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HOLOMORPHIC DIFFERENTIAL FORMS OF HIGHER DEGREE ON KUGA'S MODULAR VARIETIES

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ABSTRACT. In the paper a canonical isomorphism between the space of cusp forms $S_{w+2}(\Gamma)$ of weight $w+2$ with respect to the modular group Γ and the space of holomorphic differential forms of higher degree on Kuga's modular variety B_Γ^w is constructed.

Bibliography: 6 titles.

The main aim of this work is the proof of Theorem 5 (see [6]). The case $w = 0$ of this theorem is classical, and the case $w = 1$ has been proved by Shioda [5]. The latter paper served as a departure point for the author. At present M. Kifer and I. Skornjakov have obtained the corresponding results for the Hilbert modular group. The author expresses his gratitude to Ju. I. Manin, in whose seminar this work was done.

§0. Main results

0.1. All algebraic varieties, their morphisms, and differentials to be considered below are defined over **C**. Let B denote a nonsingular projective surface with a canonical projection $\Phi\colon B \to \Delta$ and a section $o\colon \Delta \to B$. Assume that its general fiber is an elliptic curve and Δ is a nonsingular projective curve. A point $v \in \Delta$ is called a point of *nonsingular type* if the fiber $B_v = \Phi^{-1}(v)$ is an elliptic curve. Let Δ' be the set of points of nonsingular type and $B' = B|_{\Delta'}$ (B restricted over Δ'). Then the nonsingular algebraic variety

$$(B^w)' = \underbrace{B' \times_{\Delta'} B' \times_{\Delta'} \ldots \times_{\Delta'} B'}_{w}$$

is defined for any natural number w. The variety $(B^w)'$ has a smooth canonical projective compactification B^w which is constructed in §3 proceeding from a singular projective compactification $\bar{B}^w$. The construction of $\bar{B}^w$ is given in §2. The variety B^w has a canonical projection $\Phi\colon B^w \to \Delta$ and is called *Kuga's variety* [6].

0.2. Let Γ be a subgroup of finite index in SL(2, **Z**). We will consider pairs (Γ, w), w being a natural number, such that for odd w one has

$$-E \notin \Gamma. \tag{$*$}$$

In the case $(*)$ a nonsingular projective elliptic surface B_Γ is canonically defined (see §4 of [5]). This is an elliptic surface over Δ_Γ, the corresponding modular curve. In the case

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when the condition (*) does not hold one can construct a certain nonsingular projective elliptic surface B_Γ over a modular curve Δ_Γ with the functional invariant $J_\Gamma = j \circ \varphi_\Gamma$ (see Lemma 1.5). Here j is the absolute invariant function on $\mathrm{SL}(2, \mathbf{Z}) \setminus \overline{H}$, and φ_Γ is the following canonical projection:

$$\Delta_\Gamma = \Gamma \setminus \overline{H} \to \mathrm{SL}(2, \mathbf{Z}) \setminus \overline{H}.$$

The nonsingular projective variety B_Γ^w is called *Kuga's modular variety*.

0.3. Main Theorem. a) *There are canonical isomorphisms*

$$H^0(B_\Gamma^w, \Omega^{w+1}) \simeq S_{w+2}(\Gamma), \quad H^{w+1}(B_\Gamma^w, \mathcal{O}) \simeq H^0(B_\Gamma^w, \overline{\Omega}^{w+1}) \simeq \overline{S_{w+2}(\Gamma)},$$

where $S_{w+2}(\Gamma)$ is the space of Γ-cusp forms of weight $w + 2$ (see §2.1 *of* [4]).

b)

$$\dim H^0(B_\Gamma^w, \Omega^{w+1}) = (w+1)(g-1) + \sum_{b\geqslant 1} (\nu(\mathrm{I}_b) + \nu(\mathrm{I}_b^*)) \frac{w}{2}$$
$$+ (\nu(\mathrm{II}) + \nu(\mathrm{IV}^*)) \left[\frac{w+2}{3}\right] + (\nu(\mathrm{III}) + \nu(\mathrm{III}^*)) \left[\frac{w+2}{4}\right], \text{ if } w \text{ is even};$$

$$\dim H^0(B_\Gamma^w, \Omega^{w+1}) = (w+1)(g-1) + \sum_{b\geqslant 1} \nu(\mathrm{I}_b) \frac{w}{2}$$
$$+ \sum_{b\geqslant 1} \nu(\mathrm{I}_b^*) \frac{w+1}{2} + \nu(\mathrm{IV}^*) \left[\frac{w+2}{3}\right], \quad \text{if } w \text{ is odd},$$

*where $\nu(*)$ is the number of points in Δ of type * (see* §1) *and g is the genus of the curve Δ_Γ.*

To part a) of the theorem corresponds Theorem 5.6, and to part b) Corollary 5.7. □

0.4. Let us describe as a complement to the theorem the homomorphism

$$S_{w+2}(\Gamma) \to H^0(B_\Gamma^w, \Omega^{w+1}), \quad \Phi \mapsto \omega_\Phi, \tag{0.1}$$

which is the required isomorphism in the case (*). Let $H' = H - \mathrm{SL}(2, \mathbf{Z})\{\eta\}$, $\eta = e^{2\pi i/3}$, and $\Delta' = \Gamma \setminus H' \subset \Delta_\Gamma$. Then there is a canonical isomorphism (see Proposition 5.2 b))

$$B_\Gamma^w|_{\Delta'} \simeq \Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w \setminus H' \times \mathbf{C}^w,$$

where the group $\Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w$ (semidirect product) acts on $H' \times \mathbf{C}^w$ by the following rule:

$$(\gamma, n, m) : (z, \zeta) \mapsto (\gamma z, (cz+d)^{-1}(\zeta + zn + m)) \tag{0.2}$$

for $n, m \in \mathbf{Z}^w = \mathbf{Z} \times \cdots \times \mathbf{Z}$ (w factors), $z \in H'$, $\zeta \in \mathbf{C}^w$,

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \mathrm{SL}(2, \mathbf{Z}).$$

Let $\Phi \in S_{w+2}(\Gamma)$. Then it follows directly from the definition of Φ that the differential form of degree $w + 1$

$$\Phi dz \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_w \tag{0.3}$$

is holomorphic on $H' \times \mathbf{C}^w$ ($\mathbf{C}^w = \mathbf{C} \times \cdots \times \mathbf{C}$ (w factors); $\zeta = (\zeta_1, \ldots, \zeta_w)$) and is invariant under the action of $\Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w$. Consequently, the form (0.3) defines a holomorphic differential ω_Φ of degree $w + 1$ on the open set $B_\Gamma^w|_{\Delta'} \subset B_\Gamma^w$. Since $\Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w$ acts freely, properly, and discretely on the analytic variety $H' \times \mathbf{C}^w$, the differential ω_Φ is extended to a holomorphic form on B_Γ^w and hence defines the map (0.1).

0.5. REMARKS. a) In this paper w is always assumed to be a natural number. The corresponding results for $w = 0$ have not been formulated. The author has not described them because they are classical.

b) The variety B^w is obtained from $\bar{B}^w$ by a resolution of singularities. In §4 the rationality of these singularities is proved. Therefore any resolution of $\bar{B}_\Gamma^w$ serves for Theorem 0.3. There is always some resolution defined by the results of Hironaka. In §3 we construct a canonical resolution.

§1. Elliptic surfaces

Denote by Δ a nonsingular algebraic curve. Let B denote a nonsingular algebraic surface with an elliptic structure. This means that a canonical projection $\Phi: B \to \Delta$ and a section $o: \Delta \to B$ are defined such that the general fiber of Φ is an elliptic curve.

Let E be an open subset of Δ with respect to the $\mathbf{C}$-topology (all topologies considered in the paper correspond to an analytic structure of varieties). By $B|_E$ we will denote the analytic space $\Phi^{-1}(E)$. If E is a real topological manifold with a boundary in Δ then $B|_E$ will denote the topological manifold with boundary $\Phi^{-1}(E)$. Moreover, we will denote by B_v a geometrical fiber of Φ over a point $v \in \Delta$. The *restriction* symbol | will be used also for arbitrary varieties with a projection on Δ.

Before we state the classification theorem on fibers of an elliptic surface we recall the meaning of the expression

$$B_v = \Theta_{v,0} + \sum_{i \geqslant 1} \mu_{v,i}\Theta_{v,i},$$

$\Theta_{v,i}$ are different components of the fiber, and $\mu_{v,i}$ are its multiplicities. We assume that $\Theta_{v,0}$ is the component containing the point $o(v)$, and that its multiplicity is equal to 1. Sometimes, if no confusion arises, the index v in $\Theta_{v,i}$ will be dropped. In the sequel we everywhere assume for the surface B that the $\Theta_{v,i}$ are not exceptional curves of the first kind and the functional invariant J is not constant.

1.1. THEOREM (KODAIRA). *The fiber B_v is one of the following types*:

I_0: $B_v = \Theta_0$, Θ_0 *being a nonsingular elliptic curve*;

I_1: $B_v = \Theta_0$, Θ_0 *being a rational curve with a node* q;

I_2: $B_v = \Theta_0 + \Theta_1$, Θ_0 *and* Θ_1 *being nonsingular rational curves*, $\Theta_0 \cdot \Theta_1 = q_1 + q_2$, *where* q_1 *and* q_2 *are two differential points*;

II: $B_v = \Theta_0$, Θ_0 *being a rational curve with a single singular point which is a cusp*;

III: $B_v = \Theta_0 + \Theta_1$, Θ_0 *and* Θ_1 *being nonsingular rational curves*, $\Theta_0 \cdot \Theta_1 = 2q$;

IV: $B_v = \Theta_0 + \Theta_1 + \Theta_2$, Θ_0, Θ_1 *and* Θ_2 *being nonsingular rational curves*, $\Theta_0 \cdot \Theta_1 = \Theta_0$ $\Theta_2 = \Theta_1 \cdot \Theta_2 = q$.

In the remaining cases all components Θ_i are nonsingular rational curves such that two different components Θ_i and Θ_j ($i < j$) meet each other at at most one point. Below only nontrivial intersection indices will be noted.

I_b: $B_v = \Theta_0 + \Theta_1 + \ldots + \Theta_{b-1}$ $(b \geqslant 3)$, $(\Theta_0 \cdot \Theta_1) = (\Theta_1 \cdot \Theta_2) = \ldots = (\Theta_{b-2} \cdot \Theta_{b-1}) = (\Theta_{b-1} \cdot \Theta_0) = 1$;

I_b^*: $B_v = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4 + \ldots + 2\Theta_{b+4}$ $(b \geqslant 0)$, $(\Theta_0 \cdot \Theta_4) = (\Theta_1 \cdot \Theta_4) = (\Theta_4 \cdot \Theta_5) = \ldots = (\Theta_{b+3} \cdot \Theta_{b+4}) = (\Theta_{b+4} \cdot \Theta_2) = (\Theta_{b+4} \cdot \Theta_3) = 1$;

II^*: $B_v = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 5\Theta_4 + 6\Theta_5 + 4\Theta_6 + 3\Theta_7 + 2\Theta_8$, $(\Theta_0 \cdot \Theta_1) = (\Theta_1 \cdot \Theta_2) = (\Theta_2 \cdot \Theta_3) = (\Theta_4 \cdot \Theta_5) = (\Theta_5 \cdot \Theta_6) = (\Theta_5 \cdot \Theta_7) = (\Theta_6 \cdot \Theta_8) = 1$;

III^*: $B_v = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 3\Theta_4 + 2\Theta_5 + 2\Theta_6 + \Theta_7$, $(\Theta_0 \cdot \Theta_1) = (\Theta_1 \cdot \Theta_2) = (\Theta_2 \cdot \Theta_3) = (\Theta_4 \cdot \Theta_5) = (\Theta_3 \cdot \Theta_6) = (\Theta_5 \cdot \Theta_7) = 1$;

IV^*: $B_v = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 2\Theta_3 + 2\Theta_4 + \Theta_5 + \Theta_6$, $(\Theta_0 \cdot \Theta_1) = (\Theta_1 \cdot \Theta_2) = (\Theta_2 \cdot \Theta_3) = (\Theta_2 \cdot \Theta_4) = (\Theta_3 \cdot \Theta_5) = (\Theta_4 \cdot \Theta_5) = 1$. □

The theorem is a direct corollary of the existence of the section o and of Theorem 6.2 of [3]. A fiber of type I_0 will be called *nonsingular*; other fibers are called *singular*. Moreover, a point $v \in \Delta$ will be named by the type of B_v. There exists a finite subset $\Sigma \subset \Delta$ which contains all points of singular type. In the sequel we assume furthermore that $J(\Sigma) \cap \{0, 1, \infty\} = \varnothing$, where J is the functional invariant of B.

In §§2 and 3 projective varieties $\bar{B}^w$ and B^w (the latter is nonsingular) will be constructed. Now we will define the surfaces $\bar{B}^1$ and B^1. B^1 is constructed from B by a sequence of monoidal transformations centered at points of fibers of types II, III and IV. In a fiber of type II we make the monoidal transformation centered at the singular point q. The new fiber over this point is of the form $\Theta_0 + 2\Theta_1$, the Θ_i being nonsingular rational curves with $\Theta_0 \cdot \Theta_1 = 2q'$. Making the monoidal transformation centered at q', we obtain a fiber of the form $\Theta_0 + 2\Theta_1 + 3\Theta_2$, the Θ_i being nonsingular rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_0 \cdot \Theta_2 = q''$. The monoidal transformation centered at q'' defines a surface B^1 over a point of type II. In the case of a fiber of type III we first make the monoidal transformation centered at q, the point of intersection of the components. Thus we obtain a fiber of the form $\Theta_0 + \Theta_1 + 2\Theta_2$, the Θ_i being nonsingular rational curves with $\Theta_0 \cdot \Theta_1 = \Theta_1 \cdot \Theta_2 = \Theta_0 \cdot \Theta_2 = q'$. The monoidal transformation centered at this point defines the surface B^1 over a point of type III. For points of type IV the surface B^1 is defined by the monoidal transformation centered at the point of intersection of the components. B^1 has a canonical projection $\varphi^1: B^1 \to \Delta$. The proof of the following proposition follows from the construction of the surface B^1.

1.2. PROPOSITION. *B^1 is a nonsingular projective surface. A fiber B_v^1 depending on the type of the point v has the following form*:

If v is of type I_b $(b \geqslant 0)$, I_b^* $(b \geqslant 0)$, II^*, III^* or IV^*, then $B_v^1 = B_v$.

In other cases all components Θ_i are nonsingular rational curves. Writing only nontrivial intersection indices, we have, if v is of type

$$\mathrm{II}: \ B_v^1 = \Theta_0 + 2\Theta_1 + 3\Theta_2 + 6\Theta_3,\ (\Theta_0 \cdot \Theta_3) = (\Theta_1 \cdot \Theta_3) = (\Theta_2 \cdot \Theta_3) = 1;$$
$$\mathrm{III}: \ B_v^1 = \Theta_0 + \Theta_1 + 2\Theta_2 + 4\Theta_3,\ (\Theta_0 \cdot \Theta_3) = (\Theta_1 \cdot \Theta_3) = (\Theta_2 \cdot \Theta_3) = 1;$$
$$\mathrm{IV}: \ B_v^1 = \Theta_0 + \Theta_1 + \Theta_2 + 3\Theta_3,\ (\Theta_0 \cdot \Theta_3) = (\Theta_1 \cdot \Theta_3) = (\Theta_2 \cdot \Theta_3) = 1. \ \square$$

The following proposition defines a normal projective surface $\bar{B}^1$.

1.3. PROPOSITION. *There exists a unique normal projective surface $\bar{B}^1$ which is obtained by blowing down connected components of the following curves (the type of the fiber B_v^1 corresponds to the type of the point v) on B^1:*

$\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_3$ *in a fiber of type* I_b^* $(b > 0)$;
$\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Theta_4 \cup \Theta_6 \cup \Theta_7 \cup \Theta_8$ *in a fiber of type* II*;
$\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_4 \cup \Theta_5 \cup \Theta_6 \cup \Theta_7$ *in a fiber of type* III*;
$\Theta_0 \cup \Theta_1 \cup \Theta_3 \cup \Theta_4 \cup \Theta_5 \cup \Theta_6$ *in a fiber of type* IV*;
$\Theta_0 \cup \Theta_1 \cup \Theta_2$ *in fibers of type* II, III, IV. □

In the notation of [3] (Chapter 8) we have

1.4. COROLLARY. a) *For points* v *of type* I_0, I_0^*, II, III, IV, II*, III* *or* IV* *there is an isomorphism of analytic spaces*

$$\overline{B}^1|_E \simeq C\diagdown F, \tag{1.1}$$

where C *is a cyclic group of order* $\kappa = \max\{\mu_{v,i}\}$

b) *For points* v *of type* I_b^* $(b \geqslant 1)$

$$\overline{B}^1|_E \simeq \{t\}\diagdown F, \tag{1.2}$$

where t *is an involution. In both cases* E *denotes a sufficiently small disc centered at the point* v.

PROOF. The construction of Chapter 8, iv, v, of [3] immediately implies the normality of the analytic spaces $C \setminus F$ and $\{t\} \setminus F$. Therefore the process of constructing $\overline{B}^1$ is inverse to the resolution of singularities (Chapter 8, iii, of [3]) of $C \setminus F$ and $\{t\} \setminus F$. On the other hand, the process of blowing down exceptional curves in fibers of type II, III and IV is inverse to the construction of the surface $\overline{B}^1$. □

The local descriptions of the surface $\overline{B}^1$ directly imply the existence of an analytic, and by §1.3 also an algebraic projection $\overline{\Phi}^1$ with the following commutative diagram:

$$\begin{tikzcd} B^1 \arrow[rr, "\Psi^1"] \arrow[dr, "\Phi^1"'] & & \overline{B}^1 \arrow[dl, "\overline{\Phi}^1"] \\ & \Delta & \end{tikzcd},$$

where Ψ^1 denotes the blowing down of Proposition 1.3.

Let G be the homological invariant of an elliptic surface B. Fix some point $u_0 \in \Delta' = \Delta - \Sigma$. Let us consider a closed path β on Δ' originating at u_0. Then the natural connection on $B|_{\Delta'}$ defines a homomorphism of the homology groups

$$s_\beta : H_1(B_{u_0}, \mathbf{Z}) \to H_1(B_{u_0}, \mathbf{Z}), \tag{1.3}$$

which corresponds to this path. Fix some negative definite basis e_1, e_2 in the group $H_1(B_{u_0}, \mathbf{Z})$ (that corresponds to the choice of periods $z \in H$, 1 of the elliptic curve B_{u_0}). Write the homomorphism s_β in the basis e_1, e_2 as a right action of a matrix $S_\beta \in \mathrm{SL}(2, \mathbf{Z})$. The given matrices define a representation

$$S : \pi_1(\Delta') \to \mathrm{SL}(2, \mathbf{Z}) \tag{1.4}$$

of the fundamental group $\pi_1(\Delta') = \pi_1(\Delta', u_0)$. Representation (1.4) uniquely defines the sheaf G. Each matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = S_\beta$, where β is a small positive circle centered at some point $v \in \Delta$, is conjugated in $\mathrm{SL}(2, \mathbf{Z})$ to one of the following matrices:

$$\pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} (b \geqslant 0), \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The last matrix is determined uniquely by the point v and is denoted by A_v. A_v will be called the *normal monodromy form* of an elliptic surface in the point $v \in \Delta$. The matrix

A_v determines the type of the point v (see Table 1), which is a direct corollary of the construction of Chapter 8 of [3].

TABLE 1

Type of a point v	I_b	I_b^*	II	III	IV	II*	III*	IV*
Normal form A_v	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$-\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

The arguments of the beginning of §8 of [3] easily imply the following result.

1.5. LEMMA. *For any rational function $J \in \mathbf{C}(\Delta) \setminus \mathbf{C}$ on Δ there exists an elliptic surface B over Δ with the functional invariant J.* □

PROOF OF PROPOSITION 1.3. By Artin's criterion (Theorem 2.9B of [1]) it suffices to prove the negative definiteness of the matrix $\|\Theta_i \cdot \Theta_j\|$, where the Θ_i are connected components of the blowing down Ψ^1. After a suitable permutation of Θ_i the intersection matrix has the following form:

$$\left.\begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots \;-2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix}\right\} n.$$

The corresponding quadratic form ($n \geqslant 1$)

$$-x_1^2 - x_n^2 - \sum_{k=1}^{n-1} (x_k - x_{k+1})^2$$

is negative definite. □

§2. Construction of the variety $\overline{B}^w$

Let w be some natural number. In this section a projective algebraic variety $\overline{B}^w$ equipped with a canonical projection $\overline{\Phi}^w\colon \overline{B}^w \to \Delta$ is constructed from B and w. In §§2.1 and 2.2 an analytic construction of $\overline{B}^w$ and $\overline{\Phi}^w$ is described. Theorem 2.3, using a noncanonical algebraic construction of $\overline{B}^w$, proves the projectivity.

2.1. Let U' be the universal covering of Δ' and $\tilde{z}\colon U' \to \Delta'$ a canonical projection (base point u_0). There is a multi-valued analytic function $z(u)$ defined on Δ' such that $j(z(u)) = J(u)$, where j is the absolute invariant. A choice of a negative definite basis e_1, e_2 of the group $H_1(B_{u_0}, \mathbf{Z})$ determines the choice of a branch of $z(u)$ at the point u_0. Let $z\colon U' \to H$, $\tilde{u} \mapsto z(\tilde{u})$ be the corresponding single-valued function on U'. Let S be the representation (1.4), and denote by $\beta\colon \tilde{u} \mapsto \beta\tilde{u}$, $\beta \in \pi_1(\Delta')$, the action of the fundamental group on the universal covering U'. Then

$$z(\beta\tilde{u}) = \frac{az(\tilde{u}) + b}{cz(\tilde{u}) + d} = S_\beta z(\tilde{u}), \tag{2.1}$$

where S_β + $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\beta \in \pi_1(\Delta')$. For $\beta \in \pi_1(\Delta')$ the analytic function

$$f_\beta(\tilde{u}) = (cz(\tilde{u}) + d)^{-1}$$

is defined on U', for which

$$f_{\beta\gamma}(\tilde{u}) = f_\beta(\gamma\tilde{u}) \cdot f_\gamma(\tilde{u})$$

for any $\beta, \gamma \in \pi_1(\Delta')$. Define an action of the group $\mathcal{G}^w = \pi_1(\Delta') \times \mathbf{Z}^w \times \mathbf{Z}^w$ (semidirect product) on the analytic variety $U' \times \mathbf{C}^w$ as follows:

$$(\beta, n, m) : (\tilde{u}, \zeta) \longmapsto (\beta\tilde{u}, f_\beta(\tilde{u})(\zeta + z(\tilde{u})n + m)), \tag{2.2}$$

where $\beta \in \pi_1(\Delta')$, $n, m \in \mathbf{Z}^w$, $\tilde{u} \in U'$ and $\zeta \in \mathbf{C}^w$. The group $\mathcal{G}^w$ acts properly, discretely and without fixed points. Hence the analytic variety

$$\bar{B}^w|_{\Delta'} = \mathcal{G}^w \backslash U' \times \mathbf{C}^w$$

is defined, which has a canonical projection $\bar{\Phi}^w|_{\Delta'}$ on Δ' induced by the map $(\tilde{u}, \zeta) \mapsto (\tilde{u})$. It is easy to prove the existence of the following isomorphism (the horizontal arrow) with a commutative diagram (Chapter 8, I, of [3]):

$$\begin{tikzcd} \bar{B}^w|_{\Delta'} \arrow[rr, "\simeq"] \arrow[dr, "\bar{\Phi}^w|_{\Delta'}"'] & & \underbrace{B|_{\Delta'} \times_{\Delta'} \cdots \times_{\Delta'} B|_{\Delta'}}_{w} \arrow[dl, "\underbrace{\Phi|_{\Delta'} \times_{\Delta'} \cdots \times_{\Delta'} \Phi|_{\Delta'}}_{w}"] \\ & \Delta & \end{tikzcd}$$

The rest of the construction of $\bar{B}^w$ consists in a compactification of $\bar{B}^w|_{\Delta'}$ over points of Σ and an analytic continuation of $\bar{\Phi}^w|_{\Delta'}$.

2.2. Let $u_i \in \Sigma = \{u_1, \ldots, u_t\}$. Denote by τ a local parameter at the point u_i and by E a small disk $|\tau| < \varepsilon$ on Δ; set $E' = E - u_i$.

(i) If u_i is one of the types I_0, I_0^*, II, II*, III, III*, IV or IV*, then the analytic variety F (see Corollary 1.4 and Chapter 8, ii, iv, of [3]) is an elliptic fibration over a disc $D = \{\sigma \in \mathbf{C} | \, |\sigma|^\kappa < \varepsilon\}$. Following Kodaira [3], we will denote points of F by $[\sigma, \zeta]$, where $\zeta \in \mathbf{C}$ is considered modulo the lattice $\mathbf{Z} + z(\sigma)\mathbf{Z}$ over D. Let F^w be the analytic variety

$$F^w = \underbrace{F \times_D \cdots \times_D F}_{w}.$$

We will also denote by $[\sigma, \zeta]$, $\zeta \in \mathbf{C}^w$, the points of F^w. For suitable τ, ζ and σ the action of the group C has the following form:

$$e_\kappa : [\sigma, \zeta] \mapsto [e_\kappa\sigma, f_1(\sigma)\zeta], \tag{2.3}$$

where $e_\kappa = e^{2\pi i/\kappa}$ is a generator of C and $f_1(\sigma)$ is an analytic function from Table 2. The functions $z(\sigma)$ describe the lattice which defines F^w.

TABLE 2

Fiber type $\to$	I_0	I_0^*	II	II*	III	III*	IV	IV*
$\varkappa$	1	2	6	6	4	4	3	3
$z(\sigma)$	—	—	$\frac{\eta - \eta^2\sigma^{2h}}{1-\sigma^{2h}}$ $h \equiv 1 \pmod 3$	$\frac{\eta - \eta^2\sigma^{2h}}{1-\sigma^{2h}}$ $h \equiv 2 \pmod 3$	$\frac{i + i\sigma^{2h}}{1-\sigma^{2h}}$ $h \equiv 1 \pmod 2$	$\frac{i + i\sigma^{2h}}{1-\sigma^{2h}}$ $h \equiv 1 \pmod 2$	$\frac{\eta - \eta^2\sigma^{h}}{1-\sigma^{h}}$ $h \equiv 2 \pmod 3$	$\frac{\eta - \eta^2\sigma^{h}}{1-\sigma^{h}}$ $h \equiv 1 \pmod 3$
$f_1(\sigma)$	1	-1	$-z(\sigma)^{-1}$	$(z(\sigma)+1)^{-1}$	$-z(\sigma)^{-1}$	$z(\sigma)^{-1}$	$-(z(\sigma)+1)^{-1}$	$z(\sigma)^{-1}$
$J(v)$	C	C	0	0	1	1	0	0

Here $h = \operatorname{ord}_v(J - J(v))$ and $\eta = e^{2\pi i/3}$. Formula (2.3) defines also an action of the group C on F^w in the coordinates $[\sigma, \zeta]$. C acts on F^w properly, discretely and with a finite number of fixed points.

Therefore the analytic space $\bar{B}^w|_E = C \setminus F^w$ is normal with a finite number of singular points. The function σ^κ, which is regular and invariant on F^w, induces a projection $\bar{\Phi}^w|_E$: $\bar{B}^w|_E \to E$. There exists an isomorphism (the horizontal arrow) with a commutative diagram:

$$\begin{tikzcd} \bar{B}^w|_{\Delta'}|_{E'} \arrow[rr, "\sim"] \arrow[dr, "\bar{\Phi}^w|_{\Delta'}|_{E'}"'] & & \bar{B}^w|_E|_{E'} \arrow[dl, "\bar{\Phi}^w|_E|_{E'}"] \\ & E' & \end{tikzcd} \tag{2.4}$$

Thus a compactification of $\bar{B}^w$ and an analytic continuation of the map $\bar{\Phi}^w|_{\Delta'}$ are defined over points with finite monodromy.

(ii) If u_i is of type I_b, $b \geqslant 1$, then the analytic space

$$\bar{B}^w|_E = \underbrace{B|_E \underset{E}{\times} \cdots \underset{E}{\times} B|_E}_{w}$$

is defined with a canonical projection

$$\bar{\Phi}^w|_E = \underbrace{\Phi|_E \underset{E}{\times} \cdots \underset{E}{\times} \Phi|_E}_{w}.$$

Since in this case we also have an analytic isomorphism (2.4), $\bar{B}^w|_E$ and $\bar{\Phi}^w|_E$ define a compactification of $\bar{B}^w|_{\Delta'}$ over points of type I_b.

(iii) In the case when the point u_i is of type I_b^*, $b \geqslant 1$, we let

$$F^w = \underbrace{F \underset{D}{\times} F \underset{D}{\times} \cdots \underset{D}{\times} F}_{w}.$$

The involution t (see Corollary 1.4 and Chapter 8 of [3]) determines an analytic involution on F^w. Hence the analytic quotient space $\bar{B}^w|_E = \{t\} \setminus F^w$ is defined. Moreover, since the projection is an invariant analytic function (equal to σ^2 in the Kodaira coordinates (σ, w) [3]), a projection $\bar{\Phi}^w|_E$ is defined. The pair $\bar{B}^w|_E$, $\bar{\Phi}^w|_E$ defines a compactification of $\bar{B}^w|_{\Delta'}$ over u_i.

2.3. THEOREM. *$\bar{B}^w$ is a projective algebraic variety. In particular, $\bar{\Phi}^w$ is an algebraic morphism.*

REMARK. The varieties $\bar{B}^1$ defined in 1.3, 2.1 and 2.2 coincide.

PROOF OF THEOREM 2.3. To each point u_i there corresponds a natural number $\kappa_i = \max\{\mu_{u_i,l}\}$. Let κ_0 be the least common multiple of $\kappa_1, \ldots, \kappa_t$, and let $d = \kappa_1 \cdots \kappa_t$. Then an abelian cover $\tilde{\Psi}$: $\tilde{\Delta} \to \Delta$ of degree d, with d/κ_i ramification points of index $\kappa_i - 1$ over the point u_i $(0 \leqslant i \leqslant t)$ and nonramified over $\Delta' - u_0$, is defined. The lifting of invariants $\tilde{\Psi}^*(J, G)$ defines an elliptic surface $\tilde{B}$ over $\tilde{\Delta}$ with fibers of type I_b $(b \geqslant 1)$ only. Let

$$\tilde{B}^w = \underbrace{\tilde{B} \underset{\tilde{\Delta}}{\times} \cdots \underset{\tilde{\Delta}}{\times} \tilde{B}}_{w}.$$

The abelian cover $\tilde{\Psi}$ defines a cover $\tilde{\Psi}^w$: $\tilde{B}^w \to \bar{B}^w$ with the following commutative diagram:

$$\begin{array}{ccc} \tilde{B}^w & \to & \bar{B}^w \\ \downarrow & & \downarrow \\ \tilde{\Delta} & \to & \Delta \end{array}$$

Therefore both $\bar{B}^w \simeq \tilde{C} \setminus \tilde{B}^w$ and $\tilde{B}^w$ ($\tilde{C}$ being the finite abelian group of the cover $\tilde{\Psi}^w$) are projective algebraic varieties. $\square$

§3. Kuga's variety

In this section a nonsingular projective variety B^w is defined by resolving singularities of the variety $\bar{B}^w$ constructed in §2. B^w is called *Kuga's variety associated with an elliptic surface B*. The canonical projection Φ^w: $B^w \to \Delta$ is the composition $\bar{\Phi}^w \circ \Psi^w$, where Ψ^w is a resolution of singularities $s(\bar{B}^w)$ of the variety $\bar{B}^w$. By $s(\mathfrak{M})$ we will denote the singular locus of an algebraic variety or analytic space $\mathfrak{M}$.

3.1. (i) Let μ and w be natural numbers, $\mu > 1$. There is an action

$$e_\mu : (\sigma, \zeta) \longmapsto (e_\mu \sigma, e_\mu^{-1} \zeta)$$

of the cyclic group C of order μ on the analytic variety $\mathbf{C} \times \mathbf{C}^w$, where $e_\mu = e^{2\pi i/\mu}$ is a generator of C. Let

$$\mathcal{N}_\mu^w = C \setminus \mathbf{C} \times \mathbf{C}^w.$$

The analytic space $\mathfrak{N}_\mu^w$ has one singular point, the orbit of (0, 0). Denote by N_μ^w an analytic space which is a connected open set containing the point $s(\mathfrak{N}_\mu^w)$ of $\mathfrak{N}_\mu^w$.

(ii) Just as in (i), the action

$$e_\mu : (\sigma, \varphi) \longmapsto (e_\mu \sigma, e_\mu^{-1} \zeta)$$

of the group C on $\mathbf{C} \times \mathbf{C}^w$ defines an analytic space $N^w_{-\mu}$ with one singularity, the orbit of (0, 0).

(iii) Let w_1 and w_2 be integers, $w_1 > 1$ and $w_2 \geqslant 0$. Denote by $X_1, Y_1, \ldots, X_{w_1}, Y_{w_1}$ and $Z_1, \ldots, Z_{w_2}$ coordinates on the variety $\mathbf{C}^{2w_1} \times \mathbf{C}^{w_2}$. Then the system of equations $X_1 \cdot Y_1 = \cdots = X_{w_1} \cdot Y_{w_1}$ defines an analytic space $\mathfrak{N}^{w_1, w_2}$ in $\mathbf{C}^{2w_1} \times \mathbf{C}^{w_2}$. A connected neighborhood of the point (0, 0) in $\mathfrak{N}^{w_1, w_2}$ will be denoted by N^{w_1, w_2}.

The analytic spaces $\mathfrak{N}^w_{\pm\mu}$, $N^w_{\pm\mu}$ and $\mathfrak{N}^{w_1, w_2}$, N^{w_1, w_2} are provided with projections $\Phi^w_{\pm\mu}$ and Φ^{w_1, w_2} on a neighborhood $U \subset \mathbf{C}$ of the origin, induced by maps σ^κ, $\mu | \kappa$ and $(X_1 Y_1)^\kappa$ respectively. Let M be an analytic space with a projection Φ: $M \to \Delta$. We will say that a point $q \in M$ is of type $(\pm\mu, w)_f$ or (w_1, w_2) if the following commutative diagram is defined:

$$\begin{array}{ccc} N & \subsetneq & M \\ {\scriptstyle \tilde{\Phi}}\downarrow & & \downarrow{\scriptstyle \Phi} \\ U & \subsetneq & \Delta \end{array}$$

where horizontal arrows are isomorphic imbeddings and q is contained in the image of the upper arrow, while $N = N^w_{\pm\mu}$, $N = N^{w_1, w_2}$ and $\tilde{\Phi} = \Phi^w_{\pm\mu}$, $\tilde{\Phi} = \Phi^{w_1, w_2}$ respectively. It is easy to show that the type of a point is uniquely defined except the case $(-2, w)_f = (2, w)_f$. Letting $\mu = 1$ in (i) or (ii) and $w_1 = 1$ or 0 in (iii), we introduce nonsingular types of points $(\pm 1, w)_f$, $(1, w_2)_i$ and $(0, w_2)_i$.(1) The type $(0, w_2)_i = (\pm 1, w)_f$ describes nonsingular points of M in which the projection Φ is regular at $\kappa = 1$.

3.2. Proposition. *Table 3 expresses the dependence of the type of a point of a fiber $\bar{B}_v^w$ on the type of point $v \in \Delta$.*

TABLE 3

Type of a point $v \in \Delta$	I_0, II, III, IV	$\overset{*}{\mathrm{I}}_0$, II*, III*, IV*	I_b, $b \geq 1$	$\overset{*}{\mathrm{I}}_b$, $b \geq 1$
Type of a point $q \in \bar{B}_v^w$	$(\mu, w)_f$, $\mu \mid \varkappa$	$(-\mu, w)_f$, $\mu \mid \varkappa$	$(w_1, w_2)_i$. $w_1 + w_2 = w$	$(w_1, w_2)_i$, $w_1 + w_2 = w$; $(\pm \mu, w)_f$, $\mu \mid \varkappa$ $(\varkappa = 2)$.

Denote by $s_i(\bar{B}^w)$ the reduced variety of singular points of type $(w_1, w_2)_i$. Let Ψ_i^w: $\bar{B}_i^w \to \bar{B}^w$ be the monoidal transformation centered at $s_i(\bar{B}^w)$. We will say that a point $q \in M$ is of *normal type* if a neighborhood of this point is represented by a neighborhood of the origin in $\mathbf{C}^w$ such that the projection Φ: $M \to \Delta$ is given by the function $Y_1^{n_1} \cdots Y_w^{n_w}$, where the n_i are positive integers and the Y_i are coordinates on $\mathbf{C}^w$. If each point of a fiber M_v is normal, we will say that the fiber is of *normal type*.

3.3. Theorem. *Ψ_i^w resolves singularities of type $(w_1, w_2)_i$. $\bar{B}_i^w$ is a projective variety with fibers of normal type over points of type I_b $(b \geq 0)$. Points of the remaining fibers are of normal or of finite type $(\pm \mu, w)_f$. (The canonical projection $\bar{\Phi}_i^w = \bar{\Phi}^w \circ \Psi_i^w$.)*

Let D_v be the reduced fiber of $\bar{B}_i^w$ over a point $v \in \Delta$. By the monoidal transformation over the point v we will mean the monoidal transformation of $\bar{B}_i^w$ centered at $(\kappa - 1)D_v$ if v is of type I_b $(b \geq 0)$, II, III or IV, or the simultaneous monoidal transformation of $\bar{B}_i^w$ centered at $D_v, \ldots, (\kappa - 1)D_v$ if v is of type I_b^* $(b \geq 0)$, II*, III* or IV*. Denote by Ψ_f^w the composition of monoidal transformations over all points $v \in \Delta$. The last transformation does not touch fibers of type I_b $(b \geq 0)$. Let $\Psi^w = \Psi_i^w \circ \Psi_f^w$ and denote by B^w the image of this transformation. B^w has a canonical projection $\Phi^w = \bar{\Phi}^w \circ \Psi^w$.

3.4. Theorem. *B^w is a nonsingular projective variety with each fiber of normal type.*

Theorem 3.3 permits us to reduce the investigation of the transformation Ψ^w to the local case; that is, to consider it as being defined on an analytic space $N_{\pm\mu}^w$.

Let $((X, Y_1, \ldots, Y_w), (u_1 \colon v_1 \colon \cdots \colon v_w))$ be coordinates in the space $\mathbf{C}^{w+1} \times \mathbf{P}^w$. Then the regular map

$$\begin{gathered}\mathcal{N}_\mu^w - s(\mathcal{N}_\mu^w) \to \mathbf{C}^{w+1} \times \mathbf{P}^w, \\ \text{the orbit of } (\sigma, \zeta_1, \ldots, \zeta_w) \mapsto ((\sigma^\mu, \zeta_1^\mu, \ldots, \zeta_w^\mu), (\sigma : \zeta_1 : \ldots : \zeta_w))\end{gathered} \tag{3.1}$$

is defined. Let $\mathfrak{N}$ denote the closure of the image of the map (3.1).

3.5. Lemma. a) (3.1) *is a biregular imbedding.*
b) $\mathfrak{N}$ *is a nonsingular variety.*

Proof. a) The injectivity of (3.1) is obvious, and the regularity of the inverse map follows from the fact that any branch of the function $\sqrt[\mu]{z}$, $z \in \mathbf{C}$, is regular on a small neighborhood of a nonzero point.

(1)This is $\mathbf{C}^{w_2+1}$ with projection Z_1^κ, Z_1 being the first coordinate.

b) The charts $W_0 = \{u_1 \neq 0\}$; $W_j = \{v_j \neq 0\}$ $(1 \leqslant j \leqslant w)$ cover $\mathfrak{N}$.

In these charts $\mathfrak{N}$ is given in coordinates $X, v_j/u_1$; $Y_j, u_1/v_j, v_l/v_j$ $(1 \leqslant l \leqslant w, l \neq j)$ by the equations

$$W_0 \cap \mathcal{N}: \; Y_j = X \cdot \left(\frac{v_j}{u_1}\right)^{\mu}, \tag{3.2}$$

$$W_j \cap N: \begin{cases} X = Y_j \left(\dfrac{u_1}{v_j}\right)^{\mu} \\ Y_l = Y_j \left(\dfrac{v_l}{v_j}\right)^{\mu}. \end{cases} \tag{3.3}$$

Hence $\mathfrak{N}$ is nonsingular. $\square$

By this lemma a birational map

$$\mathcal{N} \dashrightarrow \mathcal{N}^{w}_{\mu}, \tag{3.4}$$

is defined and is inverse to (3.1). Let D be the fiber over the origin for some projection Φ^w_μ.

3.6. Lemma. *The map* (3.4) *is extended to a regular one* $\mu|\kappa$ *which is the monoidal transformation centered at* $(\kappa - 1)D$.

Consider the following map into the space $\mathbf{C}^{w+1} \times (\mathbf{P}^w)^{\mu-1}$ with coordinates $((X, Y_1, \dots, Y_w), (u_1: v_{1,1}: \dots :v_{1,w}), \dots, (u_{\mu-1}: v_{\mu-1,1}: \dots : v_{\mu-1,w}))$:

$$\begin{gathered} \mathcal{N}^{w}_{-\mu} - s\,(\mathcal{N}^{w}_{-\mu}) \to \mathbf{C}^{w+1} \times (\mathbf{P}^{w})^{\mu-1}, \\ \text{the orbit of } (\sigma, \zeta_1, \dots, \zeta_w) \longmapsto ((\sigma^{\mu}, \zeta_1^{\mu}, \dots, \zeta_w^{\mu}), (\sigma : \zeta_1^{\mu-1} : \dots : \zeta_w^{\mu-1}), \\ \dots, (\sigma^{\mu-1} : \zeta_1 : \dots : \zeta_w)). \end{gathered} \tag{3.5}$$

Denote by $\mathfrak{N}$ the closure of the image of the map (3.5) (this will not lead to confusion with the preceding or the following, since we will always indicate the map to which $\mathfrak{N}$ corresponds).

3.7. Lemma. a) (3.5) *is a biregular imbedding.*

b) $\mathfrak{N}$ *is a nonsingular variety.*

Proof. It is evident that the open sets $\{v_{ij} \neq 0\}$ $(I \leqslant j \leqslant w)$, $\{u_{i-1}v_{i,j} \neq 0\}$ $(2 \leqslant i \leqslant \mu - 1)$ and $\{u_{\mu-1} \neq 0\}$ cover $\mathfrak{N}$. It follows from the relations $(i \leqslant n \leqslant \mu - 1)$

$$u_n^i = v_{n,j}^i \cdot \left(\frac{u_i}{v_{i,j}}\right)^n \cdot Y_j^{n-i}, \quad v_{n,l}^{\mu-i} = v_{n,j}^{\mu-i} \cdot \left(\frac{v_{i,l}}{v_{i,j}}\right)^{\mu-n}$$

which hold for points of the set $\mathfrak{N} \cap \{v_{i,j} \neq 0\}$, where $1 \leqslant l \leqslant w$, $l \neq j$, that there exists an inclusion

$$\mathcal{N} \cap \{v_{i,j} \neq 0\} \subset \bigcap_{i \leqslant n \leqslant \mu-1} \{v_{n,j} \neq 0\}. \tag{3.6}$$

By analogy the relation $(1 \leqslant k \leqslant i - 1)$

$$v_{k,j}^i = u_k^{i-1} \cdot \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^k \cdot Y_j^{i-k-1}$$

for points of $\mathfrak{N} \cap \{u_{i-1} \neq 0\}$ implies the inclusion

$$\mathcal{N} \cap \{u_{i-1} \neq 0\} \subset \bigcap_{1 \leqslant k \leqslant i-1} \{u_k \neq 0\}. \tag{3.7}$$

Let us introduce the following charts on the variety $\mathbf{C}^{w+1} \times (\mathbf{P}^w)^{\mu-1}$:

$$W_j^i = \Big(\bigcap_{1 \leqslant k \leqslant i-1} \{u_k \neq 0\}\Big) \cap' \Big(\bigcap_{i \leqslant n \leqslant \mu-1} \{v_{n,j} \neq 0\}\Big).$$

Since $W_j^\mu = W_l^\mu$, for brevity we denote this chart by W^μ. The inclusions (3.6) and (3.7) imply that $\mathfrak{N}$ is covered by charts W_j^i ($1 \leqslant i \leqslant \mu - 1$) and W^μ. In each chart we distinguish some subsystem of coordinates and relations which express other coordinates of points of $\mathfrak{N}$ via the distinguished ones:

$$\begin{gathered} W_j^1:\ Y_j,\ u_1/v_{1,j},\ v_{\mu-1,l}/v_{\mu-1,j},\ X = \left(\frac{u_1}{v_{1,j}}\right)^\mu \cdot Y_j^{\mu-1},\ Y_l = \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right)^\mu \cdot Y_j, \\ \frac{u_n}{v_{n,j}} = \left(\frac{u_1}{v_{1,j}}\right)^n \cdot Y_j^{n-1},\quad \frac{v_{n,l}}{v_{n,j}} = \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right)^{\mu-n}; \end{gathered} \tag{3.8}$$

$$\begin{gathered} W_j^i\ (2 \leqslant i \leqslant \mu-1):\ v_{i-1,j}/u_{i-1},\ u_i/v_{i,j},\ v_{\mu-1,l}/v_{\mu-1,j}, \\ X = \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^{\mu-i} \cdot \left(\frac{u_i}{v_{i,j}}\right)^{\mu-i+1},\quad Y_j = \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^{i} \cdot \left(\frac{u_i}{v_{i,j}}\right)^{i-1}, \\ Y_l = \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right)^{\mu} \cdot \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^{i} \cdot \left(\frac{u_i}{v_{i,j}}\right)^{i-1}, \\ \frac{v_{k,j}}{u_k} = \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^{i-k} \cdot \left(\frac{u_i}{v_{i,j}}\right)^{i-k-1},\quad \frac{v_{k,l}}{u_k} = \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right)^{\mu-k} \cdot \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^{i-k} \cdot \left(\frac{u_i}{v_{i,j}}\right)^{i-k-1}, \end{gathered} \tag{3.9}$$

$$\begin{gathered} \frac{u_n}{v_{n,j}} = \left(\frac{v_{i-1,j}}{u_{i-1}}\right)^{n-i} \cdot \left(\frac{u_i}{v_{i,j}}\right)^{n-i+1},\quad \frac{v_{n,l}}{v_{n,j}} = \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right)^{\mu-n}; \\ W^\mu:\ X,\ v_{\mu-1,j}/u_{\mu-1},\ Y_j = X^{\mu-1} \cdot \left(\frac{v_{\mu-1,j}}{u_{\mu-1}}\right)^{\mu},\ \frac{v_{k,j}}{u_k} = X^{\mu-k-1} \cdot \left(\frac{v_{\mu-1,j}}{u_{\mu-1}}\right)^{\mu-k}. \end{gathered} \tag{3.10}$$

Hence $\mathfrak{N}$ is nonsingular. The injectivity of (3.5) is obvious, and the regularity of the inverse map follows from the fact that the image of the map (3.5) lies in charts W_j^1 and W^μ with distinguished coordinates (3.8), (3.10) and the branch of the functions $\sqrt[\mu]{Y_j}$ and $\sqrt[\mu]{X}$ is regular on the corresponding chart. □

By virtue of this lemma a birational map inverse to (3.5)

$$\mathcal{N} \dashrightarrow \mathcal{N}^w_{-\mu} \tag{3.11}$$

is defined. Let D be a fiber for some projection $\Phi^w_{-\mu}$.

3.8. LEMMA. *The map* (3.11) *can be extended to a regular one* $\mu|\kappa$ *which is the simultaneous monoidal transformation with centers at* $D, \dots, (\kappa - 1)D$.

PROOF OF LEMMAS 3.6 AND 3.8. a) Let us prove first of all that the maps (3.4) and (3.11) can be extended to regular ones. The local ring of the singular point of the spaces $\mathfrak{N}^w_\mu$ and $\mathfrak{N}^w_{-\mu}$ is generated respectively by the functions

$$\sigma^{i_0} \cdot \zeta_1^{i_1} \dots \zeta_w^{i_w},$$

where $i_0, \dots, i_w$ are positive integers with $i_0 + \cdots + i_w = \mu$, and by

$$\sigma^{\mu}, \sigma\zeta_1, \ldots, \sigma\zeta_w, \zeta_1^{i_1} \ldots \zeta_w^{i_w},$$

where $i_1, \ldots, i_w$ are positive integers with $i_1 + \cdots + i_w = \mu$. Therefore it suffices to check the regularity of these functions on $\mathfrak{N}$. The expression of the functions in charts covering $\mathfrak{N}$ produced below proves their regularity:

$$W_0 \cap \mathcal{N}: \sigma^{i_0} \cdot \zeta_1^{i_1} \ldots \zeta_w^{i_w} = X \cdot \left(\frac{v_1}{u_1}\right)^{i_1} \ldots \left(\frac{v_w}{u_1}\right)^{i_w};$$

$$W_j \cap \mathcal{N}: \sigma^{i_0} \cdot \zeta_1^{i_1} \ldots \zeta_w^{i_w} = \left(\frac{u_1}{v_j}\right)^{i_0} \ldots \left(\frac{v_{j-1}}{v_j}\right)^{i_{j-1}} \cdot Y_j \left(\frac{v_{j+1}}{v_j}\right)^{i_{j+1}} \ldots \left(\frac{v_w}{v_j}\right)^{i_w};$$

$$W_j^i \cap \mathcal{N}: \sigma^{\mu} = X, \sigma\zeta_l = \sigma\zeta_j \cdot \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right), \quad \zeta_1^{i_1} \ldots \zeta_w^{i_w} = \prod_{l \neq j, l=1}^{w} \left(\frac{v_{\mu-1,l}}{v_{\mu-1,j}}\right)^{i_l} \cdot Y_j;$$

$$W_j^1 \cap \mathcal{N}: \sigma\zeta_j = Y_j \cdot \left(\frac{u_1}{v_{1,j}}\right);$$

$$W_j^i \cap \mathcal{N} (i > 1): \sigma\zeta_j = \left(\frac{v_{i-1,j}}{u_{i-1}}\right) \cdot \left(\frac{u_i}{v_{i,j}}\right); \quad \sigma^{\mu} = X; \quad \sigma\zeta_j = X \cdot \left(\frac{v_{\mu-1,j}}{u_{\mu-1}}\right);$$

$$W^{\mu} \cap \mathcal{N}: \zeta_1^{i_1} \ldots \zeta_w^{i_w} = X^{\mu-1} \cdot \prod_{j=1}^{w} \left(\frac{v_{\mu-1,j}}{u_{\mu-1}}\right)^{i_j}.$$

Let $\mathfrak{N}' \to \mathfrak{N}_{\pm\mu}^{w}$ be the monoidal transformation corresponding to Lemmas 3.6 and 3.8. Then $\mathfrak{N}'$ is the closure of the image of $\mathfrak{N}_{\pm\mu}^{w} - D$ for the following maps relative to $(+, -)$:

$$\text{the orbit of } (\sigma, \zeta) \mapsto (\text{the orbit of } (\sigma, \zeta), (\sigma^{\varkappa} : \sigma^{\varkappa-1}\zeta_1 : \ldots : \sigma^{\varkappa-1}\zeta_w));$$

$$\text{the orbit of } (\sigma, \zeta) \mapsto (\text{the orbit of } (\sigma, \zeta), (\sigma^{f(\mu+1)} : \ldots : \sigma^{f\mu+\mu-1} \cdot \zeta_1^{k_1} \ldots \zeta_w^{k_w} : \ldots));$$

where $k_1 + \cdots + k_w = \mu - i$, $1 \leqslant i \leqslant \mu - 1$, $0 \leqslant f < \kappa/\mu$. From the relation

$$(\zeta_1^{k_1} \ldots \zeta_w^{k_w})^{\mu-i} = (\zeta_1^{\mu-i})^{k_1} \ldots (\zeta_w^{\mu-i})^{k_w}$$

follows the existence of a regular map $\mathfrak{N}' \to \mathfrak{N}$ for the following commutative diagram:

$$\begin{tikzcd} \mathcal{N}' \arrow[rr] \arrow[dr] & & \mathcal{N} \arrow[dl] \\ & \mathcal{N}_{\pm\mu}^{w} & \end{tikzcd} \tag{3.12}$$

b) All fibers of $\mathfrak{N}$ for the projection $\sigma^{\kappa} = X^{\kappa/\mu}$ are of normal type, as follows obviously from inspection of the charts (3.2), (3.3) and (3.8)–(3.10). Therefore the functorial properties of monoidal transformations yield the existence of the arrow inverse to the horizontal one in diagram (3.12). □

PROOF OF THEOREM 3.4. Theorem 3.4 is a direct corollary of Theorem 3.3, Lemmas 3.5–3.8 and the normality of fibers of $\mathfrak{N}$ at the canonical map $X^{\kappa/\mu}$ (see item b) of the proof of Lemmas 3.6 and 3.8). □

Let $(u_i^{i,j}: v_i^{i,j}: u_j^{i,j}: v_j^{i,j})$ be coordinates in the space $(\mathbf{P}^3)^{w_1(w_1-1)/2}$ (the upper indices i, j are symmetric). Then we can define the regular map

$$\mathcal{N}^{w_1,w_2} - s(\mathcal{N}^{w_1,w_2}) \to \mathcal{N}^{w_1,w_2} \times (\mathbf{P}^3)^{\frac{w_1(w_1-1)}{2}}, \tag{3.13}$$
$$(X_1, Y_1, \ldots, X_{w_1}, Y_{w_1}, Z_1, \ldots) \longmapsto ((X_1, Y_1, \ldots, Z_1, \ldots), \ldots, (X_i{:}Y_i{:}X_j{:}Y_j)),$$

where $1 \leqslant i < j \leqslant w_1$. Let $\mathfrak{N}$ denote the closure of the image of (3.13).

3.9. LEMMA. a) (3.13), *being obviously a biregular imbedding, defines a regular extension of the inverse birational map*

$$\mathcal{N} \dashrightarrow \mathcal{N}^{w_1,w_2} \tag{3.14}$$

The map (3.14) *is the monoidal transformation centered at* $s(N^{w_1,w_2})$.

b) $\mathfrak{N}$ *is a nonsingular variety.*

c) *All fibers of* $\mathfrak{N}$ *for the canonical projection* $(X_1Y_1)^\kappa$ *are of normal type.*

PROOF OF THEOREM 3.3. Theorem 3.3 is a direct corollary of Proposition 3.2 and Lemma 3.9. □

In the proof of Lemma 3.9 we may obviously assume that $w_1 = w$ and $w_2 = 0$. Let $x_0 \in \mathfrak{N}$. Define the oriented graph $\Gamma(x_0)$ whose vertices are integers $1, 2, \ldots, w$ such that vertices i and j are joined by an edge $\overrightarrow{ij}$ if $u_j^{i,j} \neq 0$ or $v_j^{i,j} \neq 0$. An ordered collection of points $i_1, \ldots, i_l \in \Gamma(x_0)$ is called *linearly ordered at the point* x_0 if $\overrightarrow{i_k i_{k+1}} \in \Gamma(x_0)$, $k = 1, \ldots, l-1$.

3.10. LEMMA. a) *For a point* $x_0 \in \mathfrak{N}$, $\overrightarrow{ij} \in \Gamma(x_0)$ *and* $\overrightarrow{jk} \in \Gamma(x_0)$ *imply* $\overrightarrow{ik} \in \Gamma(x_0)$. *More exactly, if* $u_j^{i,j} \neq 0$ *and* $u_k^{jk} \neq 0$, *then* $u_k^{i,k} \neq 0$.

b) *For any point* $x_0 \in \mathfrak{N}$ *a linearly ordered collection including all vertices of the graph* $\Gamma(x_0)$ *is defined.*

PROOF OF LEMMA 3.9. b) In view of Lemma 3.10, for any point $x_0 \in \mathfrak{N}$ a linearly ordered collection consisting of all points $1, 2, \ldots, w$ is defined. By symmetry of the pairs X_i, Y_i on $\mathfrak{N}$ we may assume that the given collection is $1, 2, \ldots, w$. By virtue of symmetry within the pair X_i, Y_i we will assume that at a point $x_0 \in \mathfrak{N}$, and hence in its neighborhood, we have $u_l^{k,l} \neq 0$, where $1 \leqslant k < l \leqslant w$ ($u_{k+1}^{k,k+1} \neq 0$ by the definition of linear order at the point x_0, and the remaining inequalities hold by Lemma 3.10 a)). Then local coordinates X_w, $u_{l-1}^{l-1,l}/u_l^{l-1,l}$ $(2 \leqslant l \leqslant w)$, $v_1^{1,2}/u_2^{1,2}$ are defined in the point x_0 of $\mathfrak{N}$. In fact, $\mathfrak{N}$ can be given in the chart $\bigcap\{u_l^{k,l} \neq 0\} \subset N^{w_1,w_2} \times (\mathbf{P}^3)^{w_1(w_1-1)/2}$ by the following relations:

$$\begin{gathered}
X_{l-1} = X_l \cdot \left(\frac{u_{l-1}^{l-1,l}}{u_l^{l-1,l}}\right), \quad Y_1 = X_2 \cdot \left(\frac{v_1^{1,2}}{u_2^{1,2}}\right), \quad Y_l = Y_{l-1} \cdot \left(\frac{u_{l-1}^{l-1,l}}{u_l^{l-1,l}}\right), \\
\frac{v_l^{k,l}}{u_l^{k,l}} = \left(\frac{v_k^{k,l}}{u_l^{k,l}}\right) \cdot \left(\frac{u_k^{k,l}}{u_l^{k,l}}\right), \quad \frac{v_k^{k,l}}{u_l^{k,l}} = \left(\frac{v_k^{k,l-1}}{u_{l-1}^{k,l-1}}\right) \cdot \left(\frac{u_{l-1}^{l-1,l}}{u_l^{l-1,l}}\right), \\
\frac{v_k^{k,k+1}}{u_{k+1}^{k,k+1}} = \left(\frac{v_{k-1}^{k-1,k}}{u_k^{k-1,k}}\right) \cdot \left(\frac{u_{k-1}^{k-1,k}}{u_k^{k-1,k}}\right) \cdot \left(\frac{u_k^{k,k+1}}{u_{k+1}^{k,k+1}}\right), \quad \frac{u_k^{k,l}}{u_l^{k,l}} = \left(\frac{u_k^{k,l-1}}{u_{l-1}^{k,l-1}}\right) \cdot \left(\frac{u_{l-1}^{l-1,l}}{u_l^{l-1,l}}\right).
\end{gathered} \tag{3.15}$$

c) Since the canonical projection is of the form

$$(X_1 \cdot Y_1)^\kappa = \left[\left(\frac{u_1^{1,2}}{u_2^{1,2}}\right) \cdot \left(\frac{v_1^{1,2}}{u_2^{1,2}}\right) \times \prod_{l=3}^{w} \left(\frac{u_{l-1}^{l-1,l}}{u_l^{l-1,l}}\right)^2 \times X_w^2\right]^\kappa, \tag{3.16}$$

in a neighborhood of the point x_0, we have that all fibers are of normal type.

a) Now note that by the monoidal transformation centered at $s(\mathfrak{N}^{w_1,w_2})$ we mean the simultaneous monoidal transformation centered at components of $s(\mathfrak{N}^{w_1,w_2})$. Then a) obviously follows from the explicit equations of the components

$$X_i = Y_i = X_j = Y_j,$$

where $1 \leqslant i < j \leqslant w_1$. □

PROOF OF PROPOSITION 3.2. a) Consider a point $v \in \Delta'$. By the definition of Δ' this point is of type I_0. From §2.1 it directly follows that the fiber $B_v^w = \overline{B}_v^w$ is of nonsingular type; that is, $(0, w)_i = (\pm 1, w)_f$.

b) Suppose now that the point $v = u_i \in \Sigma$ is one of the types I_0, I_0^*, II, II*, III, III*, IV or IV*. Let $p \in F^w$, and let $C^{\kappa/\mu}$ be the stationary subgroup of this point (see §2.2 (i)). In a neighborhood of p let us introduce coordinates (depending on the type of the point v):

$$\begin{aligned} \mathrm{I}_0, \mathrm{I}_0^*:&\ \zeta \rightsquigarrow \zeta - p, \\ \mathrm{II}, \mathrm{II}^*, \mathrm{III}, \mathrm{III}^*:&\ \zeta \rightsquigarrow (1-\sigma^{2h})(\zeta - p), \\ \mathrm{IV}, \mathrm{IV}^*:&\ \zeta \rightsquigarrow (1-\sigma^{h})(\zeta - p). \end{aligned} \tag{3.17}$$

The action of the group $C^{\kappa/\mu}$ takes the following form in the coordinates (3.17):

$$e_\varkappa^{\varkappa/\mu} = e_\mu : (\sigma, \zeta) \longmapsto (e_\mu \sigma, e_\mu^{\pm} \zeta), \tag{3.18}$$

where + corresponds to types I_0, II, III, IV, and − corresponds to types I_0^*, II*, III*, IV*. Therefore the image of p in the quotient space $C \setminus F^w = \overline{B}^w|_E$ is of the form $(\pm\mu, w)_f$. The proof of (3.18) is a direct calculation using the definition of the action (2.3) and Table 2.

c) If $v = u_i \in \Sigma$ is of type I_b ($b \geqslant 1$), then all points except a finite number (the points $q_0, \ldots, q_{b-1}$ of the intersection of components of the fiber, or of the selfintersection if $b = 1$) are of nonsingular type $(0, 1)_i$. The points q_i are of type $(1, 0)_i$. Hence the type of each point of the fiber

$$\overline{B}_v^w = \underbrace{B_v \times \ldots \times B_v}_{w}$$

is a fiber product of types $(1, 0)_i$ and $(0, 1)_i$ with projections $X_1 Y_1$ and Z_1 respectively; that is, of type $(w_1, w_2)_i$, and $w_1 + w_2 = w$. Note that points of $(B_v - \{q_i\})^w \subset \overline{B}_v^w$ are of nonsingular type $(0, w)_i$.

d) Now we analyze the case when $v = u_i \in \Sigma$ is of type I_b^* ($b \geqslant 1$). F is an elliptic family over $D = \{\sigma \in \mathbf{C} \mid |\sigma|^2 < \varepsilon\}$ with one singular fiber F_0 of type I_{2b}. The involution t has four fixed points on F such that at a suitable choice of coordinates ($\zeta_{\lambda,\nu}$; see Chapter 8, v(2), of [3]) the action of t has the following form in a neighborhood of these points:

$$t : (\sigma, \zeta_{\lambda,\nu}) \longmapsto (-\sigma, -\zeta_{\lambda,\nu}).$$

Using the definition of $\overline{B}^w|_E$ from §2.2 (iii), the case considered above (see b)) and the fact that $\kappa = 2$, we obtain the last column of Table 3. □

PROOF OF LEMMA 3.10. b) For any two points i, j of the graph $\Gamma(x_0)$ one of the edges $\overrightarrow{ij}$ or $\overrightarrow{ji}$ is defined. Item b) of the lemma is a direct corollary of this property and of item a).

Item a) will be proved in its second formulation. We have

$$u_i^{i,k} = u_k^{i,k} \cdot \left(\frac{u_j^{j,k}}{u_k^{j,k}}\right) \cdot \left(\frac{u_i^{i,j}}{u_j^{i,j}}\right); \quad v_k^{i,k} = u_k^{i,k} \cdot \left(\frac{v_k^{j,k}}{u_k^{j,k}}\right); \quad v_i^{i,k} = u_k^{i,k} \cdot \left(\frac{v_i^{i,j}}{u_j^{i,j}}\right) \cdot \left(\frac{u_j^{j,k}}{u_k^{j,k}}\right)$$

for the point $x_0 \in \mathfrak{N}$. Therefore $u_k^{i,k} \neq 0$. □

3.11. Remarks. a) It is easy to verify that the B^1 defined in §1 coincides with the one defined in this section.

b) It is easy to verify also, using local coordinates, that Φ^w has a section $o^w\colon \Delta \to B^w$, which is induced by the map $\tilde{u} \mapsto (\tilde{u}, 0)$ (see §2.1).

§4. Regular differential forms of highest degree on B^w

4.1. Definition. A differential form of degree i which is analytical in all nonsingular points of an analytic space M will be called in this paper a *regular differential form* of degree i. Denote by $H^0r(M, \Omega^i)$ the space of regular differential forms of degree i. It is evident that for analytic varieties one has a canonical isomorphism $H^0(M, \Omega^i) \rightrightarrows H^0r(M, \Omega^i)$.

Let $\Psi\colon M \to \overline{M}$ be a resolution of singularities of an analytic space $\overline{M}$. Then a canonical monomorphism

$$\Psi_*\colon H^0(M, \Omega^n) \to H^0r(\overline{M}, \Omega^n)$$

is defined. In the sequel we will assume that M and $\overline{M}$ are compact. Let $\dim M = \dim \overline{M} = n$. We will say that $\overline{M}$ has only *rational singularities* if Ψ_* is an isomorphism.

The next result shows that the definition of rationality is independent of the choice of the resolution Ψ (if such exists).

4.2. Proposition. *$\overline{M}$ has only rational singularities if and only if*

a) *$\overline{M}$ has a compact resolution, and*

b) *$\int_{\overline{M}} \omega \wedge \overline{\omega} < \infty$ for any form $\omega \in H^0r(\overline{M}, \Omega^n)$ (the integral is improper).*

This proposition can be easily deduced from the proof of Theorem 3.1 of [2]. □

The last proposition will be used by us for $\overline{M} = \overline{B}^w$.

4.3. Lemma. *If $\omega \in H^0r(\overline{B}^w, \Omega^{w+1})$, then $\int_{\overline{B}^w} \omega \wedge \overline{\omega} < \infty$.*

4.4. Corollary. *$\overline{B}^w$ has only rational singularities; that is, there is a canonical isomorphism*

$$H^0(B^w, \Omega^{w+1}) \underset{\Psi^{w*}}{\overset{\Psi^w{}_*}{\rightleftarrows}} H^0r(\overline{B}^w, \Omega^{w+1}).$$

Proof. In view of Proposition 4.2 it suffices to verify a) and b). Assertion b) is Lemma 4.3, proved below; a) is a classical result of Hironaka on the resolution of singularities, which was proved in our situation in §3. □

4.5. Let us consider an analytic surface W' with an analytic function $z\colon W' \to H$ on which the group $\mathcal{G}$ acts properly, discretely and without fixed points. We will assume additionally that a representation $S\colon \mathcal{G} \to \mathrm{SL}(2, \mathbf{Z})$ is given and z satisfies the functional equation (2.1); that is, for any $\beta \in \mathcal{G}$

$$\beta^* z(\tilde{u}) = \frac{az(\tilde{u}) + b}{cz(\tilde{u}) + d} = S_\beta z(\tilde{u}),$$

where $S_\beta = \binom{a\ b}{c\ d} \in \mathrm{SL}(2, \mathbf{Z})$ and $\tilde{u} \in W'$. Then by analogy with §2.1 for any natural w the action (2.2) of the group $\mathcal{G}^w = \mathcal{G} \times \mathbf{Z}^w \times \mathbf{Z}^w$ on $W' \times \mathbf{C}^w$ is defined. There are analytic functions z and ζ_i, where ζ_i: $W' \times \mathbf{C}^w \to \mathbf{C}$ is the projection on the ith factor of $\mathbf{C}^w$, $1 \leqslant i \leqslant w$, defined on $W' \times \mathbf{C}^w$ in a natural way. ζ_i defines the real analytic functions

$$\xi_{1,i},\ \xi_{2,i} : W' \times \mathbf{C}^w \to \mathbf{R}, \qquad \zeta_i = \xi_{1,i} + z\xi_{2,i}.$$

4.6. LEMMA. *Under the assumptions of* §4.5,

a) $dz \wedge \bigwedge_{j=1}^{w} d\zeta_j = dz \wedge \bigwedge_{j=1}^{w} (d\xi_{1,j} + zd\xi_{2,j})$.

For any $(\beta, n, m) \in \mathcal{G}^w$ *the following relations hold*:

b) $(\beta, n, m)^* (z - \bar{z}) = f_\beta \cdot \bar{f}_\beta (z - \bar{z})$,

c) $(\beta, n, m)^* dz = f_\beta^2 dz$,

d) $(\beta, n, m)^* (d\xi_{1,j} + zd\xi_{2,j}) = f_\beta (d\xi_{1,j} + zd\xi_{2,j})$.

Since $\mathcal{G}^w$ acts properly, discretely and without fixed points on $W' \times \mathbf{C}^w$, the analytic variety $\mathcal{G}^w \setminus W' \times \mathbf{C}^w$ is defined.

Let ω be a holomorphic differential form of degree $w + 1$ on this variety. Then the lifting $\tilde{\omega}$ of the differential form ω on $W' \times \mathbf{C}^w$ has the form

$$\tilde{\omega} = \Phi' d\tilde{u} \wedge \bigwedge_{j=1}^{w} d\zeta_j,$$

where Φ' is a holomorphic function on $W' \times \mathbf{C}^w$. This differential can be written also in the form

$$\tilde{\omega} = \Phi dz \wedge \bigwedge_{j=1}^{w} d\zeta_j, \tag{4.1}$$

where Φ is a meromorphic function on $W' \times \mathbf{C}^w$.

4.7. LEMMA. *For any element* $(\beta, n, m) \in \mathcal{G}^w$

$$(\beta, n, m)^* \Phi = f_\beta^{-w-2} \Phi.$$

PROOF. Lemma 4.7 is a direct corollary of the invariance of $\tilde{\omega}$ with respect to $\mathcal{G}^w$ and of Lemma 4.6. □

4.8. COROLLARY. Φ *in* (4.1) *can be viewed as a meromorphic function on* U'.

PROOF. It suffices to prove that $\Phi(\tilde{u}, \zeta_1) = \Phi(\tilde{u}, \zeta_2)$ for any $\zeta_1, \zeta_2 \in \mathbf{C}^w$ and $\tilde{u} \in U'$. In fact, by the above corollary, the definition of f_β and the action of (β, n, m) with $\beta = \mathrm{id}$ we get

$$\Phi(\tilde{u}, \zeta) = \Phi(\tilde{u}, \zeta + z(\tilde{u})n + m) \tag{4.2}$$

for any $n, m \in \mathbf{Z}^w$. Moreover, by the construction of Φ from Φ' (see the lines before §4.7) it is seen that if $\Phi(\tilde{u}, \zeta)$ is holomorphic then $\Phi(\tilde{u}, \zeta')$ is holomorphic for all $\zeta' \in \mathbf{C}^w$. This and (4.2) obviously prove the relation $\Phi(\tilde{u}, \zeta_1) = \Phi(\tilde{u}, \zeta_2)$, since $z(\tilde{u}) \in H$. Thus Φ can be viewed as a meromorphic function on U'. □

PROOF OF LEMMA 4.3. First of all we will make the reduction to the case when B has singular points only of type I_b $(b \geqslant 1)$. Consider the elliptic surface $\tilde{B}$ defined in the proof of Theorem 2.3. Then a canonical morphism

$$(\tilde{\Psi}^{w})^{*}: H^0r(\bar{B}^{w}, \Omega^{w+1}) \to H^0r(\tilde{B}^{w}, \Omega^{w+1})$$

is defined. Locally $\tilde{\Psi}^w$ is defined in §2.2, and nonsingular points of B^w at which $(\tilde{\Psi}^w)^*\omega$ is not defined for $\omega \in H^0r(\tilde{\bar{B}}^w, \Omega^{w+1})$ are isolated. This is easy to see from the proof of Proposition 3.2. Since $\dim \bar{B}^w \geqslant 2$, we have $(\tilde{\Psi}^w)^*\omega \in H^0r(B^w, \Omega^{w+1})$.

Let us prove that for any point $v \in \Delta$ there exists a neighborhood U such that

$$\int_{\bar{B}^{w}|_U} \omega \wedge \bar{\omega} < \infty, \tag{4.3}$$

where $\omega \in H^0r(\bar{B}^w, \Omega^{w+1})$. By Proposition 3.2 this is evident for points of type I_0.

Let $v \in \Delta$ be of type I_b ($b \geqslant 1$). In the notation of §2.2, denote by W' the universal covering of E'. We may assume that

$$W' = \{z \in \mathbf{C} \mid \operatorname{Im} z > \text{const} > 0\}.$$

By results of Kodaira (see Chapter 8, v, of [3]), for a suitable choice of τ there are defined:

(a) the action $\mathcal{G} = \mathbf{Z}$: n: $z \mapsto z + nb$,

(b) the representation S: $Z \to \mathrm{SL}(2, \mathbf{Z})$, $n \mapsto \begin{pmatrix} 1 & nb \\ 0 & 1 \end{pmatrix}$, and

(c) the function z: $W' \to H$, z: $z \mapsto z$,

satisfying the construction of §4.5. Moreover, there is a canonical analytic isomorphism

$$\mathbf{Z} \times \mathbf{Z}^{w} \times \mathbf{Z}^{w} \diagdown W' \times \mathbf{C}^{w} \simeq \bar{B}^{w}|_{E'},$$

where the group $\mathbf{Z} \times \mathbf{Z}^w \times \mathbf{Z}^w$ acts on $W' \times \mathbf{C}^w$ as follows:

$$(k, n, m) : (z, \zeta) \longmapsto (z + kb,\ \zeta + zn + m),$$

where $k \in \mathbf{Z}$, $n, m \in \mathbf{Z}^w$, $z \in W'$ and $\zeta \in \mathbf{C}^w$. Denote by $\tilde{\omega}$ the lifting to $W' \times \mathbf{C}^w$ of the differential $\omega \in H^0r(\bar{B}^w, \Omega^{w+1})$ restricted to E'. By Corollary 4.8,

$$\tilde{\omega} = \Phi dz \wedge \bigwedge_{j=1}^{w} d\zeta_j, \tag{4.4}$$

where Φ is a holomorphic function on W' such that $z = \tilde{u}$. The map

$$z \longmapsto \tau = e^{2\pi i z/b}, \quad \zeta_i \longmapsto u_i = e^{2\pi i \zeta_i} \tag{4.5}$$

allows us to construct a canonical isomorphism

$$\mathbf{Z}^{w} \diagdown E' \times (C^{*})^{w} \simeq \bar{B}^{w}|_{E'},$$

where the group $\mathbf{Z}^w$ acts on $E' \times (\mathbf{C}^*)^w$ as follows:

$$n : (\tau, u_1, \ldots, u_w) \longmapsto (\tau, u_1\tau^{bn_1}, \ldots, u_w\tau^{bn_w}),$$

where $n = (n_1, \ldots, n_w) \in \mathbf{Z}^w$. Let $\check{\omega}$ be the lifting to $E' \times (\mathbf{C}^*)^w$ of the differential $\omega|_{E'}$ restricted to E'. Then by (4.4), (4.5) and §4.7

$$\check{\omega} = F d\tau \wedge \bigwedge_{j=1}^{n} \frac{du_j}{u_j}, \tag{4.6}$$

where F is an analytic function on E'. $\bar{B}^w|_E$ contains the analytic subvariety

$$(\mathbf{Z}^{w} \diagdown E' \times (\mathbf{C}^{*})^{w}) \cup (\mathbf{C}^{*})^{w} \simeq B^{w}|_{E'} \cup (\mathbf{C}^{*})^{w};$$

in this connection the points of $(\mathbf{C}^*)^w$ are included in the fibers B_v^w and are of regular type $(0, w)_i = (\pm 1, w)_f$. We may take (τ, u) as local coordinates at these points (see §2.2(ii) and Chapter 8, v, of [3]). Thus $F(\tau)$ is analytic on E, since $\omega \in H^0 r(\bar{B}^w, \Omega^{w+1})$. This implies that to prove (4.3) it suffices to verify the inequality

$$\int_{U_\varepsilon} d\tau \wedge \bigwedge_{j=1}^{w} \frac{du_j}{u_j} \wedge d\bar{\tau} \wedge \bigwedge_{j=1}^{w} \frac{d\bar{u}_j}{\bar{u}_j} < \infty,$$

where the integral is taken over some fundamental domain $U_\varepsilon \subset E' \times (\mathbf{C}^*)^w$ for the action of the group $\mathbf{Z}^w$ (recall that $E = \{|\tau| < \varepsilon\}$). The last assertion is nothing more than an easy exercise in the calculus, if the fundamental domain is taken as

$$U_\varepsilon = \{(\tau, u) \mid 0 < |\tau| < \varepsilon < 1,\ |\tau|^b < |u_i| < 1\}. \quad \square$$

PROOF OF LEMMA 4.6. a) follows from the relations

$$d\zeta_j = d\xi_{1,j} + z\,d\xi_{2,j} + \xi_{2,j}\,dz,\quad 1 \leqslant j \leqslant w.$$

b) follows from (2.1) and the relation

$$(\beta, n, m)^* \bar{z} = \frac{a\bar{z} + b}{c\bar{z} + d},$$

obtained by conjugacy since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{Z})$.

c) We have

$$(\beta, n, m)^* dz = d\,(\beta, n, m)^* z = d\left(\frac{az + b}{cz + d}\right) = f_\beta^2 dz.$$

d) By virtue of b) and the relations

$$f_\beta = (cz + d)^{-1}, \qquad \xi_{2,j} = \frac{\zeta_j - \bar{\zeta}_j}{z - \bar{z}}, \quad \xi_{1,j} = \frac{\bar{\zeta}_j z - \zeta_j \bar{z}}{z - \bar{z}}, \qquad (\beta, n, m)^* \zeta_j = f_\beta(\zeta_j + zn + m)$$

we have

$$(\beta, n, m)^* \xi_{2,j} = -(\xi_{1,j} + m)\,c + (\xi_{2,j} + n)\,d.$$

Since

$$(\beta, n, m)^* z = \frac{az + b}{cz + d},$$

we also get

$$(\beta, n, m)^* \xi_{1,j} = (\xi_{1,j} + m)\,a - (\xi_{2,j} + n)\,b.$$

Hence

$$(\beta, n, m)^* (d\xi_{1,j} + z\,d\xi_{2,j}) = f_\beta\,(d\xi_{1,j} + z\,d\xi_{2,j}). \quad \square$$

§5. Kuga's modular varieties

The group Γ acts in the standard way on the upper half-plane H:

$$\gamma z = \frac{az + b}{cz + d},$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in H$. The quotient space $\Gamma \setminus H$ has a natural compactification Δ_Γ. The finite set $\Pi = \Delta_\Gamma - \Gamma \setminus H$ of points which are added to obtain the compactification is called the *set of cusp points* of Δ_Γ. Δ_Γ is a nonsingular projective algebraic curve. It is called the *modular curve corresponding to the group* Γ. If $\Gamma = \mathrm{SL}(2, \mathbf{Z})$, then $\Delta_{\mathrm{SL}(2,\mathbf{Z})} \simeq \mathbf{P}^1$, the projective line, and the choice of certain (nonhomogeneous) coordinates on this line determines the *absolute invariant function* j: $\Delta_{\mathrm{SL}(2,\mathbf{Z})} \to \mathbf{C}$. Let $\Gamma \subset \Gamma' \subset \mathrm{SL}(2, \mathbf{Z})$ be subgroups of finite index. The natural map of quotient spaces $\Gamma \setminus H \to \Gamma' \setminus H$ defines a regular morphism of projective curves

$$\Delta_\Gamma \to \Delta_{\Gamma'}. \tag{5.1}$$

Taking the composition of the map (5.1) for the pair $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ with the absolute invariant function j, we define the meromorphic function J_Γ on Δ_Γ.

5.1. In the case (*) the *elliptic modular surface B_Γ over Δ_Γ corresponding to the group* Γ is canonically defined (see [5], Definition 4.1). If (*) does not hold, then (noncanonically) by Lemma 1.5 a certain elliptic surface B_Γ with the functional invariant J_Γ is defined over Δ_Γ. We will also call the given surface the *elliptic modular surface corresponding to the group* Γ. The Kuga variety B_Γ^w associated with the elliptic surface B_Γ will be called *Kuga's modular variety of degree w corresponding to the group* Γ. Note that both the given variety and B_Γ are defined canonically only in the case (*). In the sequel the index Γ is dropped for simplicity; thus we write Δ instead of Δ_Γ, Φ^w instead of Φ_Γ^w, and so on.

We will call a non-cusp point $v \in \Delta$ an *elliptic* point if v is the orbit of a point $z \in H$ such that the stationary subgroup $\Gamma_z / \pm E$ is nontrivial. In other cases (that is, $v \in \Delta$ is neither a cusp point nor an elliptic point) we say that v is *regular*. If v is an elliptic point of Δ and $z \in H$ is the corresponding point of the orbit, then $z \in \mathrm{SL}(2, \mathbf{Z})\{i, \eta\}$. We will say accordingly that the given elliptic point v is equivalent to i or η.

Let $\tilde{\mathbf{Q}} = \mathbf{Q} \cup i\infty$, and $\overline{H} = H \cup \tilde{\mathbf{Q}}$. A continuous extension of the action of $\mathrm{SL}(2, \mathbf{Z})$ is defined on the compactification $\overline{H}$. Each cusp point $p \in \Pi$ has a representative $q \in \tilde{\mathbf{Q}}$ which is defined up to the action of Γ (see §1.3 of [4]). The stationary subgroup Γ_q of the point q is conjugate with respect to $\mathrm{SL}(2, \mathbf{Z})$ to the subgroup generated by $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & -b \\ 0 & -1 \end{smallmatrix}\right)$, $b > 0$, in the case (*). Therefore in accordance with the above we will say in the case (*) that a *cusp* point v is a point of the *first* or the *second kind.* The first assertion of the following proposition translates the classification of points $v \in \Delta$ introduced just now into the language of types of points (see §1).

5.2. PROPOSITION. a) *The notions placed in the columns of Table* 4 *are equivalent*:

TABLE 4

$v \in \Delta$	regular	cusp	elliptic equivalent to i	elliptic equivalent to η
	Nonsingular of type (I_0) or type I_0^*	of type I_b, I_b^* $(b \geqslant 1)$	of type III, III*	of type II, IV*

In the case (*), *Table* 4 *takes the simpler form* 4(*).

TABLE 4(*)

$v \in \Delta$	regular	cusp		elliptic
		1st kind	2nd kind	elliptic equivalent to η
	nonsingular type	$\mathrm{I}_b\ (b \geqslant 1)$	$\mathrm{I}_b^*\ (b \geqslant 1)$	of type IV^*

b) *In the case* (*) *there is a canonical isomorphism*

$$\overline{B}_\Gamma^w|_{\Delta-\mathrm{II}} \simeq \Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w \setminus H \times \mathbf{C}^w, \tag{5.2}$$

where $\Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w$ *acts according to* (0.2).

Let $s = s_1 + s_2$ ($t' = t_1 + t_2$) denote the number of elliptic (cusp) points of Δ, where s_1 is the number of elliptic points equivalent to η and s_2 is the same for i (the given decomposition is defined in the case (*), and t_i is the number of cusp points of kind i, $i = 1, 2$). By Proposition 5.2c), in the case (*) we have

$$t_1 = \sum_{b \geqslant 1} \nu(\mathrm{I}_b), \qquad t_2 = \sum_{b \geqslant 1} \nu(\mathrm{I}_b^*),$$

where $\nu(*)$ is the number of points of type *. For other types we define t_1 and t_2 by these equalities. Then, by virtue of §5.2, $t' = t_1 + t_2$.

5.3. COROLLARY. a)

$$t_1 = \sum_{b \geqslant 1} \nu(\mathrm{I}_b), \qquad t_2 = \sum_{b \geqslant 1} \nu(\mathrm{I}_b^*),$$

$$s_1 = \nu(\mathrm{II}) + \nu(\mathrm{IV}^*), \qquad s_2 = \nu(\mathrm{III}) + \nu(\mathrm{III}^*), \qquad \nu(\mathrm{II}^*) = \nu(\mathrm{IV}) = 0.$$

b) *In the case* (*)

$$\nu(\mathrm{I}_0^*) = \nu(\mathrm{II}) = \nu(\mathrm{II}^*) = \nu(\mathrm{III}) = \nu(\mathrm{III}^*) = \nu(\mathrm{IV}) = 0,$$

$$t_1 = \sum_{b \geqslant 1} \nu(\mathrm{I}_b), \qquad t_2 = \sum_{b \geqslant 1} \nu(\mathrm{I}_b), \qquad s = s_1 = \nu(\mathrm{IV}^*). \qquad \square$$

In view of §§2.1 and 3 we have an isomorphism $B_\Gamma^w|_{\Delta'} \simeq \mathcal{G}^w \setminus U' \times \mathbf{C}^w$. Then the function z defines the cover

$$z : U' \to H' \subset H - \mathrm{SL}(2, \mathbf{Z})\{i, \eta\},$$

where H' is the image of U' under the map z: $U' \to H$. From the definition of the absolute invariant j and the functional invariant J_Γ we immediately obtain the following commutative diagram of analytic covers:

$$\begin{array}{ccc} U' & \xrightarrow{z} & H' \\ \downarrow & & \downarrow \\ \Delta' & = & \Gamma \setminus H'. \end{array} \tag{5.3}$$

5.4. DEFINITION. Consider the following space of analytic functions:

$$S_{w+2}(\Gamma, H') = \{\Phi : H' \to \mathbf{C} \text{ is an analytic function such that } \Phi \mid [g]_{w+2} = \Phi \text{ for any } g \in \Gamma\},$$

where (see §2.1 of [4])

$$(\Phi \mid [g]_{w+2})(z) = (\det g)^{\frac{w+2}{2}} (cz+d)^{-w-2} \Phi(gz);$$

the transformation $[g]_{w+2}$ is correctly defined for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}^+(2, \mathbf{Z})$ if $gH' \subset H'$.

Consider $\Phi \in S_{w+2}(\Gamma, H')$. This function determines the analytic function Φ: $U' \times \mathbf{C}^w \to \mathbf{C}$: $\Phi((\tilde{u}, \zeta)) = \Phi(z(\tilde{u}))$, $\tilde{u} \in U'$. Thus the analytic differential form

$$\tilde{\omega}_\Phi = \Phi dz \wedge \bigwedge_{j=1}^{w} d\zeta_j$$

is defined on $U' \times \mathbf{C}^w$. By (2.1) and the definition of Φ (see §5.4) we have

$$\Phi(\beta\tilde{u}) = \Phi(z(\beta\tilde{u})) = \Phi(S_\beta z(\tilde{u})) = \Phi(z(\tilde{u}))(cz(\tilde{u}) + d)^{w+2}.$$

Therefore for any element $(\beta, n, m) \in \mathcal{G}^w$ the relation $(\beta, n, m)^*\Phi = f_\beta^{-w-2}\Phi$ (cf. §4.7) holds, and by Lemma 4.6 this gives the invariance of the differential $\tilde{\omega}_\Phi$ with respect to the group $\mathcal{G}^w$. The latter allows us to define a canonical homomorphism

$$S_{w+2}(\Gamma, H') \to H^0(B_\Gamma^w|_{\Delta'}, \Omega^{w+1}), \quad \Phi \mapsto \omega_\Phi|_{\Delta'}, \tag{5.4}$$

where $\omega_\Phi|_{\Delta'}$ is an analytic differential on $B_\Gamma^w|_{\Delta'}$ corresponding to $\tilde{\omega}_\Phi$.

5.5. LEMMA. *The homomorphism* (5.4) *is an isomorphism.*

5.6. THEOREM. a) $\omega_\Phi|_{\Delta'}$ *is extended to a holomorphic differential form* ω_Φ *of degree* $w+1$ *on Kuga's modular variety* $\overline{B}_\Gamma^w$ *if and only if* $\Phi \in S_{w+2}(\Gamma) \subset S_{w+2}(\Gamma, H')$.

b) *In particular, there is a canonical isomorphism*

$$S_{w+2}(\Gamma) \xrightarrow{\sim} H^0(\overline{B}_\Gamma^w, \Omega^{w+1}), \quad \Phi \mapsto \omega_\Phi. \tag{5.5}$$

5.7. COROLLARY.

$$\dim H^0(\overline{B}_\Gamma^w, \Omega^{w+1}) = (w+1)(g-1) + \sum_{b \geq 1} (\nu(\mathrm{I}_b) + \nu(\mathrm{I}_b^*)) \frac{w}{2} + (\nu(\mathrm{II}) + \nu(\mathrm{IV}^*)) \left[\frac{w+2}{3}\right] + (\nu(\mathrm{III}) + \nu(\mathrm{III}^*)) \left[\frac{w+2}{4}\right]$$

if w *is even, and*

$$\dim H^0(\overline{B}_\Gamma^w, \Omega^{w+1}) = (w+1)(g-1) + \sum_{b \geq 1} \nu(\mathrm{I}_b) \frac{w}{2} + \sum_{b \geq 1} \nu(\mathrm{I}_b^*) \frac{w+1}{2} + \nu(\mathrm{IV}^*) \left[\frac{w+2}{3}\right]$$

if w *is odd, where* g *is the genus of the curve* Δ_Γ.

PROOF. This is a direct corollary of the above theorem, Theorems 2.24, 2.25 of [4] and Corollary 5.3. □

PROOF OF THEOREM 5.6. a) Since $\overline{B}_\Gamma^w|_{\Delta'} = B_\Gamma^w|_{\Delta'}$, in view of Lemma 5.5 the homomorphism (5.4) defines the isomorphism

$$S_{w+2}(\Gamma, H') \xrightarrow{\sim} H^0(\bar{B}_\Gamma^w|_{\Delta'}, \Omega^{w+1}), \quad \Phi \mapsto \omega_\Phi|_{\Delta'}.$$

Therefore by virtue of Proposition 4.2 it suffices to prove that $\omega_\Phi|_{\Delta'}$ is extended to a regular differential on B_Γ^w if and only if $\Phi \in S_{w+2}(\Gamma)$. Then by Lemma 4.3 and the proof of Theorem 3.1 of [2] it suffices to establish the following equivalence for $\Phi \in S_{w+2}(\Gamma, H')$:

$$\Phi \in S_{w+2}(\Gamma) \Leftrightarrow \int_{\bar{B}_\Gamma^w|_{\Delta'}} \omega_\Phi|_{\Delta'} \wedge \bar{\omega}_\Phi|_{\Delta'} < \infty \tag{5.6}$$

(the integral on the right is improper). Obviously

$$\int_{\bar{B}_\Gamma^w|_{\Delta'}} \omega_\Phi|_{\Delta'} \wedge \bar{\omega}_\Phi|_{\Delta'} = \int_{\tilde{U}} \tilde{\omega}_\Phi \wedge \bar{\tilde{\omega}}_\Phi,$$

where the integral on the right is taken over a fundamental domain $\tilde{U} \subset U' \times \mathbf{C}^w$ with respect to the group $\mathcal{G}^w$. Let U be a fundamental domain with respect to the group Γ. Denote by U_1 some (one-to-one) lifting of U to U' using z from diagram (5.3). Then the fundamental domain $\tilde{U}$ can be taken as follows:

$$\tilde{U} = \{(\tilde{u}, \zeta) \mid \tilde{u} \in U_1, \zeta_j = t_{1,j} + t_{2,j}\ \tilde{z}(u), t_{i,j} \in [0, 1]\}.$$

Then

$$\int_{\tilde{U}} \tilde{\omega}_\Phi \wedge \bar{\tilde{\omega}}_\Phi = \int_{\tilde{U}} |\Phi|^2 dz \wedge \bigwedge_{j=1}^{w} d\zeta_j \wedge d\bar{z} \wedge \bigwedge_{j=1}^{w} d\bar{\zeta}_j$$
$$= c \int_{U_1} |\Phi|^2 (\operatorname{Im} z)^w dz \wedge d\bar{z} = c \int_{U} |\Phi|^2 (\operatorname{Im} z)^w dz \wedge d\bar{z},$$

where $c \in \mathbf{C}^*$ (an easily calculated number independent of Φ). Therefore for the proof of (5.6) it suffices to establish the following equivalence for $\Phi \in S_{w+2}(\Gamma, H')$:

$$\Phi \in S_{w+2}(\Gamma) \Leftrightarrow \int_U |\Phi|^2 (\operatorname{Im} z)^w dz \wedge d\bar{z} < \infty.$$

This can easily be deduced from the analyticity of the function Φ on H' and the isolation of its singularities.

b) Let $\Delta' \supset \Delta_1'$. Then we have the following commutative diagram of analytic covers:

$$\begin{array}{ccc} U_1' & \to & U' \\ z\downarrow & & \downarrow z \\ H_1' & \subset & H'. \end{array}$$

Consequently we have the commutative diagram

$$\begin{array}{ccc} S_{w+2}(\Gamma, H') & \to & H^0(\bar{B}_\Gamma^w|_{\Delta'}, \Omega^{w+1}) \\ \cap & & \cap \\ S_{w+2}(\Gamma, H_1') & \to & H^0(\bar{B}_\Gamma^w|_{\Delta_1'}, \Omega^{w+1}), \end{array}$$

where vertical inclusions are "restrictions". This easily implies the canonicity of isomorphism (5.5). □

PROOF OF LEMMA 5.5. It suffices to show that the function $\Phi\colon U' \to \mathbf{C}$ satisfying the equation (see §§4.7, 4.8 and 4.5, where $W' = U'$)

$$\beta^*\Phi = f_\beta^{-w-2}\Phi \tag{5.7}$$

coincides with the lifting of a function $\psi\colon H' \to \mathbf{C}$ on U'. In fact, $H' \simeq S^{-1}(\pm E) \setminus U'$. Therefore it suffices to show that $\beta^*\Phi = \Phi$ for $\beta \in S^{-1}(\pm E)$. This immediately follows from the functional equation (5.7), since $S^{-1}(\pm E) = S^{-1}(E)$ for even w by the definition of B_Γ (see [5], Definition 4.1), and condition $(*)$ holds for odd w. $\square$

PROOF OF PROPOSITION 5.2. a) Table 4$(*)$ is a simple corollary of Proposition 4.2 of [5].

Table 4 is deduced from Table 2, using elementary properties of the absolute invariant j, and also the following property of elliptic surfaces: B_v is of type I_b or I_b^* ($b \geqslant 1$) if and only if $J(v) = \infty$. For instance, let $v \in \Delta$ be an elliptic point equivalent to η. Then, since $j(\eta) = 0$, we have $J_\Gamma(v) = 0$. Hence v is of one of the types I_0, I_0^*, II, II*, IV or IV*. It is evident that I_0 and I_0^* fall away. By virtue of the relation $h = 1$, $h \equiv 1 \bmod 3$, the case II* falls away from the pair II, II*. By analogy the case IV falls away from the pair IV, IV* by virtue of the relation $h = 1$, $h \equiv 1 \bmod 3$.

If v is of type IV* or II, then by Table 2 v is an elliptic point equivalent to η, since $z(0) = \eta$.

b) From the definition of the representation S of §4 and Proposition 4.2 from [5] we obtain by §2.1 an isomorphism

$$\overline{B}_\Gamma^w|_{\Delta'} \simeq \Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w \setminus H' \times \mathbf{C}^w,$$

where $H' = H - \mathrm{SL}(2, \mathbf{Z})\{\eta\}$ and $\Delta' = \Gamma \setminus H$. The given isomorphism extends obviously to an isomorphism

$$\overline{B}_\Gamma^w|_{\Delta'} \simeq \Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w \setminus H' \times \mathbf{C}^w,$$

where Δ' is the set of regular points of Δ and H' is the inverse image of these points under the natural map $H \to \Gamma \setminus H$. Thus it remains to extend this isomorphism by Table 4$(*)$ to points v which are elliptic and equivalent to η. Let $z_0 \in H$ be a representative of this point. Then, since Γ_{z_0} is a subgroup of order three and v is a point of type IV*, we have

$$F^w \simeq \mathbf{Z}^w \times \mathbf{Z}^w \setminus U \times \mathbf{C}^w,$$

where U is a neighborhood of the point z_0, F^w is defined in §2.2(i), and the action of C corresponds to the action of some nontrivial element of Γ_{z_0} on $\mathbf{Z}^w \times \mathbf{Z}^w \setminus U \times \mathbf{C}^w$. This obviously implies the existence of an extension of the isomorphism to (5.2). $\square$

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