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LETTERS OF A BI-RATIONALIST
I. A projectivity criterion

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In memoriam - Wei-Liang Chow

In April of 1990, during an algebraic geometry conference in Bayreuth (Germany), I heard from F. Campana a conjecture that any Moishezon manifold without rational curves is projective. For 3-folds it was proven by Th. Peternell in 1986 [P1, Theorem in § 2] (cf. Example 9.5.1 below). In the non-singular case, Kollar stated due to Mori a kind of progress in the conjecture for the algebraic spaces in the positive characteristics [Kol, II.5.16-II.17.1].

Here we discuss the conjecture in a more general framework using the LMMP [Sh3, 5.1]. A conclusion is that a non-projectivity may be attributed to an existence of rational curves and as well as of certain singularities (cf. Examples 9.5.2-3).

All algebraic objects and their morphisms below are Noetherian separated of finite type and defined over an algebraically closed field k. (In the relative analytic case we are working over small neighborhoods of a compact subspace in a base space, assuming that the spaces are Moishezon over such neighborhoods. Respectively, rational curves should be replaced by meromorphic images of $\mathbb{P}^1$, etc.)

1. LOG MINIMAL CONJECTURE. *Suppose that $(X/S, B)$ is a log pair with an algebraic space X and a boundary B, such that*

1.1. X is normal (or even semi-normal);

1.2. X/S is proper;

1.3. $K_X + B$ is log canonical (or even semi-log canonical); and

1.4. there are no rational curves in X/S.

Then X/S is a weakly log canonical model, or, equivalently, K_X+B is nef/S. Each log minimal model of $(X/S, B)$ is dominant over X.

1.5. REMARK-EXAMPLE. The conjecture holds in dimension ≤ 2 even with condition 1.3 replaced by the $\mathbb{R}$-Cartier property of $K_X + B$ (cf. Example 9.5.2). Moreover, if $K_X + B$ is not so, we may check that $K_X + B$ is nef (and even

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numerically ample whenever $K_X + B$ is non-log canonical) with intersection defined as in [Sh1, § 3].

Indeed, it is enough to check for a (semi-)log canonical resolution where it is well known.

However in dimension ≥ 3 it is rather possible that $K_X + B$ is negative on a non-rational curve C in non-log canonical singularities of $K_X + B$. This allows an existence of non-projective 3-folds with $\mathbb{Q}$-Cartier $K_X + B$ negative on C (see Example 9.5.3).

1.6. REMARK ON LOG SINGULARITIES. In a definition of log canonical and similar singularities a topology plays a crucial role. This due to a local nature of the Cartier and $\mathbb{Q}$-Cartier property. Of course our results hold in the Zariski topology. But as well we may use the classical complex topology for complex algebraic spaces or etale in general. Then any non-singular algebraic space is $\mathbb{Q}$-factorial and terminal which may not hold for non-algebraic varieties in the Zariski topology.

2. THEOREM. *Conjecture 1 holds in the dimensions* $\leq n$ *if*

2.1. each X/S *with* $\dim X \leq n$ *has a log resolution* $Y \to X/S$ *which is projective*/S;

2.2. the Base Point Free Theorem and

2.3. the LMMP hold in the dimensions $\leq n$.

Essentially the prove below uses a generalization of [Sh2, Conjecture], where projective morphisms are replaced by proper ones. In general this and corollaries (cf. [Sh2]) will be discussed elsewhere.

3. LEMMA. *Under assumptions 2.1-3, let* $f : X \to S$ *be a proper morphism of normal algebraic space with a boundary* B, $\dim X \leq n$, *and such that* $K_X + B$ *is Kawamata log terminal near a subspace* E, *consisting of components of the* degenerate locus

$$\mathcal{E}xc(f/P) := \{x \in X \mid f(x) \in P \textit{ and } f \text{ is not finite at } x\}$$

over a closed subspace $P \subseteq S$, *and let* $\gamma : E - \to G/S$ *be a rational* proper *dominant morphism of algebraic spaces. Suppose also that* $-(K_X + B)$ *is nef and big over a neighborhood of* P *in* S. *Then* E *has a family (possibly disconnected) of rational curves* $\{C_\lambda\}/S$ *which* covers G/S, *i.e., so does* $\{\gamma(C_\lambda)\}$ *in* G/S.

Then [C, Théorème 1.2] implies that

3.1. X/P *(i.e., each fiber*/P*) is rationally connected.*

In the lemma rational *proper* dominant γ means that it is dominant and proper over non-empty Zariski open subset of G. In the case of connected fibers it is *quasi-fibration* [C, 0.7]. Here we present only a modest fragment. We will prove a little bit more

3.2. *a family of rational curves on* E *is the image of a such family from a resolution* Y/X.

SKETCH PROOF. First, by the very definition our statement is local in the etale topology of S. So, we may assume that S is affine. Assuming that X is normal, E is irreducible and after taking sections we suppose that E is complete with $f(E) = P = \text{pt}$.

Next we complete $K_X + B$ to a similar log divisor $K_X + B$ which is $\sim_{\mathbb{R}} 0/S$. This follows from the same claim in the projective case [Sh1, Proposition 5.5] to which we reduce the situation according to the Base Point Free Theorem [KMM, Remark 3-1-2]. Thus we replace $K_X + B$ by an $\mathbb{R}$-complement, so $(X/S, B)$ is a weakly log canonical model with S/S as its canonical model. However a new boundary B is big which allows us to replace the nef and big properties of $K_X + B$ by

3.3. $K_X + B \sim_{\mathbb{R}} 0/S$ *and* Supp $B \geq D$ *where the birational image of* D *is ample*/S *on some partial (or log) resolution* Y/X.

For projective X/S we may take $Y = X$, otherwise we apply 2.1 or even Chow's Lemma [Kn, Theorem 3.1].

Then we can use [Sh2, Heuristic Arguments] with the following changes. Since E is a component of $f^{-1}P$, we may construct $K_X + B + \varepsilon f^*H$ maximally log canonical in first closed subspace $E := E' \subseteq E$, and rational dominant/G, where H is a generic hyperplane through P in the case when $S \neq P$. Then when we consider a fiber case $i : X \to E/S = P$, use as well an induction on $\dim X$, because X is rational dominant/G, and we may take $E := X$ with $\gamma := \gamma \circ i$. In the case when $S \neq P$, new X is a divisor on old X, and we may preserve 3.2 by the Tsen's theorem. (Of course, here we assume that a resolution in 2.1 may be done by monoidal transforms. In characteristic 0 it is known by Hironaka.)

Moreover, we can suppose that (X, B) is again log minimal (in particular, projective/E and even $/S = P$ after that. In applications below we have this from the beginning).

Indeed, by 2.1 we may assume that G is projective, and X/G is projective after the LMMP (and [Sh3, Proposition 4.4]) as well. Note that by our construction and 3.3, X/G has the log Kodaira dimension 0 over the generic point of G.

But 3.3 we need to replace by

3.3'. $K_X + B \sim_{\mathbb{R}} 0/G$, *semi-negative on rather generic curves in* X, for instance, on hyperplane curve sections, *and* Supp $B \geq D$ *where* D *is of fiber type for* γ, i.e., $\gamma(D)$ is a divisor on G, *and* $\gamma(D)$ *is big.*

Then $K_X + B - \varepsilon D$ will be negative on rather generic curves which not in fibers of γ. So we continue as in [Sh2, Heuristic Arguments]. A covering family will be not in fibers/G by 3.3'.

Of course, we assume that $G \neq \text{pt}$.. Then we have non-trivial D in G and so is $D := \gamma^{-1}D$ in E. ■

3.4. (SELF-)ADVERTISEMENT. Similarly, we may derive from 2.1-2 that any relative log Fano X/S with $\dim X \leq n$ and having only log canonical singularities is rationally connected/S (cf. [KMMo, 3.11]). But about this and something else in the next letter.

PROOF OF THEOREM 2. Let $(Y/S, B_Y)$ be a log resolution of $(X/S, B)$ as in 2.1.

First, we may find such minimal but partial resolution $g : Y \to X$. It means, that $(Y/S, B_Y)$ is log minimal. Otherwise, by the LMMP we have an extremal contraction $h : Y \to Z/S$. By Lemma 3 each fiber $E = h^{-1}P, P \in Z$ has a family of rational curve/S covering $G = g(E)$. (Take $\gamma = g$.) Then by 1.4 $G =$ pt. and each $h^{-1}P$ is a cycle/X. So, h makes a contraction/X as well and it is birational.

The fliped Y^+/S is again a partial resolution of X/S. Hence, essentially by 1.4, the LMMP is compatible with a relative structure/X, and we can continue it.

According to the termination, we have finally log minimal Y/S. Using a regular hut and [Sh1, 1.5.7] we check as well that any log minimal model $(Y/S, B_Y)$ of $(X/S, B)$ is dominant over X.

Since Y/S is minimal, so is Y/X. Again by [Sh1, 1.5.7] and by 1.3, $K_Y + B_Y$ is numerically trivial/X. Therefore, X/S is a weakly log canonical model as Y/S. ∎

3.5. REMARK. In a similar way Lemma 3 makes also the following improvement of Kleiman's criterion for projectivity by Peternell [P2, Theorem 2.6.1]. Suppose that X/S has mild singularities (for instance, log terminal, or strictly log terminal in the log case), then X/S is projective when

3.5.1. $C \equiv 0$ implies $C = 0$ for any rational curve/S, and

3.5.2. $\mathrm{NE}(X) \cap -\overline{\mathrm{NE}}(X) = 0$.

Of course, it is stated up to 2.1-3. So it holds in characteristic 0 up to dimension 3 of X (cf. Example 9.5.1 below).

A restatement of Theorem 2 is

4. COROLLARY. *Under conditions 2.1-3 with $n = \dim X$, X has a rational curve/S whenever 1.1-3 hold as well as $(K_X + B.C) < 0$ for some curve C in X/S.*

4.1. PROBLEM. Does exist in Corollary 4 a rational curve C in X/S with $(K_X + B.C) < 0$? (The same for Corollaries 6 and 8.)

Note now that 2.1-2 hold in characteristic 0. 2.1 follows from Chow's Lemma and Hironaka's resolution in characteristic 0. (It works as well in the semi-normal case.) 2.2, the Base Point Free Theorem, is implied by [KMM, 3-1-2(1)] and the proof of [Sh3, 2.7]. So, we have

5. COROLLARY. *In characteristic 0 Conjecture 1 holds in the dimensions $\leq n$ if the LMMP holds in the dimensions $\leq n$.*

6. COROLLARY. *In characteristic 0 and under the LMMP in the dimensions $\leq n = \dim X$, X has a rational curve/S whenever 1.1-3 hold as well as $(K_X + B.C) < 0$ for some curve C in X/S.*

The LMMP holds $n \leq 3$ at least in characteristic 0 and for normal 3-folds [Sh3]. Here by the latter we mean 3-dimensional spaces. So,

7. COROLLARY. *In characteristic 0 Conjecture 1 holds for 3-folds.*

8. COROLLARY. *In characteristic 0, a 3-fold X has a rational curve/S whenever 1.1-3 hold as well as $(K_X + B.C) < 0$ for some curve C in X/S.*

9. PROJECTIVITY CRITERION. *Under assumptions 1.1-4, 2.1-3, suppose moreover that*

9.1. $K_X + B$ *is Kawamata log terminal, and*

9.2. there exists a big ($\mathbb{R}$*-)Cartier divisor* D/S*; or just*

9.1'-2'. X *has only log terminal singularities and* $\mathbb{Q}$*-factorial in etale topology, for instance,* X *is non-singular.*

Then X *is projective/*S*, and* X *over its log canonical model* Y/S *or the Iitaka morphism* $I : X \to Y/S$ *has equi-dimensional fibers,* i.e., for each $y \in Y$,

$$\dim I^{-1}y = \dim X - \dim Y.$$

Each weakly log canonical model is dominated over Y *(cf. Theorem 2 for log minimal models). In particular,* $Y = X$ *is the log minimal model/*S *if and only if* X/S *is of log general type.*

PROOF-COMMENTARY. First, condition 9.1 implies that X normal and irreducible if connected. (If X does not connected, the statements holds for each connected component of X.)

Let D be big/S and $\mathbb{R}$-Cartier. Then we may assume that D is effective [Sh3, Lemma 6.17]. So, by Theorem 2 $K_X + B + \varepsilon D$ is nef, big and still satisfies 9.1. Therefore according to the Base Point Free Theorem, for $K_X + B + \varepsilon D$, there exists the canonical model Y/S with the Iitaka contraction $I : X \to Y/S$.

By 3.1 and 1.4, $I = \mathrm{id}$, $Y = X$ and X is projective/S.

In addition, if the Iitaka contraction exists for $K_X + B$, it will be projective and by 1.4 with [Sh2, Corollary 3], I has no degeneracy or equi-dimensional.

If 9.1'-2' hold, then K_X is Kawamata log terminal and any big divisor/S is $\mathbb{R}$-Cartier, which gives 9.2. ■

9.3. REMARK. So, if X has the Iitaka contraction, then B does not intersects generic fibers $E = I^{-1}y, y \in Y$, and they are projective varieties with $K_E \equiv 0$. So, if X has no such subvarieties/S then $K_X + B$ is ample/S. Presumably, it will take place whenever X is the Brody hyperbolic/S [Br]. Maybe, in an algebraic version we need to replace maps of $\mathbb{C} \to X$ by maps of $\mathbb{P}^1$ and of some special varieties E to X with $K_E \equiv 0$ (with semi-elliptic E in the analytic case as well as), e.g., by maps of Abelian varieties, Calabi-Yau 3-folds, and even elliptic curves sometimes (cf. 14 and 14.2).

In addition, correspondence $P \mapsto I(P)$ for prime divisors P, gives an isomorphisms between Weil divisors/Y and that of Y, as well so is for $\mathbb{Q}$-Cartier/Y, equivalently, for $\mathbb{Q}$-Cartier$\equiv 0/Y$. So, $K_X + B \equiv f^*(K_Y + B_Y)$ for some divisor B_Y. Presumably, we may chose B_Y as a boundary with Kawamata log terminal $K_Y + B_Y$. Respectively, Y is $\mathbb{Q}$-factorial and log terminal whenever so is X.

9.4. *If, in addition,* $-(K_X + B)$ *is nef/*S*, then any effective and, in particular, big (*$\mathbb{R}$*-)Cartier divisor* D *is nef/*S*. Hence by Theorem 2* $K_X \equiv B \equiv 0/S$*, whenever* X *is* $\mathbb{Q}$*-Gorenstein.*

In characteristic 0, we have Criterion 9 and 9.3-4, without assumptions 2.1-2. In dimension ≤ 3 without assumption 2.3 as well.

9.5. EXAMPLES.

9.5.1. Let X be a complete 3-fold in characteristic 0 having only $\mathbb{Q}$-factorial log terminal singularities (for instance, non-singular) and having no rational curves. Then X is projective with nef K_X. This explains constructions of non-projective 3-folds [III, VI.2.3].

If, in addition, $-K_X$ is nef, then $K_X \equiv 0$ and the cone of the effective divisors D coincides with that of nef. Both such D's will be semi-ample.

So, every non-projective X with above singularities and $K_X \equiv 0$ (for instance, Bogomolov-Calabi-Yau) will have a rational curve.

9.5.2. Let $X \to Y$ be a birational contraction of a non-singular complete surface X to a non-projective normal algebraic surface Y [N2, Th. 1]. Then after replacing X by finite (cyclic) coverings we may suppose, according to Bogomolov, that X and Y have no rational curves. (Indeed, then X be of general type with $c_1^2(X) > c_2(X)$.) However Y is not a weakly log canonical model for any boundary B because Y is non-projective. More precisely, $K_Y + B$ will not be $\mathbb{R}$-Gorenstein.

However in dimension ≥ 3 we have similar examples with $\mathbb{R}$-Gorenstein $K_X + B$ (and even with $B = 0$).

9.5.3. Now let $X \to Y$ be a birational extremal contraction of two non-singular curves C_1 and C_2 in a non-singular 3-fold, so that $C_1 \equiv aC_2$ for a rational multiple $a \neq 0$, and a flip exists in both of them. (For instance, they arise after blow-ups in two non-singular curves in a 3-fold, having a normal crossing in two points, cf. [III, ib].) We may suppose that X is not projective after a flip in one of these curve while Y is still projective. Finite coverings preserve all these conditions, except for singularities, and after that we may assume that X has no rational curves. Since these coverings may be chosen Galois, K_X will be $\mathbb{Q}$-Cartier. Moreover, we may suppose that fliped curves for C_i's will be again non-rational. Thus after these flips we will have no rational curves in fliped X as well.

Finally we may chose a covering with a ramification in a Cartier divisor D sufficiently negative on C_1. Then $K_X.C_1 < 0$ after the covering.

Note that in this case we may construct X as an algebraic variety.

For higher dimensions we may take a product of X by a projective non-singular variety without rational curves, e. g., by an Abelian.

Condition 1.4 in conjunction with mild singularities is anti-birational as shown by the following results. (Cf. [CKM, 1.1-6].)

10. INDETERMINACY LOCUS THEOREM. *Let $g : X \dashrightarrow Y/S$ be a rational morphism of algebraic spaces where X/S satisfies 1.2, 9.1 (or 9.1', i.e., X has only log terminal singularities), and Y/S is proper. Then for each closed $P \subseteq X$, $g(P)$ is covered by rational curves/S, whenever 2.1-3.*

SKETCH COMMENT-PROOF. First, remark that $g(P)$ here denotes the *indeterminacy* locus of P:

$$g(P) = q(p^{-1})P \text{ for a regular hut}$$

$$\begin{array}{ccccc} & & W & & \\ & p & & q & \\ & \swarrow & & \searrow & \\ X & & & & Y \end{array}$$

(for example, a graph diagram). We may assume also that $(W/X, B_W)$ is strictly log terminal. Then we use the LMMP/X (as well the the Base Point Free Theorem) with Lemma 3. In the latter $\gamma = q$. (Cf. the proof of Theorem 2 and Criterion 9.) ■

10.1. ADVERTISEMENT. Similarly we can prove that $g(P = \text{pt.})$ is rationally connected/S which is especially interesting when g is the inverse to a birational contraction $g^{-1} : Y \to X$ (for instance, a resolution) with $g(P) = (g^{-1})^{-1}P$. This gives an affirmative answer to a Borisov's question.

10.2. REMARK-COROLLARY. However Theorem 10 implies the following generalization of an Abhyankar result [Ab, Proposition 4] [CKM, Proposition 1.2]:

10.2.1. *Under the assumptions of Theorem 10, let g be the inverse to a birational contraction $g^{-1} : Y \to X$. Then any fiber of Y/X is covered by rational curves.*

Using this and the Inverse of Adjunction [Sh1, 3.4] we may generalize two fundamental theorems of Matsusaka and Mumford [MM, Theorems 1 and 2] to the case of varieties with mild singularities. For instance, one of the results will assert that

10.2.2. *Two projective (log) varieties, which are isomorphic as polarized varieties, remain isomorphic after specializations over a discreet valuation, whenever they have satisfies 1.2, 9.1 (or 9.1', i.e., they have only log terminal singularities), and at least one of them is non-uniruled.*

The required model over the ring will satisfies 1.2, 9.1 (or 9.1') as well.

10.3. REGULARITY COROLLARY. *Under the assumptions of Theorem 10, suppose that Y/S has no rational curves/S. Then g is regular.*

Or its restatement

10.4. COROLLARY. *Under the assumptions of Theorem 10, suppose that g is not regular. Then Y has a rational curve/S.*

10.5. REMARK-EXAMPLE. [CKM, 1.5] does not hold in general. Take $f : Y \to Z$ being a contraction of a connected subspace on Y into a point z_1, and $g = \text{id} : Y \to X = Y$. Then $g(f^{-1}z_0) = \text{pt.}$ if and only if $z_0 \neq z_1$. Note that the corresponding set $Z \setminus z_1$ is not closed.

However [CKM, 1.6] may use the Negativity of a Birational Contraction [Sh1, 1.1]. But essentially it follows from Corollary 10.4.

11. COROLLARY. *Let $g : X - \to Y/S$ be a birational morphism of algebraic spaces where $X, Y/S$ satisfy 1.2, 9.1(or 9.1'), and both $X, Y/S$ do not have rational curves/S. Then g is biregular, whenever 2.1-3 hold.*

In particular, each birational automorphism of X/S is biregular.

Again, in characteristic 0, we have Theorem 10 and Corollaries 10.2-4, 11 without assumptions 2.1-2. In dimension ≤ 3 without assumption 2.3 as well.

In most of the above statements, we may even drop 1.2.

12. QUASI-PROJECTIVITY CRITERION. *Without 1.2 in Conjecture 1 (in the analytic case X/S is a Zariski open subspaces of a Moishezon space/S) Theorems 2, 9, 9.4, 10, Corollaries 4-8, 10.2-4, Remark 9.3 hold with the following changes:*

12.1. By a weakly log canonical model X/S we mean a Zariski open subset of an appropriate completion $\overline{X}/S$, whereas B is extended to $\overline{B}$ by a complement divisor aF with $F = 0_{\overline{X}} = \overline{X} \setminus X$ and some $0 \leq a \leq 1$. So, we suppose that $K_{\overline{X}} + \overline{B}$ is log canonical or has the same properties as $K_X + B$.

12.2. If a curve C/S is non-complete, i.e., intersects F, we consider $(K_{\overline{X}}+\overline{B}.C)$ instead of $K_X + B$. However then we assume conditions 9.1-2 or 9.1'-2'.

12.3. Dominant means a proper morphism of a Zariski open subset, for instance, of a log minimal model, when it dominates, etc.

12.4. The projectivity we replace by the quasi-projectivity.

12.5. However, Y/S is still supposed to be proper in Theorem 10, Corollaries 10.2-4.

Moreover, in Remark 9.3 and according to it, I is proper and even projective on X/S, and 9.4 takes place only for projective X/S.

Finally, here by a rational curve$/S$ we mean a curve $C \subseteq X/S$ with a rational complement.

COMMENT-PROOF. Use an existence of a completion X by Chow's Lemma [Kn, Theorem 3.1] (cf. with Nagata's [N1]).

Note that now in 12.2 of Theorem 2 we need the Semi-Ampleness [Sh3, Conjecture 2.6] but in a big case which in characteristic 0 follows from the Base Point Free Theorem. ■

13. ARBITRARY FIELDS. *All derivations and results hold over any base field k.*

By a rational curve we mean here a 1-dimensional subspace C/S which consists of rational curve over an algebraic closure $\overline{k}$, i.e., C is geometrically uniruled.

COMMENT-PROOF. The existence of rational curves in X/S is equivalent to that of in $X_{\overline{k}}/S$. The latter can be descend to X in terms of conjugations. ■

We know that in the proper case (with algebraically closed k) a rational curve is the image of regular non-constant map $\mathbb{P}^1 \to X$. In general, case we assume the map is only rational. But maybe we can improve it.

14. ALGEBRAIC HYPERBOLICITY QUESTION. *Whether it is true Conjecture 1 and etc, if we mean by a rational curve an image of a regular non-constant map of $\mathbf{A}^1_k = K \to X$.* Equivalently, it extended to $\mathbb{P}^1$, or log $(\mathbb{P}^1, B = \infty)$ whereas $(\mathbb{P}^1, B = \infty) \to (\overline{X}, F)$ maps only ∞ into F.

To strength results, to log canonical one, we need perhaps to exclude more: *the elliptic curves as regular maps, and the log proper maps of* $(\mathbb{P}^1, B = 0 + \infty)$. Such type of curves are compatible with the etale topology. In particular, Theorem 2 in that case should state that *$(X/S, B)$ is* quasi-*log canonical model, i.e., a Zariski open subset of a log canonical model $(\overline{X}/S, B_{\overline{X}})$ and so X/S is quasi-projective.* This should be a generalization of the Satake compactification (cf. 14.2 below).

Even we may change the target X/S by a finite cover $\widetilde{X} \to X$, because in the given case, the projectivity and quasi-projectivity is compatible with finite maps by [H, Exercise 5.7 (d) in Ch. III] and the regular property of the automorphisms of the log canonical model $(\overline{X}/S, F)$.

If the answer is negative we may replace $\mathbb{P}^1$ and $\mathbf{A}^1$ by log varieties (Y, B_Y) with nef $-(K_Y + B_Y)$ or according to Bogomolov-Yau by log Abelian, simplectic and SU-varieties.

14.1. PREMATURE-EXAMPLE. Suppose that $\mathcal{M}$ is a fine moduli space of algebraic spaces of given topological type. Then they will be quasi-projective whenever we have a rigidity

14.1.1. Any family of such spaces over $\mathbb{A}^1_k = k$ is locally trivial (in the etale topology).

A similar condition, for $\mathbb{P}^1$ (and for an elliptic curve) instead of $\mathbb{A}^1$, is fulfilled in some cases. For instance, when we consider moduli of projective non-singular varieties X with ample canonical polarization K_X, it is true according to Kovacs [Kov]. If the moduli space is not fine, we can often add an extra structure to make this which gives a finite covering and we have again 14.1.1 on it. Of course, the quasi-projectivity of these moduli is known by other reasons.

For the quasi-projectivity, we may replace 4.1.1 also by weaker but a sufficient condition:

14.1.2. Any family of such spaces over any log varieties (Y, B_Y) with nef $-(K_Y + B_Y)$ or even with $K_Y + B_Y \equiv 0$ (cf. Remark 9.3 and Question 14) is etale locally trivial.

14.2. ANALYTIC CASE. In the analytic case the exclusions in Conjecture 14 are covered by holomorphic maps $\mathbb{C} \to X$ which means the Brody hyperbolicity.

So, in particular, we anticipate that Brody hyperbolic Zariski open subset of a Moishezon space are quasi-projective.

For instance, this is true for free algebraic quotients of bounded domains with the Satake compactification. For other quotients we may use a finite covering given by a subgroup with a free action.

In the opposite case when $-(K_{\overline{X}} + \overline{B})$ is nef$/S$, whether X is covered (or a non-empty open subset of X) by holomorphic curves $\mathbb{C} \to X/S$ (cf. Remark 9.3).

There exists an analytic interpretation of the Moishezon property [ShSh].

References

[Ab] Abhyankar S. S., *On the valuations centered in a local domain*, Amer. J. Math. **76** (1956), 321-348.

[Br] Brody R., *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213-219.

[C] Campana F., *Connexité rationelle des variétés de Fano*, Ann. scient. Éc. Norm. Sup., 4 série, t. **25**, 1992, 539-545.

[CKM] Clemens H., Kollár J., Mori S., *Higher dimensional complex geometry*, Astérisque vol. **166**, Soc. Math. France, 1988.

[H] Hartshorne R., *Algebraic geometry*, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[KMM] Kawamata Y., Matsuda K. and Matsuki, *Introduction to the minimal model problem*, in Algebraic Geometry Sendai 1985, Adv. St. Pure Math. **10** (1987), 283-360.

[Kn] Knutson D., *Algebraic spaces*, Lecture Notes in Mathematics, **203**, 1971.

[KMMo] Kollar J., Miyaoka Y., Mori S., *Rationally connected varieties*, J. Alg. Geom. **1**(1992), 429-448.

[Kol] Kollar, J., *Rationale curves on algebraic varieties*, to appear.

[Kov] Kovacs S., *Smooth families over rational and elliptic curves*, J. Algebraic Geom. **5** (1996), no. 2, 369-385.

[MM] Matsusaka T., Mumford D., *Two fundamental theorems on deformations of polarized varieties*, Amer. J. Math. **86** (1964), 668-684.

[N1] Nagata M., *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ., **2** (1962), 1–10.

[N2] Nagata M., *On the imbeddings of abstract surface in projective varieties*, Mem. Coll. Sci. Kyoto (A), **30** (1957), 231–235.

[P1] Peternell Th., *Rational curves in Moišezon 3-folds*, Lecture Notes in Math. **1194** (1986), 133–144.

[P2] Peternell Th., *Moishezon manifolds and rigidity theorems*, preprint.

[Ш] Шафаревич И.Р., Основы алгебраической геометрии, том. 2, Москва "Наука", 1988.

[ShSh] Shanyu J., Shiffman B., *Properties of compact complex manifolds carrying closed positive currents*, the Journal of Geometric Analysis, Volume **3** (1993), 37-61.

[Sh1] Шокуров В.В. Трехмерные логперестройки// Изв. АН СССР, сер. мат. 1992. Т. 56. С. 105–203.

[Sh2] Shokurov, V.V., *Anticanonical boundedness for curves*, appendix to V.V. Nikulin "Hyperbolic reflection group methods and algebraic varieties" in Higher dimensional complex varieties: Proceedings of the international conference, held in Trento, Italy, June 15–24, 1994 /ed. M. Andreatta; T. Peternell. – Berlin: New York: de Gruyter, 1996, pp. 321–328.

[Sh3] Shokurov, V.V., *3-fold log models*, (in print).

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