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Изв. Акад. Наук СССР
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V. V. ŠOKUROV

ABSTRACT. This paper studies integrals of the form $\int_\alpha^{i\infty} \Phi z^k\,dz$ on the upper half-plane, where α is a rational number, $0 < k < w$ is integral, and Φ is a cusp form of weight $w + 2$ with respect to some modular group $\Gamma \subset SL(2, \mathbf{Z})$. The main result is that if Γ is a congruence subgroup and Φ is an eigenvector of all the Hecke operators, then all these integrals are representable as linear combinations of two complex numbers with coefficients in some field of algebraic numbers.

Bibliography: 13 titles. Figures: 11.

This paper completes the proof, begun in [12] and [13], of a series of results on the periods of cusp forms, Kuga varieties, and modular symbols, which were announced in [10] and [11]. The author expresses his gratitude to Ju. I. Manin, in whose seminar this work was completed.

§0. Main results

Integrals on the upper half-plane of the form

$$\int_\alpha^{i\infty} \Phi z^k dz, \tag{0.1}$$

where the parameter α is a rational number, $0 \leqslant k \leqslant w$ is an integer and Φ is a cusp form of weight $w + 2$ for some modular group $\Gamma \subset SL(2, \mathbf{Z})$, are called *Shimura integrals*. The goal of this paper is to study the Shimura integrals of a cusp form Φ for some congruence subgroup which are of weight $\geqslant 2$ and are eigenvectors of all Hecke operators.

0.1. Our notation for the theory of modular forms agrees with that of [8]. Recall that in [8] Γ' denotes the congruence subgroup

$$\Gamma' = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid a \in \mathfrak{h},\ b \equiv 0 \bmod (t),\ c \equiv 0 \bmod (N)\right\}, \tag{0.2}$$

where N is a natural number, t is a positive divisor of N, and $\mathfrak{h}$ is a subgroup of $(\mathbf{Z}/N\mathbf{Z})^*$. The Hecke operators $T'(n)_{w+2}$ (see §3.5 of [8]) act on the space $S_{w+2}(\Gamma')$ of cusp forms for Γ' of weight $w + 2$.

The main result of this paper is the following theorem.

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0.2. THEOREM ON PERIODS. *Let $\Phi \in S_{w+2}(\Gamma')$ be an eigenvector of all the Hecke operators*:

$$T'(n)_{w+2}\Phi = \lambda_n \Phi.$$

Then

$$\dim_K \left(\sum_{\substack{\alpha \in \mathbf{Q} \\ 0 \leqslant k \leqslant w}} K \int_\alpha^{i\infty} \Phi z^k dz \right) \leqslant 2,$$

where K denotes the field of algebraic numbers $\mathbf{Q}(\lambda_1, \lambda_2, \dots)$.

Further, essentially equivalent, variants of this theorem are given in Theorems 5.2 and 5.4 of §5.

First of all we indicate the geometric interpretation of Shimura integrals. This construction will be made for any modular group $\Gamma \subset SL(2, \mathbf{Z})$. The condition

$$-E \notin \Gamma, \qquad (*)$$

which was required in [11] in order to simplify the construction, is not obligatory here. A natural restriction on the pair (Γ, w) is that examined in [10]: w is a natural number and is even if $(*)$ does not hold (otherwise the integrals (0.1) and the forms are trivial). In [12] we constructed, from the pair (Γ, w), a nonsingular projective variety B_Γ^w over the field of complex numbers. This variety is called a *Kuga modular variety*. It projects naturally onto the modular curve $\Delta_\Gamma = \overline{\Gamma \setminus H}$:

$$\Phi^w : B_\Gamma^w \to \Delta_\Gamma.$$

The fiber of the Kuga variety B_Γ^w over a general point of the base Δ_Γ is the wth power of an elliptic curve. The set of cusps of the modular curve Δ will be denoted by Π. The Shimura integrals are interpreted as integrals of holomorphic forms of highest weight over the relative cycles of the pair $(B_\Gamma^w, B_\Gamma^w|_\Pi)$, where $B_\Gamma^w|_\Pi = (\Phi^w)^{-1}(\Pi)$ is the restriction of B_Γ^w to Π (see the beginning of §1 in [12]).

An algebro-geometric interpretation of cusp forms is given by the canonical isomorphism

$$S_{w+2}(\Gamma) \simeq H^0(B_\Gamma^w, \Omega^{w+1}), \quad \Phi \mapsto \omega_\Phi \qquad (0.3)$$

defined in [12] (see §0.3 of [12], and for the case $w = 1$ see Shioda [9], Theorem 6.1).

Integrals of the type (0.1) are interpreted, via (0.3), as integrals of the holomorphic form ω_Φ corresponding to $\Phi \in S_{w+2}(\Gamma)$ over a relative cycle on B_Γ^w of dimension $w + 1$ whose boundary lies over a pair of cusps of Δ. Of course it would be more natural to integrate over absolute cycles of B_Γ^w, as in the case of abelian integrals, but then most of the integrals of type (0.1) would not be included in this set-up. The use of absolute cycles allows one to obtain arbitrary Shimura integrals, but it has its defects, which will be indicated below.

Let us describe in more detail the interpretation of the Shimura integrals in the case of an effective action of the group Γ on the upper half-plane $H = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ (that is, $E \notin \Gamma$). This effectiveness condition is satisfied, for example, by $\Gamma(N)$, the principal congruence subgroup of $SL(2, \mathbf{Z})$ of level N (see §1.6 of [8]) when $N \geqslant 3$. The set of general points (a Zariski-open subset) of the Kuga modular variety B_Γ^w has the following evident description. Denote by H' the set $H - \mathrm{SL}(2, \mathbf{Z})\{\eta\}$, $\eta = e^{2\pi i/3}$. On the analytic manifold $H' \times \mathbf{C}^w$ we have a free, proper, and discrete action of the group $\Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w$ (semidirect product) given by

$$(\gamma, n, m) : (z, \zeta) \mapsto (\gamma z, (cz + d)^{-1}(\zeta + zn + m)),$$
$$n, m \in \mathbf{Z}^w = \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{w}, \quad z \in H', \quad \zeta \in \mathbf{C}^w = \mathbf{C} \times \cdots \mathbf{C},$$
$$\gamma z = \frac{az + b}{cz + d}, \tag{0.4}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset \mathrm{SL}(2, \mathbf{Z})$. Then there is an analytic isomorphism

$$B_\Gamma^w|_{\Delta'} \simeq \Gamma \times \mathbf{Z}^w \times \mathbf{Z}^w \backslash H' \times \mathbf{C}^w,$$

where $\Delta' = \Gamma \setminus H' \subset \Delta_\Gamma$ (see the start of the proof of 5.2.6 in [12]). For the interpretation we describe a cycle such that if we integrate over it the differential form ω_Φ corresponding to the cusp form $\Phi \in S_{w+2}(\Gamma)$, we get integrals on the upper half-plane of the form

$$\int_\alpha^\beta \Phi \prod_{i=1}^{w} (n_i z + m_i)\, dz, \tag{0.5}$$

where $n = (n_1, \ldots, n_w)$ and $m = (m_1, \ldots, m_w)$ are integral vectors and $\alpha, \beta \in \tilde{\mathbf{Q}} = \mathbf{Q} \cup i\infty$, the set of rational numbers with an infinity adjoined. Let $\widetilde{\alpha\beta}$ be a path joining α to β on H' and approaching the cusps α and β vertically ($\tilde{\mathbf{Q}}$ is considered as the set of points compactifying H to $\overline{H} = H \cup \tilde{\mathbf{Q}}$). Over each point $z \in \operatorname{Int} \widetilde{\alpha\beta}$ we consider the cell

$$\{\zeta \in \mathbf{C}^w \mid \forall i = 1, \ldots, w\ \exists t_i \in [0, 1] \subset \mathbf{R} \quad \text{such that} \quad \zeta_i = t_i(zn_i + m_i)\}.$$

The closure of the symmetrized union of these cells defines the desired relative cycle on B_Γ^w with boundaries over the cusps of Δ that correspond to α and β. The homology class of the given cycle does not depend on the choice of the admissible path $\widetilde{\alpha\beta}$ in $H_{w+1}(B_\Gamma^w, B_\Gamma^w|_\Sigma, \mathbf{Q})$, where $\Sigma = \Delta - \Delta'$. This homology class equals $GR_{1,w}\{\alpha, \beta, n, m\}_\Gamma$. The map

$$GR_{1,w} : H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w) \to H_{w+1}(B_\Gamma^w, B_\Gamma^w|_\Sigma, \mathbf{Q})$$

is the restriction of the geometric realization mapping ([13], §3) to the subspace $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w) \subset H_1(\Delta, \Sigma, R_w\Phi_*^w\mathbf{Q})$ (see [13], 2.5a and (3.3)). $\{a, \beta, n, m\}_\Gamma$ is a *modular symbol* (see 1.2).

The following proposition (for a proof see §1) gives a geometric interpretation of integrals of type (0.5) and, therefore, of the integrals (0.1).

0.3. PROPOSITION (on Shimura integrals).

$$\int_\alpha^\beta \Phi \sum_{i=1}^{w} (n_i z + m_i)\, dz = \int_{GR_{1,w}\{\alpha,\beta,n,m\}_\Gamma} \omega_\Phi.$$

This proposition enables us to transfer the study of the integrals (0.1) to the geometry of the Kuga modular variety B_Γ^w. One of the questions investigated is to determine how completely the Eichler-Shimura relations describe the periods of cusp forms.

Let $I = \Gamma \setminus \mathrm{SL}(2, \mathbf{Z})$ be the set of right cosets. By the periods of the cusp form Φ we mean the following finite set of complex numbers:

$$r(j, k, \Phi) = \int_0^{i\infty} (\Phi \mid [g]_{w+2}) z^k dz, \tag{0.6}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j \in I$, $(\Phi|[g]_{w+2})(z) = (cz + d)^{-w-2}\Phi(gz)$, and $0 \leqslant k \leqslant w$ is an integer. Denote by s and t the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ respectively.

0.4. *The Eichler-Shimura relations.* The set of periods uniquely determines the cusp form Φ and satisfies the following system of relations (for the proof see §3):

$$\begin{gathered} r(j, k, \Phi) + (-1)^k r(js, w-k, \Phi) = 0; \\ r(j, k, \Phi) + \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} r(jt, w-k+i, \Phi) \\ + \sum_{i=0}^{w-k} (-1)^{i+w} \binom{w-k}{i} r(jt^2, i, \Phi) = 0. \end{gathered} \tag{0.7}$$

A geometric treatment of the Eichler-Shimura relations in the style of Theorem 1.9 of [5] is given in §2. The system of relations (0.7) is also satisfied by the periods of the forms in the complex conjugate space $\overline{S_{w+2}(\Gamma)}$:

$$r(j, k, \Phi) = \int_0^{i\infty} \overline{(\Phi \mid [g]_{w+2})}\, \bar{z}^k d\bar{z}.$$

Therefore it is more natural to study the periods of forms in the space $S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$, which, along with its first summand, will be called a *space of cusp forms* in the sequel. In spite of this extension of the space of forms, the Eichler-Shimura relations do not provide a complete system of relations among the periods. However, in the case of a congruence subgroup Γ the system (0.7) can be supplemented by new relations for the periods, which are also linear and homogeneous with integral coefficients.

The appearance of new relations is connected with the theory of the Hecke operators $T'(n)_{w+2}$ on the spaces $S_{w+2}(\Gamma')$ of cusp forms. In this connection an important role is played by the geometric interpretation of the Hecke operators indicated in §4. The cause of the incompleteness of the Eichler-Shimura system is that the Shimura integrals, and in particular the periods

$$r(j, k, \Phi) = \int_{g(0)}^{g(i\infty)} \Phi\, (dz - b)^k (a - cz)^{w-k} dz, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in j,$$

are integrated as integrals over relative cycles. The method of using Hecke operators to obtain absolute cycles from relative cycles with boundaries over cusps, the idea of which is contained in Manin's paper [5], evidently was first described by Drinfel'd [2]. This device is also used here (see §4), and enables us to get a series of new relations, which, together with (0.7), completely determine the values of the periods of the forms in the space $S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$. These additional relations, connected with the theory of Hecke operators, evidently were first stated by Manin [6]. In this paper they appear in

§5. In addition to the general relations, every form Φ that is an eigenvector for some system of Hecke operators

$$T'(n)_{w+2}\Phi = \lambda_n \Phi$$

satisfies an additional system of linear, homogeneous relations with coefficients in the field of algebraic numbers generated by the field $\mathbf{Q}$ of rational numbers and the eigenvalues λ_n of the Hecke operators.

The importance of studying the periods of cusp forms and their distribution among the other Shimura integrals is revealed in the following result, which holds for any modular subgroup Γ.

0.5. Theorem. *For any rational number $\alpha \in \mathbf{Q}$ and any integer $0 \leqslant k \leqslant w$, there exist integers $h(j, l)$ such that*

$$\int_{\alpha}^{i\infty} \Phi z^k \, dz = \sum_{\substack{0 \leqslant l \leqslant w \\ j \in I}} h(j, l) r(j, l, \Phi)$$

for any cusp form $\Phi \in S_{w+2}(\Gamma)$.

This theorem immediately follows from the results of §§1 and 3. These results are connected with the concept of the *modular symbol* $\{\alpha, \beta, n, m\}_\Gamma$ of a homology class (see §1) corresponding to arbitrary $\alpha, \beta \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$. More precisely, $\{\alpha, \beta, n, m\}_\Gamma$ is an element of the homology space $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$ of a pair (Δ, Π) with coefficients in the sheaf $(R_1\Phi_*\mathbf{Q})^w$. $(R_1\Phi_*\mathbf{Q})^w$ is a symmetric tensor power of the sheaf $R_1\Phi_*\mathbf{Q} = G \otimes_{\mathbf{Q}} \mathbf{Q}$, a "rational" homological invariant of the elliptic surface B_Γ. Modular symbols were introduced by Manin [5] and Swinnerton-Dyer and Mazur [3], [4] as a means of calculating the periods of cusp forms of weight 2.

The construction of the modular symbol is guided by an investigation of the properties of the homology space $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$ introduced in [13].

The formulations of some theorems announced in [11] are incomplete in comparison with this paper, and also possess some imprecision, which disappears if one considers the results of [11] for a group Γ containing no elliptic elements. Moreover, the modular symbol of [11] is different from the monomorphism $GR_{1,w}$ considered in §1. The monomorphism $GR_{1,w}^{-1}$ translates Theorems 1, 2 and 3 of [11] into Theorems 1.5, 2.3 and 4.3, respectively. And Theorems 5 and 6 and the corollary of [11] correspond to Theorems 3.4, 5.7 and 0.5, 0.2.

§1. Modular symbols and marked classes

1.1. We first define the mapping

$$\{,\,\}_\Gamma : \tilde{\mathbf{Q}} \times \mathbf{Z}^w \times \mathbf{Z}^w \to H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w),$$
$$(\alpha, n, m) \mapsto \{\alpha, n, m\}_\Gamma.$$

The element $\{\alpha, n, m\}_\Gamma$ will be called a *boundary symmetric modular symbol of weight* $w + 2$, or, for short, "the modular symbol $\{\alpha, n, m\}_\Gamma$" (we shall omit the index Γ when this will not lead to a misunderstanding). Below we shall canonically identify the space $H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w)$ with a subspace of $H_0(\Sigma, R_w\Phi_*^w\mathbf{Q})$ (see [13], 3.1).

Let $D \subset H'$ be a connected, simply connected domain such that $gD \cap D = \varnothing$ for all $g \in \Gamma - \{\pm \mathrm{id}\}$. In this case D can also be considered as embedded in Δ'; and $z|_D$, the restriction to D of the natural coordinate on the upper half-plane H, can be considered a branch of the many-valued function z (see [12], 2.1). In the group of sections $G|_D$(¹) choose a basis e_1, e_2 such that $e_1|_{z_0}, e_2|_{z_0} \in G|_{z_0} = H_1(B_{z_0}, \mathbf{Z})$ is a basis and

$$\int_{e_1|z_0} \omega \Big/ \int_{e_2|z_0} \omega = z_0,$$

where ω is a form of the first kind on the elliptic curve B_{z_0}, $z_0 \in D$. This basis is easily seen to be uniquely determined up to the transformations $\pm$id. In case (*),

$$B_\Gamma|_D \simeq \bigcup_{z_0 \in D} \mathbf{Z} z_0 + \mathbf{Z} \diagdown \mathbf{C} \subset \Gamma \times \mathbf{Z} \times \mathbf{Z} \diagdown H' \times \mathbf{C} \subset B_\Gamma.$$

The last embedding here is canonically determined by the isomorphism (5.2) of [12]. Hence in this case a basis e_1, e_2 can be uniquely specified: $e_1|_{z_0}$ corresponds to the path $[0, 1] \times z_0 \subset \mathbf{C}$, and $e_2|_{z_0}$ to $[0, 1] \times 1 \subset \mathbf{C}$. At each point $v \in \Delta$ fix a local parameter τ; all the discs about the point v considered below are taken with respect to this local parameter. Let $\alpha \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$. Denote by $p_0 \in \Pi$ the cusp corresponding to α. Then there is a direct sum decomposition:

$$H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) = \bigoplus_{p \in \Pi} H_0(p, (R_1\Phi_*\mathbf{Q})^w). \tag{1.1}$$

By definition the modular symbol $\{\alpha, n, m\}_\Gamma$ is trivial on the summands $H_0(p, (R_1\Phi_*\mathbf{Q})^w)$ of (1.1) for $p \neq p_0$. Therefore, the symbol $\{\alpha, n, m\}_\Gamma$ can be defined as an element of the homology space $H_0(p_0, (R_1\Phi_*\mathbf{Q})^w)$. Consider a small disc E about p_0. Denote by $\tilde{U}_\alpha$ a neighborhood of α in H' that covers E. We denote this covering, as well as its Galois group, by $\tilde{\Gamma}_\alpha$. We choose a point $z_E \in \tilde{U}_\alpha$ and identify it with the point $v_E = \tilde{\Gamma}_\alpha z_E$. By what has been said above this identification corresponds to a canonical choice of basis in the group $G|_{v_E} \subset R_1\Phi_*\mathbf{Q}|_{v_E}$ in case (*); and if (*) fails, the basis is determined up to $\pm$id. We include the point v_E as a 0-cell in some cellular decomposition of the disc E (all the cellular decompositions in this paper that are needed to compute the homology of the sheaf $(R_1\Phi_*\mathbf{Q})^w$ are such that every point of special type is contained within a 2-cell). Denote by $\{z_E, n, m\}_\Gamma^E \subset H_0(E, (R_1\Phi_*\mathbf{Q})^w)$ the cohomology class of a 0-chain–namely, of the cycle

$$\prod_{j=1}^{w} (n_j e_1 + m_j e_2) v_E. \tag{1.2}$$

The correctness of this definition in the case where (*) fails follows from the evenness of w. Taking the projective limit of the projective system of spaces $H_0(E, (R_1\Phi_*\mathbf{Q})^w)$ linked by the morphisms

$$H_0(E', (R_1\Phi_*\mathbf{Q})^w) \to H_0(E, (R_1\Phi_*\mathbf{Q})^w),$$

where $E' \subset E \subset \Delta$ are small discs (see [13], 3.1), we define

$$\{\alpha, n, m\}_\Gamma = \varprojlim_E \{z_E, n, m\}_\Gamma^E \in H_0(E, (R_1\Phi_*\mathbf{Q})^w). \tag{1.3}$$

To confirm the correctness of this definition it suffices to show that $\{z_E, n, m\}_\Gamma^E = \{z'_E, n, m\}_\Gamma^E$ for all z_E and z'_E. Indeed, from this it easily follows that the limit (1.3) exists

(¹) For a sheaf of coefficients $\mathcal{F}$ we denote by $\mathcal{F}|_D$, where D is a subset of Δ, the *group of sections* $\Gamma(\mathcal{F}, D)$ *not* the restriction of $\mathcal{F}$ to D.

and is independent of the choice of z_E. Moreover, this shows that the limit (1.3) is independent of the choice of the local parameter at p_0. First, consider "near" points z_E, $z'_E \in \tilde{U}_\alpha$. This means that these points can be joined by a path $\widetilde{z_E z'_E}$ (directed from z_E to z'_E) which is isomorphically embedded in $E' = E - p_0$ by the mapping $\tilde{\Gamma}_\alpha$. $\tilde{\Gamma}_\alpha$ identifies the path $\widetilde{z_E z'_E}$ with its image $\widetilde{v_E v'_E} = \tilde{\Gamma}_\alpha(\widetilde{z_E z'_E})$.

Moreover, we shall consider that $\widetilde{v_E v'_E}$ is a cell; that is, the path $\widetilde{z_E z'_E}$ is homeomorphic to the segment [0, 1]. In this situation one can define a connected, simply connected domain $\widetilde{z_E z'_E} \subset D \subset \tilde{U}_\alpha$ of the type described above. Let e_1, e_2 be the corresponding basis of the group of sections $G|_D \subset R_1\Phi_*\mathbf{Q}|_D$. Consider a cellular decomposition of E that includes the cell $\widetilde{v_E v'_E}$. Then the boundary of the 1-chain $\prod_{j=1}^{w}(n_j e_1 + m_j e_2)\widetilde{v_E v'_E}$ equals

$$\prod_{j=1}^{w} (n_j e_1|_{v'_E} + m_j e_2|_{v'_E}) v'_E - \prod_{j=1}^{w} (n_j e_1|_{v_E} + m_j e_2|_{v_E}) v_E.$$

Therefore, $\{z_E, n, m\}_\Gamma^E = \{z'_E, n, m\}_\Gamma^E$ for "near" $z_E z'_E$. For arbitrary points z_E, $z'_E \in \tilde{U}_\alpha$ one can define a finite sequence of points $z_E^1, \ldots, z_E^n \in \tilde{U}_\alpha$ such that the pairs (z_E, z_E^1), $(z_E^1, z_E^2), \ldots, (z_E^{n-1}, z_E^n)$, (z_E^n, z'_E) consist of "near" points. Hence the relation $\{z_E, n, m\}_\Gamma^E = \{z'_E, n, m\}_\Gamma^E$ immediately follows from the same relation for "near" points.

1.2. LEMMA-DEFINITION. *There exists a unique mapping*

$$\tilde{\mathbf{Q}} \times \tilde{\mathbf{Q}} \times \mathbf{Z}^w \times \mathbf{Z}^w \to H_1(\Delta_\Gamma, \Pi, (R_1\Phi_*\mathbf{Q})^w),$$
$$(\alpha, \beta, n, m) \mapsto \{\alpha, \beta, n, m\}_\Gamma,$$

whose images $\{\alpha, \beta, n, m\}_\Gamma$ are called modular symbols of weight $w + 2$ and which have the following properties:

a) $\partial\{\alpha, \beta, n, m\}_\Gamma = \{\beta, n, m\}_\Gamma - \{\alpha, n, m\}_\Gamma$, *where*

$$\partial : H_1(\Delta_\Gamma, \Pi, (R_1\Phi_*\mathbf{Q})^w) \to H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w)$$

is the boundary mapping of the pair (Δ, Π) (*see* [13], (3.3), *with* $\Pi' = \varnothing$).

b) *For any cusp forms* $\Phi_1, \Phi_2 \in S_{w+2}(\Gamma)$

$$(\{\alpha, \beta, n, m\}_\Gamma, (\Phi_1, \overline{\Phi_2})) = \int_\alpha^\beta \Phi_1 \prod_{j=1}^{w} (n_j z + m_j)\, dz + \int_\alpha^\beta \overline{\Phi_2} \prod_{j=1}^{w} (n_j \bar{z} + m_j)\, d\bar{z},$$

where $\alpha, \beta \in \tilde{\mathbf{Q}}$, $n = (n_1, \ldots, n_w)$, $m = (m_1, \ldots, m_w) \in \mathbf{Z}^w$, *and* (,) *is the canonical pairing constructed in* [13] (*see* 0.1).

Moreover, modular symbols have the following properties:

c) $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w) = \sum_{\alpha,\beta \in \tilde{\mathbf{Q}};\, n,m \in \mathbf{Z}^w} \mathbf{Q}\{\alpha, \beta, n, m\}_\Gamma$.

d) $H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) = \sum_{\alpha \in \tilde{\mathbf{Q}};\, n,m \in \mathbf{Z}^w} \mathbf{Q}\{\alpha, n, m\}_\Gamma$.

e) *For any* $n, m \in \mathbf{Z}^w$ *and* $\alpha, \beta, \gamma \in \tilde{\mathbf{Q}}$

(e_1) $\{\alpha, \beta, n, m\}_\Gamma + \{\beta, \gamma, n, m\}_\Gamma + \{\gamma, \alpha, n, m\}_\Gamma = 0$;

(e_2) *for all* $1 \leqslant j \leqslant w$

$$\{\alpha, \beta, n, m\}_\Gamma = n_j\{\alpha, \beta, n(j, 1), m(j, 0)\}_\Gamma + m_j\{\alpha, \beta, n(j, 0), m(j, 1)\}_\Gamma,$$
$$\{\alpha, n, m\}_\Gamma = n_j\{\alpha, n(j, 1), m(j, 0)\}_\Gamma + m_j\{\alpha, n(j, 0), m(j, 1)\}_\Gamma,$$

where $n(j, *) = (n_1, \ldots, n_{j-1}, *, n_{j+1}, \ldots, n_w)$;

(e$_3$) *for any element* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$\{g(\alpha), g(\beta), dn-cm, -bn+am\}_\Gamma = \{\alpha, \beta, n, m\}_\Gamma,$$
$$\{g(\alpha), dn-cm, -bn+am\}_\Gamma = \{\alpha, n, m\}_\Gamma;$$

(e$_4$) *for any permutation* $s \in A_w$ *of* w *elements*

$$\{\alpha, \beta, s(n), s(m)\}_\Gamma = \{\alpha, \beta, n, m\}_\Gamma,$$
$$\{\alpha, s(n), s(m)\}_\Gamma = \{\alpha, n, m\}_\Gamma,$$

where $s(n) = (n_{s(1)}, \ldots, n_{s(w)})$.

1.3. COROLLARY. *For any* $\alpha, \beta \in \mathbf{Q}$ *and* $n, m \in \mathbf{Z}^w$

a. $\{\alpha, \alpha, n, m\}_\Gamma = 0$, *and*

b. $\{\alpha, \beta, n, m\}_\Gamma = -\{\beta, \alpha, n, m\}_\Gamma$.

In the sequel we shall often omit the index Γ from the modular symbol, assuming that the group Γ is fixed.

PROOF OF COROLLARY 1.3. a. This immediately follows from 1.2(e$_1$) with $\alpha = \beta = \gamma$, since 3 is invertible in the field $\mathbf{Q}$.

b. This follows from 1.2(e$_1$) with $\gamma = \alpha$ and from 1.3a. ∎

On the set $\tilde{\mathbf{Q}} \times \tilde{\mathbf{Q}} \times \mathbf{Z}^w \times \mathbf{Z}^w$ the group

$$GL^+(2, \mathbf{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \mid a, b, c, d \in \mathbf{Z}, ad-bc>0\}$$

acts from the left by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\alpha, \beta, n, m) \to (g(\alpha), g(\beta), dn-cm, -bn+am).$$

This action induces an "action" on the modular symbols:

$$g \mid \{\alpha, \beta, n, m\}_\Gamma = \{g(\alpha), g(\beta), dn-cm, -bn+am\}_\Gamma. \tag{1.3$'$}$$

$\{\alpha, \beta, n, m\}_\Gamma$ is a homology class in the space of one-dimensional homology $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$ which is written in the form of the given modular symbol. The "action" (1.3$'$) is defined not simply for the homology class but for its representation as the modular symbol $\{\alpha, \beta, n, m\}_\Gamma$. The image of this "action" is also not a homology class, but its representation as a modular symbol $\{g(\alpha), g(\beta), dn - cm, -bn + am\}_\Gamma$ obtained from the initial symbol $\{\alpha, \beta, n, m\}_\Gamma$. Likewise, there is an "action" of the group $GL^+(2, \mathbf{Z})$ on the boundary modular symbols

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| \{\alpha, n, m\}_\Gamma = \{g(\alpha), dn-cm, -bn+am\}_\Gamma. \tag{1.4}$$

In the sequel the "actions" (1.3$'$) and (1.4) will be called *transformations of modular symbols* by elements of the group $GL^+(2, \mathbf{Z})$. Moreover, a permutation $s \in A_w$ defines the following transformations:

$$s \mid \{\alpha, \beta, n, m\} = \{\alpha, \beta, s(n), s(m)\}_\Gamma,$$
$$s \mid \{\alpha, n, m\} = \{\alpha, s(n), s(m)\}_\Gamma.$$

By (e_3) and (e_4) of Lemma-Definition 1.2 the transformations $g|$ and $s|$, for $g \in \Gamma$ and $s \in A_w$, of a modular symbol are equal to the original symbol. The transformation also possesses the following general properties for all $\alpha, \beta \in \tilde{\mathbf{Q}}$, $n, m \in \mathbf{Z}^w$, $g, g' \in GL^+(2, \mathbf{Z})$ and $s, s' \in A_w$:

(tp_1) $g|g'| = gg'|$ and $s|s'| = s's|$.

(tp_2) $g|s| = s|g|$.

(tp_3) For all $1 \leqslant k \leqslant w$

$$g|\{\alpha, \beta, n, m\}_\Gamma = n_k(g|\{\alpha, \beta, n(k, 1), m(k, 0)\}_\Gamma) + m_k(g|\{\alpha, \beta, n(k, 0), m(k, 1)\}_\Gamma),$$

which is proved by 1.2(e_2).

1.4. Definition. Let $I = \Gamma \setminus SL(2, \mathbf{Z})$ be the set of right cosets $j = \Gamma g$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$. For pairs (j, k), where $j = \Gamma g$ is a right coset and $0 \leqslant k \leqslant w$ is an integer, we define a homology class $\xi(j, k)$ in the space $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$:

$$\xi(j, k) = g|\{0, i\infty, 1_k, 1_w - 1_k\}_\Gamma,$$

where 1_k is an integral vector whose entries are 0 and 1, the number of ones being k; for example, we can take

$$1_k = (\underbrace{\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0}_{w}).$$

The correctness of the definition follows from (tp_1), (tp_2), 1.2(e_3) and (e_4). The element $\xi(j, k) \in H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$ will be called a *marked class*.

1.5. Theorem. a.

$$H_1(\Delta_\Gamma, \Pi, (R_1\Phi_*\mathbf{Q})^w) = \sum_{\substack{j \in I \\ 0 \leqslant k \leqslant w}} \mathbf{Q}\xi(j, k).$$

b. *For any modular symbol* $\{\alpha, \beta, n, m\}_\Gamma$, *where* $\alpha, \beta \in \tilde{\mathbf{Q}}$ *and* $n, m \in \mathbf{Z}^w$, *there are integers* $h(j, k)$ *such that*

$$\{\alpha, \beta\ n, m\}_\Gamma = \sum_{\substack{j \in I \\ 0 \leqslant k \leqslant w}} h(j, k)\xi(j, k). \tag{1.5}$$

Proof. a. This is an immediate corollary of 1.2c) and of point b of Theorem 1.5, proved below.

b. For any $\alpha, \beta \in \tilde{\mathbf{Q}}$ there exists a finite sequence of points $\gamma_1, \ldots, \gamma_h \in \tilde{\mathbf{Q}}$ and elements $g_1, \ldots, g_{h+1}$ of the group $SL(2, \mathbf{Z})$ such that

$(\alpha, \gamma_1) = (g_1(0), g_1(i\infty)), \ldots, (\gamma_i, \gamma_{i+1}) = (g_{i+1}(0), g_{i+1}(i\infty)), \ldots,$

$(\gamma_h, \gamma_\beta) = (g_{h+1}(0), g_{h+1}(i\infty))$

(the proof of this is contained in the proof of Theorem 1.6 of [5]). From 1.2(e_1) and 1.3 it is easy to derive the relation

$$\{\alpha, \gamma, n, m\} = \{\alpha, \beta, n, m\} + \{\beta, \gamma, n, m\}$$

for any $\alpha, \beta, \gamma \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$. Therefore, an induction can prove that

$$\{\alpha, \beta, n, m\} = \{\alpha, \gamma_1, n, m\} + \sum_{i=1}^{h-1} \{\gamma_i, \gamma_{i+1}, n, m\} + \{\gamma_h, \beta, n, m\}.$$

This reduces the proof of 1.5b to the case $(\alpha, \beta) = (g(0), g(i\infty))$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z})$. In this case we have the representation

$$\{\alpha, \beta, n, m\} = g | \{0, i\infty, n', m'\}, \tag{1.6}$$

where $n' = an + cm$ and $m' = dm + bn$, since $g \in \mathrm{SL}(2, \mathbf{Z})$.

Let

$$\prod_{i=1}^{w} (n_i x + m_i y) = \sum_{k=0}^{w} h_k(n, m) x^k y^{w-k}, \tag{1.7}$$

where $h_k(n, m)$ are integral coefficients uniquely determined by the vectors $n, m \in \mathbf{Z}^w$. From 1.2(e_2) and (e_4) we then get

$$\{0\ \ i\infty, n, m\} = \sum_{k=0}^{w} h_k(n, m) \{0, i\infty, 1_k, 1_w - 1_k\}. \tag{1.8}$$

To obtain the decomposition we use the operations (e_2) and (e_4) of 1.2, which, by (tp_2) and (tp_3), commute with the transformations $g|$. Therefore, for any $g \in \mathrm{GL}^+(2, \mathbf{Z})$

$$g | \{0, i\infty, n, m\} = \sum_{k=0}^{w} h_k(n, m) (g | \{0, i\infty, 1_k, 1_w - 1_k\}). \tag{1.9}$$

From (1.6) we then get

$$\{\alpha, \beta, n, m\} = \sum_{k=0}^{w} h_k(n', m') \xi(j, k), \quad j = \Gamma g \in I. \ \blacksquare$$

We shall compute some concrete decompositions. Let $g \in \mathrm{GL}^+(2, \mathbf{Z})$, $0 \leqslant k \leqslant w$, $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ (this notation for the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ will be maintained throughout). By (1.6) and (tp_1) we then have

$$\begin{aligned} g | \{i\infty, 0, 1_k, 1_w - 1_k\} &= gs | \{0, i\infty, 1_w - 1_k, -1_k\}, \\ g | \{i\infty, 1, 1_k, 1_w - 1_k\} &= gt | \{0, i\infty, 1_w, -1_k\}, \\ g | \{1, 0, 1_k, 1_w - 1_k\} &= gt^2 | \{0, i\infty, 1_w - 1_k, -1_w\}. \end{aligned} \tag{1.10}$$

The corresponding polynomials (1.7) are of the form

$$\begin{aligned} x^{w-k}(-y)^k &= (-1)^k x^{w-k} y^k, \\ (x-y)^k x^{w-k} &= \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} x^{w-k+i} y^{k-i}, \\ (x-y)^{w-k} (-y)^k &= \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} x^i y^{w-i}. \end{aligned}$$

Therefore, by (1.9) and (1.10),

$$g|\{i\infty, 0, 1_k, 1_w - 1_k\} = (-1)^k (gs|\{0, i\infty, 1_{w-k}, 1_w - 1_{w-k}\}),$$
$$g|\{i\infty, 1, 1_k, 1_w - 1_k\} = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} (gt|\{0, i\infty, 1_{w-k+i}, 1_w - 1_{w-k+i}\}), \tag{1.11}$$
$$g|\{1, 0, 1_k, 1_w - 1_k\} = \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} (gt^2|\{0; i\infty, 1_i, 1_w - 1_i\}),$$

and in case $g \in \mathrm{SL}(2, \mathbf{Z})$ we find that

$$g|\{i\infty, 0, 1_k, 1_w - 1_k\} = (-1)^k \xi(js, w-k),$$
$$g|\{i\infty, 1, 1_k, 1_w - 1_k\} = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \xi(jt, w-k+i), \tag{1.12}$$
$$g|\{1, 0, 1_k, 1_w - 1_k\} = \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} \xi(jt^2, i),$$

where $j = \Gamma g \in I$.

Proof of Lemma-Definition 1.2. *Uniqueness*. Let $\alpha, \beta \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$. Assume that we have two modular symbols $\{\alpha, \beta, n, m\}$ and $\{\alpha, \beta, n, m\}'$, and put $\sigma = \{\alpha, \beta, n, m\} - \{\alpha, \beta, n, m\}'$. Then $\sigma \in H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w)$ by a. By b, $(\sigma, (\Phi_1, \overline{\Phi_2})) = 0$ for $(\Phi_1, \overline{\Phi_2}) \in S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$. Since the pairing (,) is nondegenerate (see [13], 0.2) we find that $\sigma = 0$.

We first prove (e_1) and (e_2). The second identities of (e_2) and (e_4) evidently follow from (1.2) and (1.3). In the remaining identities in (e_1) and (e_2) there are modular symbols of the form { , , , } that are as yet undefined. Therefore, the proof of these identities is subsumed under the proof of the existence of the third symbol, if at least two of the symbols in the identity are known to exist. The third symbol is defined as a homology class in the space $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$. To prove the existence of the third symbol, properties a and b must be verified. The methods of verifying the various cases are similar. As an example we take the case of the homology class $\{\alpha, \beta, n, m\}$ defined by the first identity of (e_2), assuming known the existence of the modular symbols $\{\alpha, \beta, n(j, 1), m(j, 0)\}$ and $\{\alpha, \beta, n(j, 0), m(j, 1)\}$. By the second identity of (e_2) we then have

$$\partial\{\alpha, \beta, n, m\} = n_j \partial\{\alpha, \beta, n(j, 1), m(j, 0)\} + m_j \partial\{\alpha, \beta, n(j, 0), m(j, 1)\}$$
$$= n_j\{\beta, n(j, 1), m(j, 0)\} - n_j\{\alpha, n(j, 1), m(j, 0)\} + m_j\{\beta, n(j, 0), m(j, 1)\}$$
$$- m_j\{\alpha, n(j, 0), m(j, 1)\} = \{\beta, n, m\} - \{\alpha, n, m\}.$$

Moreover,

$$(\{\alpha, \beta, n, m\}, (\Phi_1, \overline{\Phi_2})) = n_j(\{\alpha, \beta, n(j, 1), m(j, 0)\}, (\Phi_1, \overline{\Phi_2}))$$
$$+ m_j(\{\alpha, \beta, n(j, 0), m(j, 1)\}, (\Phi_1, \overline{\Phi_2})) = \int_\alpha^\beta \Phi_1 \prod_{i=1}^{w} (n_i z + m_i)\, dz$$
$$+ \int_\alpha^\beta \overline{\Phi_2} \prod_{i=1}^{w} (n_i \bar{z} + m_i)\, d\bar{z}.$$

Therefore, the modular symbol $\{\alpha, \beta, n, m\}$ exists. The following statements may be proved in the same style.

(e_1) If the modular symbols $\{\alpha, \beta, n, m\}$ and $\{\beta, \gamma, n, m\}$ exist, then so does the symbol $\{\alpha, \gamma, n, m\}$, and we have the identity $\{\alpha, \gamma, n, m\} = \{\alpha, \beta, n, m\} + \{\beta, \gamma, n, m\}$, where $\alpha, \beta, \gamma \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$ are arbitrary.

The change in statements (e_1), (e_2) proved in this part, together with the uniqueness and existence of the modular symbols $\{\alpha, \beta, n, m\}$ which will be proved below, complete the proof of (e_1) and (e_2). (e_4) is proved similarly.

Existence. Step 1 (*reduction of the construction of the modular symbol* $\{\alpha, \beta, n, m\}_\Gamma$ *for arbitrary* $\alpha, \beta \in \tilde{\mathbf{Q}}$ *and* $n, m \in \mathbf{Z}^w$ *to the case of the construction of the modular symbols* $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}$, *where* $g \in \mathrm{SL}(2, \mathbf{Z})$ *and* $0 \leqslant k \leqslant w$). Choose a finite sequence of points $\gamma_1, \ldots, \gamma_h \in \tilde{\mathbf{Q}}$ between α and β, as at the start of b) in the proof of Theorem 1.5. Using induction and (e_1), we reduce the construction of the symbol $\{\alpha, \beta, n, m\}$ to the construction of symbols of the form $\{g(0), g(i\infty), n, m\}$, where $g \in \mathrm{SL}(2, \mathbf{Z})$. By ($e_2$) and ($e_4$), to prove the existence of arbitrary symbols it suffices to establish the existence of the symbols of the form $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}$, where $g \in \mathrm{SL}(2, \mathbf{Z})$ and $0 \leqslant k \leqslant w$.

Step 2 (*construction of the cellular decomposition* L'). Let L be the cellular decomposition of Δ_Γ constructed in 1.9b of [5]. The cellular decomposition L' of Δ is gotten from L by the following construction. Consider a point $v_0 \in \Delta$ that is not a cusp but is a 0-cell of L. Then v_0 is the orbit of some point $z_0 = gi \in H$, where $g \in \mathrm{SL}(2, \mathbf{Z})$.

(cd_1) If v_0 is not an elliptic point, then we kill the 0-cell v_0, joining the two cells $e_1(j)$ and $e_1(js)$ into one: $e_1'(j) = e_1(js) - e_1(j)$ (see Figure 1). Evidently $e_1'(js) = -e_1'(j)$, where $e_1'(js)$ is considered as a 1-chain with coefficients in $\mathbf{Z}$, and $j = \Gamma g \in I$.

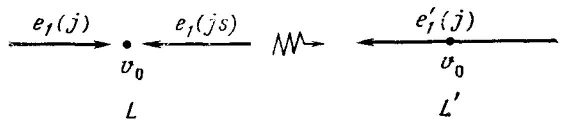

FIGURE 1

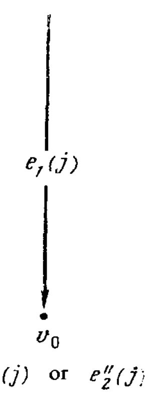

FIGURE 2

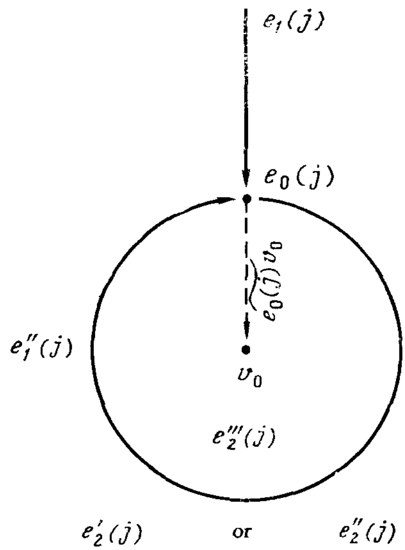

FIGURE 3

(cd_2) If v_0 is an elliptic point, denote by E a small disc centered at the point v_0. The cellular decomposition L locally at the point v_0 has the following form (see Figure 2): 0-cell v_0, 1-cell $e_1(j) = e_2(js)$, 2-cell $e_2'(j)$ or $e_2''(j)$ (our notation for the cells of the complex L is taken from [5]). The following local change at the point v_0 determines L' locally at v_0 (see Figure 3): 0-cell $e_0(j) = \partial E \cap e_1(j)$, 1-cells $e_1''(j) = -\partial e_2'''(j)$, $e_1(j)$ (denotes the cell $e_1(j)$ of the decomposition L with the excised part $\widetilde{e_0(j)v_0}$), 2-cells $e_2'''(j) = E$ (with its natural orientation), $e_2'(j)$ or $e_2''(j)$ (denotes the cell $e_2'(j)$ or $e_2''(j)$ of the decomposition L with the excised part E). Note that in the process of constructing the cell complex L', the original 2-cell $e_2'(j)$ or $e_2''(j)$ of the decomposition L can undergo from one to three excisions.

Step 3 (*construction of the cellular decomposition L_ε' for sufficiently small $\varepsilon \geqslant 0$*). By definition, $L_0' = L'$. Let $\tau = e^{2\pi i z}$ be the natural local parameter at the unique cusp of the curve $\Delta_{\mathrm{SL}(2,\mathbf{Z})}$. Consider some cusp $v_0 \in \Pi$ on the curve $\Delta = \Delta_\Gamma$. This point is of type I_b or I_b^* ($b \geqslant 1$). The canonical projection

$$\Psi : \Delta \to \Delta_{\mathrm{SL}(2,\mathbf{Z})}$$

maps the point v_0 to the cusp of $\Delta_{\mathrm{SL}(2,\mathbf{Z})}$, and the order of ramification at the point v_0 is b. Therefore, we can take $\tau_{v_0} = \sqrt[b]{\tau \circ \Psi}$, choosing one branch of the root $\sqrt[b]{\ }$, as a fixed local parameter at v_0. Put

$$F^\varepsilon = \bigcup_{v_0 \in \Pi} E_{v_0}^\varepsilon, \quad E_{v_0}^\varepsilon = \{|\tau_{v_0}| \leqslant \varepsilon\}.$$

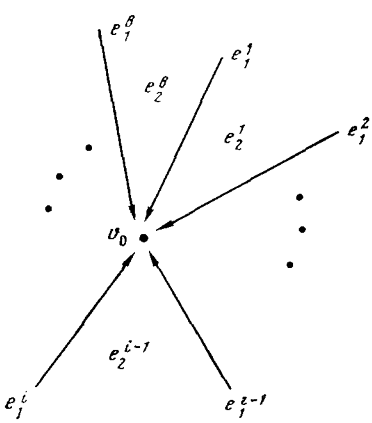

FIGURE 4

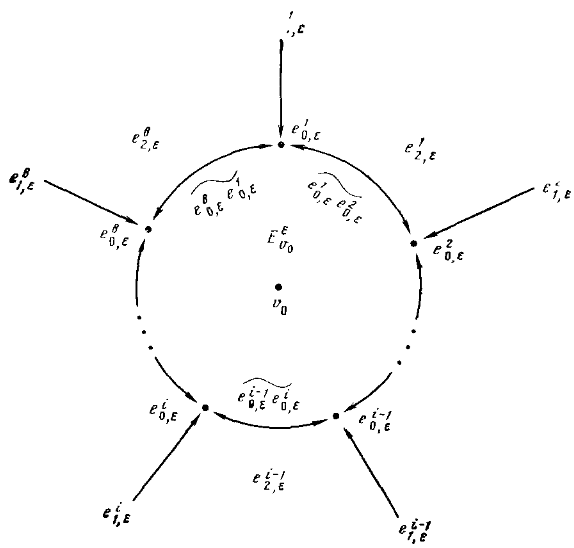

FIGURE 5

The cellular decomposition L_ε' will be a decomposition of the pair (Δ, F^ε). The cellular decomposition L' locally at v_0 has the following form (see Figure 4): 0-cell v_0, 1-cells e_1^i ($1 \leqslant i \leqslant b$) (equal to $e_1(j)$ or $e_1'(j)$), 2-cells e_2^i ($1 \leqslant i \leqslant b$) (equal to $e_2'(j)$ or $e_2''(j)$). The following local change at the point v_0 determines L_ε' locally for $0 < \varepsilon$ at v_0 (see Figure 5): 0-cells $e_{0,\varepsilon}^i$ ($1 \leqslant i \leqslant b$), equal to the points of the intersection $\partial E_{v_0}^\varepsilon \cap e_1^i$, 1-cells $e_{1,\varepsilon}^i$ ($1 \leqslant i \leqslant b$) (they denote the cell e_1^i with the excised part $\widetilde{e_{0,\varepsilon}^i v_0}$), $\widetilde{e_{0,\varepsilon}^1 e_{0,\varepsilon}^2}, \ldots, \widetilde{e_{0,\varepsilon}^{b-1} e_{0,\varepsilon}^b}$, $\widetilde{e_{0,\varepsilon}^b e_{0,\varepsilon}^1}$, 2-cells $E_{v_0}^\varepsilon$ and $e_{2,\varepsilon}^i$ ($1 \leqslant i \leqslant b$) (they denote the cell e_2^i with the excised two-dimensional simplex $\Delta v_0 e_{0,\varepsilon}^i e_{0,\varepsilon}^{i+1}$, where $b + 1 = 1$). The 1- and 2-cells of L_ε' gotten by excision from the cells of the decomposition L' will be denoted by $e_1(j)_\varepsilon$, $e'_1(j)_\varepsilon$, $e'_2(j)_\varepsilon$ and $e''_2(j)_\varepsilon$. The cells $e_0(j)$, $e''_1(j)$ and $e'''_2(j)$ (see (cd$_2$)) are not changed in constructing L'_ε. We maintain our previous notation for them. The remaining cells of L'_ε lie in F^ε. By (cd$_1$) the range of variation of j for the 1-cells $e'_1(j)_\varepsilon$, $e'_1(j)_0 = e'_1(j)$, consists of the j satisfying the inequality $js \neq j$, since the inequality $js \neq j$ is equivalent to $js \neq \pm j$. Therefore, the range of variation of j for the 1-cells $e_1(j)_\varepsilon$, $e_1(j)_0 = e_1(j)$, and $e''_1(j)$ is given by the equality $js = j$. Therefore, a 1-chain of the pair (Δ, F^ε) for the cellular decomposition L'_ε has the form (with coefficients in the sheaf $(R_1\Phi_*\mathbf{Q})^w$)

$$\sigma_1 = \sum_{js\neq j} \alpha'_j e'_1(j)_\varepsilon + \sum_{js=j} (\alpha_j e_1(j)_\varepsilon + \alpha''_j e''_1(j)), \tag{1.13}$$

where

$$\alpha'_j \in (R_1\Phi_*\mathbf{Q})^w|_{e'_1(j)_\varepsilon}, \quad \alpha_j \in (R_1\Phi_*\mathbf{Q})^w|_{e_1(j)_\varepsilon},$$
$$\alpha''_j \in (R_1\Phi_*\mathbf{Q})^w|_{e''_1(j)}$$

(as usual, the symbol for an i-cell denotes a homeomorphic image of the interior of the i-dimensional sphere). The 2-cells $e'''_2(j)$ and the 0-cells $e_0(j)$, as well as $e_1(j)$, are numbered by the j such that $js = j$. According to 1.9b in [5], the 2-cells $e''_2(j)_\varepsilon$, $e''_2(j)_0 = e''_2(j)$, are numbered by the j for which $j \neq -jt$, since the inequality $j \neq -jt$ is equivalent to $j \neq \pm jt$ (we have the condition $j \neq \pm jt$ instead of the condition $j \neq jt$ of [5] because here we are considering a group $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$, not $\Gamma/\pm E \subset \mathrm{SL}(2, \mathbf{Z})/\pm E$). Consequently, the 2-cells $e'_2(j)_\varepsilon$, $e'_2(j)_0 = e'_2(j)$, are numbered by the j such that $-jt = j$. Then a 2-chain with coefficients in the sheaf $(R_1\Phi_*\mathbf{Q})^w$ for the pair (Δ, F^ε) with respect to the cellular decomposition L'_ε has the form

$$\sigma_2 = \sum_{-jt\neq j} \beta''_j e''_2(j)_\varepsilon + \sum_{-jt=j} \beta'_j e'_2(j)_\varepsilon + \sum_{js=j} \beta'''_j e'''_2(j), \tag{1.14}$$

where

$$\beta'_j \in (R_1\Phi_*\mathbf{Q})^w|_{e'_2(j)_\varepsilon}, \quad \beta''_j \in (R_1\Phi_*\mathbf{Q})^w|_{e''_2(j)_\varepsilon},$$
$$\beta'''_j \in (R_1\Phi_*\mathbf{Q})^w|_{e'''_2(j)}.$$

In contrast to (2.5) of [13], this description of the chains σ_1 and σ_2 does not use a noncanonical choice of a point $u_0 \in \Delta'$ or the identification $(R_1\Phi_*\mathbf{Q})^w|_e \hookrightarrow (R_1\Phi_*\mathbf{Q})^w|_{u_0}$ for the cells e of the decomposition L'_ε. The boundary operator on chains αe with coefficients in $(R_1\Phi_*\mathbf{Q})^w$, where e is some cell, acts as follows:

$$\partial(\alpha e) = \sum \alpha|_{e_i} \cdot e_i, \tag{1.15}$$

where $\partial e = \Sigma e_i$, and $\alpha|_{e_i}$ denotes the continuous prolongation of the coefficient $\alpha \in (R_1\Phi_*\mathbf{Q})^w|_{e_i}$ to the boundary e_i.

Step 4 (*construction of bases of the group of sections* $G|_e$ *for the* 0- *and* 1-*cells* $e = e'_1(j)$, $e_1(j)$, $e''_1(j)$ and $e_0(j)$). Let $j = \Gamma g \in I$.

(cb$_1$) $e = e'_1(j)$, $js \neq j$. In this case there exists a domain $D \supset g(\widehat{0\, i\infty})$ ($\widehat{0\, i\infty}$ is the imaginary semiaxis) of the type described in 1.1. In case (*) we have defined a canonical basis $e'_{1,1}(g)$, $e'_{1,2}(g)$ of the group $G|_{e'_1(j)}$ (the first lower index indicates the dimension of the cell over which the sections are taken, and the second numbers the basis). In case (*) fails, the basis $e'_{1,1}(g)$, $e'_{1,2}(g)$ is determined up to $\pm$id (see the note in §2 to step 7 (cg_2) of the proof of 2.3).

(cb_2) $e = e_1(j)$, $js = j$. In this case there exists a domain $D \supset g(\widetilde{i\infty\ i})$ satisfying the conditions of 1.1. By (cd_2) we have defined an identification $e_1(j) \subset g(\widetilde{i\infty})z_0 \subset g(\widetilde{i\infty\ i})$ (see Figure 6) for some uniquely determined point $z_0 \in H$. Therefore, as in (cb_1), in the group of sections $G|_{e_1(j)}$ we have defined the basis $e_{1,1}(g)$, $e_{1,2}(g)$. Making a similar construction for $gs \in j$, we get a basis $e_{1,1}(gs)$, $e_{1,2}(gs)$ in the same group $G|_{e_1(j)}$. In the last case $e_1(j)$ is identified with $\widetilde{g(0)z'_0} \subset \widetilde{g(0\ i\infty)}$ (see Figure 6), and the point z'_0 is uniquely determined.

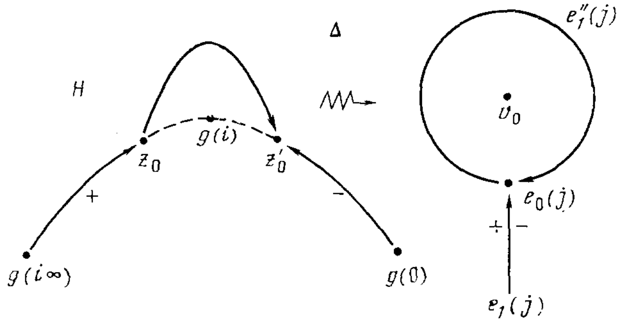

FIGURE 6

(cb_3) $e = e''_1(j)$, $js = j$. Denote by $\widetilde{z_0z'_0}$ a path with orientation of the cell $e''_1(j)$ on H (see Figure 6). This path is simple. There exists a domain $D \supset \widetilde{z_0z'_0}$ (as usual, $\widetilde{z_0z'_0}$ denotes the interior points of the path from z_0 to z'_0) of type 1.1. Therefore, as above, identifying $e''_1(j)$ with $\widetilde{z_0z'_0}$, we get a basis $e''_{1,1}(g)$, $e''_{1,2}(g)$ in the group $G|_{e''_1(j)}$.

(cb_4) $e = e_0(j)$, $js = j$. Identifying z_0 with $e_0(j)$, we correspondingly find bases $e_{0,1}(g)$, $e_{0,2}(g)$ and $e_{0,1}(gs)$, $e_{0,2}(gs)$ in the group $G|_{e_0(j)}$.

Note that cases (cb_{2-4}) are possible only if (*) fails, in which case there can exist elliptic points equivalent to i. In the case where (*) fails, w is even. Therefore, the symmetric products

$$(n, m)^{**}_{*}(g) = \prod_{i=1}^{w} (n_i e^{**}_{*,1}(g) + m_i e^{**}_{*,2}(g)) \in (R_1\Phi_*\mathbf{Q})^w|_{e^{**}_{*}(j)} \tag{1.16}$$

are uniquely determined by g and $n, m \in \mathbf{Z}^w$, where ** and * denote the indices ′, ″, 1 or 2, or no index. By the construction of the bases e_1, e_2 in 1.1 and (cb_{2-4}) we have

$$\begin{aligned}(m, n)_1(g)|_{e_0(j)} &= (m, n)''_1(g)|_{-e_0(j)} = (m, n)_0(g),\\ (m, n)''_1(g)|_{e_0(j)} &= (m, n)_1(gs)|_{e_0(j)} = (m, n)_0(gs).\end{aligned} \tag{1.17}$$

For brevity, in the sequel we shall write $\alpha \cdot e_\varepsilon$ instead of $\alpha|_{e_\varepsilon} \cdot e_\varepsilon$, for $\alpha \in (R_1\Phi_*\mathbf{Q})^w|_{e_\varepsilon}$. By (1.13) and (1.15)–(1.17), and because all the 0-cells, except $e_0(j)$, lie in F^e, all the 1-cycles of the pair (Δ, F^ε) with coefficients in the sheaf $(R_1\Phi_*\mathbf{Q})^w$ can be represented in the form of a finite sum:

$$\sigma_1 = \sum_{js\neq j}\sum_{g\in j}\sum_{k=0}^{w} \alpha'_{j,g,k}(1_k, 1_w - 1_k)'_1(g)\cdot e_1(j)_\varepsilon$$

$$+ \sum_{js=j}\sum_{g\in j}\sum_{k=0}^{w} \alpha_{j,g,k}(((1_k, 1_w - 1_k)_1(g)$$

$$-(1_k, 1_w - 1_k)_1(gs))\cdot e_1(j)_\varepsilon + (1_k, 1_w - 1_k)''_1(g)\cdot e''_1(j)), \tag{1.18}$$

where $\alpha_{j,g,k}, \alpha'_{j,g,k} \in \mathbf{Q}$.

Denote by $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon \in H_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$ the homology class of the cycle

$$(1_k, 1_w - 1_k)_1'(g)e_1'(j)_\varepsilon$$

if $js \neq j$, or the cycle

$$((1_k, 1_w - 1_k)_1(gs) - (1_k, 1_w - 1_k)_1(g))e_1(j)_\varepsilon - (1_k, 1_w - 1_k)_1''(g)\cdot e_1''(j)$$

if $js = j$, where $j = \Gamma g$.

Step 5 (*properties of the homology class* $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon$).

a_ε.

$$\partial\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon = \{z'_\varepsilon, 1_k, 1_w - 1_k\}^\varepsilon - \{z_\varepsilon, 1_k, 1_w - 1_k\}^\varepsilon$$
$$\in H_0(F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w),$$

where z'_ε, z_ε in the case $js \neq j$ are determined by the oriented identification $\widetilde{z_\varepsilon z'_\varepsilon} \subset g(\widetilde{0\, i\infty})$, and in case $js = j$ are determined by the identifications $\widetilde{z'_\varepsilon z_0} \subset \widetilde{g(i\infty)z_0}$ and $\widetilde{z_\varepsilon z'_0} \subset \widetilde{g(0)z'_0}$. This property follows immediately from the construction of the basis e_1, e_2 in 1.1, (1.15), $(\mathrm{cb}_{1\text{-}4})$ and (1.2).

b_ε. Denote by $\sigma_\varepsilon \in H_{1+j}(B^w, B^w_{F^\varepsilon}, \mathbf{Q})$ the image of the class $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon$ under the mapping (3.5) of [13] for $F = F^\varepsilon$. Then

$$\int_{\sigma_\varepsilon} \omega_{\Phi_1} + \overline{\omega}_{\Phi_2} = \int_{z_\varepsilon}^{z'_\varepsilon} \Phi_1 z^k dz + \int_{z_\varepsilon}^{z'_\varepsilon} \overline{\Phi_2 z^k dz}. \tag{1.19}$$

To prove this, consider the identification of one of the 1-cells $e_1(j)_\varepsilon$, $e'_1(j)_\varepsilon$ or $e''_1(j)_\varepsilon$ with $e \subset H$. Suppose e_1, e_2 is a basis of the group $G|_e$, and $D \supset e$ is an open set chosen according to 1.1. Then, in case $(*)$, by (5.2) of [12] there is a canonical isomorphism

$$B^w_\Gamma|_D \cong \mathrm{id}\times\mathbf{Z}^w\times\mathbf{Z}^w\backslash D\times\mathbf{C}^w, \tag{1.20}$$

where $\mathrm{id}\times\mathbf{Z}^w\times\mathbf{Z}^w$ acts according to (0.4). In case $(*)$ fails, by 2.1 of [12] and the choice of the function $z(\tilde{u})$ there is also an isomorphism (1.20), but it is determined only up to the automorphism $(\pm\mathrm{id}, 0, 0)$. Changing the sign of the basis e_1, e_2 if necessary, we can consider that $e_1|_{z_0}$ and $e_2|_{z_0}$ correspond to $z_0 \times [0, 1]$ and $[0, 1]$ (see the beginning of §3.7 in [13]) for $z_0 \in D$. In case $(*)$ this normalization is introduced in 1.1. By 3.7 of [13] and the description of the embedding

$$(R_1\Phi_*\mathbf{Q})^w \subsetneq (R_1\Phi_*\mathbf{Q})^{\otimes w}$$

in 2.2 of [13] we have the following fundamental set $\tilde{\sigma} \subset D \times \mathbf{C}^w$ with orientation for the chain $[w!e_1^k e_2^{w-k} e]$: $\tilde{\sigma}$ is projected onto $e \subset D$ and each fiber over $z_0 \in e$ has the form

$$\tilde{\sigma}|_{z_0} = \bigcup_{(n,m)} \prod_{i=1}^{w} ([0, 1] \times z_0 n_i \oplus [0, 1] m_i) \subset \mathbf{C}^w, \tag{1.21}$$

where (n, m) runs through all the permutations $(s(1_k), s(1_w - 1_k))$ for $s \in A_w$ (the group of permutations of w elements). The definition of the differential form ω_Φ by 5.5 and (5.4) of [12] in the general case and by (0.1) of [12] in the case (*) gives an evident form of the lifting of the holomorphic form $\omega_\Phi|_D$ and the antiholomorphic form $\bar{\omega}_\Phi|_D$ to $D \times \mathbf{C}^w$:

$$\begin{aligned} \tilde{\omega}_\Phi &= \Phi dz \wedge d\zeta_1 \wedge \dots \wedge d\zeta_w, \\ \tilde{\bar{\omega}}_\Phi &= \bar{\Phi} d\bar{z} \wedge d\bar{\zeta}_1 \wedge \dots \wedge d\bar{\zeta}_w. \end{aligned} \tag{1.22}$$

Then, since

$$\int_{[0,1]z_0} d\zeta_i = z_0, \quad \int_{[0,1]\bar{z}_0} d\bar{\zeta}_i = \bar{z}_0, \quad \int_{[0,1]} d\zeta_i = \int_{[0,1]} d\bar{\zeta}_i = 1,$$

by (1.21) and (1.22) we have

$$\int_{w!e_1^k e_2^{w-k}\cdot e} \omega_{\Phi_1} + \bar{\omega}_{\Phi_2} = w! \left(\int_{z_0}^{z_0'} \Phi_1 z^k dz + \int_{z_0}^{z_0'} \bar{\Phi}_2 \bar{z}^k d\bar{z} \right), \tag{1.23}$$

where $\partial e = z_0' - z_0$ on H. From the definition of the homology class $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon$, 3.7 of [13], (1.23) and the additivity of the integral with respect to the domain of integration we get

$$\int_{w!\sigma_\varepsilon} \omega_{\Phi_1} + \bar{\omega}_{\Phi_2} = w! \left(\int_{z_\varepsilon}^{z_\varepsilon'} \Phi_1 z^k dz + \int_{z_\varepsilon}^{z_\varepsilon'} \bar{\Phi}_2 \bar{z}^k d\bar{z} \right).$$

By cancelling $w!$ we prove (1.19) for any cusp forms $\Phi_1, \Phi_2 \in S_{w+2}(\Gamma)$.

Step 6 (*construction of the modular symbol* $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}_\Gamma$ *for* $g \in \mathrm{SL}(2, \mathbf{Z})$ *and* $0 \leqslant k \leqslant w$). The pairs (Δ, F^ε) form a cofinal system in the projective system 3.1 of [13], and the homomorphism (3.1) of [13] for $F' = F^{\varepsilon'}$, $F = F^\varepsilon$, $0 < \varepsilon' < \varepsilon$ and $\mathcal{F} = (R_1\Phi_*\mathbf{Q})^w$ takes the homology class $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^{\varepsilon'}$ to $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon$. Therefore, the limit

$$\begin{aligned} \{g(0), g(i\infty), 1_k, 1_w - 1_k\}_\Gamma &= \varprojlim_{F^\varepsilon} \{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon \\ &\in H_1(\Delta_\Gamma, \Pi, (R_1\Phi_*\mathbf{Q})^w) \end{aligned}$$

is defined. Passing to the limit in properties a_ε and b_ε gives a and b. Consequently, $\{g(0), g(i\infty), 1_k, 1_w - 1_k\}_\Gamma$ is a modular symbol. By step 1 this proves the existence of the modular symbols $\{\alpha, \beta, n, m\}_\Gamma$ for arbitrary $\alpha, \beta \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$.

By (1.18)

$$H_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w) = \sum_{\substack{g \in \mathrm{SL}(2,\mathbf{Z}) \\ 0 \leqslant k \leqslant w}} \mathbf{Q}\{g(0), g(i\infty), 1_k, 1_w - 1_k\}^\varepsilon,$$

whence, by (3.2) of [13], we get c. By 2.56 of [13] we have

$$H_0(\Delta, (R_1\Phi_* \mathbf{Q})^w) = 0 \quad \text{for} \quad w \geqslant 1.$$

Point d is then gotten immediately from c and the surjectivity of the mapping ∂ in the exact sequence (3.3) of [13] for $\mathfrak{F} = (R_1\Phi_*\mathbf{Q})^w$ and $\Pi' = \varnothing$. Properties ($e_{1,2}$) and (e_4) have been proved above.

We prove (e_3). From the definition of a modular symbol and of cusp forms Φ_1, $\Phi_2 \in S_{w+2}(\Gamma)$, by substituting $g(z)$ in the integral

$$\int\limits_{g(\alpha)}^{g(\beta)} \Phi_1 \prod_{i=1}^{w} ((dn_i - cm_i) z + (-bn_i + m_i))\, dz + \int\limits_{g(\alpha)}^{g(\beta)} \bar{\Phi}_2 \prod_{i=1}^{w} ((dn_i - cm_i) \bar{z} + (-bn_i + m_i))\, d\bar{z},$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we obtain

$$(\{g(\alpha),\ g(\beta),\ dn - cm, -bn + am\},\ (\Phi_1,\ \bar{\Phi}_2)) = (\{\alpha,\ \beta,\ n,\ m\},\ (\Phi_1,\ \bar{\Phi}_2))$$

for all Φ_1, $\Phi_2 \in S_{w+2}(\Gamma)$. From the second identity in (e_3) and the uniqueness we then get the third identity. To prove the second identity it suffices to establish that

$$\{z_E,\ n,\ m\}^E_\Gamma = \{g(z_E),\ dn - cm,\ -bn + am\}^E_\Gamma \tag{1.24}$$

by 1.1 and (1.3). z_E and $z'_E = g(z_E)$ belong to one orbit $v_E \in \Delta'$. In agreement with 1.1 we denote by e_1, e_2 and e'_1, e'_2 the bases in the group of sections $G|_{v_E}$ that correspond to the identification of v_E with z_E and z'_E. By definition of the representation (1.4) of [12], by the continuity along a path in H' of the construction of the basis 1.1, and by (2.1) of [12] we have

$$(e_1',\ e_2') = \pm (e_1,\ e_2)\,(g). \tag{1.25}$$

The sign $\pm$ is absent in case (*); and when (*) fails it can be left out of account because w is even; that is, we shall consider that

$$e_1' = ae_1 + be_2, \qquad e_2' = ce_1 + de_2. \tag{1.26}$$

Consequently, by (1.2), $\{z'_E, dn - cm, -bn + am\}^E$ is the homology class of the 0-cycle

$$\prod_{i=1}^{w} ((dn_i - cm_i)(ae_1 + be_2) + (-bn_i + am_i)(ce_1 + de_2))\, v_E = \prod_{i=1}^{w} (n_i e_1 + m_i e_2)\, v_E,$$

which proves (1.24). ■

1.6. By (e_2) and (e_4) it is easy to prove, using the notation of (1.8), the following identity:

$$\{\alpha,\ \beta,\ n,\ m\} = \sum_{k=0}^{w} h_k(n,\ m)\{\alpha,\ \beta,\ 1_k,\ 1_w - 1_k\} \tag{1.27}$$

for any $\alpha, \beta \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$. From (1.27) for $\alpha = g(0)$, $\beta = g(i\infty)$ and $g \in SL(2, \mathbf{Z})$, and from (1.16), (1.7) and the linearity of the limit $\underleftarrow{\lim}$, we then get

$$\{g(0), g(i\infty), n, m\}_\Gamma = \underleftarrow{\lim}\{g(0), g(i\infty), n, m\}^\varepsilon \tag{1.28}$$

for any $n, m \in \mathbf{Z}^w$, where $\{g(0), g(i\infty), n, m\}^\varepsilon \in H_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$ is the homology class of the cycle $(n, m)_1'(g) \cdot e_1'(j)_\varepsilon$ when $js \neq j$ and of the cycle

$$((n, m)_1(gs) - (n, m)_1(g))e_1(j)_\varepsilon - (n, m)_1''(g)e_1''(j)$$

when $js = j, j = \Gamma g$. Put

$$\tilde{\xi}(j, k)_\varepsilon = \begin{cases} ((d+c)1_k - c1_w, (-b-a)1_k + a1_w)_1'(g)\, e_1(j)_\varepsilon' & \text{if } js \neq j; \\ ((d+c)1_k - c1_w, (-b-a)1_k + a1_w)_1(gs) - ((d+c)1_k - c1_w, \\ (-b-a)1_k + a1_w)_1(g)) \cdot e_1(j)_\varepsilon - ((d+c)1_k - c1_w, (-b-a) \\ \quad \times 1_k + a1_w)_1''(g)\, e_1''(j) & \text{if } js = j, \end{cases}$$

for arbitrary $j = \Gamma g \in I$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $0 \leqslant k \leqslant w$. By (1.16) and (1.18), $\tilde{\xi}(j, k)_\varepsilon \in Z_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$, the group of 1-cycles of the pair (Δ, F^ε) with coefficients in the sheaf for the cellular decomposition L_ε'. Moreover, it is evident that $\tilde{\xi}(j, k)_\varepsilon$ is independent of the choice of the vector $1_k \in \mathbf{Z}^w$ for fixed k. To prove the correctness of the definition of the cycle $\tilde{\xi}(j, k)_\varepsilon$ it is also necessary to verify that the definition is independent of the choice of $g \in j$. Suppose

$$g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tilde{g}g \in j$$

is another choice, where

$$\tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \Gamma.$$

Reasoning as we did above in the case of the points z_E and z_E', we get a relation for the bases that is analogous to (1.26) if we make a suitable normalization of $\pm$ in case $(*)$ fails:

$$e_{*,1}^{**}(g') = \tilde{a}e_{*,1}^{**}(g) + \tilde{b}e_{*,2}^{**}(q), \quad e_{*,2}^{**}(g') = \tilde{c}e_{*,1}^{**}(g) + \tilde{d}e_{*,2}^{**}(g) \tag{1.29}$$

for all possible $**$ and $*$. Therefore, by (1.16),

$$(n, m)_*^{**}(\tilde{g}g) = (\tilde{a}n + \tilde{c}m, \tilde{b}n + \tilde{d}m)_*^{**}(g) \tag{1.30}$$

for all $\tilde{g} \in \Gamma$, $g \in SL(2, \mathbf{Z})$ and $n, m \in \mathbf{Z}^w$. Since

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

from (1.30) it is easy to derive the identity

$$\begin{aligned} &((d' + c')1_k - c'1_w, (-b' - a')1_k + a'1_w)_*^{**}(g') \\ &= ((d + c)1_k - c1_w, (-b - a)1_k + a1_w)_*^{**}(g). \end{aligned} \tag{1.31}$$

In the same way, from the relations (1.30) for the pairs $g's = \tilde{g}gs$ and gs we get (1.31) with the substitutions $g' \mapsto g's$ and $g \mapsto gs$, which proves the correctness of the definition of the cycle $\tilde{\xi}(j, k)_\varepsilon$. Denote by $\xi(j, k)_\varepsilon \in H^1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$ the homology class of $\tilde{\xi}(j, k)_\varepsilon$. From (1.28), (1.3$'$) and the definition of $\xi(j, k)$ we then have

$$\xi(j, k) = \varprojlim \xi(j, k)_\varepsilon \tag{1.32}$$

as $\varepsilon \to 0$.

Proof of Proposition 0.3 (on Shimura integrals). This immediately follows from the definition of the pairing (,) (see 0.3 of [13]) and Lemma-Definition 1.2 for $\Phi_1 = \Phi$ and $\Phi_2 = 0$. ■

Denote by $\sigma \subset B_\Gamma^w$ the set of points with coefficient $1/w!$ that was constructed with respect to $\widetilde{\alpha\beta} \subset H'$ and $n, m \in \mathbf{Z}^w$ just before Proposition 0.3. There σ was called a relative cycle of the pair B_Γ^w, $B_\Gamma^w|_{p_0, p_1}$. To prove this one must show that $\operatorname{Supp}\sigma \subset B_\Gamma^w$ is a subcomplex of some cellular decomposition of B_Γ^w. The last claim is evident for arbitrary intersections $\operatorname{Supp}\sigma \cap B_\Gamma^w|_{\Delta - E_{p_0}\cup E_{p_1}}$, where E_{p_0} and E_{p_1} are small discs about the cusps p_0 and p_1 corresponding to α and β. That is, the verification of the cellularity of the embedding of $\operatorname{Supp}\sigma$ is reduced to a local problem. This problem is not trivial; it is solved by studying the character of the approach of $\operatorname{Supp}\sigma$ to the fibers $B_\Gamma^w|_{p_0}$ and $B_\Gamma^w|_{p_1}$. This investigation will not be carried out here. The cycle σ was introduced with an illustrative goal: to graphically represent the homology class $GR_{1,w}\{\alpha, \beta, n, m\}_\Gamma$ in §0, where we had not given a complete description of the geometric realization mapping or a definition of the modular symbol. We shall show, under the assumption that the embedding $\operatorname{Supp}\sigma \subset B_\Gamma^w$ is cellular, that the homology class of the cycle σ in $H_{w+1}(B_\Gamma^w, B_\Gamma^w|_\Sigma, \mathbf{Q})$ is $GR_{1,w}\{\alpha, \beta, n, m\}_\Gamma$. For this it suffices, by 1.2, the definition of the pairing (,), and 3.2a, b of [13], to verify the following properties:

a. $\partial\sigma = GR_{0,w}(\{\beta, n, m\}_\Gamma - \{\alpha, n, m\}_\Gamma)$,

b. $\int_\sigma \omega_{\Phi_1} + \bar{\omega}_{\Phi_2} = \int_\alpha^\beta \Phi_1 \prod_{j=1}^w (n_j z + m_j)\, dz + \int_\alpha^\beta \bar{\Phi}_2 \prod_{j=1}^w (n_j \bar{z} + m_j)\, d\bar{z}$; $\Phi_1, \Phi_2 \in S_{w+2}(\Gamma)$.

The last follows from the description of a fundamental set for $\operatorname{Supp}\sigma$ and the lifting of the form ω_Φ on $H' \times \mathbf{C}^w$. By the stabilization of the limit (3.8) of [13], the definition of $\{**, n, m\}$, and 3.6 of [13], to prove point a it suffices to establish the homology

$$\sigma \cap B_{v_E}^w \sim \left[\prod_{j=1}^w (n_j e_1 + m_j e_2)\right], \qquad v_E = \operatorname{orbit}(z_E),$$

where $z_E \in \widetilde{\alpha\beta}$ is a point near $**$; $** = \alpha, \beta$. The last is evident from the choice of the basis e_1, e_2 in case (*) (see 1.1), the description of the embedding $(R_1\Phi_*\mathbf{Q})^w \hookrightarrow (R_1\Phi_*\mathbf{Q})^{\otimes w}$, 3.6 of [13] and the definition of σ. ■

§2. The Eichler-Shimura relations

Theorem 1.5a proves that the marked classes $\xi(j, k)$ generate the space $H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$. In 2.1 we indicate a system of relations among the marked classes $\xi(j, k)$ which are called the Eichler-Shimura relations. This is a complete system of relations in the sense of Theorem 2.3.

2.1. The marked classes $\xi(j, k)$ satisfy the following system of linear homogeneous relations over $\mathbf{Z}$:

$$\begin{gathered} \xi(j, k) + (-1)^k \xi(js, w-k) = 0, \\ \xi(j, k) + \sum_{i=0}^{k} (-1)^{k+1} \binom{k}{i} \xi(jt, w-k+i) + \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} \xi(jt^2, i) = 0, \end{gathered} \tag{2.1}$$

where $j \in I_\Gamma = \Gamma \setminus SL(2, \mathbf{Z})$, the right cosets of the subgroup Γ (for brevity we shall often simply write I), and $0 \leqslant k \leqslant w$ are integers. The relations in the system (2.1) are called the *Eichler-Shimura relations*. Let $j = \Gamma g$, $g \in SL(2, \mathbf{Z})$. By (1.3), 1.3b and 1.2(e_1) we have

$$g\,|\,\{0,\ i\infty,\ 1_k,\ 1_w-1_k\}+g\,|\,\{i\infty,\ 0,\ 1_k,\ 1_w-1_k\}=0,$$
$$g\,|\,\{0,\ i\infty,\ 1_k,\ 1_w-1_k\}+g\,|\,\{i\infty,\ 1,\ 1_k,\ 1_w-1_k\} + g\,|\,\{1,\ 0,\ 1_k,\ 1_w-1_k\}=0,$$

whence, by (1.12), we immediately get (2.1).

2.2. Let $C_\Gamma = \bigoplus \mathbf{Q}(j, k)$ be the free vector space over $\mathbf{Q}$ generated by all the pairs (j, k), where $j \in I_\Gamma$ and $0 \leqslant k \leqslant w$. Then there is a uniquely determined linear mapping

$$\xi : C_\Gamma \to H_1(\Delta,\ \Pi;\ (R_1\Phi_*\mathbf{Q})^w),$$

taking the generator (j, k) to $\xi(j, k)$.

2.3. THEOREM. *The kernel of the mapping ξ is generated by the vectors*

$$(j,\ k)+(-1)^k(js,\ w-k), \tag{2.2}$$
$$(j, k)+\sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}(jt,\ w-k+i)+\sum_{i=0}^{w-k}(-1)^{w+i}\binom{w-k}{i}(jt^2,\ i), \tag{2.3}$$
$$j\in I,\quad 0\leqslant k\leqslant w,\quad k\in\mathbf{Z}.$$

PROOF. *Step* 1 (*reduction to the case of a homomorphism* ξ_ε). The homology classes $\xi(j, k)_\varepsilon$ (see 1.6) determine a $\mathbf{Q}$-linear mapping

$$\xi_\varepsilon : C \to H_1(\Delta,\ F^\varepsilon,\ (R_1\Phi_*\mathbf{Q})^w),$$
$$\xi_\varepsilon : (j,\ k) \mapsto \xi(j,\ k)_\varepsilon.$$

By (1.32), $\xi = \varprojlim \xi_\varepsilon$. By the stabilization of the limit $\varprojlim$, Theorem 2.3 is equivalent to the theorem with the same formulation but with the mapping ξ_ε in place of ξ, for sufficiently small ε.

Step 2 ($(j, k) + (-1)^k(js, w - k) \in \operatorname{Ker} \tilde{\xi}_\varepsilon$ *if* $js \neq j$). The cycles $\tilde{\xi}(j, k)$ (see 1.6) determine a $\mathbf{Q}$-linear mapping

$$\tilde{\xi}_\varepsilon : C \to Z_1(\Delta,\ F^\varepsilon,\ (R_1\Phi_*\mathbf{Q})^w).$$

Let $j \in I$, $js \neq j$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j$. Since $g(\widetilde{0\, i\infty}) = gs(0\, i\infty)$, in case (*) ($cb_1$) yields

$$e'_{1,1}(gs) = e'_{1,1}(g), \quad e'_{1,2}(gs) = e'_{1,2}(g). \tag{2.4}$$

When (*) fails it is also possible to assume (2.4); for this it suffices to choose an appropriate sign $\pm$, which does not influence the subsequent computation because w is even. By (cd_1) and the subsequent construction of the decomposition L'_ε, it is evident that $e'_1(js)_\varepsilon = -e'_1(j)_\varepsilon$. By the definition of $\tilde{\xi}_\varepsilon$, since $gs = \left(\begin{smallmatrix} b & -a \\ d & -c \end{smallmatrix}\right)$ we have

$$\begin{aligned}&\tilde{\xi}_\varepsilon((j,\ k)+(-1)^k(js,\ w-k))\\ &= ((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)'_1\,(g)\,e_1(j)'_\varepsilon\\ &+ (-1)^{k+1}((-c+d)\,1_{w-k} - d1_w,\ (a-b)\,1_{w-k}+b1_w)'_1\,(gs)\,e_1(j)'_\varepsilon,\end{aligned}$$

whence, by (1.16) and (2.4), we find that $(j, k) + (-1)^k(js, w - k) \in \operatorname{Ker} \tilde{\xi}_\varepsilon$ if $js \neq j$.

Step 3 (*a remark on bases of symmetric powers*). Let e_1, e_2 be a basis of the free $\mathbf{Z}$-module $\mathbf{Z}e_1 \oplus \mathbf{Z}e_2$. It is then easy to show that the following vectors in the symmetric tensor product $(\mathbf{Q}e_1 \oplus \mathbf{Q}e_2)^w$ form a basis of it over $\mathbf{Q}$:

$$(de_1 - be_2)^k(-ce_1 + ae_2)^{w-k} = ((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)$$

for integers $0 \leqslant k \leqslant w$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{Z})$ is some fixed matrix.

Step 4 ($\tilde{\xi}_\varepsilon$ *is an epimorphism and* Ker $\tilde{\xi}_\varepsilon$ *is generated by the vectors* (2.2) *with* $js \neq j$). By the construction of the cellular decomposition L'_ε (see 1.9b of [5], step 2.3 of the proof of 9.2) the relation

$$\sum q'_j e'_1(j)_\varepsilon + \sum q''_j e''_1(j)$$

for $q'_j, q''_j \in \mathbf{Q}$ is generated by the relations

$$e'_1(j)_\varepsilon + e'_1(js)_\varepsilon = 0$$

for $js \neq j$. We choose a finite set $I' \subset I$ such that for any $j \in I$ there exists a unique $j' \in I'$ for which $j = j's^k$. By the preceding, by (1.18), step 3, and by the definition of $\tilde{\xi}(j, k)_\varepsilon$, the vectors $\tilde{\xi}(j, k)_\varepsilon$, for $j \in I'$ and integers $0 \leqslant k \leqslant w$, form a basis of the space $Z_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$. Denote by C' the subspace of C spanned by the vectors (j, k), where $j \in I'$ and $0 \leqslant k \leqslant w$. Then $\tilde{\xi}_\varepsilon|_{C'}$ is an isomorphism. It is evident that every vector in C is distinguished from some vector of C' by a linear combination of vectors of the form (2.2) for $js \neq j$ with coefficients in $\mathbf{Q}$. Consequently, by step 2, Ker $\tilde{\xi}_\varepsilon$ is generated by vectors of the form (2.2), where $j \in I, js \neq j$, and $0 \leqslant k \leqslant w$. Moreover, from the fact that $\tilde{\xi}_\varepsilon|_{C'}$ is an isomorphism it follows that $\tilde{\xi}_\varepsilon$ is an epimorphism.

Step 5 (*reduction to the proof that the subspace of* 1-*boundaries* $B_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$ *is generated by the vectors*

$$\tilde{\xi}(j, k)_\varepsilon + (-1)^k\,\tilde{\xi}(js, w-k)_\varepsilon, \quad js = j, \tag{2.5}$$

$$\tilde{\xi}(j, k)_\varepsilon + \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i}\tilde{\xi}(jt, w-k+i)_\varepsilon + \sum_{i=0}^{w-k}(-1)^{w+i}\binom{w-k}{i}\tilde{\xi}(jt^2, i)_\varepsilon, \tag{2.6}$$

where $j \in I$ *and* $0 \leqslant k \leqslant w$). By the definition of ξ_ε we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\xi}_\varepsilon} & Z_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w) \\ & {}_{\xi_\varepsilon}\searrow & \downarrow \\ & & H_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w) \end{array}$$

where the vertical arrow is the natural epimorphism mapping a cycle to its homology class. The kernel of the vertical arrow, $B_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$, is the space of 1-boundaries of the pair (Δ, F^ε) with coefficients in the sheaf $(R_1\Phi_*\mathbf{Q})^w$ for the cellular decomposition L'_ε. Therefore, the variant of Theorem 2.3 that has ξ_ε in place of ξ reduces, by the preceding step, to the proof that the vectors (2.5) and (2.6) generate the space $B_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$.

Step 6 (*generators of the space* V^S *for an elliptic mapping* S). Let V be a linear space of dimension $w + 1$ over $\mathbf{Q}$ with basis $e_0, \ldots, e_w$. By ellipticity of S we understand that the Jordan form of S on $V \otimes \mathbf{C}$ is of the form

$$\begin{pmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_w \end{pmatrix},$$

where the ε_i are roots of 1 of degree l. Then $S^l = E$, and a simple verification proves that the vectors $\sum_{i=0}^{l-1} S^i e_k$, $0 \leqslant k \leqslant w$, generate the space V^S of S-invariant vectors.

Step 7 (*construction of generators of the group of sections of* $(R_1\Phi_*\mathbf{Q})^w|_{e_2}$ *for the* 2-*cells* $e_2 = e_2'(j), e_2''(j), e_2'''(j)$). Let $j = \Gamma g \in I$.

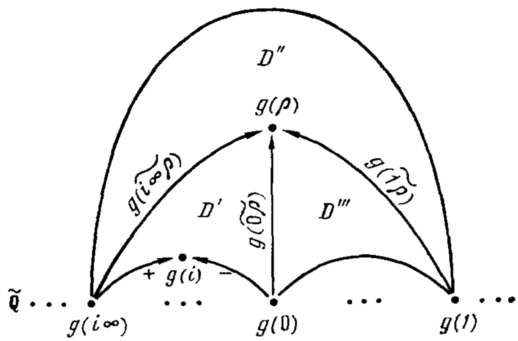

FIGURE 7

(cg$_1$) $e_2 = e_2'(j)$, $-jt = j$. Let $D' = g(E')$, $D'' = gt(E')$ and $D''' = gt^2(E')$ (see Figure 7), where E' is a triangular domain with Poincaré segments $\widetilde{i\infty\, 0}$, $\widetilde{0\rho}$ and $\widetilde{\rho\, i\infty}$. The domains D', D'' and D''' satisfy the conditions 1.1 and are identified with the open subset $D \subset e_2'(j)$. According to 1.1, in the group of sections $G|_D$ we construct bases $e_{2,1}'(g)$, $e_{2,2}'(g)$ and $e_{2,1}'(gt)$, $e_{2,2}'(gt)$ and $e_{2,1}'(gt^2)$, $e_{2,2}'(gt^2)$, identifying D', D'' and D''' respectively with D. Let v_0 and v_1 be the orbits of the points $g(\rho) = gt(\rho)$ and $g(i\infty)$ under Γ, where $\rho = \eta + 1 = e^{\pi i/3}$. Since $-jt = j$, the point $v_1 \in \Delta$ is also the orbit of the points $g(1) = gt(i\infty)$ and $g(0) = gt^2(i\infty)$.

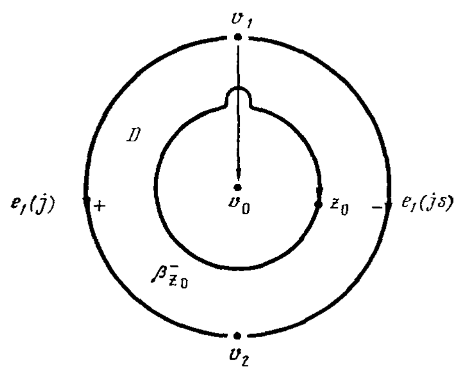

FIGURE 8

The cell $e_2'(j)$ decomposes into cells (see Figure 8): 0-cell v_0, 1-cells $\widetilde{v_1 v_0}$ (the image of $g(\widetilde{0\rho})$ or $g(\widetilde{i\infty\,\rho})$ or $g(\widetilde{1\rho})$ under the factorization of H' by Γ), and 2-cell D. In the cell

$e_2'(j)$ there is only one point of special type v_0, an elliptic point, equivalent to η; that is, by 5.2a of [12], of type II or IV*. For points $z_0 \in D$ we denote by $\beta_{z_0}^-$ a negative simple circuit around v_0 along the open ring $e_2'(j) - v_0$. The continuous prolongation of a section $u \in (R_1\Phi_*\mathbf{Q})^w|_D$ along the path $\beta_{z_0}^-$ defines a $\mathbf{Q}$-linear mapping

$$S_-: (R_1\Phi_*\mathbf{Q})^w|_D \to (R_1\Phi_*\mathbf{Q})^w|_D.$$

It is then evident that we have a canonical isomorphism (induced by restriction)

$$(R_1\Phi_*\mathbf{Q})^w|_{e_2'(j)-v_0} \xrightarrow{\sim} (R_1\Phi_*\mathbf{Q})^w|_D^{S_-}$$

because of the local constancy of the sheaf $(R_1\Phi_*\mathbf{Q})^w$ on $e_2'(j) - v_0$. Therefore, by definition of the sheaf $(R_1\Phi_*\mathbf{Q})^w$ (see 2.1 of [13]) we get a canonical isomorphism

$$(R_1\Phi_*\mathbf{Q})^w|_{e_2'(j)} \xrightarrow{\sim} (R_1\Phi_*\mathbf{Q})^w|_D^{S_-}. \tag{2.7}$$

Moreover, by the local constancy of the sheaf $(R_1\Phi_*\mathbf{Q})^w|_{e_2'(j)-v_0}$ and the definition of S_-, as well as (1.3) of [12], we have the following commutative diagram (the horizontal isomorphisms are induced by restriction):

$$\begin{array}{ccc} (R_1\Phi_*\mathbf{Q})^w|_D & \xrightarrow{\sim} & (R_1\Phi_*\mathbf{Q})^w|_{z_0} \\ {\scriptstyle S_-}\downarrow & & \downarrow{\scriptstyle s_{\beta_{z_0}^-}} \\ (R_1\Phi_*\mathbf{Q})^w|_D & \xrightarrow{\sim} & (R_1\Phi_*\mathbf{Q})^w|_{z_0}, \end{array} \qquad z_0 \in D,$$

where the linear mapping $s_{\beta_{z_0}^-}$ is induced by the mapping* $s_{\beta_{z_0}^-}$ for $w = 1$. Therefore, to prove S_- elliptic it suffices to establish that $s_{\beta_{z_0}^-}$ is elliptic when $w = 1$. By definition of the normal form of monodromy A_{v_0} (see [12], §1), since v_0 is of type IV* or II (in case (*) we have IV*), in a suitable basis of $G|_{z_0} \subset R_1\Phi_*\mathbf{Q}|_{z_0}$ the mapping $s_{\beta_{z_0}^-}$ can be written as the action of a right matrix $\pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$ (see [12], Table 1). This proves S_- elliptic. Moreover, since the eigenvalues of the matrix $\pm\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ are $\pm\eta$ and $\pm\eta^{-1}$, we have $S_-^3 = \pm E$. According to 1.1 and the definition of the mapping S_- in case (*) we get

$$\begin{aligned} S_- e_{2,1}'(g) &= e_{2,1}'(gt), & S_- e_{2,2}'(g) &= e_{2,2}'(gt), \\ S_-^2 e_{2,1}'(g) &= e_{2,1}'(gt^2), & S_-^2 e_{2,2}'(g) &= e_{2,2}'(gt^2). \end{aligned} \tag{2.8}$$

In case (*) fails we shall also assume (2.8). For this it suffices to choose a suitable normalization of $\pm$ for the bases $e_{2,1}'(gt)$, $e_{2,2}'(gt)$ and $e_{2,1}'(gt^2)$, $e_{2,2}'(gt^2)$ which does not influence the subsequent choice of generators because w is even. By steps 3 and 6, and by (2.7) and (2.8), as generators of the space $(R_1\Phi_*\mathbf{Q})^w|_{e_2'(j)}$ we can take the vectors

$$-\sum_{i=0}^{2} ((d+c)\,1_k - c\,1_w,\ (-b-a)\,1_k + a\,1_w)_2'(gt^i) \tag{2.9}$$

for integers $0 \leqslant k \leqslant w$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in j$ and the $(n, m)_2'(*)$ are defined by (1.16).

(cg_2) $e_2 = e_2''(j)$, $-jt \neq j$. Let $D = g(E'')$, where E'' is the domain defined in 1.9b of [5]. The domain D satisfies conditions 1.1 and is identified with the cell $e_1''(j)$. Therefore, according to 1.1, in the group of sections $G|_{e_2''(j)}$ we have defined a basis $e_{2,1}''(g)$, $e_{2,2}''(g)$.

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* *Editor's note.* The apposed symbolism, which is apparently in error, is reproduced from the original.

Then, by step 3, in the space $(R_1\Phi_*\mathbf{Q})^w|_{e_2''(j)}$ we have defined a basis

$$-((d+c)1_k - c1_w, (-b-a)1_k + a1_w)_2''(g) \tag{2.10}$$

for integers $0 \leqslant k \leqslant w$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j$ and the $(n, m)_2''(*)$ are defined by (1.16).

REMARK. The construction of bases in 1.1 is carried out for domains $D \subset H'$. The same construction can be generalized to the case of an arbitrary connected, simply connected domain $D \subset H$ with the condition $gD \cap D = \varnothing$ for $g \in \Gamma - \{\pm E\}$. This is necessary, for example, in (cb_1) and (cg_2). For the construction we embed D into Δ and consider an arbitrary connected, simply connected subdomain $D' \subset D \cap H'$. The given domain D does not contain elliptic points of the group Γ. Consequently, by 5.2a of [12] all the points of D are of nonspecial type. By the connectivity and simple connectivity of D we then have an isomorphism (induced by restriction)

$$G|_D \simeq G|_{D'}.$$

Therefore, the choice of a basis of $G|_D$ reduces to the choice of a basis of $G|_{D'}$. Denote by e_1, e_2 a basis of $G|_D$ such that the basis $e_1|_{D'}, e_2|_{D'}$ of $G|_{D'}$ is constructed according to 1.1 for D'. By (5.2) of [12] the canonicity of this basis is evident, as is its independence of the choice of D' in case $(*)$. In the case where $(*)$ fails, by the continuity of the construction of 1.1 along a path, the choice of basis does not depend on D' and is defined up to $\pm\mathrm{id}$. This means that for every domain $D' \subset D$ satisfying conditions 1.1 the basis $e_1|_{D'}, e_2|_{D'}$ is constructed by 1.1.

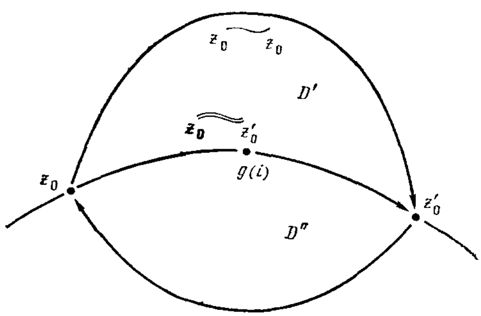

FIGURE 9

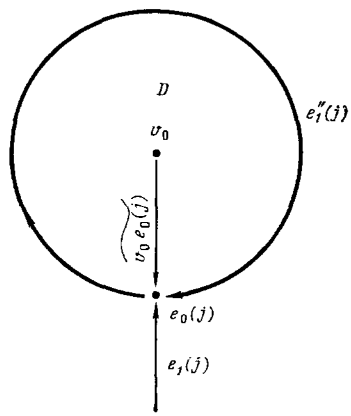

FIGURE 10

(cg_3) $e_2 = e_2'''(j), js = j$ (see step 3 of the proof of 1.2). Denote by D' the 2-gon with sides $\widetilde{z_0z_0'}$ and $\widetilde{\widetilde{z_0z_0'}}$ (see Figure 9), where the path $\widetilde{z_0z_0'}$ is defined in step 4 (cb_3) of the proof of 1.2 (see Figure 6), and $\widetilde{\widetilde{z_0z_0'}}$ is a path on $g(\widehat{0\,i\,\infty})$ from z_0 to z_0'. Put $D'' = gsg^{-1}D'$ (see Figure 9); this is a domain obtained from D' by turning about $g(i)$. The domains D' and D'' are identified with a domain $D \subset e_2'''(j)$. According to 1.1, in the group of sections $G|_D$ we construct bases $e_{2,1}'''(g), e_{2,2}'''(g)$ and $e_{2,1}'''(gs), e_{2,2}'''(gs)$, identifying D' and D'', respectively, with D. The image of the path $\widetilde{\widetilde{z_0z_0'}}$ on Δ will be the cell $\widehat{v_0e_0(j)}$ (see Figures 10 and 6). The cell $e_2'''(j)$ decomposes into cells (see Figure 10): 0-cell v_0, 1-cell $\widehat{v_0e_0(j)}$, and 2-cell D. In the cell $e_2'''(j)$ there is only one point of special type III or III*.

Just as in (cg_1) the paths $\beta^-_{z_0}$ determine a **Q**-linear mapping S_-, and there is a canonical isomorphism (induced by restriction)

$$(R_1\Phi_*\mathbf{Q})^w\big|_{e_2'''(j)} \xrightarrow{\sim} (R_1\Phi_*\mathbf{Q})^w\big|_D^{S_-}. \tag{2.11}$$

Since $A_{v_0} = \pm\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ by Table 1 of [12], we can use the methods of (cg_1) to prove that S_- is elliptic and that for $w = 1$ the mapping has eigenvalues i and $-i$. Therefore, the eigenvalues of S_- are $i^\alpha(-i)^{w-\alpha} = (-1)^\alpha(-i)^w$. Now note that, by Table 4 (*) of [12], the cells $e_2'''(j)$ are possible only in case (*) fails, since v_0 is an elliptic point equivalent to i. Therefore w is even, and the eigenvalues of S_- are $(-1)^{\alpha+w/2}$, where $0 \leqslant \alpha \leqslant w$ is an integer. This means that $S_-^2 = E$. According to 1.1, the definition of the bases $e_{2,1}'''(*)$, $e_{2,2}'''(*)$, and the definition of the mapping S_-, we find that, for a suitable normalization of $\pm$,

$$S_-e'''_{2,1}(g) = e'''_{2,1}(gs), \quad S_-e'''_{2,2}(g) = e'''_{2,2}(gs). \tag{2.12}$$

By steps 3 and 6, and (2.11) and (2.12), as generators of the space $(R_1\Phi_*\mathbf{Q})^w|_{e_2'''(j)}$ we can take the vectors

$$\sum_{i=0}^{1}((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)_2'''(gs^i) \tag{2.13}$$

for integers $0 \leqslant k \leqslant w$, where $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in j$ and the $(n, m)_2'''(*)$ are defined by (1.16).

Step 8 (*reduction of step* 5 *to the proof that the boundaries*

$$\partial\left(-\sum_{i=0}^{2}((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)_2'(gt^i)\,e_2'(j)_\varepsilon\right),$$

$$\partial(-((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)_2''(g)\,e_2''(j)_\varepsilon),$$

$$\partial\left(\sum_{i=0}^{1}((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)_2'''(gs^i)\,e_2'''(j)\right)$$

are respectively the vectors (2.6) *for* $-jt = j$; (2.6) *for* $-jt \neq j$; (2.5); *where* $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in j$). The proof of this reduction follows from the fact that the boundaries indicated in the formulation of this step generate the space $B_1(\Delta, F^\varepsilon, (R_1\Phi_*\mathbf{Q})^w)$. The last follows from (1.14) and step 7, since $e_2'(j)_\varepsilon \subset e_2'(j)$ and $e_2''(j)_\varepsilon \subset e_2''(j)$. Therefore, to prove Theorem 2.3 it remains to establish the relations

$$\begin{aligned}\partial\left(-\sum_{i=0}^{2}((d+c)\,1_k - c1_w,\ (-b-a)\,1_k + a1_w)_2'(gt^i)\,e_2'(j)_\varepsilon\right) \\ = \tilde{\xi}(j, k)_\varepsilon + \sum_{i=0}^{k}(-1)^{k+1}\binom{k}{i}\tilde{\xi}(jt, w-k+i)_\varepsilon \\ + \sum_{i=0}^{w-k}(-1)^{w+i}\binom{w-k}{i}\tilde{\xi}(jt^2, i)_\varepsilon\end{aligned} \tag{2.14}$$

for $-jt = j$,

$$\partial(-((d+c)1_k - c1_w, (-b-a)1_k + a1_w)''_2(g)\, e''_2(j)_\varepsilon)$$
$$= \tilde{\xi}(j, k)_\varepsilon + \sum_{i=0}^{k} (-1)^{k+1} \binom{k}{i} \tilde{\xi}(jt, w-k+i)_\varepsilon$$
$$+ \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} \tilde{\xi}(jt^2, i)_\varepsilon \tag{2.15}$$

for $-jt \neq j$, and

$$\partial\left(\sum_{i=0}^{1} ((d+c)1_k - c1_w, (-b-a)1_k + a1_w)'''_2 (gs^i)\, e'''_2(j)\right)$$
$$= \tilde{\xi}(j, k)_k + (-1)^k \tilde{\xi}(js, w-k)_\varepsilon \tag{2.16}$$

for $js = j$, where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j \in I$.

Step 9 (*proof of the relations* (2.14)). Note that in this case $-jt = j$.

a. *Case* $js = j$. Note that in this case $\Gamma = SL(2, \mathbf{Z})$. By formula (8) of [5], (cd_2) and step 3 of the proof of 1.2 we have

$$\partial e'_2(j)_\varepsilon = e_1(j)_\varepsilon - e_1(j)_\varepsilon + e''_1(j) \tag{2.17}$$

(1-cells of the form $\widetilde{e^i_{0,\varepsilon} e^{i+1}_{0,\varepsilon}} \subset F^\varepsilon$ (see Figure 5) are not taken into account, since the differential of the pair (Δ, F^ε) is computed). To compute the differentials of chains with coefficients in a sheaf, by (1.15) it is necessary to know that $\alpha|_{e_i}$ is the restriction with respect to a corresponding approach to the boundary e_i. By definition of the bases $e_{1,1}(g)$, $e_{1,2}(g)$; $e_{1,1}(gs)$, $e_{1,2}(gs)$ and $e''_{1,1}(g)$, $e''_{1,2}(g)$ in (cb_{2-3}) of the proof of 1.2 (see Figure 6), as well as by the definition of the bases $e'_{2,1}(g)$, $e'_{2,2}(g)$ in step 7 (cg_1) (see Figure 7) and the continuity of the construction of the bases in 1.1, we have, for a suitable normalization of ± 1 (for a picture see Figures 7 and 8, where + and − indicate the corresponding approaches to $e_1(j)$ and to $-e_1(j)$, and v_2 is the orbit of the point $g(i)$),

$$(e'_{2,1}(g), e'_{2,2}(g))|_{e_1(j)} = (e_{1,1}(g), e_{1,2}(g)),$$
$$(e'_{2,1}(g), e'_{2,2}(g))|_{-e_1(j)} = (e_{1,1}(gs), e_{1,2}(gs), \tag{2.18}$$
$$(e'_{2,1}(g), e'_{2,2}(g))|_{\pm e''_1(j)} = (e''_{1,1}(g), e''_{1,2}(g))$$

for any $g \in j$. In the last relation $|_{\pm e''_1(j)}$, since the cell $e''_1(j) \subset D$ and the boundary therefore does not depend on the approach. In the first two relations (2.18) the cell $e_1(j)$ is considered as the boundary of a cell of D (see (cg_1) and Figure 8, $e_1(js) = e_1(j)$). Note that $gt \in j$ and $gt^2 \in j$, since (*) fails in this case. Moreover, $e^*_1(jt^i)_\varepsilon = e^*_1(j)_\varepsilon$ since $jt^i = j$. Therefore, by (2.9), (2.7), (1.16), (2.18), (2.17) and (1.15) the left side of (2.14) equals

$$\sum_{i=0}^{2} (((d+c)1_k - c1_w, (-b-a)1_k + a1_w)_1 (gt^i s)$$
$$- ((d+c)1_k - c1_w, (-b-a)1_k + a1_w)_1 (gt^i))\, e_1(jt^i)_\varepsilon$$
$$- ((d+c)1_k - c1_w, (-b-a)1_k + a1_w)''_1 (gt^i) \cdot e''_1(jt^i). \tag{2.19}$$

From the relations

$$(x-y)^k x^{w-k} = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} x^{w-k+i} y^{k-i},$$

$$(x-y)^{w-k} (-y)^k = \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} x^i y^{w-i}$$

by substituting

$$x = -ce^{**}_{*,1}(\tilde{g}) + ae^{**}_{*,2}(\tilde{g}), \quad y = -(c+d)\, e^{**}_{*,1}(\tilde{g}) + (a+b)\, e^{**}_{*,2}(\tilde{q});$$
$$x = (-c-d)\, e^{**}_{*,1}(\tilde{g}) + (a+b)\, e^{**}_{*,2}(\tilde{g}), \quad y = -de^{**}_{*,1}(\tilde{g}) + be^{**}_{*,2}(\tilde{g})$$

correspondingly, from (1.16) we get the identities

$$((d+c)\, 1_k - c1_w,\ (-b-a)\, 1_k + a1_w)^{**}_*(\tilde{g})$$
$$= \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} (d1_{w-k+i} - (c+d)\, 1_w,\ -b1_{w-k+i} + (a+b)\, 1_w)^{**}_*(\tilde{g}), \quad (2.20)$$

$$((d+c)\, 1_k - c1_w,\ (-b-a)\, 1_k + a1_w)^{**}_*(\tilde{g})$$
$$= \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} (-c1_i - d1_w,\ a1_i + b1_w)^{**}_*(\tilde{g}) \quad (2.21)$$

for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\tilde{g} \in \mathrm{SL}(2, \mathbf{Z})$. For any $j \in I$ such that $jt^is = jt^i$ with a fixed integer $0 \leqslant i \leqslant 2$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j$ and an integer $0 \leqslant k \leqslant w$, we denote by $T(jt^i, k)_\varepsilon$ the ith addend of the sum (2.19). Then from (2.21), (2.20) and the definition of the cycle $\tilde{\xi}(jt^i, k)_\varepsilon$ we get the identities

$$T(j, k)_\varepsilon = \tilde{\xi}(j, k)_\varepsilon,$$
$$T(jt, k)_\varepsilon = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} \tilde{\xi}(jt, w-k+i)_\varepsilon, \quad (2.22)$$
$$T(jt^2, k)_\varepsilon = \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} \tilde{\xi}(jt^2, i)_\varepsilon,$$

since

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad gt = \begin{pmatrix} a+b & -a \\ c+d & -c \end{pmatrix}, \quad gt^2 = \begin{pmatrix} b & -a-b \\ d & -c-d \end{pmatrix}.$$

From (2.19) and (2.22), (2.14) immediately follows in case $js = j$.

b. *Case* $js \neq j$. By (8) of [5], by (cd_1) and step 3 of the proof of 1.2 (see Figures 8 and 1; the nonelliptic point v_0 of Figure 1 corresponds to the point v_2 of Figure 8) we have

$$\partial e_2'(j)_\varepsilon = -e_1'(j)_\varepsilon. \quad (2.23)$$

By the definition of the bases $e'_{1,1}(g)$, $e'_{1,2}(g)$ in (cb_1) of the proof of 1.2, and $e'_{2,1}(g)$, $e'_{2,2}(g)$ in step 7 (cg_1) (see Figure 7; $g(\widehat{0\, i\infty})$ is a boundary of D'), and also by the continuity of the construction of the bases 1.1 (for the correctness see the remark after (cg_2)) we have, for a suitable normalization of ± 1, in case (*) fails for any $g \in \pm j$,

$$(e'_{2,1}(g),\ e'_{2,2}(g))\big|_{\pm e'_1(j)} = (e'_{1,1}(g),\ e'_{1,2}(g)). \quad (2.24)$$

The $\pm$ upon restriction of (2.24) to the boundary $e_1'(j)$ of the cell D indicates the simplicity of the approach. By general considerations accompanying the establishment of (2.19), by (2.23), and also by (2.24) for $g \in \pm j$, $gt \in \pm j$ and $gt^2 \in \pm j$ we find that the left member of (2.14) equals

$$\sum_{i=0}^{2} ((d+c)\,1_k - c1_w, (-b-a)\,1_k + 1_w)_1' (gt^i)\, e_1'(jt^i)_\varepsilon, \tag{2.25}$$

since $e_1'(\pm jt^i) = e_1'(j(-t)^i) = e_1'(j)$ as long as $-jt = j$. We extend the definition of the 1-cycle $T(jt^i, k)_\varepsilon$ in the following way. For any $j \in I$ such that $jt^is \neq jt^i$ for a fixed integer $0 \leqslant i \leqslant 2$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j$ and an integer $0 \leqslant k \leqslant w$, we denote by $T(jt^i, k)_\varepsilon$ the ith summand of the sum (2.25). By (2.20), (2.21) and the definition of the cycles $\tilde{\xi}(jt^i, k)_\varepsilon$ it is easy to verify (2.22) for the cycles $T(jt^i, k)_\varepsilon$ just introduced. Therefore, the cycle $T(jt^i, k)_\varepsilon$ is now defined for any $j \in I$, $0 \leqslant i \leqslant 2$ and $0 \leqslant k \leqslant w$, and is connected with the cycles $\tilde{\xi}(j, k)_\varepsilon$ by (2.22). It is evident that (2.25) and (2.22) imply (2.14) in case $js \neq j$.

Step 10 (*proof of* (2.15)). In this case $-jt \neq j$. By (9) of [5] and steps 2 and 3 of the proof of 1.2,

$$\partial e_2''(j)_\varepsilon = \sum_{i=0}^{2} \begin{cases} e_1(jt^i)_\varepsilon - e_1(jt^i)_\varepsilon + e_1''(jt^i) & \text{if } jt^is = jt^i, \\ -e_1'(jt^i) & \text{if } jt^is \neq jt^i. \end{cases} \tag{2.26}$$

By (cg$_2$), one of the boundaries of the domain D is $g(\widetilde{0\, i\infty})$ (in Figure 7 the domain D is represented as the interior of the triangle with Poincaré segments joining the vertices $g(i\infty)$, $g(0)$, $g(1)$). Therefore, for any $g \in \pm j$, by the construction of the bases in step 7 (cg$_2$), and also by the continuity of the construction of the bases 1.1 we have, for a suitable normalization in case (*) fails,

$$(e_{2,1}''(g), e_{2,2}''(g))\,|_{\mp e_1'(j)} = (e_{1,1}'(g), e_{1,2}'(g)) \tag{2.27}$$

if $js \neq j$, and

$$\begin{aligned} (e_{2,1}''(g), e_{2,2}''(g))\,|_{e_1(j)} &= (e_{1,1}(g), e_{1,2}(g)), \\ (e_{2,1}''(g)\; e_{2,2}''(g))\,|_{-e_1(j)} &= (e_{1,1}(gs), e_{1,2}(gs)), \\ (e_{2,1}''(g), e_{2,2}''(g))\,|_{\pm e_1''(j)} &= (e_{1,1}''(g), e_{1,2}''(g)) \end{aligned}$$

if $js = j$, since $e_{*,1}^{**}(\pm g) = e_{*,1}^{**}(g)$ and $e_{*,2}^{**}(\pm g) = e_{*,2}^{**}(g)$. Note that, by the construction in step 7 (cg$_2$), since $gt^i(E'') = D$ for any integers i, for a suitable normalization of $\pm$ in case (*) fails

$$(e_{2,1}''(gt^i), e_{2,2}''(gt^i)) = (e_{2,1}''(g), e_{2,2}''(g)) \tag{2.28}$$

for any integer i. Then, by the general considerations accompanying the deduction of (2.19), by (2.26), (2.28) and (2.27) for $g \in j$, $gt \in jt$ and $gt^2 \in jt^2$, and also by the definition of the cycles $T(jt^i, k)_\varepsilon$, introduced in step 9, we find that the left member of (2.15) equals

$$\sum_{i=0}^{2} T(jt^i, k)_\varepsilon,$$

from which (2.15) follows, by (2.22).

Step 11 (*proof of* (2.16)). In this case $js = j$. Using (1.16) and the definition of $\tilde{\xi}(j, k)_\varepsilon$ and $\tilde{\xi}(js, w - k)_\varepsilon$ from the matrices $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j$ and $gs = \left(\begin{smallmatrix} b & -a \\ d & -c \end{smallmatrix}\right) \in j$ (see 1.6), and also by the equalities $e^{**}_{*,1}(\pm g) = e^{**}_{*,1}(g)$ and $e^{**}_{*,2}(\pm g) = e^{**}_{*,2}(g)$, we get

$$\begin{aligned}
&\tilde{\xi}(j, k)_\varepsilon + (-1)^k \tilde{\xi}(js, w-k)_\varepsilon \\
&= ((de_{1,1}(gs) - be_{1,2}(gs))^k (-ce_{1,1}(gs) + ae_{1,2}(gs))^{w-k} \\
&\quad - (de_{1,1}(g) - be_{1,2}(g))^k (-ce_{1,1}(g) + ae_{1,2}(g))^{w-k} \\
&\quad + (-1)^k (-ce_{1,1}(g) + ae_{1,2}(g))^{w-k} (-de_{1,1}(g) + be_{1,2}(g))^k \\
&\quad - (-1)^k (-ce_{1,1}(gs) + ae_{1,2}(gs))^{w-k} (-de_{1,1}(gs) + be_{1,2}(gs))^k) e_1(j)_\varepsilon \\
&\quad - ((de''_{1,1}(g) - be''_{1,2}(g))^k (-ce''_{1,1}(g) + ae''_{1,2}(g))^{w-k} \\
&\quad + (-1)^k (-ce''_{1,1}(gs) + ae''_{1,2}(gs))^{w-k} (-de''_{1,1}(gs) + be''_{1,2}(gs))^k) e''_1(j) \\
&= -\sum_{i=0}^{1} ((d+c) 1_k - c 1_w, (-b-a) 1_k + a 1_w)''_1 (gs^i) e''_1(j).
\end{aligned}$$

Therefore, for the proof of (2.16) it suffices to establish the equality

$$\begin{aligned}
&\partial\left(\sum_{i=0}^{1} ((d+c) 1_k - c 1_w, (-b-a) 1_k + a 1_w)'''_2 (gs^i) e'''_2(j)\right) \\
&= -\sum_{i=0}^{1} ((d+c) 1_k - c 1_w, (-b-a) 1_k + a 1_w)''_1 (gs^i) e''_1(j). \qquad (2.29)
\end{aligned}$$

By (cd_2) of the proof of 1.2 (see Figure 3)

$$\partial e'''_2(j) = -e''_1(j). \qquad (2.30)$$

Moreover, by definition of the bases $e''_{1,1}(g)$, $e''_{1,2}(g)$ and (cb_3) of the proof of 1.2 (see Figures 6 and 9), $e'''_{2,1}(g)$, $e'''_{2,2}(g)$ in step 7 (cg_3) (see Figures 9 and 10), and also by the continuity of the construction of the bases 9.1, for a suitable normalization of $\pm$ we get

$$(e'''_{2,1}(g), e'''_{2,2}(g))\big|_{\pm e''_1(j)} = (e''_{1,1}(g), e''_{1,2}(g)) \qquad (2.31)$$

for any $g \in j$, from which, by (2.11), (1.15), (1.16), (2.30) and (2.31) for $g \in j$ and $gs \in j$ we get (2.29). ■

§3. The period mapping of cusp forms

We recall the following notation (see [8], §2.1):

$$(\Phi \mid [g]_{w+2})(z) = (\det g)^{\frac{w+2}{2}} (cz + d)^{-w-2} \Phi(gz);$$

the transformation $[g]_{w+2}$ is defined for arbitrary $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL^+(2, \mathbf{Z})$, Φ is a function on the upper half-plane, and $(\det g)^{(w+2)/2}$ is an arithmetic root.

3.1. Proposition. *For any* $g \in j \in I_\Gamma$, *an integer* $0 \leqslant k \leqslant w$, *and a form* $\Phi = \Phi_1 + \overline{\Phi}_2 \in S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$ *we have*

$$(\xi(j, k), \Phi) = \int_0^{i\infty} (\Phi_1 \mid [g]_{w+2}) z^k \, dz + \int_0^{i\infty} \overline{(\Phi_2 \mid [g]_{w+2})} \bar{z}^k \, d\bar{z},$$

where the pairing (,) *is defined in* [12].

3.2. The complex numbers

$$r(j, k, \Phi) = \int_0^{i\infty} (\Phi_1 \mid [g]_{w+2}) z^k \, dz + \int_0^{i\infty} (\overline{\Phi_2 \mid [g]_{w+2}}) \bar{z}^k d\bar{z}$$

for $g \in j \in I$ and integers $0 \leqslant k \leqslant w$ are called the *periods of the cusp form* $\Phi = \Phi_1 + \Phi_2 \in S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$.

By Proposition 3.1,

$$r(j, k, \Phi) = (\xi(j, k), \Phi). \tag{3.1}$$

By (3.1) we then get a system of relations for the periods of a cusp form:

$$\begin{gathered} r(j, k, \Phi) + (-1)^k r(js, w-k, \Phi) = 0, \\ r(j, k, \Phi) + \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} r(jt, w-k+i, \Phi) \\ + \sum_{i=0}^{w-k} (-1)^{i+w} \binom{w-k}{i} r(jt^2, i, \Phi) = 0, \end{gathered} \tag{3.2}$$

where $j \in I$ and $0 \leqslant k \leqslant w$ are integers. The relations (3.2) are called the *Eichler-Shimura relations for periods*.

PROOF OF 0.4. This is an immediate consequence of (3.2) for

$$\Phi \in S_{w+2}(\Gamma) \subset S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$$

and 3.4a. ∎

As usual, by C^* we denote the dual space of C (see 2.2), and by $(j, k)^*$ we denote the basis of C^* dual to (j, k);

$$((j, k), (j', k')^*) = \begin{cases} 0 & \text{if } j \neq j' \text{ or } k \neq k', \\ 1 & \text{if } j = j' \text{ and } k = k'. \end{cases}$$

Moreover, for any subfield $K \subset \mathbf{C}$ we put

$$C(K) = C \otimes_{\mathbf{Q}} K$$

and dually

$$C^*(K) = (C(K))^* = C^* \otimes_{\mathbf{Q}} K.$$

Consider the subspace of $C^*(K)$ consisting of the vectors

$$\sum_{\substack{j \in I, \\ 0 \leqslant k \leqslant w}} r(j, k)(j, k)^*$$

that satisfy the Eichler-Shimura relations; that is, $r(j, k) \in K$ and

$$\begin{gathered} r(j, k) + (-1)^k r(js, w-k) = 0, \\ r(j, k) + \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} r(jt, w-k+i) \\ + \sum_{i=0}^{w-k} (-1)^{w+i} \binom{w-k}{i} r(jt^2, i) = 0 \end{gathered} \tag{3.3}$$

for all $j \in I_\Gamma$ and $0 \leqslant k \leqslant w$. We denote this subspace by $R_{w+2}(\Gamma, K)$. It is evident that for subfields $K' \subset K \subset \mathbf{C}$ we have the inclusion $R_{w+2}(\Gamma, K') \hookrightarrow R_{w+2}(\Gamma, K)$. Moreover, since equations (3.3) are defined over $\mathbf{Q}$, the embedding determines a canonical isomorphism of K-spaces

$$R_{w+2}(\Gamma, K') \otimes_{K'} K \rightleftarrows R_{w+2}(\Gamma, K), \qquad x \otimes k \to k \cdot x. \tag{3.4}$$

3.3. DEFINITION. The space $R_{w+2}(\Gamma, \mathbf{C})$ is called the space of periods of cusp forms of weight $w + 2$ for the group Γ. The $r(j, k, \Phi)$, the periods of the cusp forms $\Phi \in S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$ (see 3.2), determine a $\mathbf{C}$-linear mapping

$$r: S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)} \to R_{w+2}(\Gamma, \mathbf{C}),$$
$$r: \Phi \mapsto \sum_{\substack{j \in I \\ 0 \leqslant k \leqslant w}} r(j, k, \Phi)(j, k)^*.$$

In fact, the $\mathbf{C}$-linearity of r follows from the $\mathbf{C}$-linearity of the pairing (,) in the second argument and from (3.1), and $r(\Phi) \in R_{w+2}(\Gamma, \mathbf{C})$ by (3.2) and (3.3). The mapping r will be called the *period mapping*.

3.4. THEOREM. a. *The period mapping r is an embedding.*

b. codim Im $r = t_1 + t_2\delta(w)$, *where* $\delta(w) = 1$ *for even* w *and is* 0 *otherwise* (*for the definition of* t_i *see* [12], 5.3).

PROOF. a. Suppose $r(\Phi) = 0$ for $\Phi = \Phi_1 + \overline{\Phi}_2 \in S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$. By definition of the mapping r and by (3.1), $(\xi(j, k), \Phi) = 0$ for any $j \in I$ and $0 \leqslant k \leqslant w$. Therefore, from the nondegeneracy of the pairing (,) (see [13], 0.2), and also since the $\xi(j, k)$ by 1.5a generate $H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w) \subset H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$, we get $\Phi = 0$.

b. For any field $K \subset \mathbf{C}$ duality gives us an isomorphism

$$R_{w+2}(\Gamma, K)^* \simeq H_1(\Delta, \Pi, (R_1\Phi_*K)^w). \tag{3.5}$$

This duality is induced by the natural pairing (,) of the spaces $C(K)$ and $C^*(K)$. The proof of this is an immediate consequence of Theorem 2.3, the definition of $R_{w+2}(\Gamma, K)$ and the relation

$$H_1(\Delta, \Pi, (R_1\Phi_*K)^w) = H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w) \otimes_{\mathbf{Q}} K. \tag{3.6}$$

The last relation is the formula for "change of scalars": $R_1\Phi_*K = R_1\Phi_*\mathbf{Q} \otimes_{\mathbf{Q}} K$. Moreover, we note that by the definition of r we have

$$(\sigma, r(\Phi)) = (\sigma, \Phi) \tag{3.7}$$

for any $\sigma \in H_1(\Delta, \Pi, (R_1\Phi_*K)^w)$ and $\Phi \in S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$.

The duality (3.5) and the injectivity of r enable us to know the dimension of $R_{w+2}(\Gamma, \mathbf{C})$ and of Im r. Namely,

$$\dim_{\mathbf{C}} R_{w+2}(\Gamma, \mathbf{C}) = \dim_{\mathbf{C}} H_1(\Delta, \Pi, (R_1, \Phi_*\mathbf{C})^w) = \dim_{\mathbf{Q}} H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w),$$
$$\dim_{\mathbf{C}} \operatorname{Im} r = 2\dim_{\mathbf{C}} S_{w+2}(\Gamma) = \dim_{\mathbf{C}} H_1(\Delta, (R_1\Phi_*\mathbf{C})^w) = \dim_{\mathbf{Q}} H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w)$$

(see [13], 6.1). Therefore,

$$\operatorname{codim}_{\mathbf{C}} \operatorname{Im} r = \dim_{\mathbf{Q}} H_1(\Delta, \Pi\, (R_1\Phi_*\mathbf{Q})^w) - \dim_{\mathbf{Q}} H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w).$$

Consider the exact sequence of pairs

$$0 \to H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w) \to H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w) \to H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) \to H_0(\Delta, (R_1\Phi_*\mathbf{Q})^w) \to \cdots \tag{3.8}$$

(see (3.3) of [13] with $\mathcal{F} = (R_1\Phi_*\mathbf{Q})^w$ and $\Pi' = \varnothing$). Since $H_0(\Delta, (R_1\Phi_*\mathbf{Q})^w) = 0$ for $w \geqslant 1$ by 2.5b of [13], we have

$$\operatorname{codim}_{\mathbf{C}} \operatorname{Im} r = \dim_{\mathbf{Q}} H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) = \sum_{p\in\Pi} \dim_{\mathbf{Q}} H_0(p, (R_1\Phi_*\mathbf{Q})^w).$$

Therefore, in order to compute $\operatorname{codim}_{\mathbf{C}} \operatorname{Im} r$ it remains to prove that

$$\dim_{\mathbf{Q}} H_0\big(p, (R_1\Phi_*\mathbf{Q})^w\big) = \begin{cases} 1 & \text{if } p \text{ is of type } \mathrm{I}_b\ (b \geqslant 1), \\ \delta(w) & \text{if } p \text{ is of type } \mathrm{I}_b^*\ (b \geqslant 1). \end{cases} \tag{3.9}$$

By stabilization of the limit (3.8) of [13] for sufficiently small discs $E \ni p$ and points $u_0 \in \partial E$ we have

$$H_0(p, (R_1\Phi_*\mathbf{Q})^w) \simeq H_0(E, (R_1\Phi_*\mathbf{Q})^w) \simeq (R_1\Phi_*\mathbf{Q})^w|_{u_0}^{\mathrm{coinv}}. \tag{3.10}$$

The coinvariants in the last relation are taken for the monodromy about the point p. The monodromy matrix in an appropriate basis of the space $(R_1\Phi_*\mathbf{Q})^w|_{u_0}$ is of the form (2.9) of [13] (see the proof of Lemma 2.7), from which by (3.10) it is easy to get (3.9). ∎

PROOF OF PROPOSITION 3.1. By the definition of $\xi(j, k)$ (see 1.4) and by 1.2b we have

$$(\xi(j,k), \Phi) = \int_{g(0)}^{g(i\infty)} \Phi_1 (dz - b)^k (-cz + a)^{w-k}\, dz + \int_{g(0)}^{g(i\infty)} \overline{\Phi}_2 (d\bar{z} - b)^k (-c\bar{z} + a)^{w-k}\, d\bar{z}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in j$, $\det g = 1$. Substituting $z \rightsquigarrow g(z)$ in the integrals and applying the definition of $\Phi_i|[g]_{w+2}$, we get the needed relation. ∎

§4. The Hecke operators of the space $R_{w+2}(\Gamma, K)$

The embedding r transfers the action of the Hecke operator T_{w+2} from the space $S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}$ to the subspace $\operatorname{Im} r \subset R_{w+2}(\Gamma, \mathbf{C})$. In this section we construct a natural extension of this operator T_{w+2} on the space $R_{w+2}(\Gamma, \mathbf{C})$. The use of this operator enables us to compute the equation of the subspace $\operatorname{Im} r \subset R_{w+2}(\Gamma, \mathbf{C})$.

4.1. Let M be a subset of matrices in $\mathrm{GL}^+(2, \mathbf{Z})$ having the following properties:

a. $\Gamma_1 M \Gamma_2 = M$.

b. M has a finite decomposition into right cosets

$$M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$$

for some modular groups Γ_1 and Γ_2.

Then the following **C**-linear map is defined:

$$[M]_{w+2}: S_{w+2}(\Gamma_1) \oplus \overline{S_{w+2}(\Gamma_1)} \dashrightarrow S_{w+2}(\Gamma_2) \oplus \overline{S_{w+2}(\Gamma_2)},$$

$$[M]_{w+2}: \Phi = \Phi_1 + \overline{\Phi_2} \mapsto \sum_{\tilde{g}} (\det \tilde{g})^{\frac{w}{2}} (\Phi_1 \mid [\tilde{g}]_{w+2} + \overline{\Phi_2 \mid [\tilde{g}]_{w+2}}) \tag{4.1}$$

(see (3.4.1) of [8]). That this definition is independent of the choice of representatives of $\tilde{g}$ follows from the property $\|[gg']_{w+2} = \|[g]_{w+2}\|[g']_{w+2}$ for any $g, g' \in \mathrm{GL}^+(2, \mathbf{Z})$ (see [8], §2.1) and the definition of cusp forms. Therefore, to verify the correctness of the definition of $[M]_{w+2}$ it remains to verify the inclusion

$$[M]_{w+2}(\Phi) \in S_{w+2}(\Gamma_2) \oplus \overline{S_{w+2}(\Gamma_2)}.$$

For this it suffices to establish the inclusion $[M]_{w+2}(S_{w+2}(\Gamma_1)) \subset S_{w+2}(\Gamma_2)$. The last is easily inferred from Proposition 3.37 of [8]; for this one must use the decomposition $M = \bigcup_g \Gamma_1 g \Gamma_2$ into a finite number of double cosets, which is possible by points a and b and the evident relations $[M]_{w+2} = \Sigma_g [\Gamma_1 g \Gamma_2]_{w+2}$.

4.2. LEMMA-DEFINITION. *Under the assumptions and with the notation of the preceding subsection for admissible* (Γ_1, w) *and* (Γ_2, w) *the following assertions are true*:

a. *There exists a unique* **Q**-*linear mapping*

$$[M]^*_{w+2}: H_1(\Delta_{\Gamma_2}, \Pi_2, (R_1\Phi_*\mathbf{Q})^{w}) \to H_1(\Delta_{\Gamma_1}, \Pi_1, (R_1\Phi_*\mathbf{Q})^{w}) \tag{4.2}$$

such that

$$[M]^*_{w+2}: \{\alpha, \beta, n\ \ m\}_{\Gamma_2} \mapsto \sum_{\tilde{g}} \tilde{g} \mid \{\alpha, \beta, n, m\}_{\Gamma_1}$$

for any $\alpha, \beta \in \tilde{\mathbf{Q}}$ *and* $n, m \in \mathbf{Z}^w$, *and also any decompositions* $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$.

b. *There is a unique* **Q**-*linear mapping*

$$[M]^*_{w+2}: H_0(\Pi_2, (R_1\Phi_*\mathbf{Q})^{w}) \to H_0(\Pi_1, (R_1\Phi_*\mathbf{Q})^{w})$$

such that

$$[M]^*_{w+2}: \{\alpha, n, m\}_{\Gamma_2} \to \sum_{\tilde{g}} \tilde{g} \mid \{\alpha, n, m\}_{\Gamma_1}$$

for any $\alpha \in \tilde{\mathbf{Q}}$ *and* $n, m \in \mathbf{Z}^w$, *and also any decompositions* $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$.

The mappings in a *and* b *have the following properties*:

c. *The following diagram is commutative*:

$$\begin{array}{ccc} H_1(\Delta_{\Gamma_2}, \Pi_2 (R_1\Phi_*\mathbf{Q})^{w}) & \xrightarrow{\partial} & H_0(\Pi_2, (R_1\Phi_*\mathbf{Q})^{w}) \\ {\scriptstyle [M]^*_{w+2}} \downarrow & & \downarrow {\scriptstyle [M]^*_{w+2}} \\ H_1(\Delta_{\Gamma_1}, \Pi\ \ (R_1\Phi_*\mathbf{Q})^{w}) & \xrightarrow{\partial} & H_0(\Pi_1, (R_1\Phi_*\mathbf{Q})^{w}) \end{array}$$

d. *For any* $\sigma \in H_1(\Delta_{\Gamma_2}, \Pi_2, (R_1\Phi_*\mathbf{Q})^w)$ *and any* $\Phi \in S_{w+2}(\Gamma_1) \oplus \overline{S_{w+2}(\Gamma_1)}$

$$([M]^*_{w+2}\sigma, \Phi) = (\sigma, [M]_{w+2}\Phi), \tag{4.3}$$

which justifies writing the operator $[M]^*_{w+2}$ *as the conjugate of* $[M]_{w+2}$.

Let Γ' be the congruence subgroup defined in §0 (see (0.2)). Put

$$\Delta' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+(2, \mathbf{Z}) \mid a \in \mathfrak{h},\ b \equiv 0 \bmod (t),\ c \equiv 0 \bmod (N) \right\}$$

in the notation of 0.1. Moreover, we consider the following sets of matrices:

$$M_n' = \{g \in \Delta' \mid \det g = n\},$$

defined for arbitrary integers n. It is easy to check that $\Gamma' M_n' \Gamma' = M_n'$; and according to Proposition 3.36 of [8], M_n' satisfies 4.1b. Therefore, there are defined a $\mathbf{C}$-endomorphism $[M_n']_{w+2}$ of the space $S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$ and a $\mathbf{Q}$-endomorphism of the space $H_1(\Delta_{\Gamma'}, \Pi, (R_1\Phi_*\mathbf{Q})^w) - [M_n']^*_{w+2}$. The endomorphism $[M_n']_{w+2}$ is also called a *Hecke operator* and denoted by $T'(n)_{w+2}$. The mapping $[M_n']^*_{w+2}$ is also called a *Hecke operator*, and we shall denote it by $T'(n)^*_{w+2}$. $T'(n)^*_{w+2}$ defines a K-endomorphism of the space $H_1(\Delta_{\Gamma'}, \Pi, (R_1\Phi_*K)^w)$ for any field $K \subset \mathbf{C}$ by 3.6. By the duality (3.5) a K-endomorphism $T'(n)_{w+2}$ of the space $R_{w+2}(\Gamma', K)$ is determined. This mapping will also be called a *Hecke operator*. We shall check that $T'(n)_{w+2}$ extends the Hecke operator $T'(n)_{w+2}$ defined on $\operatorname{Im} r \subset R_{w+2}(\Gamma', \mathbf{C})$ by the embedding r; that is, the following diagram is commutative:

$$\begin{array}{ccc} S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')} & \overset{r}{\hookrightarrow} & R_{w+2}(\Gamma', \mathbf{C}) \\ {\scriptstyle T'(n)_{w+2}} \downarrow & & \downarrow {\scriptstyle T'(n)_{w+2}} \\ S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')} & \overset{r}{\hookrightarrow} & R_{w+2}(\Gamma', \mathbf{C}) \end{array} \tag{4.4}$$

To prove this it suffices to verify the relation

$$(\sigma, rT'(n)_{w+2}\Phi) = (T'(n)^*_{w+2}\sigma, r\Phi)$$

for any $\Phi \in S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$ and $\sigma \in H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^w)$. The last follows from (4.3), since $(\ ,r) = (\ ,\ )$ by (3.7).

The following subgroup of Γ' with $t = N$ and $\mathfrak{h} = E$

$$\Gamma_N = \Gamma(N) = \{g \in SL(2, \mathbf{Z}) \mid g \equiv E \bmod N\}$$

is called the *principal congruence subgroup* (of $SL(2, \mathbf{Z})$) *of level* N. The corresponding Hecke operators will be denoted by $T_N(n)_{w+2}$ and $T_N(n)^*_{w+2}$.

4.3. THEOREM. a. *For any prime* $p \equiv 1 \bmod (N)$

$$(T_N(p)^*_{w+2} - 1 - p^{w+1}) H_1(\Delta_{\Gamma(N)}, \Pi, (R_1\Phi_*\mathbf{Q})^w) \subset H_1(\Delta_{\Gamma(N)}, (R_1\Phi_*\mathbf{Q})^w).$$

b. *For any cusp* $\mathfrak{p} \in \Pi \subset \Delta_{\Gamma(N)}$ *there is a unique (up to multiplication by* $\mathbf{Q}^*$*) nonzero homology class* $e_{\mathfrak{p}}^{w+2} \in H_1(\Delta_{\Gamma(N)}, \mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w)$ *such that*

$$T_N(p)^*_{w+2} e_{\mathfrak{p}}^{w+2} = (1 + p^{w+1}) e_{\mathfrak{p}}^{w+2} \tag{4.5}$$

for any prime $p \equiv 1 \bmod (N)$.

4.4. COROLLARY. *For any congruence subgroup* Γ *there is a unique subspace* $\mathcal{E}_\Pi^{w+2} \subset H_1(\Delta_\Gamma, \Pi, (R_1\Phi_*\mathbf{Q})^w)$ *with the following properties*:

a. $(\mathcal{E}_\Pi^{w+2}, S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}) = 0$, *and*

b. $\partial \mathcal{E}_\Pi^{w+2} = H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w)$.

4.5. COROLLARY. *Im r is defined by a homogeneous system of linear equations over $\mathbf{Q}$ in the basis $(j, k)^*$ of the space $C^*(\mathbf{C})$. More precisely, there is a unique subspace $S_{w+2}(\Gamma, \mathbf{Q}) \subset R_{w+2}(\Gamma, \mathbf{Q})$ such that $S_{w+2}(\Gamma, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$ is canonically isomorphic to $S_{w+2}(\Gamma, \mathbf{C}) = \operatorname{Im} r$ under the isomorphism* (3.4) (*Γ is a congruence subgroup*).

PROOF. Let $\mathfrak{S}_{\Pi}^{w+2}$ be the space constructed according to 4.4. Put

$$S_{w+2}(\Gamma, K) = \left(\mathfrak{S}_{\Pi}^{w+2} \otimes_{\mathbf{Q}} K\right)^{\perp} = \left(\mathfrak{S}_{\Pi}^{w+2}\right)^{\perp} \otimes_{\mathbf{Q}} K,$$

where the orthogonal complement is taken with respect to the pairing (,) that establishes the duality (3.5). Then, by 4.4a,

$$S_{w+2}(\Gamma, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \supset \operatorname{Im} r. \tag{4.6}$$

The exact sequence (3.8) for $w \geqslant 1$ determines an exact (by 2.5b of [13]) triple

$$0 \to H_1(\Delta, (R_1\Phi_*\mathbf{Q})^{w}_{..}) \to H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^{w}) \xrightarrow{\partial} H_0(\Pi, (R_1\Phi_*\mathbf{Q})^{w}) \to 0. \tag{4.7}$$

By 4.4b and (4.7) we get

$$\begin{aligned}\dim_{\mathbf{C}} S_{w+2}(\Gamma, \mathbf{C}) = \dim_{\mathbf{Q}} S_{w+2}(\Gamma, \mathbf{Q}) \leqslant \dim_{\mathbf{Q}} H_1(\Delta, \Pi, (R_1\Phi_*\mathbf{Q})^{w}) \\ - \dim_{\mathbf{Q}} H_0(\Pi, (R_1\Phi_*\mathbf{Q})^{w}) = \dim_{\mathbf{C}} R_{w+2}(\Gamma, \mathbf{C}) - t_1 - t_2\delta(w).\end{aligned}$$

The last holds by (3.9). Therefore, by 3.4b, $\dim_{\mathbf{C}} S_{w+2}(\Gamma, \mathbf{C}) \leqslant \dim_{\mathbf{C}} \operatorname{Im} r$, from which, by (4.6), we obtain the existence of the space $S_{w+2}(\Gamma, \mathbf{Q})$. Uniqueness is evident. ■

PROOF OF COROLLARY 4.4. *Uniqueness.* From (0.2) of [13] and 4.4a it follows that

$$\mathfrak{S}_{\Pi}^{w+2} \cap H_1\left(\Delta, \left(R_1\Phi_*\mathbf{Q}\right)^{w}\right) = 0.$$

From the exactness of the sequence (4.7) and from 4.4b we then have a direct sum decomposition

$$H_1\left(\Delta, \Pi, \left(R_1\Phi_*\mathbf{Q}\right)^{w}\right) = \mathfrak{S}_{\Pi}^{w+2} \oplus H_1\left(\Delta, \left(R_1\Phi_*\mathbf{Q}\right)^{w}\right). \tag{4.8}$$

Moreover, it is evident that if ${}'\mathfrak{S}_{\Pi}^{w+2}$ is another space with properties 4.4a, b, then $\mathfrak{S}_{\Pi}^{w+2} + {}'\mathfrak{S}_{\Pi}^{w+2}$ also has these properties. Hence, because of (4.8), it easily follows that $\mathfrak{S}_{\Pi}^{w+2} = {}'\mathfrak{S}_{\Pi}^{w+2}$.

Existence. 1. *The case of a principal congruence subgroup* $\Gamma = \Gamma_N$. We first prove the invertibility of the operator $T_N(p)_{w+2} - 1 - p^{w+1}$ on the space $S_{w+2}(\Gamma_N) \oplus \overline{S_{w+2}(\Gamma_N)}$ for a sufficiently large prime $p \gg 0$. For this, of course, it suffices to establish its invertibility on $S_{w+2}(\Gamma_N)$. The last leads to a proof that $\operatorname{Ker}(T_N(p)_{w+2} - 1 - p^{w+1}) = 0$ for a prime $p \gg 0$. Consider the direct sum decomposition

$$S_{w+2}(\Gamma_N) = \bigoplus_{\psi} S_{w+2}(\Gamma_0', \psi),$$

where Γ_0' is defined by (3.5.1) of [8] for $t = N$, and ψ are characters of the group $(\mathbf{Z}/N\mathbf{Z})^*$. According to (3.5.6) of [8]

$$T_N(p)_{w+2}\big|_{S_{w+2}(\Gamma_0', \psi)} = T'(p)_{w+2,\psi}.$$

Therefore, it suffices to establish the triviality of the kernels of the operators $T'(p)_{w+2,\psi} - 1 - p^{w+1}$ for sufficiently large $p \gg 0$. This is easily deduced from Lemma 3.62 of [8] for $\Gamma = \Gamma_N$, $k = w + 2$, and the relations $\lambda_p c(l) = c(pl)$ for $T'(p)_{w+2,\psi} f = \lambda_p f$, $\lambda_p \in \mathbf{C}$, $(l, p) = 1$, $f \in S_{w+2}(\Gamma_0', \psi)$.

By (4.5), (4.3) and the bilinearity of (,), the invertibility of the operators $T_N(p)_{w+2} - 1 - p^{w+1}$ follows from the orthogonality

$$(e_{\mathfrak{p}}^{w+2}, S_{w+2}(\Gamma_N) \oplus \overline{S_{w+2}(\Gamma_N)}) = 0 \tag{4.9}$$

for the vector $e_{\mathfrak{p}}^{w+2}$ from 4.3b. It is only necessary to verify the nonemptiness of the set of prime numbers $p \equiv 1 \bmod N$, $p \gg 0$. The last follows from Dirichlet's theorem on primes in an arithmetic progression. By (4.7) and 0.2 of [13] we find that $\partial e_{\mathfrak{p}}^{w+2} \neq 0$. Moreover, $\dim H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w) = 1$ (see (3.9)). Therefore,

$$\partial(\mathfrak{S}_{\Pi}^{w+2}) = H_0\big(\Pi, (R_1\Phi_*\mathbf{Q})^w\big) \quad \text{for } \mathfrak{S}_{\Pi}^{w+2} = \sum_{\mathfrak{p}\in\Pi} \mathcal{Q}e_{\mathfrak{p}}^{w+2}.$$

By (4.9), $\mathfrak{S}_{\Pi}^{w+2}$ satisfies 4.4a. This proves the existence of $\mathfrak{S}_{\Pi}^{w+2}$ in case $\Gamma = \Gamma_N$.

2. *The case of an arbitrary congruence subgroup.* Let Γ be a congruence subgroup. Then, from the definition, for some natural number N we have an inclusion $\Gamma_N \subset \Gamma$. Note that if the pair (Γ, w) is admissible, then the same holds for any subgroup $\Gamma' \subset \Gamma$ with the same w. Denote by $\mathfrak{S}_{\Pi,N}^{w+2}$ the subspace of $\mathfrak{S}_{\Pi}^{w+2}$ constructed with respect to the group Γ_N according to 4.4 (see case 1). Consider the double coset $M = \Gamma \cdot E \cdot \Gamma_N = \Gamma \cdot E$. Put

$$\mathfrak{S}_{\Pi}^{w+2} = [M]_{w+2}^{*}\mathfrak{S}_{\Pi,N}^{w+2}.$$

By (4.1)

$$[M]_{w+2}S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)} = S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)} \subset S_{w+2}(\Gamma_N) \oplus \overline{S_{w+2}(\Gamma_N)}.$$

Therefore, from (4.3) and from the definition of $\mathfrak{S}_{\Pi,N}^{w+2}$ we find that 4.4a also holds for $\mathfrak{S}_{\Pi}^{w+2}$. That 4.4b also holds for $\mathfrak{S}_{\Pi}^{w+2}$ follows from the commutativity of the diagram in 4.2c and the fact that $[M]_{w+2}^{*}$ is an epimorphism. The last is easily gotten from 1.2d, since by 4.2b

$$[M]_{w+2}^{*}\{\alpha, n, m\}_{\Gamma_N} = \{\alpha, n, m\}_{\Gamma}. \ \blacksquare$$

Proof of Theorem 4.3. a. By the commutativity of the diagram in 4.2c it suffices to establish that

$$(T_N(p)_{w+2}^{*} - 1 - p^{w+1}) H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) = 0. \tag{4.10}$$

Let $g \in SL(2, \mathbf{Z})$ be an arbitrary matrix. Then

$$\Gamma_N g \Gamma_N = \Gamma_N g = g\Gamma_N,$$

since Γ_N is a normal subgroup of $SL(2, \mathbf{Z})$. The operator $[\Gamma_N g \Gamma_N]_{w+2}^{*}$ is invertible, and $[\Gamma_N g^{-1}\Gamma_N]_{w+2}^{*}$ is its inverse. Moreover, $[\Gamma_N g\Gamma_N]_{w+2}^{*}$ commutes with the operator $T_N(p)_{w+2}^{*}$, since $\Gamma_N g M_p' = M_p' g \Gamma_N$, where M_p' is a double coset constructed in the case $t = N$, $\mathfrak{h} = 1$. Consider the following decomposition into a direct sum:

$$H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) = \bigoplus_{\mathfrak{p}\in\Pi} H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w).$$

The automorphism $[\Gamma_N g\Gamma_N]_{w+2}^{*}$ permutes the components of this sum (see 4.2b). The component of a cusp $\mathfrak{p} = \Gamma_N\alpha$ is mapped to the component of the point $\mathfrak{p}' = g\Gamma_N\alpha = \Gamma_N g\alpha$. The cusp $\Gamma_{N\,i\infty}$ can be mapped to any other cusp. Therefore, since the Hecke operator $T_N(p)^*_{w+2}$ commutes with the automorphism $[\Gamma_N g \Gamma_N]^*_{w+2}$, by (4.10) it suffices to show that

$$(T_N(p)^*_{w+2} - 1 - p^{w+1}) H_0(\Gamma_{N\, i\infty}, (R_1\Phi_*\mathbf{Q})^w) = 0. \tag{4.11}$$

To prove the last fact it is necessary to require that p be prime and that $p \equiv 1 \bmod (N)$. In fact, by Proposition 3.36 of [8] we then have

$$M'_p = \Gamma_N \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cup \bigcup_{b=0}^{p-1} \Gamma_N \begin{pmatrix} 1 & bN \\ 0 & p \end{pmatrix}.$$

Therefore, by 4.2b and 1.2(e_2), (e_4),

$$\begin{aligned} &T_N(p)^*_{w+2} \{i\infty, 1_w, 0\}_{\Gamma(N)} \\ &= \left(\sum_{b=0}^{p-2} \begin{pmatrix} 1 & bN \\ 0 & p \end{pmatrix}\right|\{i\infty, 1_w, 0\}\Bigg) + \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\Bigg|\{i\infty, 1_w, 0\} \\ &= (p^{w+1} + 1)\{i\infty, 1_w, 0\}_{\Gamma(N)}, \end{aligned} \tag{4.12}$$

since $\{i\infty, 1_k, 1_w - 1_k\}_{\Gamma(N)} = 0$ for $0 \leqslant k \leqslant w$. The last is proved by induction on k using the relation

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}\Bigg|\{i\infty, 1_k, 1_w - 1_k\}_{\Gamma(N)} = \{i\infty, 1_k, 1_w - 1_k\}_{\Gamma(N)} \tag{4.13}$$

(see 1.2(e_3)). In fact, by (1.7) and (1.8) (in the last formula one must replace $\{0, i\infty, *, **\}$ by $\{i\infty, *, **\}$) we find that the left member of (4.13) equals

$$\begin{aligned} &\{i\infty, 1_k, 1_w - 1_k\}_{\Gamma(N)} - Nk\{i\infty, 1_{k-1}, 1_w - 1_{k-1}\}_{\Gamma(N)} \\ &\quad + \sum_{l=0}^{k-2} \alpha_l \{i\infty, 1_l, 1_w - 1_l\}_{\Gamma(N)}, \end{aligned}$$

where $\alpha_l \in \mathbf{Z}$. From this we get what we need, using an induction on k, $1 < k \leqslant w$. By 1.2($e_{2\text{-}4}$) the modular symbols $\{i\infty, 1_k, 1_w - 1_k\}_{\Gamma(N)}$ for $0 \leqslant k < w$ generate the space $H_0(\Gamma_{N\, i\infty}, (R_1\Phi_*\mathbf{Q})^w)$. Then (4.11) follows immediately from (4.12) and the triviality of the modular symbols $\{i\infty, 1_k, 1_w - 1_k\}_{\Gamma(N)}$ for $0 \leqslant k \leqslant w$.

b. The relation

$$[XY]^*_{w+2} = [X]^*_{w+2}[Y]^*_{w+2}$$

is gotten in the notation of Proposition 3.38 of [8] by analogous methods by 4.2a, b. Therefore, by Theorem 3.34 of [8] the operators $T_N(p)^*_{w+2}$ and $T_N(p')^*_{w+2}$ commute for primes $p \equiv p' \equiv 1 \bmod (N)$. Choose p so large that the operator $T_N(p)_{w+2} - 1 - p^{w+1}$ is invertible on $S_{w+2}(\Gamma_N) \oplus \overline{S_{w+2}(\Gamma_N)}$. That this is possible was shown at the start of the analysis of case 1 in the existence part in the proof of 4.4. From (4.11), with the aid of the operator $[\Gamma_N g \Gamma_N]^*_{w+2}$ permuting the summands of $H_0(\Pi, (R_1\Phi_*\mathbf{Q})^w) = \bigoplus_{\mathfrak{p}\in\Pi} H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w)$ we find that

$$(T_N(p)^*_{w+2} - 1 - p^{w+1}) H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w) = 0$$

for all $\mathfrak{p} \in \Pi$, since $p \equiv 1 \bmod (N)$. Therefore, for all $\mathfrak{p} \in \Pi$

$$(T_N(p)^*_{w+2} - 1 - p^{w+1}) H_1(\Delta, \mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w) \subset H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w). \tag{4.14}$$

The exact sequence (3.3) of [13] for $\mathcal{F} = (R_1\Phi_*\mathbf{Q})^w$, $w \geqslant 1$, $\Pi' = \varnothing$ and $\Pi = \{\mathfrak{p}\}$ by 2.5b determines an exact sequence

$$0 \to H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w) \to H_1(\Delta, \mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w) \xrightarrow{\partial} H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w) \to 0. \quad (4.15)$$

The group Γ_N satisfies (*) for $N \geqslant 3$, but not for $N = 1$ or 2. Moreover, it is well known that all the cusps of Γ_N ($N \geqslant 3$) are of the first kind (type I_N). Consequently, by (3.9), $\dim H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w) = 1$. Denote by $e_\mathfrak{p}^{w+2} \neq 0$ a vector of the space

$$H_1\big(\Delta, \mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w\big)$$

for which

$$T_N(p)^*_{w+2} e_\mathfrak{p}^{w+2} = (1 + p^{w+1})\, e_\mathfrak{p}^{w+2}. \quad (4.16)$$

The existence of this vector follows from (4.14) and from the exactness of the sequence (4.15). From (4.3), (4.16), the bilinearity of $(\ ,\ )$ and the invertibility of the operator $T_N(p)_{w+2} - 1 - p^{w+1}$ on $S_{w+2}(\Gamma_N) \oplus \overline{S_{w+2}(\Gamma_N)}$ follows the orthogonality of (4.9). Therefore by (0.2) of [13] we have that $e_\mathfrak{p}^{w+2} \notin H_1(\Delta, (R_1\Phi_*\mathbf{Q})^w)$; that is, by (4.15)

$$\partial e_\mathfrak{p}^{w+2} \neq 0. \quad (4.17)$$

Consequently, by the one-dimensionality of the space $H_0(\mathfrak{p}, (R_1\Phi_*\mathbf{Q})^w)$, the vector $e_\mathfrak{p}^{w+2}$ is determined up to multiplication by $\mathbf{Q}^*$. We shall show that $e_\mathfrak{p}^{w+2}$ satisfies (4.5) for any $p' \equiv 1 \bmod (N)$. In fact, since $T_N(p)^*_{w+2}$ and $T_N(p')^*_{w+2}$ commute, by (4.16) we have

$$T_N(p)^*_{w+2} T_N(p')^*_{w+2}\, e_\mathfrak{p}^{w+2} = (1 + p^{w+1})\, T_N(p')^*_{w+2} e_\mathfrak{p}^{w+2}.$$

Then, by the proof of uniqueness,

$$T_N(p')^*_{w+2}\, e_\mathfrak{p}^{w+2} = \alpha e_\mathfrak{p}^{w+2}$$

for some $\alpha \in \mathbf{Q}$. Applying the operator ∂ to both sides of the last equality and using the commutativity of the diagram in 4.2c as well as (4.14), we find that $\alpha\partial e_\mathfrak{p}^{w+2} = (1 + p'^{w+1})\partial e_\mathfrak{p}^{w+2}$. Then, by (4.17), $\alpha = 1 + p'^{w+1}$. ∎

PROOF OF LEMMA-DEFINITION 4.2. We begin with points c, d and e.

c follows from 1.2c, a, (1.3–4) and 4.2a, b.

d follows from c and the exactness of the sequence (4.7).

e. By 4.2a, 1.2b and (1.3′) we have

$$([M]^*_{w+2}\{\alpha, \beta, n, m\}_{\Gamma_2}, \Phi) = \sum_{\tilde g} \int_{\tilde g\alpha}^{\tilde g\beta} \Phi_1 \prod_{i=1}^{w} ((\tilde d n_i - \tilde c m_i) z$$

$$+ (-\tilde b n_i + \tilde a m_i))\, dz + \int_{\tilde g\alpha}^{\tilde g\beta} \overline{\Phi}_2 \prod_{i=1}^{w} ((\tilde d n_i - \tilde c m_i)\bar z + (-\tilde b n_i + \tilde a m_i))\, d\bar z$$

for any $\alpha, \beta \in \tilde{\mathbf{Q}}$, $n, m \in \mathbf{Z}^w$ and $\Phi = \Phi_1 + \overline{\Phi}_2 \in S_{w+2}(\Gamma_1) \oplus \overline{S_{w+2}(\Gamma_1)}$, where $\tilde g = \left(\begin{smallmatrix}\tilde a & \tilde b\\ \tilde c & \tilde d\end{smallmatrix}\right)$. Changing z to $\tilde g z$ in the integrals of this sum, we get, by the definition of $\Phi|[g]_{w+2}$ and $[M]_{w+2}$,

$$([M]^*_{w+2}\{\alpha, \beta, n, m\}_{\Gamma_2}, \Phi) = \int_\alpha^\beta [M]_{w+2}\Phi_1 \prod_{i=1}^{w} (n_i z + m_i)\, dz$$

$$+ \int_\alpha^\beta \overline{[M]_{w+2}\Phi_2} \prod_{i=1}^{w} (n_i \bar z + m_i)\, d\bar z = (\{\alpha, \beta, n, m\}_{\Gamma_2}, [M]_{w+2}\Phi).$$

The last we get by 1.2b, since $[M]_{w+2}\Phi = [M]_{w+2}\Phi_1 + \overline{[M]_{w+2}\Phi_2} \in S_{w+2}(\Gamma_2) \oplus \overline{S_{w+2}(\Gamma_2)}$. Now (4.3) follows from what has been proved, by 1.2.

The uniqueness of the mappings of a and b follows from 1.2c, d and (e_3).

Existence. a. By the linearity of $[M]^*_{w+2}$ and by the decomposition

$$\{\alpha, \beta, n, m\}_{\Gamma_2} = \{\alpha, \gamma_1, n, m\}_{\Gamma_2} + \{\gamma_h, \beta, n, m\}_{\Gamma_2} + \sum_{i=1}^{h-1} \{\gamma_i, \gamma_{i+1}, n, m\}_{\Gamma_2}$$

(see the proof of 1.5b) and (1.6) it suffices to establish the existence of a mapping $[M]^*_{w+2}$ such that

$$[M]^*_{w+2} g \,|\, \{0, i\infty, n, m\}_{\Gamma_2} = \sum_{\tilde{g}} \tilde{g} g \,|\, \{0, i\infty, n, m\}_{\Gamma_1}$$

for any $g \in SL(2, \mathbf{Z})$ and $n, m \in \mathbf{Z}^w$, and also for any decompositions $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$. Then by (1.9) and by the linearity of the mapping $[M]^*_{w+2}$ it suffices to establish the existence of an $[M]^*_{w+2}$ which is defined on marked classes of the following form:

$$[M]^*_{w+2} \xi(j, k)_{\Gamma_2} = \sum_{\tilde{g}} \tilde{g} g \,|\, \{0, i\infty, 1_k, 1_w - 1_k\}_{\Gamma_1} \tag{4.18}$$

for any $g \in SL(2, \mathbf{Z})$ and integers $0 \leqslant k \leqslant w$, and also for any decompositions $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$, where $j = \Gamma_2 g$. By 1.2(e_3) it is easy to verify that the right-hand member depends only on the pair (j, k) in (4.18). Therefore, we can define a $\mathbf{Q}$-linear mapping

$$[\widetilde{M}]^*_{w+2} : C_{\Gamma_2} \to H_1(\Delta_{\Gamma_1}, \Pi_1, (R_1 \Phi_* \mathbf{Q})^w),$$
$$[\widetilde{M}]^*_{w+2} : (j, k)_{\Gamma_2} \to \sum_{\tilde{g}} \tilde{g} g \,|\, \{0, i\infty, 1_k, 1_w - 1_k\}_{\Gamma_1},$$

where $j = \Gamma_2 g$. Therefore, by Theorem 2.3, to prove the existence of $[M]^*_{w+2}$ it suffices to establish that the mapping $[\tilde{M}]^*_{w+2}$ takes the vectors (2.2) and (2.3) to 0. In fact, by the linearity of $[\tilde{M}]^*_{w+2}$ and by (1.11) (the first relation), for representatives $g \in j$ and $gs \in js$, we find that $[\tilde{M}]^*_{w+2}$ maps (2.2) to

$$\sum_{\tilde{g}} \tilde{g} g \,|\, \{0, i\infty, 1_k, 1_w - 1_k\}_{\Gamma_1} + \tilde{g} g \,|\, \{i\infty, 0, 1_k, 1_w - 1_k\}_{\Gamma_1} = 0.$$

We get the same thing for the vector (2.3) by (1.11) (the last two relations) and 1.2(e_1), taking representatives $g \in j$, $gt \in jt$ and $gt^2 \in jt^2$.

b. Consider the decomposition

$$H_0(\Pi_2, (R_1 \Phi_* \mathbf{Q})^w) = \bigoplus_{\mathfrak{p} \in \Pi_2} H_0(\mathfrak{p}, (R_1 \Phi_* \mathbf{Q})^w).$$

Since $\{\alpha, n, m\}_{\Gamma_2} \in H_0(\Gamma_2 \alpha, (R_1 \Phi_* \mathbf{Q})^w)$, it suffices to establish the existence of a $\mathbf{Q}$-linear mapping

$$[M]^*_{w+2} : H_0(\mathfrak{p}, (R_1 \Phi_* \mathbf{Q})^w) \to H_0(\Pi_1, (R_1 \Phi_* \mathbf{Q})^w) \tag{4.19}$$

for any $\mathfrak{p} \in \Pi_2$ such that

$$[M]^*_{w+2} \{\alpha, n, m\}_{\Gamma_2} = \sum_{\tilde{g}} \tilde{g} \,|\, \{\alpha, n, m\}_{\Gamma_1} \tag{4.20}$$

for any $\alpha \in \mathfrak{p}$ (that is, $\mathfrak{p} = \Gamma_2\alpha$) and $n, m \in \mathbf{Z}^w$, and for any decompositions $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$. It is easy to show that if we are given a decomposition $M = \bigcup \Gamma_1 \tilde{g}$ then $M = \bigcup \Gamma_1 \tilde{g}g$ is also a decomposition for any $g \in \Gamma_2$. For any $\alpha \in \tilde{\mathbf{Q}}$ and $n, m \in \mathbf{Z}^w$ we have

$$\{g\alpha, n, m\} = g \mid \{\alpha, an + cm, dm + bn\},$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{Z})$. Consequently, it suffices to construct a mapping (4.19) such that (4.20) holds for some fixed $\alpha \in \mathfrak{p}$. Fix $\alpha \in \tilde{\mathbf{Q}}$ and $\mathfrak{p} = \Gamma_2\alpha \in \Pi_2$. Then, by 1.2($e_3$), ($e_4$) and by the linearity of (4.19), it suffices to establish the existence of a mapping (4.19) such that

$$[M]^*_{w+2}\{\alpha, 1_k, 1_w - 1_k\}_{\Gamma_2} = \sum_{\tilde{g}} \tilde{g} \mid \{\alpha, 1_k, 1_w - 1_k\}_{\Gamma_1} \tag{4.21}$$

for any integer $0 \leqslant k \leqslant w$, and also for any decompositions $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$. In the notation of 1.1, by 1.2(e_{3-4}) we can define a linear mapping

$$[\widetilde{M}]^*_{w+2} : Z_0(E, (R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w) \to H_0(\Pi_1, (R_1\Phi_*\mathbf{Q})^w),$$

$$[\widetilde{M}]^*_{w+2} e_1^k e_2^{w-k} v_E = \sum_{\tilde{g}} \tilde{g} \mid \{\alpha, 1_k, 1_w - 1_k\}_{\Gamma_1}$$

for any integers $0 \leqslant k \leqslant w$ and for any decompositions $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g}$, where $Z_0(E, (R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w)$ is the space of 0-cycles with coefficients in the sheaf $(R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w$ for a cellular decomposition of $E \subset \Delta_{\Gamma_2}$ (see Figure 11): 0-cell v_E, 1-cell dE and 2-cell E with standard orientation. By definition of the mapping $[\tilde{M}]^*_{w+2}$, by the stabilization of the limit $\varprojlim H_0(E, (R_1\Phi_*\mathbf{Q})^w)$, and by (1.3) and (4.21), to establish the existence of the mapping (4.19) it suffices to check that

$$[\widetilde{M}]^*_{w+2} B_0(E, (R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w) = 0, \tag{4.22}$$

where $B_0(E, (R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w)$ is the space of 0-boundaries over E with coefficients in the sheaf $(R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w$ for a cellular decomposition of E described above, for a sufficiently small disc E. Identify $z_E \in H'$ with v_E, and $z_E z'_E$ with ∂E. Denote by $e_{1,1}$, $e_{1,2}$ a basis of the group of sections $G|_{\partial E}$ chosen according to 1.1, and denote by e_1, e_2 and e'_1, e'_2 a basis of $G|_{v_E}$ for the identification of z_E with v_E and z'_E with v_E, respectively. Evidently, for a suitable normalization of $\pm$ in the case where $(*)$ fails we have $(e_{1,1}, e_{1,2})|_{-v_E} = (e_1, e_2)$ and $(e_{1,1}, e_{1,2})|_{v_E} = (e'_1, e'_2)$.

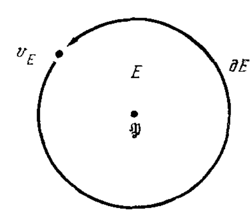

FIGURE 11

Therefore, by (1.15), since $\partial\partial E = v_E - v_E$, the space $B_0(E, (R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w)$ is generated by the vectors $\varepsilon_k = ((e'_1)^k (e'_2)^{w-k} - e_1^k e_2^{w-k}) \cdot v_E$ for integers $0 \leqslant k \leqslant w$. By (1.7) and

(1.26) for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_2$ such that $gz_E = z'_E$ we get

$$\varepsilon_k = \sum_{i=0}^{w} h_i((a-c)1_k + c1_w, (b-d)1_k + d1_w)\, e_1^i e_2^{w-i} v_E - e_1^k e_2^{w-k} v_E.$$

Hence, by the definition of $[\tilde{M}]^*_{w+2}$ and the relation

$$\tilde{g}\,|\{\alpha, n, m\} = \sum_{k=0}^{w} h_k(n, m)\tilde{g}\,|\{\alpha, 1_k, 1_w - 1_k\},$$

which holds for any $\tilde{g} \in GL^+(2, \mathbf{Z})$ and $n, m \in \mathbf{Z}^w$ (proved just like (1.9)), it follows that

$$[\tilde{M}]^*_{w+2}\varepsilon_k = \sum_{\tilde{g}} \tilde{g}\,|\{\alpha, (a-c)1_k + c1_w, (b-d)1_k + d1_w\}_{\Gamma_1} - \tilde{g}\,|\{\alpha, 1_k, 1_w - 1_k\}_{\Gamma_1}.$$

Since $g\alpha = \alpha$, we have

$$\{\alpha, (a-c)1_k + c1_w, (b-d)1_k + d1_w\} = g^{-1}\,|\{\alpha, 1_k, 1_w - 1_k\}.$$

Consequently,

$$[\tilde{M}]^*_{w+2}\varepsilon_k = \sum_{\tilde{g}} \tilde{g}g^{-1}\,|\{\alpha, 1_k, 1_w - 1_k\}_{\Gamma_1} - \tilde{g}\,|\{\alpha, 1_k, 1_w - 1_k\}_{\Gamma_1}.$$

It is easy to show that $M = \bigcup_{\tilde{g}} \Gamma_1 \tilde{g} g^{-1}$ is also a decomposition. Therefore

$$[\tilde{M}]^*_{w+2}\varepsilon_k = 0.$$

The vectors ε_k generate the space $B_0(E, (R_1\Phi_{\Gamma_2 *}\mathbf{Q})^w)$, which proves (4.22). ∎

§5. Theorems on the periods of cusp forms

In this section, unless stated otherwise, by $\Phi \in S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$ we denote a cusp form that is an eigenvector for all the Hecke operators $T'(n)_{w+2}$ with eigenvalues λ_n; that is,

$$T'(n)_{w+2}\Phi = \lambda_n\Phi \tag{5.1}$$

for all natural numbers $n \geqslant 1$ and $\lambda_1 = 1$. The group Γ' is defined by (0.2). By K we shall mean the field $\mathbf{Q}(\lambda_1, \lambda_2, \dots) \subset \mathbf{C}$. It is well known that K is a field of algebraic numbers. Let $\mathcal{O}$ be the ring of algebraic integers of the field K. Since $\mathcal{O} \subset \mathbf{C}$, $\mathbf{C}$ can be considered as an $\mathcal{O}$-module. Therefore, it makes sense to speak of an $\mathcal{O}$-submodule of $\mathbf{C}$.

5.1. MAIN THEOREM. *There exist a free $\mathcal{O}$-submodule $\mathfrak{M} \subset \mathbf{C}$ of rank $\leqslant 2$ such that*

$$\int_\alpha^\beta \Phi_1 z^k\, dz + \int_\alpha^\beta \overline{\Phi}_2 \bar{z}^k\, d\bar{z} \in \mathfrak{M}$$

for all $\alpha, \beta \in \tilde{\mathbf{Q}}$, all integers $0 \leqslant k \leqslant w$, and $\Phi = \Phi_1 + \overline{\Phi}_2 \in S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$.

Since the Hecke operators $T'(n)_{w+2}$ on $S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$ are extensions of the Hecke operators $T'(n)_{w+2}$ on $S_{w+2}(\Gamma')$ (see (4.1)), from Theorem 5.1 we get the following result.

5.2. THEOREM. *If $\Phi \in S_{w+2}(\Gamma')$ is an eigenvector of all the Hecke operators $T'(n)_{w+2}$, i.e. if* (5.1) *holds, then there exists a free $\mathcal{O}$-submodule $\mathfrak{M} \subset \mathbf{C}$ of rank $\leqslant 2$ such that*

$$\int_\alpha^\beta \Phi z^k\, dz \in \mathfrak{M}$$

for all $\alpha, \beta \in \tilde{\mathbf{Q}}$ and integers $0 \leqslant k \leqslant w$. ∎

The following weakening of Theorem 5.1 is essentially equivalent to it (see 5.6).

5.3. THEOREM. *The following inequality holds (under the assumptions and with the notation of* 5.1):

$$\dim_K\left(\sum_{\substack{\alpha,\beta\in\tilde{\mathbf{Q}}\\ 0\leqslant k\leqslant w}} K\left(\int_\alpha^\beta \Phi_1 z^k\,dz+\int_\alpha^\beta \overline{\Phi}_2\bar{z}^k\,d\bar{z}\right)\right)\leqslant 2.$$

Just as 5.2 followed from 5.1, the following result is immediately obtained from 5.3.

5.4. THEOREM. *Under the assumptions and with the notation of* 5.2,

$$\dim_K\left(\sum_{\substack{\alpha,\beta\in\tilde{Q}\\ 0\leqslant k\leqslant w}} K\int_\alpha^\beta \Phi z^k\,dz\right)\leqslant 2. \quad ■$$

PROOF OF 0.2. This follows immediately from 5.4. ■

The following lemma holds for any modular subgroup Γ.

5.5. LEMMA. *For arbitrary $\alpha,\beta\in\tilde{\mathbf{Q}}$ and any integer $0\leqslant k\leqslant w$, there exist integers $h(j,l)$ such that*

$$\int_\alpha^\beta \Phi_1 z^k\,dz+\int_\alpha^\beta \overline{\Phi}_2\bar{z}^k\,d\bar{z}=\sum_{\substack{0\leqslant l\leqslant w\\ j\in I}} h(j,l)\,r(j,l,\Phi)$$

for any arbitrary cusp form $\Phi=\Phi_1+\overline{\Phi}_2\in S_{w+2}(\Gamma)\oplus\overline{S_{w+2}(\Gamma)}$.

PROOF OF 0.5. This is an immediate consequence of Lemma 5.5. ■

5.6. LEMMA. *The following assertions about the form Φ are equivalent*:

a. *Theorem* 5.1.

b. *Theorem* 5.3.

c.

$$\dim_K\left(\sum_{\substack{j\in I'\\ 0\leqslant l\leqslant w}} K r(j,l,\Phi)\right)\leqslant 2,$$

where $I'=\Gamma'\setminus SL(2,\mathbf{Z})$.

d. *There exists a free $\mathcal{O}$-submodule $\mathfrak{M}\subset\mathbf{C}$ of rank $\leqslant 2$ such that $r(j,l,\Phi)\in\mathfrak{M}$ for all $j\in I'$ and $0\leqslant l\leqslant w$.*

5.7. PROPOSITION. *In the notation of* 5.6,

$$\dim_K\left(\sum_{\substack{j\in I'\\ 0\leqslant l\leqslant w}} K r(j,l,\Phi)\right)\leqslant 2.$$

PROOF OF THEOREMS 5.1 AND 5.3. These assertions immediately follow from Lemma 5.6 and Proposition 5.7. ■

PROOF OF PROPOSITION 5.7. $r(\Phi)\in\operatorname{Im} r$, and the vector $r(\Phi)$, by the commutativity of the diagram (4.4), gives a system of equalities over K: $T'(n)_{w+2}r(\Phi)=\lambda_n r(\Phi)$. Since

Im r, by Corollary 4.5, is defined by a system of linear equations over $\mathbf{Q}$ (equations in the preceding and in the last case are taken in the basis $(j, k)^*$), it suffices to prove the following inequality:

$$\dim_{\mathbf{C}}\{x \in \operatorname{Im} r \mid T'(n)_{w+2}x = \lambda_n x,\ n = 1, 2, \ldots\} \leqslant 2.$$

This inequality, by the commutativity of (4.4), is equivalent to

$$\dim_{\mathbf{C}}\{\Phi \in S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')} \mid T'(n)_{w+2}\Phi = \lambda_n\Phi,\ n = 1, 2, \ldots\} \leqslant 2. \quad (5.2)$$

If $\Phi = \Phi_1 + \overline{\Phi}_2 \in S_{w+2}(\Gamma') \oplus \overline{S_{w+2}(\Gamma')}$ satisfies (5.1), then by (12.1)

$$T'(n)_{w+2}\Phi_1 = \lambda_n\Phi_1, \quad n = 1, 2 \ldots, \quad T'(n)_{w+2}\Phi_2 = \overline{\lambda}_n\Phi_2, \quad n = 1, 2, \ldots.$$

Therefore, to prove (5.2) it suffices to show that

$$\dim_{\mathbf{C}}\{\Phi \in S_{w+2}(\Gamma') \mid T'(n)_{w+2}\Phi = \mu_n\Phi,\ n = 1, 2, \ldots\} \leqslant 1$$

for any $\mu_i \in \mathbf{C}$, $i \in N$. The last inequality follows from 3.53 and 3.44 of [8] (and also (3.5.6) of [8]).

PROOF OF LEMMA 5.6. The implication a $\Rightarrow$ b is evident. The implication b $\Rightarrow$ c is gotten from (3.1) and from the relation in the proof of Proposition 3.1. From the finiteness of the set I' the implication c $\Rightarrow$ d evidently follows. The implication d $\Rightarrow$ a follows from Lemma 5.5. ■

PROOF OF LEMMA 5.5. The left-hand member of the relation to be proved, by 1.2b, equals $(\{\alpha, \beta, 1_k, 1_w - 1_k\}, \Phi)$. Therefore, this lemma is an immediate consequence of 1.5b and (3.1). ■

5.8. REMARK (*on new forms*). Any new cusp form (see [1] or [7]) is an eigenvector of all the Hecke operators. Therefore, for any new form Theorems 5.2 and 5.3 will hold.

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