [11]; and

(9) the Thaddeus principle [27] [5].

However in general, for higher dimensions, one can hardly imagine an effective and explicit *geometric* construction (for instance, a chain of certain blow-ups and blow-downs) for *flips*, even for log ones, except for very special situations with extra structures, e.g., as in (6) (8), and for moduli spaces as in (9). On the other hand, we hope that the log flips exist and this can be established in a more formal and algebraic way. Recently, this was done for the log flips up to dimension 4 [26, Corollary 1.8]. Since these flips were obtained without the use of any classification or concrete geometry of them, it is worthwhile in the aftermath to get some of the aforementioned geometrical facts. This is a goal of the note which we pursue in a more general situation. For the convenience of the reader, we put the list of notation and terminology at the end of the letter.

Definition. A birational transform $X - \to X^+/Z$ between two birational contractions $f : X \to Z$ and $f^+ : X^+ \to Z$ of normal algebraic varieties is called a (directed) *D-quasi-flip*$/Z$ or, shortly, *-qflip*$/Z$, for a Weil $\mathbb{R}$-divisor D on X, when there is a semiample$/Z$ $\mathbb{R}$-Cartier divisor D^+ on X^+ such that $f^+_* D^+ \sim_{\mathbb{R}} f_* D$.

A D-qflip$/Z$ can be given in a *log form* or, shortly, in *lf*, that is, in terms of log structures on X/Z and X^+/Z, namely, there are Weil $\mathbb{R}$-divisors B and B^+ on X and X^+ respectively such that:

- $f^+_* B^+ = f_* B$; and
- $D = K + B$ and $D^+ = K_{X^+} + B^+$.

The qflip is a *log* one if in addition:

- B and B^+ are boundaries; and
- pairs (X, B) and (X^+, B^+) are log canonical.

Note that up to an $\mathbb{R}$-linear equivalence of D and/or D^+ we can assume that $f^+_* D^+ = f_* D$ in the definition. Then any D-qflip is a qflip in lf for some B and B^+ (but maybe not a log qflip). (We always take all canonical divisors K, K_{X^+}, etc. on modifications of X given by the same differential form, or by the same bi-divisor [23, Example 1.1.3].) Any D-flip is a D-qflip with D^+ as the birational transform of D. The inverse holds when X^+/Z is small and D^+ is ample/Z; for example, by Monotonicity below, for any qflip in lf, X^+/Z is small if X/Z is small, $-D = -(K+B)$ is nef/Z, and (X,B) is terminal in codimension 2, that is, $a(Y) > 1$ for any subvariety $Y \subset X$ of codimension ≥ 2 in notation below. Thus a log qflip is a log flip under the last conditions, and with ample $D^+ = K_{X^+} + B^+/Z$. Log qflips, with nonsmall X^+/Z and a boundary B^+, are naturally induced by log flips on the reduced part of B (cf. the proof of [26, Special termination 2.2]).

Even always assuming that the characteristic of base field k is 0, we expect that most of the results and statements below hold without such an assumption, e.g., the following generalizations of [14, Lemma 5-1-17] and Monotonicity [19, (2.13.3)] – our basic tools.

Lemma. *Let $X - \to X^+/Z$ be a D-qflip for an $\mathbb{R}$-Cartier divisor D on X such that:*

1. *X/Z is a D-*contraction, that is, $-D$ is numerically ample/Z [23, Section 5]*, and*
2. *X/Z is a nonisomorphisms.*

Then

$$^+c \leq d + 1$$

where

- *$d = d(X/Z)$ is the* minimal *dimension of the irreducible components of the exceptional locus E of X/Z; and*
- *^+c is the* minimal *codimension in X^+ of the irreducible components of the* rational transform *^+E of E in X^+/Z.*

If E has the pure *dimension d*, that is, each irreducible component of E has the dimension d*, then ^+c can be taken as the* maximal *codimension.*

Warning 1. In general, ^+E is quite different from the exceptional locus $E^+ = E(X^+/Z)$. However $E^+ \subseteq {}^+E$ whenever $X - \to X^+$ is an isomorphism on $X \setminus E$; for instance, the latter holds for the D-flips of D-contractions but not for all qflips.

Remark 1. The minimal dimensions and codimensions can be replaced in the dual form of the lemma by maximal ones, namely, $c \leq {}^+d + 1$ in its *maximal* form. The lemma itself, in the maximal form, is dual to its symmetric statement $c \leq {}^+d + 1$ in the *minimal* form holding for the same E and ${}^+E$.

Note also that taking the minimal dimension and maximal codimension we consider only nonempty components, in particular, such (co-)dimensions are defined only for nonempty subvarieties. This explains Condition 2.

Proof. After a birational contraction of X^+/Z given by D^+ we can assume that X^+/Z is $-D^+$-contraction; this change only increases ${}^+c$.

Then the ampleness of $p_1^*(-D) + p_2^*D^+$ on $X \times_Z X^+/Z$ and [14, Proof of Lemma 5-1-17] imply that $X \times_Z X^+$ is divisorial over Z (see also [21, Negativity 1.1]). Moreover, for each irreducible component Y of E and its rational image ${}^+Y \subseteq {}^+E \subset X^+$,

$$\dim Y + \dim {}^+Y \geq \dim Y \times_Z {}^+Y = \dim X - 1.$$

That gives the required inequality.

The last statement for the pure d follows from the maximal case mentioned in Remark 1. □

Monotonicity. *Let $(X, B) - \to (X^+, B^+)/Z$ be a qflip in lf with nef $-(K + B)/Z$. Then, for each prime bi-divisor P of X,*

$$a(X^+, B^+, P) \geq a(X, B, P).$$

Moreover,

$$a(X^+, B^+, P) > a(X, B, P)$$

for each P with $\mathrm{center}_X P \subseteq E$ when $-(K + B)$ is numerically ample/Z; the equivalent inclusion is $\mathrm{center}_{X^+} P \subseteq {}^+E$.

Proof. As for [19, (2.13.3)]. □

Let $(X/Z, B)$ be a log pair with a boundary B such that:

(BIR) $f : X \to Z$ is a *birational* contraction which we always consider locally over some fixed point in Z; and

(WLF) the pair is a *weak log canonical Fano* contraction, that is, (X, B) is log canonical, and $-(K + B)$ is nef/Z;

it is said to be a *log canonical Fano* contraction or *log contraction* when $-(K+X)$ is numerically ample/Z. Note that $-(K+B)$ is big/Z by (BIR). The log pairs include, in particular, the birational *log contractions* of LMMP (Log Minimal Model Program), which are birational contractions X/Z of extremal faces numerically negative/Z with respect to $K+B$ [23, 5.1.1b]. However in this letter we do not touch fibred contractions [2].

The most fundamental questions in geometry concern *dimensions*. In our situation they are

- $n = \dim X$; and
- the *minimal* dimension $d = d(X/Z)$ for the exceptional locus E of X/Z.

Other more modern numerical invariants:

- the *(log) length* $l = l(X/Z, B)$ of $(X/Z, B)$, that is the minimal $-(K+B.C)$ for generic curves C in the covering families of *contracted* locus E (this is the exceptional locus whenever X/Z is birational, and $E = X$ otherwise) of X/Z; and
- the *m.l.d. (minimal log discrepancy)* $a = a(E) = a(X, B, E)$ of (X, B) *in* E, that is the minimal log discrepancy $a(X, B, P)$ at prime bidivisors P having the center *in* E; the latter means that $\operatorname{center}_X P \subseteq E$.

It is known that the dimension d depends on the length [24, Theorem], and on the singularities [4, Théorème 0]. Sometimes a more subtle interaction occurs.

Example 1. Suppose that (X, B) has only canonical (terminal) singularities in codimension 2, and a curve C is an irreducible component of the exceptional locus E of projective X/Z. Then the *existence of log flips in dimension* $n \geq 3$ *in the formal/*$\overline{k}$*, or analytic category when* $\overline{k} = \mathbb{C}$*, implies that* $(K+B.C) \geq -1$ *(respectively,* > -1*).* More precisely, for $n \geq 2$, we can assume just $a(C) \geq 1$ (respectively > 1). In other words, this means that the length l of the contraction is ≤ 1 (respectively < 1) whenever $d = 1$ and $a(C) \geq 1$ (respectively > 1). One can drop the existence of the log flip in dimension $n \leq 4$.

We verify that $l \leq 1$ in the canonical case; the terminal case is similar (cf. [7, Lemma 3.4]). Indeed, suppose that $l > 1$ and $a(C) \geq 1$. Then over a small neighborhood of $f(C)$ in the classical complex topology for $\overline{k} = \mathbb{C}$ (or formally over arbitrary algebraically closed $\overline{k}$), there exists a rather generic hyperplane section H/Z that intersects C transversely in a single point. In addition, changing the contraction over such a neighborhood we can assume that $E = C$. So, locally$/Z$, $(X/Z, B + H)$ is again a log pair under (BIR) and (WLF). Since the exceptional locus $E = C$ is a *proper* subvariety in X, the new m.l.d. $a := a(X, B + H, E) = a(X, B, E) \geq 1$, too.

On the other hand, according to our assumptions, there exists a flip $X - \to X^+/Z$ with respect to $K + B + H$. Actually it is also the flip for $K + B$ and is the $(-H)$-flip [26, Corollary 3.4]. So, the flip transform ${}^+H$ of H is the birational transform H^+ of H and numerically negative$/Z$ on the exceptional locus E^+ of the flipped contraction X^+/Z. Hence $E^+ \subseteq {}^+E \subset H^+$, and $E^+ = {}^+E$ unless $n = 2$ with $E^+ = \emptyset$ (cf. Warning 1). Moreover, ${}^+E$ has the minimal codimension $c^+ = 2$ by the lemma, and the m.l.d. ${}^+a = a(X^+, B^+ + H^+, {}^+E) \leq 1$; it is enough to establish the latter for $n = 2$, when it is well-known [23, Example 4.2.1]. But this contradicts to the assumption $a \geq 1$ because $a < {}^+a \leq 1$ by Monotonicity.

Remark 2. In the last paragraph we proved a little bit more. Let $(X/Z, B)$ be a purely log terminal pair with the reduced divisor H. Then each flip of $(X/Z, B)$ with $a \geq 1$ gives the flip on $(H/f(H), B_H)$ where B_H is given by the adjunction. (So, then $d \geq 2$ by the lemma when $H \sim_{\mathbb{R}} -h(K+B-H)$, with $h \in \mathbb{R}$, is numerically ample and $\not\equiv 0/Z$ as in the example.) Therefore, for a purely log terminal and canonical in codimension 2 pair $(X/T, B)$, LMMP with only *flipping* contractions $(X/Z/T, B)$ induces LMMP on $(H/f(H), B_H)$ (cf. the proof of [26, Special Termination 2.3]). Moreover, the same holds for any chain of *birational* contractions in LMMP for $(X/T, B)$ unless one of them contracts a component of $\operatorname{Supp} B$.

Advertisement 1. A generalization and applications of the improvement in Remark 2 will be treated in one of the following letters.

In the 3-dimensional terminal case with $B = 0$, Example 1 implies the Benveniste result [4, Théorème 0][2]; now without linear systems arguments. But it looks difficult to apply his arguments in higher dimensions; even in dimension 4. Deformation arguments in any dimension gives the weaker

[2]In general the strict inequality in the theorem fails in presence of canonical singularities along curve C, e.g., when C is obtained by the contraction along the second factor of a surface $\mathbb{P}^1 \times \mathbb{P}^1$ in a nonsingular 3-fold X with the normal bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2)$ where $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection on i-th factor.

inequality $l < 2$ in Example 1. In general, $d > l/2$ even for more difficult singularities [24, Theorem]. On the other hand, we expect

Conjecture 1. Under conditions (BIR) and (WLF), suppose that X/Z is *projective*. Then $d \geq a - 1$ ($>$ in the *log Fano* case), or, equivalently, $d \geq \lceil a - 1 \rceil$.

In addition, if $d = \lceil a - 1 \rceil, E \neq \emptyset$, and $(X/Z, B)$ is a *log contraction, that is, X/Z is a D-contraction for $D = K + B$*, then, for any log qflip $X - \to X^+/Z$, the transform ${}^+E$ satisfies the following properties:

(CDM) ${}^+c = d + 1$;

(NSN) each irreducible component of ${}^+E$ of the minimal codimension $d+1$ is *nonsingular* as a scheme point of X^+; and

(PDM) if d is the *pure dimension*, then ${}^+E$ is also of the pure codimension $d + 1$.

Moreover, for the log flip $X - \to X^+/Z$, ${}^+E = E^+$ is the exceptional locus of X^+/Z.

It is enough to establish the conjecture when $E \neq \emptyset$. Otherwise we put $d = -\infty$ and $a = -\infty$ as it used to be.

Example 2. In particular, if X is nonsingular and $B = 0$ then $a = n - d$ and Conjecture 1 implies that $d \geq (n-1)/2$ (respectively $d > (n-1)/2$. This is Wiśniewski's inequality [29, Theorem 1.1, p. 147] in a *single* formula).

Perhaps there is a relation between the length l and the m.l.d. a. I am not sure. But definitely, $l/2 \geq a - 1$ does not hold always.

Example 3. Francia's flip corresponds to the contraction, which is obtained from the relative model Y after contraction of its plane E in [8, Section 2], and it has $a = 3/2$ and $l = 1/2$. So, the above inequality $d > l/2$ is less sharp than that of in Conjecture 1.

Conjecture 1 can be derived from the conjecture on existence of log flips [23, Conjecture 5.1.2] and another conjecture on the m.l.d. [20, Problem 5a]:

Conjecture 2. For any (scheme) point $P \in X$ the m.l.d. $a(X, B, P) \leq \operatorname{codim} P$, with $=$ holds only when P is nonsingular in X and $B = 0$ near P. Taking hyperplane sections, it is enough to prove that $a(X, B, P) \leq \dim X$ and the $=$ case for closed points.

Moreover, the nonsingularity of X still holds when we replace $\operatorname{codim} P$ by $\operatorname{codim} P - \varepsilon$ with any $0 \le \varepsilon < 1$. Equivalently, $\lceil a(X, B, P) \rceil \le \operatorname{codim} P$ with $=$ holds only when P is nonsingular in X.

Warning 2. In the conjecture we still assume that B is a boundary!

Conjecture 2 was proven for $\operatorname{codim} P \le 3$ (after a $\mathbb{Q}$-factorialization, follows from [23, Corollary 3.3] with a final step by Markushevich [16, Theorem 0.1]). The stronger form with $\lceil \quad \rceil$ follows from the weaker one, the covering trick and LMMP. From LMMP, we need only the existence of $\mathbb{Q}$-factorializations. (It is expected that the next case with $a(X, 0, P) = \operatorname{codim} P - 1$ corresponds to the higher dimensional cDV singularities; cf. [1, Proposition 3.3].)

Since the m.l.d. measures the singularity, it is natural to expect that it decreases under the specialization that is stated in Ambro's conjecture [1, Conjecture 0.2]. It implies Conjecture 2, and is proved up to dimension 3 [1, Theorem 0.1] and for toric varieties by [1, Theorem 4.1]. The former gives again $\operatorname{codim} P \le 3$ and the latter gives the toric case.

Example 4. Conjecture 2 holds for toric varieties with invariant B.

Theorem. *The* existence of log flips in dimension n *and* Conjecture 2 *in dimension* m *implies* Conjecture 1 *in dimension* n *for any* $d \le m - 1$.

Actually, log flips can be weaken to log qflips. More precisely, it is enough to have the existence of log qflips for log contractions $(X/Z, B)$ *instead of log flips.*

Proof. According to the closing remark in Conjecture 1, we can assume that $E \ne \emptyset$. In particular, $n \ge 2$.

We can suppose also that $a > 0$. Otherwise, $a = 0$ and $d \ge 1 > -1 = a - 1 = \lceil a - 1 \rceil$.

Since X/Z is projective, after perturbation of B we can assume that X/Z is a log contraction. Note that taking quite a small perturbation, which increases B and decreases a, we preserve $\lceil a - 1 \rceil$. If the original B gave the log contraction we do not change B.

Now we can apply the lemma. Let $X - \rightarrow X^+/Z$ be a log qflip of $(X/Z, B)$ with a boundary B^+ on X^+. Then by the lemma there is an irreducible component $Y \subseteq {}^+E$ such that $\operatorname{codim} Y \le d + 1$. Hence Conjecture 2 implies that $a(X^+, B^+, Y) \le d + 1$, and Monotonicity

$$a = a(X, B, E) \le a(X^+, B^+, Y) \le d + 1$$

gives the required inequality.

Under additional assumptions in Conjecture 1, if $\operatorname{codim} Y \leq d$ then, according to the same inequalities, $a \leq d$ and $\lceil a-1 \rceil \leq d-1 < d$. This proves (CDM) because, according to these assumptions, $d = \lceil a-1 \rceil$ or $\lceil a \rceil = \lceil a(X^+, B^+, Y) \rceil = d+1$. So, (NSN) follows from Conjecture 2. The last statement in the lemma implies (PDM). If $X- \to X^+/Z$ is the log flip, then $E^+ = {}^+E$ (cf. Warning 1). Indeed, for small X/Z, the inverse transform is the anti-flip. Otherwise $a \leq 1, d = n-1 = \lceil a-1 \rceil \leq 0$, and $n \leq 1$, which contradicts to our assumptions. □

Corollary 1. *Conjecture 1 holds for the toric contractions X/Z, with only canonical singularities and $B = 0$, which are numerically negative with respect to K.*

A more general case we consider in one of our future letters (see Advertisement 2). It would be interesting to know whether the combinatorics behind this statement were known. In particular, how important is the projectivity in this statement (cf. Question 1)?

Proof. Immediate by Example 4 and the existence of toric flips [17, Theorem 0.2]. We do not need to perturb $B = 0$ since K itself is numerically negative. □

Corollary 2. *The theorem holds without* Conjecture 2 *for all $d \leq m = 2$.*

Proof. Immediate by [23, Corollary 3.3] and [16, Theorem 0.1]. □

The main inequality $d \geq a-1$ in Conjecture 1 can be established for rather high dimensions d without flips.

Example 5. For all $d \geq (n-1)/2$, Conjecture 1 follows form Conjecture 2 for $m \leq (n+1)/2$; in particular, up to $n = 6$ we can drop Conjecture 2. Indeed, let Y be an irreducible component of E then $a \leq \operatorname{codim} Y \leq (n+1)/2 \leq d+1$. Moreover, $d = \lceil a-1 \rceil$ only if $d = (n-1)/2$, the dimension is pure and E is nonsingular in each of its irreducible components as a scheme point; the additional statements (CDM), (NSN) and (PDM) in Conjecture 1 hold by the lemma and our hypothesis (cf. the proof of the theorem). Otherwise $d \geq n/2$ is integral, and $\lceil a \rceil \leq n/2 = d < d+1$.

Corollary 3. *In dimensions $n \leq 6$ one can drop* Conjecture 2 *in the* theorem.

Proof. In Example 5 it was proven for all $d \geq (n-1)/2$ because $m = (n+1)/2 \leq (6+1)/2 = 7/2$. For $d \leq 2 < (n-1)/2 \leq (6-1)/2 = 5/2$, the corollary was proven in Corollary 2. □

Corollary 4. *In dimension $n \leq 4$ one can drop both conjectural hypotheses in the* theorem, namely, the existence of log flip and Conjecture 2*.*

Proof. Immediate by Corollary 3 because the log flips exists. The latter was proven in [26, Corollary 1.8] when (X, B) is Kawamata log terminal. The other log flips also exist due to [26, Special termination 2.3] and the local log semiampleness (cf. [23, Conjecture 2.6], and see the proof of [23, Log Flip Theorem 6.13]).

Actually, by Example 5 it is enough to consider the case with pure $d = 1$. Then $a \leq 2$. Indeed, otherwise after a strict log terminal resolution we can assume that each reduced component H in B is $\mathbb{Q}$-Cartier, and intersects properly the curves C of E. Indeed, we can construct the log terminal resolution$/X$ using Kawamata log terminal flips and [26, Special termination 2.3] as in [21, Reduction 6.5] (cf. the proof of [26, Proposition 10.6]). This is not an isomorphism only over a finite set of points in C because X is nonsingular in the generic points of C by our assumption. This is impossible by the Kawamata log terminal case (cf. Example 1) because C is contractible at least in the formal or analytic category, $a > 2$ can only be increased after the construction, and now the flip in C exists (see [26, Remark 1.12]). □

Example 6 (Minimal contractions). Under conditions (BIR) and (WLF), suppose that X/Z is projective and dimension $d = \lceil a-1 \rceil$ is *pure* (or maximal). Then a log pair $(X/Z, B)$ will be called a *minimal (log) contraction* (respectively, when $B \neq 0$).

In particular, a minimal log contraction with $d = a-1$ is possible only when $(X/Z, B)$ is a 0-log pair, that is, $K + B \equiv 0/Z$ in our situation. This follows from a more subtle version of $>$ in Monotonicity under $\not\equiv 0/Z$ by [21, Negativity 1.1], or LMMP including the log termination and our theorem. If X/Z is projective under hypotheses of the theorem (cf. its proof) there exists a nonidentical directed flop $X - \to X^+/Z$ with the transform ${}^+E$ of pure codimension and satisfying the properties (CDM), (NSN), and (PDM) of

Conjecture 1 (perturb B by D as a negative to a polarization). In particular, such a flop is unique when X is $\mathbb{Q}$-factorial and X/Z is *formally extremal*, that is, the *formal* (in the formal or analytic category) relative Picard number of X/Z is 1. Moreover, for dimension $d = (n-1)/2 = a - 1$, it is expected that $(X/Z, B)$ is nonsingular in the irreducible components of E as scheme points, $f(E)$ is a *closed point* (take a general hyperplane section of $f(E)$ when $\dim f(E) \geq 1$), and each directed flop (qflop) $X - \to X^+/Z$ should be "symmetric", that is, the exceptional $E^+ = {}^+E$ of pure dimension $d = (n-1)/2$ with the nonsingular irreducible components of E^+ as scheme points (cf. Questions 1 and 2 below). The same follows from LMMP for any projective flop $(X^+/Z, B)$ of $(X/Z, B)$ as a composition of directed ones.

For instance, if X is nonsingular and $B = 0$, then it is expected that $d \geq (n-1)/2$ (cf. Example 2). So, by Conjecture 2 $(X/Z, 0)$ is minimal only when $d = a - 1 = (n-1)/2$ and the dimension is pure. Again by Conjecture 2 it is expected that any directed or/and projective flop X^+/Z is nonsingular with the same number of irreducible components of E^+ as for E (the number of exceptional prime divisors over E or E^+ with the log discrepancy $a = (n+1)/2$) whenever X^+/Z is *formally* $\mathbb{Q}$*-factorial* (in the formal or analytic category; cf. Question 2 below). The latter should hold for any nontrivial flop when X/Z is formally extremal. In this case the flop is unique when it exists and will be called *minimal formally extremal*. The LMMP implies that each of the directed and projective flops to X^+/Z is a composition of (formally) extremal contractions and flops; only such flops are enough when both X and X^+/Z are (formally) $\mathbb{Q}$-factorial.

An elementary example with nonsingular X, $B = 0$, and $E = \mathbb{P}^d$ belongs to the toric geometry. Its invariant divisors are $(n+1)/2 = d+1$ numerically negative D_i^- (intersecting E up to the linear equivalence by a negative to its hyperplane; their intersection in X is E itself) and $(n-1)/2 + 1 = d + 1$ numerically positive D_j^+ (intersecting E in hyperplane sections in a general position – an anti-canonical divisor in total). The construction of such a contraction X/Z see in [9, Example 3.12.2(iii)] with $r = (n+1)/2 = d + 1$, and $a_1 = \ldots = a_r = 1$. This contraction and its flop are *formally or analytically toric* (cf. a conjecture after the proof of [25, Theorem 6.4]) and *semistable* (cf. (6)); in this situation the latter means that there exists a nonsingular hypersurface H passing through E and $\equiv 0/Z$. Thus they induce a flop on this hypersurface as in Example 7 below with E/pt. Moreover, it is a symplectic one (cf. Question 4). Each toric contraction $(X/Z, B)$ is algebraically (analytically or formally) log i-symplectic for any $0 \leq i \leq$

$n = \dim X$ with B as invariant divisor (take a general linear combination of invariant i-forms $\wedge^i(dz_j)/z_j)$. In our situation, we can take 3-form (3-symplectic structure)

$$\frac{d(f_1^- f_1^+)}{f_1^- f_1^+} \wedge (\sum_{2\le i,j\le d+1} \frac{d(f_i^-)d(f_j^+)}{f_i^- f_j^+})$$

where f_i^- and f_j^+ are respectively local sections/Z of $\mathcal{O}_X(-D_i^-)$ and $\mathcal{O}_X(-D_j^+)$. Then after a generic perturbation of products $f_1^- f_1^+$ and $f_i^- f_j^+$ into g_{11}, with $H = \{g_{11} = 0\}$ vanishing on $\mathbb{P}^d$, and g_{ij} (the latter does not vanishing on $\mathbb{P}^d$ at all), the 3-form

$$\frac{dg_{11}}{g_{11}} \wedge (\sum_{2\le i,j\le d+1} \frac{d(f_i^-)d(f_j^+)}{g_{ij}})$$

induces by its residue a symplectic 2-form on H. Such flops will be called *induced toric*.

Note that, according to A. Borisov (a private communication) the *top* m.l.d. in dimension n for the toric *isolated* $\mathbb{Q}$*-factorial singularities* P is $a(P) = n/2 < (n+1)/2$. The latter is the mld for the minimal contraction of nonsingular X with $B = 0$. Actually, *for any toric isolated* $\mathbb{Q}$*-Gorenstein singularity* P*,* $a(P) \le (n+1)/2$*, and* $a(P) = (n+1)/2$ *only for the above toric contraction.* For such a toric singularity $P \in Z$, a toric projective $\mathbb{Q}$-factorialization X/Z is small birational, with the exceptional locus $E = f^{-1}P$, and crepant; it exists by [17, Theorem 0.2] because we can assume that $a(P) = a(Z, 0, P) > 1$ (otherwise $a(P) \le 1 < (n+1)/2$ for $n \ge 2$). Thus $a(P) = a(Z, 0, P) = a = a(X, 0, E)$. We can suppose also that X is nonsingular and $d \ge 1$ since otherwise, by the above $\mathbb{Q}$-factorial case, $a \le n/2$. If, in addition, it is extremal then, by Example 2, the arguments in the proof of our Theorem, Example 4, and the existence of any D-flip (flop) [26, Example 3.5.1], $d \ge (n-1)/2$ (cf. Corollary 1), and $a \le n-d \le (n+1)/2$. The case $a = (n+1)/2$ is possible only when $d = (n-1)/2$, and this is a minimal contraction. Since the latter is toric and extremal, E is a projective nonsingular toric variety with only ample invariant divisors, that is, $E = \mathbb{P}^d$ (the Fano variety of the maximal index $d+1$), and this is the above contraction. Suppose now that the relative Picard number of X/Z is 2. Then X/Z has two extremal contractions/Z (it is known in the toric geometry, and follows from the Cone Theorem [14, Theorem 4-2-1] after a perturbation of $B = 0$). They are small and, by the induction on dimension

n, give $a \leq n/2$ if one of them does not contract the exceptional locus into a point. Thus, if $a \geq (n+1)/2$, we can assume that both are again the above contractions of $E_1 = \mathbb{P}^d \neq E_2 = \mathbb{P}^d$ with $d = (n-1)/2$, and, by the extremal properties of both contractions $E_1 \cap E_2$ is at most a point. Since E is connected and all numerically negative invariant divisors on E_i/Z give E_i in intersection, there is one of them, say D_i, which is numerically positive/Z on the other E_{3-i}. According to the previous description $D_1 + D_2 \equiv 0/Z$, and their intersection $D_1 \cap D_2$ gives the induction in dimension n. However, such a contraction is impossible when $n = 3$. Indeed, for $n = 3$, there is no invariant divisor D which is numerically negative/Z on two curves in E, because these curves form a connected exceptional sublocus of E in D and each of its components is a (-1)-curve. In this situation, each E_i is an intersection of two pairwise distinct invariant divisors. So, $E = E_1 \cup E_2$ because each invariant divisor passing through a curve of E is negative on this curve only. This gives a contradiction because 4 invariant divisors passing through one of curves E_i pass through the intersection point $E_1 \cap E_2$. In particular, we prove that $a \leq n/2$ when the relative Picard number is 2. The same holds for higher Picard numbers by the last case and the induction when we consider a contraction of 2 dimensional face of the Kleiman-Mori cone for X/Z. This completes our proof and explains what are the minimal toric contractions.[3]

So, in dimension $n = 3$, Atiyah's flops (2-3) and their contractions are the only *nonsingular toric minimal* ones.

Other elementary examples of minimal semistable flops are Reid's pagodas and their possible higher dimensional generalizations (cf. La Torre Pendante [11, 8.12]). For $n \geq 5$, the induced flop on H can be different from the toric induced (or symplectic) one. All nonsingular minimal flops and their contractions for 3-fold are semistable (that is, their contractions Z have the cDV type). They are *absolutely* extremal, that is, formally/$\overline{k}$ or analytically when $\overline{k} = \mathbb{C}$, if and only if $E = \mathbb{P}^1$ is irreducible, and the flops are pagodas – (1-3) (5) above. Other minimal 3-fold flops are their composite over $\overline{k}$, e.g., modifications of Kulikov's model for K3 surfaces semistable degenerations.

Question 1. Does Conjecture 1 hold for nonprojective X/Z? Or at least in the nonsingular case of Example 6? It is interesting and nontrivial for $n \geq 4$ because in dimension 3 each small X/Z is at least formally or analytically projective.

[3] A. Borisov knows a pure combinatorial proof of this fact.

Question 2. (Cf. Example 6.) What do minimal contractions of nonsingular X with $n \geq 5$ look like? Their flops? Absolutely extremal? Are they still semistable? Their combinatorics? Is E irreducible and normal ($= \mathbb{P}^d$) for absolutely extremal contraction X/Z?

Example 7 (Almost minimal contractions). Under conditions (BIR) and (WLF), the next important class having pure $d = a$ is *almost minimal*.

Suppose also that X is nonsingular and $B = 0$ then, for such a contraction, $d = a = n/2$, and the dimension is pure, but either 0-log or non0-log pairs $(X/Z, 0)$ are possible, e.g., Mukai's flop and Kawamata's flip for $n = 4$. By the theorem it is expected that $f(E) \leq 1$, and $f(E)$ is a nonsingular curve in $P \in f(E)$ only when $(X/Z, 0)$ is locally over Z a 0-log pair with absolutely extremal X/Z near P; then it is expected to be a minimal nonsingular contraction over the transversal hyperplane sections of $f(E)$ through P. The directed or projective flop is fibred. In particular, this should be a nonsingular (nonsymplectic even locally over the contraction, for example, with $\mathbb{P}^{d-1}$-fibration because then the lines have $2d-4+1 = 2d-3 = n-3 < n-2$ parameters and this contradicts to Ran's estimation [6, Lemma 2.3]) flop.

The case with the point $f(E)$ is more complicated. First, suppose that $(X/Z, 0)$ is a non0-log pair (this is the *minimal* case *among* the *non*0-*contractions*; cf. Question 3 below). Then in dimension 4 the MMP implies that the flip $X - \to X^+$ exists, and X^+/Z is nonsingular with a curve $^+E = E^+$. The absolutely extremal components of such flips are Kawamata's flips (7). One can hope for a similar picture in higher dimensions.

Now suppose that $f(E) = pt.$ is a closed point, and $(X/Z, 0)$ is a 0-log pair. Such contractions appear as the toric ones induced in Example 6 and also as the *nonsingular* birational *symplectic* contractions with pure dimension $\leq n/2$ for E (then the pure dimension of E is $n/2$ and as we know E/pt). In these cases, for $n \geq 4$, one can expect the existence of a nontrivial nonsingular "symmetric" direct and/or projective flop. It should have $E^+ = {}^+E$ of the pure dimension $d = n/2$ with the same number of irreducible components as E (the number exceptional prime divisors over E or over E^+ with the log discrepancy $a = n/2$) in the formal or analytic case. For instance, conjecturally the Mukai flop is the only nonsingular almost minimal symplectic flop in dimension $n = 4$ (cf. Question 4 below).

But there are also singular flops. For instance, in dimension 4, an extremal flop $X - \to X^+/Z$ which transforms $E = \mathbb{P}^2$ into ${}^+E = E^+ = \mathbb{P}^1$ with a single simple singularity having the m.l.d. $= 2$.

In dimension 2, the almost minimal contractions $(X/Z, 0)$, which are 0-log pairs, are the minimal resolutions of Du Val singularities $P = f(E) \in Z$. They are always symplectic, but toric only of type $\mathbb{A}_*$, and, by Example 6, induced toric only of type $\mathbb{A}_1$ (correspond to the ordinary double singularity).

Question 3. Is the nonsingular non0-log pair case with $d = n/2$ in higher dimensions similar to dimension $n = 4$? In particular, is the flip X^+ always nonsingular? $E = \mathbb{P}^d$ for formally or analytically extremal contractions?

Question 4. Classify the nonsingular birational almost minimal 0-log pairs $(X/Z, 0)$ with $f(E) = pt.$ and their flops. In particular, such nonsingular symplectic flops. Is any such flop induced toric when $n \geq 4$ (see Example 6)?

Example 8 (More minimal contractions). A *nonidentical* contraction with the m.l.d. $a \leq 1$ is never minimal. For the next (terminal) segment $a \in (1, 2]$, the contraction is minimal only when $d = 1$ and E has only curve components C. Moreover, X/Z is *small* for $n \geq 3$. In particular, in dimension 3, $E = C$ is a curve.

Such contractions for terminal 3-folds with $B = 0$ appear in the MMP as 0-log pairs when $a = 2$ and non0-log pairs with $a = (m+1)/m$ where $m \geq 2$ is the index of K in $E = C$. So, $a \leq 3/2$ in the latter case. In addition, Francia's flip of Example 3 corresponds to (the only one with difficulty 1) an extremal terminal minimal log contraction $(X/Z, 0)$ with $a = 3/2$ or of the index 2; it also has the maximal length $l = 1/2$ among the index 2 contractions (cf. Example 1). For 3-folds with locally complete intersection singularities, we have only 0-log pairs with terminal Gorenstein singularities and their flops that are well-known.

Flops similar to the latter in dimension $n = 4$ are still not classified (even the absolutely extremal amongst them; cf. Question 4). However flips of some (maybe nonminimal) terminal Gorenstein contractions are explicitly known (8); they have $d = 2$ by Example 1.

Advertisement 2. An opposite class of *maximal* contractions and its application to the termination of log flips will be given in one of our future letters.

List of notation and terminology

$a = a(E) = a(X, B, E)$, the m.l.d. of (X, B) *in* the exceptional locus E

$a(Y) = a(X, B, Y)$, the *m.l.d. (minimal log discrepancy)* of (X, B) *in* subvariety Y, that is, the minimal log discrepancy $a(X, B, P)$ at the prime bi-divisors P having the center *in* Y; the latter means that $\operatorname{center}_X P \subseteq Y$; this assumes that $Y \neq \emptyset$ (otherwise we can put $a(Y) = -\infty$; cf. Conjecture 1)

$a(X, B, P)$, for a prime bi-divisor P (prime divisors on some model of X [23, p. 2668]), the log discrepancy of (X, B) or $K + B$ at P [21, p. 98]; P is considered here as its general or scheme point but not as a subvariety

B, a Weil $\mathbb{R}$-divisor on X; usually a boundary (except for Monotonicity), that is, all its multiplicities $0 \leq b_i \leq 1$, and $K + B$ is $\mathbb{R}$-Cartier

B^+, a Weil $\mathbb{R}$-divisor on X^+ such that $f_*^+ B^+ = f_* B$; usually a boundary (except for Monotonicity); for the log flips, the birational transform of B

(BIR), the condition on p. 4 which we assume afterwards, e.g., in Conjecture 1

${}^+c$, the minimal codimension in X^+ of the irreducible components of the rational transform ${}^+E$ of E in X^+/Z; the codimension is *pure* when all the irreducible components are of the same codimension

$\operatorname{center}_X P$, for a prime bi-divisor P (prime divisors on some model of X [23, p. 2668]), its center in X [23, p. 2669];P is considered here as a subvariety

(CDM), the property on p. 7 under additional assumptions in Conjecture 1

$\operatorname{codim} P$, for a scheme point $P \in X$, its codimension $\dim X - \dim P$ in X, e.g., $\operatorname{codim} P = \dim X$ if and only if P is a closed point in X

$d = d(X/Z)$, the minimal dimension of the irreducible components of the exceptional locus E of X/Z; this assumes that $E \neq \emptyset$ (otherwise we can put $d = -\infty$; cf. Conjecture 1); the dimension is *pure* when all the irreducible components are of the same dimension

D, a Weil $\mathbb{R}$-divisor D on X

D^+, a semiample/Z $\mathbb{R}$-Cartier divisor on X^+ [23, Definition 2.5] such that $f_*^+ D^+ \sim_{\mathbb{R}} f_* D$ (see Definition on p. 2); for the log flips, D^+ is the birational transform of D [21, p. 98]

$E = E(X/Z)$, the exceptional locus E of X/Z, that is, the union of contractible curves

$E^+ = E(X^+/Z)$, the exceptional locus of X^+/Z; thus E means here to be exceptional; in general, ${}^+E$ is quite different from the rational transform ${}^+E$

${}^+E$, the *rational* or *complete birational transform* of E in X^+/Z (see ${}^+Y$)

$f : X \to Z$, a birational contraction of normal algebraic varieties over k; sometimes we use such contractions in the analytic or formal category (cf. Example 1), and most of the statements work in them

$f^+ : X^+ \to Z$, its qflip or log flip

K, K_{X^+}, canonical divisors respectively on X and X^+ given by the same differential form, or by the same bi-divisor [23, Example 1.1.3]

$l = l(X/Z, B)$, the *(log) length* of log pair $(X/Z, B)$; see p. 5

LMMP, the log minimal model program and its conjectures [23, 5.1]

$n = \dim X$, the dimension of X

(NSN), the property on p. 7 under additional assumptions in Conjecture 1

(PDM), the property on p. 7 under additional assumptions in Conjecture 1

pt., a closed point

(WLF), the condition on p. 4 which we assume afterwards, e.g., in Conjecture 1

${}^+Y$, the *rational* or *complete birational transform* of Y in X^+/Z; it is defined for any subvariety $Y \subseteq X$ and any rational map $g : X - \to Y$ as $g(Y) = \psi \circ \phi^{-1}(Y)$, where

$$g = \psi \circ \phi^{-1} : X \overset{\phi}{\leftarrow} W \overset{\psi}{\to} Y$$

with a birational contraction ϕ, and independent on the decomposition

$X - \to X^+/Z$, either a D-qflip/Z, or log qflip (see Definition on p. 2), or a log flip [23, p. 2684]

$(X, B) - \to (X^+, B^+)/Z$, in Monotonicity, a qflip in lf (see Definition on p. 2) with possibly nonboundaries B and B^+

$(X/Z, B)$, a log pair which usually satisfies (BIR) and (WLF) on p. 4

a 0-log pair is a log pair $(X/Z, B)$ such that (X, B) is log canonical and $K + B \equiv 0/Z$ (cf. [26, Remark 3.27, (2)])

$\sim_{\mathbb{R}}$, the $\mathbb{R}$-linear equivalence [23, Definition 2.5]

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Department of Mathematics,
Johns Hopkins University,
Baltimore, MD–21218, USA
e-mail: shokurov@math.jhu.edu