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1985 Math. USSR Izv. 24 193

(http://iopscience.iop.org/0025-5726/24/1/A08)

Изв. Акад. Наук СССР
Сер. Матем. Том 48(1984), № 1

Math. USSR Izvestiya
Vol. 24(1985), No. 1

ON THE CLOSED CONE OF CURVES OF ALGEBRAIC 3-FOLDS

UDC 512.7

V. V. SHOKUROV

ABSTRACT. In this paper the author establishes, under natural conditions, the local polyhedrality of the closed cone of curves of a three-dimensional algebraic variety in the part that is negative with respect to the canonical class. In particular, it is shown that there always exists an extremal ray giving a contraction. The results can be used in three-dimensional birational geometry.

Bibliography: 10 titles.

X denotes throughout a *normal projective* 3-*fold* defined over an algebraically closed field k of characteristic 0. We recall the terminology of Mori [4] and Kawamata [3]. There are two real vector spaces associated with the variety X,

$$N(X) = (\{\text{1-cycles on } X\}/\equiv) \otimes \mathbf{R}$$

and

$$N(X)^0 = (\{\text{Cartier divisors on } X\}/\equiv) \otimes \mathbf{R},$$

where $\equiv$ denotes numerical equivalences; the intersection pairing

$$(\cdot): N(X)^0 \times N(X) \to \mathbf{R}$$

is nondegenerate by definition of $\equiv$. On $N(X)$ and $N(X)^0$ we fix a Euclidean norm $\| \; \|$. This defines the *closed cone of curves* $\overline{NE}(X) \subset N(X)$, which is the closure with respect to $\| \; \|$ of the cone $NE(X)$ of effective 1-cycles on X. This cone is obviously independent of the choice of $\| \; \|$.

K_X denotes the canonical Weil divisor of X [5]. By definition $\mathcal{O}_{\mathrm{Reg}(X)}(K_X) = \Omega^3_{\mathrm{Reg}(X)}$, where $\mathrm{Reg}(X) = X - \mathrm{Sing}(X)$ is the set of nonsingular points of X.

By a **Q**-*Cartier divisor* we mean a linear combination of Cartier divisors with rational coefficients. We suppose furthermore that X is **Q**-*factorial*. This means that every Weil divisor D on X is a rational multiple of a Cartier divisor; that is, there exists an integer n such that nD is a Cartier divisor on X. On such a variety each Weil divisor D corresponds to a **Q**-Cartier divisor, and has a numerical class $(D) \in N(X)^0$. In particular, we can take the intersection of Weil divisors with 1-cycles. The Weil divisor K_X defines a **Q**-Cartier

1980 *Mathematics Subject Classification*. Primary 14E30, 14J30; Secondary 14J99.

divisor which we continue to denote by K_X. We recall that X is said to have *canonical singularities* (respectively *terminal* singularities) if for some resolution $h\colon X' \to X$,

$$K_{X'} \equiv h^*K_X + \sum a_i E_i,$$

where the E_i are the exceptional divisors of h, and all the a_i are $\geqq 0$ (respectively > 0). It is easy to check that this definition is independent of the resolution h. We assume from now on that X is a *variety with canonical singularities.*

We let $\overline{NE}(X)^-$ denote the cone $\{Z \in \overline{NE}(X) | Z \cdot K_X < 0\}$.

By an *extremal ray* of $NE(X)^-$ we mean a ray $R \subset \overline{NE}(X)^-$ such that

(i) If $Z_1, Z_2 \in \overline{NE}(X)$ and $Z_1 + Z_2 \in R$, then $Z_1, Z_2 \in R$.

A ray R is said to be *locally polyhedral* if there exists a divisor $D \in N(X)^0$ and a finite collection of curves $C_1, \dots, C_r$ on X such that $\overline{NE}(X) = \overline{NE}(X, D)^+ + \sum_1^r \mathbf{R}_+(C_i)$ and $D \cdot Z < 0$ for all $Z \in R - \{0\}$; here $\overline{NE}(X, D)^+ = \{Z \in \overline{NE}(X) | D \cdot Z \geqq 0\}$. In this case the ray R satisfies Mori's conditions, namely

(ii) R is rational; that is, $R = \mathbf{R}_+(C_R)$ for some curve $C_R \subset X$.

(iii) $R^\perp = \{D \in N(X)^0 | D \cdot R = 0\}$ contains an open subset of numerically effective divisors $D \in N(X)^0$ for which $D^\perp \cap \overline{NE}(X) = R$.

To a locally polyhedral extremal ray $R \subset \overline{NE}(X)^-$ we can apply Kawamata's technique [3], and so R determines a morphism $\mathrm{cont}_R\colon X \to Y$ contracting the extremal ray R. (Kawamata's preprint in fact assumes that X has terminal singularities, but this condition is not used in an essential way in his proof; see [7].)

We say that R is a ray of type (a) (respectively of type (b)) if R is a locally polyhedral extremal ray of $\overline{NE}(X)^-$ such that the morphism $\mathrm{cont}_R\colon X \to Y$ is birational and contracts a surface E of X (respectively contracts only a finite seet of curves of X). Compare [3], Theorem 4.

MAIN THEOREM. *Let X be a projective normal $\mathbf{Q}$-factorial 3-fold with canonical singularities, and suppose that any compact subset of the cone $\overline{NE}(X)^-$ has at most a finite number of extremal rays of type* (b). *Then $\overline{NE}(X)$ is locally polyhedral in $\overline{NE}(X)^-$; that is, for any ample divisor A and any $\varepsilon > 0$ there exists a finite set of curves $C_1, \dots, C_r$ such that*

$$\overline{NE}(X) = \overline{NE}_\varepsilon(X, A) + \sum_{i=1}^{r} \mathbf{R}_+(C_i),$$

where $\overline{NE}_\varepsilon(X, A) = \{Z \in \overline{NE}(X) | ((K_X + \varepsilon A) \cdot Z) \geqslant 0\}$.

COROLLARY. *If K_X is not numerically effective, then $\overline{NE}(X)^-$ always contains a locally polyhedral extremal ray R.*

This result was proved independently (but later) by Reid [7], using a closely related method.*

§2. The main lemma

2.1. LEMMA. *Let X be a projective normal $\mathbf{Q}$-factorial 3-fold with canonical singularities, let A be an ample Cartier divisor, and suppose that for some $\alpha \in \mathbf{R}$, $D \in (A + \alpha K)$ is a numerically effective divisor such that*

(i) *the face of $\overline{NE}(X)$ given by $R = D^\perp \cap \overline{NE}(X)$ satisfies $R \subset \overline{NE}(X)^-$, and*

(ii) *either $D^3 > 0$ or $-D^2K_X > 0$.*

**Translator's note.* Much progress has been made on this problem in recent months; see [8], [9] and [10]. Both the Contraction Theorem and the Theorem on the Cone are now known in all dimensions.

Then either $\overline{NE}(X)$ is locally polyhedral in a neighborhood of R (that is, there exist a finite set of curves $C_1, \ldots, C_r$ and an $\varepsilon > 0$ such that

$$\overline{NE}(X) = \overline{NE}_\varepsilon(X, D) + \sum_{i=1}^{r} \mathbf{R}_+(C_i), \tag{2.2}$$

where $\overline{NE}_\varepsilon(X, D) = \{Z \in \overline{NE}(X) | (D + \varepsilon K_X \cdot Z) \geqslant 0\})$, or there exists a morphism φ: $X \to Y$ making X into a conic fibration, such that $(C) \in R$ for a general fiber $C = \varphi^{-1}(y)$.

PROOF. Let $\alpha = m/n - \delta$, where m and n are positive integers and $0 < \delta \leqslant 1/n$. Then

$$D \equiv A + (m/n)K_X - \delta K_X.$$

From (i), the divisor $D_{m/n} = A + (m/n)K_X$ is numerically negative on R. By virtue of the proof of Theorem 1 in [3], in order to establish the decomposition (2.2) it is enough to check that, for some integer $N > 0$,

$$|ND_{m/n}| \neq \varnothing. \tag{2.4}$$

We will prove this using Riemann-Roch and vanishing; consider a resolution h: $X' \to X$ which is the standard resolution along the curves of canonical singularities, and is otherwise arbitrary. Then the exceptional divisors E_i which map to curves of X have discrepancy $a_i = 0$. We also assume that all exceptional divisors of h are nonsingular and interest transversally. Set

$$\bar{h}(mK_X) = -\left[-mK_{X'} + \sum(m-1)a_iE_i\right] = mK_{X'} - \sum[(m-1)a_i]E_i,$$

where $[\ ]$ denotes the integral part of a number or a divisor. For $n \gg 0$ the divisor $D - (1/n - \delta)K_X \equiv A + ((m-1)/n)K_X$ will satisfy the hypothesis of the Kawamata-Viehweg vanishing theorem [1], except in the case $D^3 = 0$ and $\delta = 1/n$. However, in this case, by (ii), D is a $\mathbf{Q}$-Cartier divisor with $D^3 = 0$ and $-D^2K_X > 0$; then D defines a conic fibration $\varphi_{|ND|}$: $X \to Y$. This is proved in [2] and [3] assuming terminal singularities, and in general using Kawamata's technique in [6] and [7]. The general fiber $C = \varphi^{-1}(y)$ obviously has class $(C) \in R$ (by the definition of R; see (i)). In this case we have one of the conclusions of the lemma, so that from now on we can assume that it does not occur. Then, by Kawamata-Viehweg vanishing,

$$h^i(X', \mathcal{O}_{X'}(nh^*A + \bar{h}(mK_X))) = h^i(X', \mathcal{O}_{X'}(-[-nh^*A - (m-1)h^*K_X] + K_{X'})) = 0$$

for all $i > 0$. Hence

$$h^0(X', \mathcal{O}_{X'}(nh^*A + \bar{h}(mK_X))) = \chi(\mathcal{O}_{X'}(nh^*A = \bar{h}mK_X)) = \text{R-R expression}.$$

Now note that

$$\begin{aligned} \bar{h}(mK_X) &= mK_{X'} - \sum(m-1)a_jE_j - \sum\{(m-1)a_jE_j\} \\ &\equiv mh^*K_X + \sum\left(a_j - \{(m-1)a_j\}\right)E_j. \end{aligned} \tag{2.5}$$

Hence

$$nh^*A + \bar{h}(mK_X) \equiv h^*(nA + mK_X) + \sum b_jE_j,$$

where $b_i = O(1)$ as $n \gg 0$. By (2.3), $nA = mK_X = nD + \delta nK_X$. Writing down only the cubic and quadratic terms in the Riemann-Roch formula, and using the fact that $|\delta n| \leqslant 1$, we get

$$h^0\left(X', \mathcal{O}_{X'}\left(nh^*A = \bar{h}(mK_X)\right)\right) = \tfrac{1}{6}(nD + \delta nK_X)^3 - \tfrac{1}{4}\left(h^*(nD + \delta nK_X)\right)^2 K_{X'} + \cdots, \quad (2.6)$$

where the dots denote terms bounded by a linear function of n. We now prove that the right-hand side of (2.6) is strictly positive if $n \gg 0$. If $D^3 > 0$ this is obvious. Suppose then that $D^3 = 0$ and $-D^2K_X > 0$. If α is rational, we have seen above that D defines a conic fibration of X, and since we are assuming that X is not a conic fibration, α is irrational. Then letting m/n be a continued fraction approximation of α, we can assume that $\delta n \leqq 1/n$, and then for $n \gg 0$ we get

$$h^0\left(X', \mathcal{O}_{X'}\left(nh^*A + \bar{h}(mK_X)\right)\right) = -\tfrac{1}{4}n^2D^2K_X + \cdots > 0,$$

with the dots as before. Thus $|nh^*A + \bar{h}(mK_X)| \neq \varnothing$ for suitable $n \gg 0$, and using (2.5) we get the required nonemptiness assertion (2.4). ■

§3. Proof of the main theorem

3.1. *Choice of the curves C_i.* The cone $\overline{NE}(X)^-$ can have at most a finite set of extremal rays of type (a) which "contract to a point", since the exceptional surfaces E corresponding to these rays are disjoint in pairs, so that their classes in $N(X)^0$ are linearly independent. We also have outside $\overline{NE}_\varepsilon(X, A)$ a finite set of extremal rays of type (a) which "contract onto a curve", since there is a curve C in such rays with $CK_X = -1$. So first of all we assume that $\{C_i\}$ includes a finite set of curves C_i giving the extremal rays $\mathbf{R}_+(C_i)$ of type (a) outside $\overline{NE}_\varepsilon(X, A)$.

We can also see that the cone $\{Z \in \overline{NE}(X) | (K_X + \varepsilon A \cdot Z) \leqq 0\}$ can have at most a finite set of rays of the form $\mathbf{R}_+(C)$ where $C = \varphi^{-1}(y)$ is the general fiber of a conic fibration $\varphi\colon X \to Y$. Indeed, then $CK_X = -2$, so that, assuming $(K_X + \varepsilon A \cdot C) < 0$, the degree $(A \cdot C) < 2/\varepsilon$ is bounded, so that such curves belong to a bounded family. We include in $\{C_i\}$ a finite set of curves which exhausts this set of rays.

By hypothesis, the half-cone $\{Z \in \overline{NE}(X) | (K_X + \varepsilon A \cdot Z) < 0\}$ has only a finite number of rays of type (b), and we add to $\{C_i\}$ the curves corresponding to these.

Now consider the cone

$$V = \overline{NE}_\varepsilon(X, A) + \sum_{i=1}^{r} \mathbf{R}_+(C_i) \subset \overline{NE}(X).$$

If $V = \overline{NE}(X)$ then the theorem is proved. Otherwise $\overline{NE}(X)$ contains a rational ray $Z = \mathbf{R}_+(C) \not\subset V$, and obviously $(C \cdot K_X) < 0$.

$$V_Z = \overline{NE}_\varepsilon(X, A) + \sum_{i=1}^{r} \mathbf{R}_+(C_i) + Z \subset \overline{NE}(X),$$

so that Z is an edge of V_Z, and take a Cartier divisor D such that the hyperplane $D^\perp$ passes through this edge, with $D^\perp \cap V_Z = Z$. Corresponding to D we have an affine line $[D, K_X] \subset N(X)^0$, and this line contains a divisor L_1 such that $L_1^\perp$ is a supporting hyperplane of $\overline{NE}(X)$, with L_1 numerically effective and positive on V; this L_1 can be written as a combination $L_1 = D + \alpha K_X$, with $\alpha > 0$. By construction the cone $R = L_1^\perp \cap \overline{NE}(X)$ is nonempty and is contained strictly inside the half-cone $\overline{NE}(X)^-$. Moreover, a suitable small neighborhood of R does not contain any of the rays $\mathbf{R}_+(C_i)$, and the divisor

$mL_1 - K_X$ is ample for $m \gg 0$. It follows that $L_1^3 \geqq 0$. If $L_1^3 > 0$ then it follows from the main lemma that $\overline{NE}(X)$ is locally polyhedral in a neighborhood of R. Then L_1 can be taken to be $\mathbf{Q}$-rational; but then R contains an extremal ray R' of type (a) or (b), which is impossible by construction. Hence $L_1^3 = 0$. Then $-L_1^2K_X \geqq 0$. If $-L_1^2K_X > 0$ then again using the main lemma we see that either R contains a ray of the form $\mathbf{R}_+(C)$ where $C = \varphi^{-1}(y)$ is the general fiber of a conic fibration, which is impossible by construction, or $\overline{NE}(X)$ is locally polyhedral in a neighborhood of R. In this final case we again get either a ray of type (a) or (b), or a ray corresponding to a conic fibration, any of which are impossible by construction. Hence $-L_1^2K_X = 0$.

3.2. We have thus arrived at the situation that $L_1^3 = L_1^2K_X = 0$. Using Mori's argument from [4], §6, we see that $L_1^2 \equiv 0$. If $\rho(X) \geqq 3$ then there exists another L_2 so that $L_2^\perp$ is a supporting hyperplane of $\overline{NE}(X)$ similar to L_1, but L_2 are not proportional. Again $L_2^2 \equiv 0$. By the numerical effectivity of L_1 and L_2 we have $L_1L_2 \in \overline{NE}(X)$. On the other hand, $L_1^2L_2 = L_1L_2^2 = 0$, and hence $L_1L_2 \in R_1 \cap R_2$. If $R_1 \cap R_2 = 0$, then $L_1L_2 \equiv 0$, sot hat by Mori's arguments it follows that L_1 and L_2 are proportional, which is impossible by assumption. If $R_1 \cap R_2 \neq 0$ then $L_1 + L_2$ again satisfies the same conditions as L_1, and then $(L_1 + L_2)^2 \equiv 0$. Hence $L_1L_2 \equiv 0$, which again leads to a contradiction.

3.3. Finally it remains to consider the case $L_1^3 = L_1^2K_X = 0$ and $\rho(X) = 2$, the case $\rho(X) = 1$ being trivial. Here the extremality condition is trivial, and according to the results of [3] we need only the rationality of L_1. Indeed, if L_1 is rational, then by Kawamata's results $L_1^\perp$ is a supporting hyperplane for a ray R specifying a fibration of del Pezzo surfaces. But as with the rays giving conic fibrations, there are only a finite number of such rays outside $\overline{NE}_\varepsilon(X, A)$. Thus we could have added to $\{C_i\}$ the classes of curves C_i of general del Pezzo's surfaces in such fibrations having $-(C_iK_X) \leqq 9$.

Thus L_1 is an irrational divisor, so that we can assume that $L_1 = D + \alpha K_X$ with α irrational, and D an ample Cartier divisor. The equations $L_1^3 = L_1^2K_X = 0$ give polynomial equations of degree $\leqslant 3$ and 2 in α. Hence α is a quadratic irrationality. Let α' be the conjugate irrationality, and $L_2 = D + \alpha' K_X$. Now L_2 must satisfy both the equations, since they have rational coefficients. It is easy to check that the cycle L_1L_2 is rational. But $L_1^2L_2 = 0$. Hence $L_1L_2 \equiv 0$, since otherwise by irrationality of α we would have $L_1L_2K_X = 0$; but if $ZK_X = ZL_1 = 0$ then $Z \equiv 0$. The relation $L_1L_2 \equiv 0$ again leads to a contradiction, since L_1 and L_2 are not proportional, so that $D = \beta L_1 + \gamma L_2$, and hence $D^3 = (\beta L_1 + \gamma L_2)^3 = 0$. This contradiction completes the proof of the main theorem. ■

Received 4/MAR/83

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Translated by M. REID

*** Added by translator.

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