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The first main theorem on complements: from global to local

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Abstract. The purpose of this paper is to explain and generalize the methods of [24] (see also [18] and [19]). We establish that for local Fano contractions the existence of complements can be reduced to the existence of complements for projective Fano varieties of smaller dimension.

Introduction

The purpose of this paper is to explain and generalize the methods of [24], § 7 (see also [18] and [19]). We prove that for local Fano contractions the existence of complements can be reduced to the existence of complements for projective Fano varieties of smaller dimension. The main conjecture on n-complements (Conjecture 1.3 in [24]) states that they are bounded in every given dimension.

Numerous results obtained in [1], [6], [7], [9], [10], [14], [19], [21], [23] and [24] show that this conjecture is plausible. Any n-complement is actually a "good" element in a multiple antilog-canonical linear system. It was noted in [23] that complements have some good properties compatible with restrictions of linear systems and the Kawamata-Viehweg vanishing theorem. The latter circumstance explains a non-trivial property of n-complements that can serve as a definition (cf. (1.1) below). In the main conjecture we consider log pairs $(X/Z, D)$ consisting of Fano contractions X/Z and boundaries D. To use induction in the proof of the conjecture, we have to divide log pairs and their complements into two classes according to the dimension of the base Z: local (if $\dim(Z) > 0$) and global (otherwise). Equivalently, in the global case Z is a point and X is a projective log Fano variety. We shall prove the existence of n-complements for local log Fano contractions, where n belongs to some set N determined by a class of projective log Fano varieties of smaller dimension. We call this the *first main theorem on complements* (see Theorem 3.1 below): from the global to the local case. To prove it we run the log Minimal Model Program (logMMP). Hence, our proof is conditional in dimensions $n = \dim(X) \geqslant 4$ and complete for $n \leqslant 3$. The main idea is to extend an n-complement from the central fibre of a good modification for $(X/Z, D)$

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(cf. the proof of Theorem 5.6 and Example 5.2 in [23]). Moreover, this approach enables us to control certain numerical invariants of complements, for example, the indices, the type (exceptional or non-exceptional), and the regularity (cf. [24], § 7).

The *second theorem* (from local to global) will be discussed in a future paper. A prototype is the global case considered in [24], where local and inductive complements were used (see [24], § 2, and the definition of "tiger" in [9]). Theorem 5.1 is an elementary but fairly general case of the second theorem and is a modification of the first theorem. This step also shows that the main difficulty is concealed in the Borisov–Alexeev conjecture (see Conjecture 1.8) and concerns ε_d-log-terminal log Fano varieties (namely, that they are bounded for some $\varepsilon_d > 0$ depending on the dimension d). In particular, $\varepsilon_2 = 6/7$ in dimension 2.

The paper is organized as follows: § 1 is auxiliary, in § 2 we make a very important definition of exceptional pairs, in § 3 we prove the main result (Theorem 3.1), and in § 4 we discuss some corollaries and applications. Finally, in § 5 we present a global version of Theorem 3.1.

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§ 1. Preliminaries

Let $K(X)$ be the field of rational functions on a variety X. The formula $D_1 \approx D_2$ means that the prime divisors D_1 and D_2 define the same discrete valuation on $K(X)$. The (Weil) canonical divisor will be denoted by K_X (or simply by K, if there is no danger of ambiguity).

We consider algebraic varieties over the field $\mathbb{C}$ of complex numbers. A *contraction* $f\colon Y \to X$ is defined to be a projective morphism of a normal variety such that $f_*\mathcal{O}_Y = \mathcal{O}_X$. *Blow-ups* and *blow-downs* are birational contractions. We use the standard definitions, abbreviations and notation of the Minimal Model Program, for example, MMP, log-canonical, Kawamata log-terminal and purely log-terminal singularities, $\equiv$, $\sim$, $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$, $\{\cdot\}$, $\overline{\mathrm{NE}}(X/Z)$, $a(E,D)$, $\operatorname{discr}(X,D)$, $\operatorname{totaldiscr}(X,D)$ (see [8], [13], [11]). Unless otherwise stated, by a *boundary* we always mean a $\mathbb{Q}$-boundary, that is, a $\mathbb{Q}$-Weil divisor $D = \sum d_i D_i$ such that $0 \leqslant d_i \leqslant 1$ for all i. A *log variety* (*log pair*) $(X/Z \ni o, D)$ is defined to be a contraction $X \to Z$ together with the boundary D on X, considered locally near the fibre over $o \in Z$. By the dimension of a log pair $(X/Z \ni o, D)$ we always mean the dimension of the total space X.

Definition 1.1 [23]. Let $(X/Z, D)$ be a log variety. Then

(i) a *numerical complement* is an $\mathbb{R}$-boundary $D' \geqslant D$ such that $K + D'$ is log-canonical and numerically trivial,

(ii) an $\mathbb{R}$-*complement* is an $\mathbb{R}$-boundary $D' \geqslant D$ such that $K+D'$ is log-canonical and $\mathbb{R}$-linearly trivial,

(iii) a $\mathbb{Q}$-*complement* is a $\mathbb{Q}$-boundary $D' \geqslant D$ such that $K+D'$ is log-canonical and $\mathbb{Q}$-linearly trivial,

(iv) if $D = S + B$, where $S = \lfloor D \rfloor$ and $B = \{D\}$, then an n-*complement* is a $\mathbb{Q}$-boundary D^+ such that $K + D^+$ is log-canonical, $n(K + D^+) \sim 0$ and

$$nD^+ \geqslant nS + \lfloor (n+1)D \rfloor . \tag{1.1}$$

Note that $\mathbb{R}$-complements can be regarded as n-complements with $n=\infty$, since passage to the limit (as $n\to\infty$) in inequality (1.1) yields the relation $D'\geqslant D$. All these definitions remain valid in the more general situation when D is an $\mathbb{R}$-subboundary (that is, an $\mathbb{R}$-divisor $D=\sum d_iD_i$ with $d_i\leqslant 1$ for all i).

It is obvious that the following implications hold:

$$\exists \text{ a } \mathbb{Q}\text{-complement} \Longrightarrow \exists \text{ an } \mathbb{R}\text{-complement} \Longrightarrow \exists \text{ a numerical complement.}$$

A simple example (see below) shows that an n-complement is not necessarily a $\mathbb{Q}$-complement (or even a numerical one).

Example 1.2. Let P_1, P_2 and P_3 be different points of $\mathbb{P}^1$. We put

$$D:=P_1+\left(\frac{1}{2}+\varepsilon\right)P_2+\left(\frac{1}{2}-\varepsilon\right)P_3,\qquad D':=P_1+\frac{1}{2}P_2+\frac{1}{2}P_3,$$

where $0<\varepsilon\ll 1$. Then $K+D'$ is a 2-complement of the log divisor $K+D$, but the inequality $D'\geqslant D$ is false, that is, $K+D'$ is not a $\mathbb{Q}$-complement of $K+D$.

Under additional restrictions on the coefficients of D (for example, if they are *standard*, see item A below) the inequality $D^+\geqslant D$ does hold (see [24], Lemma 2.7, or [20], § 4.2). Hence, D^+ is a $\mathbb{Q}$-complement in this case.

The question of existence of complements arises naturally for varieties of Fano or Calabi–Yau type, that is, for varieties with numerically effective antilog-canonical divisors, but the numerical efficiency of $-(K+D)$ is not sufficient for the existence of complements ([24], Example 1.1).

Proposition 1.3 ([23], Proposition 5.5). *Let $(X/Z\ni o,D)$ be a log variety. Assume that (X,D) is a log-canonical pair and the divisor $-(K+D)$ is (semi-)ample over Z. Then there is a $\mathbb{Q}$-complement of the log divisor $K+D$ near the fibre over o.*

We shall use the following notation.

A. Let $\Phi\subset[0,1]$ be fixed. We write $D\in\Phi$ if the coefficients of the divisor D are contained in Φ. For example, we can consider $\Phi=\Phi_{\mathbf{sm}}:=\{1-1/m\mid m\in\mathbb{N}\cup\{\infty\}\}$ (the case of so-called *standard coefficients*). However, some of our assertions and conjectures can be stated for other choices of Φ (see, for example, (1.3) below).

B. Let (X,D) be a projective log variety such that

(i) the pair (X,D) is log-canonical,

(ii) $D\in\Phi$,

(iii) the divisor $-(K_X+D)$ is numerically effective and big,

(iv) there is a $\mathbb{Q}$-complement of the log divisor K_X+D (this condition holds if the divisor $-(K_X+D)$ is semi-ample: for example, Theorem 3-1-2 in [8] implies that this condition holds if K_X+D is Kawamata log-terminal).

Such pairs are called *log Fano varieties*.

In the notation of item B we define the minimal complementary number to be

$$\operatorname{compl}(X,D):=\min\{m\mid K_X+D\quad\text{is } m\text{-complementary}\}\tag{1.2}$$

and consider the set

$$\begin{aligned}N_d(\Phi):=\{m\in\mathbb{N}\mid\exists\quad&\text{a log Fano variety } (X,D) \text{ of dimension } d\\ &\text{such that } D\in\Phi \text{ and } \operatorname{compl}(X,D)=m\}.\end{aligned}$$

For example, $N_1([0,1]) = \{1,2,3,4,6\}$ (see [23]). By considering the product of X and $\mathbb{P}^1$ we can show that $N_{d-1}(\Phi) \subset N_d(\Phi)$. Let $N_0([0,1]) = \{1,2\}$. We define by induction

$$\begin{aligned} \Phi^1_{\mathbf{m}} &:= \Phi_{\mathbf{sm}}, & N_1 &:= \max N_1(\Phi^1_{\mathbf{m}}),\\ \Phi^d_{\mathbf{m}} &:= \Phi_{\mathbf{sm}} \bigcup \left[1-\frac{1}{N_{d-1}+1}, 1\right], & N_d &:= \max\left(\bigcup_{k=1}^{d} N_d(\Phi^k_{\mathbf{m}})\right). \end{aligned} \tag{1.3}$$

We do not exclude the case when $N_d = \infty$ (in this case $\Phi^d_{\mathbf{m}} := \Phi_{\mathbf{sm}}$), but we hope that $N_d < \infty$ (see Conjecture 1.7 below). According to Example 5.2 in [23], we have

$$N_1 = 6, \qquad \Phi^2_{\mathbf{m}} = \Phi_{\mathbf{sm}} \cup [6/7, 1].$$

It was proved in [24] that N_2 is finite. Our definition implies that $N_d \geqslant N_{d'}$ and $\Phi^d_{\mathbf{m}} \subset \Phi^{d'}_{\mathbf{m}}$ if $d \geqslant d'$.

Lemma 1.4 (cf. [24], Lemma 2.7). *If $\alpha \in \Phi^d_{\mathbf{m}}$, then*

$$\lfloor (n+1)\alpha \rfloor \geqslant n\alpha$$

for any $n \leqslant N_{d-1}$.

Proof. If $\alpha \in \Phi_{\mathbf{sm}}$, then $\alpha = 1 - 1/m$ for some $m \in \mathbb{N}$. In this case we write $n\alpha = q + k/m$, where $q = \lfloor n\alpha \rfloor$ and $k/m = \{n\alpha\}$, $k \in \mathbb{Z}$, $0 \leqslant k \leqslant m-1$. We have

$$\lfloor (n+1)\alpha \rfloor = \lfloor q + k/m + 1 - 1/m \rfloor = \begin{cases} q & \text{if } k = 0,\\ q+1 & \text{otherwise.} \end{cases}$$

In both cases we have $\lfloor (n+1)\alpha \rfloor \geqslant q + k/m = n\alpha$. Assume that $\alpha \notin \Phi_{\mathbf{sm}}$. Then

$$\alpha > 1 - \frac{1}{n_{d-1}+1}$$

and

$$\lfloor (n+1)\alpha \rfloor \geqslant \left\lfloor n + 1 - \frac{n+1}{n_{d-1}+1} \right\rfloor \geqslant n \geqslant n\alpha.$$

Corollary 1.5. *Let (X, D) be a log pair such that $D \in \Phi^d_{\mathbf{m}}$, and let D^+ be an n-complement with $n \leqslant N_{d-1}$. Then $D^+ \geqslant D$.*

Lemma 1.6 (cf. [23], Lemma 4.2). *Let (X, D) be a log-canonical log variety, and let $S := \lfloor D \rfloor$ and $B := \{D\}$. Assume that the divisor $K + S$ is purely log-terminal and $D \in \Phi^d_{\mathbf{m}}$ for some d ($D \in \Phi_{\mathbf{sm}}$). Then* $\operatorname{Diff}_S(B) \in \Phi^d_{\mathbf{m}}$ ($\operatorname{Diff}_S(B) \in \Phi_{\mathbf{sm}}$).

Proof. We write $B = \sum b_j B_j$, $0 < b_j < 1$. Let α be the coefficient of $\operatorname{Diff}_S(B)$. Then (see [23], Corollary 3.10)

$$\alpha = \frac{m-1}{m} + \sum_j \frac{b_j n_j}{m}, \tag{1.4}$$

where $m \in \mathbb{N}$ and $n_j \in \mathbb{N} \cup \{0\}$. Since the pair $(S, \operatorname{Diff}_S(B))$ is log-canonical (see [13], Theorem 17.7), we have $\alpha \leqslant 1$. We can assume that $\alpha < 1$. Using the inequality $b_j \geqslant 1/2$, we can easily show that $\sum n_j \leqslant 1$ in (1.4) (see [23], Lemma 4.2). If $n_j = 0$ in (1.4) for all j, then it is obvious that $\alpha \in \Phi_{\mathbf{sm}}$. Otherwise $n_{j_0} = 1$ for some j_0 and in (1.4) we have $n_j = 0$ for $j \neq j_0$. Then

$$\alpha = (m - 1 + b_{j_0})/m.$$

If $b_{j_0} \in \Phi_{\mathbf{sm}}$, then $b_{j_0} = 1 - 1/n$, $n \in \mathbb{N}$, and $\alpha = \frac{mn-1}{mn} \in \Phi_{\mathbf{sm}}$, but if $b_{j_0} \geqslant 1 - \frac{1}{N_{d-1}+1}$, then

$$\alpha \geqslant b_{j_0} \geqslant 1 - \frac{1}{N_{d-1}+1}.$$

In both cases $\alpha \in \Phi^d_{\mathbf{m}}$.

Conjecture 1.7. *In the notation of item* B *the set* $N_d(\Phi)$ *is finite.*

The proof of Conjecture 1.7 in dimension two given in [24] is largely based on the boundedness theorems for log del Pezzo surfaces [2] (see also [17]). For arbitrary dimensions we have the following conjecture.

Conjecture 1.8. *Let* $\varepsilon > 0$ *be fixed. Let* (X, D) *be a normal projective log variety such that*

(i) *the divisor* $K + D$ *is* $\mathbb{Q}$*-Cartier*,

(ii) $\operatorname{totaldiscr}(X, D) > -1 + \varepsilon$,

(iii) *the divisor* $-(K_X + D)$ *is numerically effective and big.*

Then the variety X *belongs to one of finitely many algebraic families.*

It is well known that this conjecture is true if $\dim(X) = 2$. For $\dim(X) \geqslant 3$ there are only isolated results [3], [4]. A new approach to the proof of Conjecture 1.8 was suggested in [9], § 9.

Conjecture 1.9 (Inductive Conjecture). *Let the assumptions of item* B *hold for the pair* (X, D) (*in particular*, $D \in \Phi$). *Assume that there is a* $\mathbb{Q}$*-complement of* $K + D$ *that is not Kawamata log-terminal. Then* $K + D$ *has an* n*-complement if* $n \in N_{d-1}(\Phi)$. *Moreover, this new complement can be chosen in such a way as not to be Kawamata log-terminal.*

One might expect that Conjecture 1.9 is true for $\Phi = \Phi_{\mathbf{sm}}$ or $\Phi = \Phi^d_{\mathbf{m}}$, where $d = \dim(X)$. However, in the general case it is false: see [24], Example 2.4, and [20], § 8.1. This conjecture has nevertheless been proved (in an even stronger form) for $\dim(X) = 2$ and $\Phi = \Phi^2_{\mathbf{m}}$: see [24], § 2.

§ 2. Exceptionality

Definition 2.1. We say that the contraction $f\colon X \to Z$ is of *local type* if $\dim(Z) > 0$. Otherwise (that is, if Z is a point) it is of *global type.*

Hence, contractions of local type can be either birational or fibred. In both cases we consider the structure of $f\colon X \to Z$ near a fixed fibre $f^{-1}(o)$, $o \in Z$. We usually assume that X is a sufficiently small neighbourhood of the fibre over o.

Definition 2.2 (see [23], § 5, [24], Definition 1.5). Let $(X/Z \ni o, \Delta)$ be a log variety of local type. Assume that $K + \Delta$ has at least one $\mathbb{Q}$-complement near the fibre over o. The log variety $(X/Z \ni o, \Delta)$ is said to be *exceptional* if for any $\mathbb{Q}$-complement[1] $K + \Delta^+$ of $K + \Delta$ near the fibre over o there is at most one divisor E of the field $K(X)$ with $a(E, \Delta^+) = -1$.

It is clear that exceptionality depends on the choice of the base point $o \in Z$.

Lemma 2.3. *Let* $(X/Z \ni o, \Delta)$ *and* $(X'/Z \ni o, \Delta')$ *be log varieties* (*of local or global type*) *and let* $f\colon X \to X'$ *be a contraction over* Z. *Assume that the divisor* $K_{X'} + \Delta'$ *is* $\mathbb{Q}$-*Cartier and* Δ *is the crepant pullback of* Δ' (*that is,* $f^*(K_{X'} + \Delta') = K_X + \Delta$ *and* $f_*\Delta = \Delta'$). *Then* $(X/Z \ni o, \Delta)$ *is exceptional if and only if* $(X'/Z \ni o, \Delta')$ *is exceptional.*

The proof follows from [11], Proposition 3.10.

Proposition 2.4. *Let* $(X/Z \ni o, \Delta)$ *be a log variety of local type, and let* D *and* D' *be* $\mathbb{Q}$-*complements such that* $K + D$ *and* $K + D'$ *are not Kawamata log-terminal. Let* S *and* S' *be prime divisors of the field* $K(X)$ *such that* $a(S, D) = -1$ *and* $a(S', D') = -1$. *Assume that* $S \not\approx S'$. *Then there is a* $\mathbb{Q}$-*complement* G *of the log divisor* $K + \Delta$ *such that* $a(S, G) = a(E, G) = -1$ *for some divisor* $E \not\approx S$ *of* $K(X)$.

Proof (cf. [14], Proposition 2.7, [7], Proposition 2.4). Note that the divisor $D' - D$ is $\mathbb{Q}$-Cartier and numerically trivial over Z. Put $D(\alpha) := D + \alpha(D' - D)$. Then $D(0) = D$, $D(1) = D'$, and $K + D(\alpha)$ is a $\mathbb{Q}$-complement for all $0 \leqslant \alpha \leqslant 1$ (since the property of being log-canonical is convex; see [23], § 1.4.1, or [13], Proposition 2.17). We fix an effective Cartier divisor L on Z (passing through o) and put $F := f^*L$. For $0 \leqslant \alpha \leqslant 1$ we consider the function

$$\varsigma(\alpha) := \sup\{\beta \mid K + D(\alpha) + \beta F \quad \text{is log-canonical}\}$$

and put $T(\alpha) := D(\alpha) + \varsigma(\alpha)F$. We fix some log resolution of $(X, D + D' + F)$. Let $\sum E_i$ be the union of the exceptional divisor and the proper transform of $\operatorname{Supp}(D + D' + F)$. Then the function $\varsigma(\alpha)$ can be computed as follows:

$$\varsigma(\alpha) = \max_{E_i}\{\beta \mid a(E_i, D(\alpha) + \beta F) \geqslant -1\}$$

(see, for example, [8], § 0-2-12). In particular, $\varsigma(\alpha) \in \mathbb{Q}$. Therefore, the log divisor $K + T(\alpha)$ is a $\mathbb{Q}$-complement. The above arguments show that $\beta = \varsigma(\alpha)$ can be determined from the linear inequalities $a(E_i, D(\alpha) + \beta F) \geqslant -1$, where the E_i range over some finite set of prime divisors. Therefore, the function $\varsigma(\alpha)$ is piecewise-linear and continuous in α. The coefficients of the divisor $T(\alpha)$ also have these properties. The pair $(X, T(\alpha))$ is not Kawamata log-terminal for any α, $0 \leqslant \alpha \leqslant 1$. We claim that $a(S, T(0)) = -1$. Indeed, $T(0) = D + \varsigma(0)F \geqslant D$. Therefore, $a(S, T(0)) \leqslant a(S, D) = -1$. Since the divisor $K + T(0)$ is log-canonical, we have $a(S, T(0)) = -1$. Further, let

$$\alpha_0 := \sup\{\alpha \mid a(S, T(\alpha)) = -1\}.$$

[1] We can assume that this condition holds for any $\mathbb{R}$-complement.

The above arguments show that α_0 is rational (and $a(S, T(\alpha_0)) = -1$). If $\alpha_0 = 1$, then we put $G := T(1)$ and $E = S'$. Otherwise $a(S, T(\alpha)) > -1$ for all $\alpha > \alpha_0$. Therefore, there is a divisor $E \not\approx S$ of the field $K(X)$ such that $a(E, T(\alpha)) = -1$. Once again we can choose E among the components of $\sum E_i$. Hence, E does not depend on α if $0 < \alpha - \alpha_0 \ll 1$. It is obvious that $a(E, T(\alpha_0)) = -1$, and we can put $G := T(\alpha_0)$.

Corollary 2.5. *Let $(X/Z \ni o, \Delta)$ be a non-exceptional log variety of local type and let $D \geqslant \Delta$ be a $\mathbb{Q}$-complement such that the pair (X, D) is not Kawamata log-terminal. Let S be a divisor of the field $K(X)$ such that $a(S, D) = -1$. Then there is a $\mathbb{Q}$-complement $G \geqslant \Delta$ such that $a(S, G) = a(E, G) = -1$ for some divisor $E \not\approx S$ of $K(X)$.*

Proof. Since the pair $(X/Z \ni o, \Delta)$ is non-exceptional, there is a $\mathbb{Q}$-complement $D' \geqslant \Delta$ such that $a(S', D') = -1$ for some $S' \not\approx S$. We complete the proof, using Proposition 2.4.

Corollary 2.6. *Let $(X/Z \ni o, \Delta)$ be an exceptional log variety of local type. Then there is a uniquely determined divisor S of the field $K(X)$ such that for any $\mathbb{Q}$-complement D we have $a(E, D) > -1$ if $E \not\approx S$ in $K(X)$.*

The divisor S defined in Corollary 2.6 will be called the *central divisor* of the exceptional log pair $(X/Z \ni o, \Delta)$.

Corollary 2.7. *Let $(X/Z \ni o, \Delta)$ be an exceptional log variety of local type, and let S be the central divisor. Then the centre of S on X is contained in the fibre over o.*

Proof. Let $K+D$ be a $\mathbb{Q}$-complement such that $a(S, D) = -1$ and let H be a general hyperplane section of Z through o. If f^*H does not contain the centre of S, then $\operatorname{mult}_S f^*H = 0$ and $a(S, D) = a(S, D + cf^*H) = -1$ for all c. Consider a c such that $K + D + cf^*H$ is maximally log-canonical. As in the proof of Proposition 2.4, we obtain that $a(E, D + cf^*H) = -1$ for some $E \not\approx S$, a contradiction.

Example 2.8. Consider a log-canonical singularity $X \ni o$ (that is, $X = Z$ and $\Delta = 0$). This singularity is exceptional if and only if for any boundary B on X such that the pair (X, B) is log-canonical, there is at most one divisor E of $K(X)$ such that $a(E, B) = -1$. For example, a two-dimensional log-terminal singularity is exceptional if and only if it belongs to one of the types $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ (see [23], Example 5.2.3, [14]).

In the global case Definition 2.2 has a somewhat different form.

Definition 2.9. Let (X, Δ) be a log variety of global type. Assume that the divisor $K + \Delta$ has at least one n-complement. Then the pair (X, Δ) is said to be *exceptional* if any $\mathbb{Q}$-complement $K + \Delta^+$ of $K + \Delta$ is Kawamata log-terminal (that is, $a(E, \Delta^+) > -1$ for any divisor E of $K(X)$).

Example 2.10. (i) Let $X = \mathbb{P}^1$, let Z be a point, and let

$$\Delta = \sum_{i=1}^{r} (1 - 1/m_i) P_i, \qquad m_i \in \mathbb{N},$$

where $P_1,\dots,P_r$ are distinct points. The divisor $-(K+\Delta)$ is numerically effective if and only if $\sum_{i=1}^r(1-1/m_i)\leqslant 2$. In this case the set $(m_1,\dots,m_r)$ yields an exceptional pair if and only if it coincides (up to a permutation) with one of

$$\begin{array}{llllll} E_6: & (2,3,3), & E_7: & (2,3,4), & E_8: & (2,3,5), \\ \widetilde{E}_6: & (3,3,3), & \widetilde{E}_7: & (2,4,4), & \widetilde{E}_8: & (2,3,6), \\ \widetilde{D}_4: & (2,2,2,2). & & & & \end{array}$$

(ii) Let $X=\mathbb{P}^d$, let Z be a point, and let

$$\Delta=\sum_{i=1}^{d+2}(1-1/m_i)\Delta_i, \qquad m_i\in\mathbb{N},$$

where $\Delta_1,\dots,\Delta_{d+2}$ are hyperplanes in $\mathbb{P}^d$. The log divisor $-(K+\Delta)$ is numerically effective if and only if $\sum 1/m_i\leqslant 1$. If the log pair (X,Δ) is exceptional, then the divisor $-\bigl(K+\Delta_j+\sum_{i\neq j}(1-1/m_i)\Delta_i\bigr)$ is not numerically effective for any j. Therefore, $\sum_{i\neq j}1/m_i>1$. Now it is easy to show that there are constants $\mathrm{Const}(d)$ such that $m_j\leqslant\mathrm{Const}(d)$ for all j (cf. [11], Example 8.16). Hence, there are only finitely many possibilities for exceptional sets $(m_1,\dots,m_{d+2})$.

These examples and many other facts (see [24], [14], [7], [19], [6], [21]) show that in the general situation one might expect that the following principles hold:

(1) for non-exceptional pairs the linear system $|-m(K+D)|$ has good properties for some small m,

(2) the exceptional pairs may be classified.

§ 3. Fano contractions

In this section we prove Theorem 3.1. A two-dimensional version of this theorem was proved by the second author in [23] and generalized in [24], [19].

Theorem 3.1 (the local case). *Let $\Phi:=\Phi^d_{\mathbf{m}}$ (or $\Phi:=\Phi_{\mathbf{sm}}$), and let $(X/Z\ni o,D)$ be a d-dimensional log variety of local type such that*

(i) $D\in\Phi$,

(ii) *the pair (X,D) is Kawamata log-terminal,*

(iii) *the divisor $-(K+D)$ is numerically effective and big over Z.*

Let $f\colon X\to Z$ be a structural morphism. Assume that the logMMP holds in dimension d. Then for some integer $n\in N_{d-1}(\Phi)$ there is an n-complement of $K+D$ near $f^{-1}(o)$ that is not Kawamata log-terminal. Moreover, if the pair $(X/Z\ni o,D)$ is non-exceptional and Conjecture 1.9 holds in dimensions $d'\leqslant d-1$ for $\Phi=\Phi^d_{\mathbf{m}}$ (or $\Phi=\Phi_{\mathbf{sm}}$), then the divisor $K+D$ is n-complementary near the fibre $f^{-1}(o)$ if $n\in N_{d-2}(\Phi)$. Moreover, there is a non-exceptional complement.

In the non-exceptional case we expect more exact results, in which the singularities of the complement depend on the topological structure of the essential exceptional divisor (see [24], § 7).

Example 3.2. Let $(Z \ni o)$ be a two-dimensional Du Val singularity (rational double point), let $D = 0$, and let f be the identity map. For some $n \in N_1(\Phi_{\mathbf{sm}}) = \{1, 2, 3, 4, 6\}$ there is an n-complement of the divisor K_Z that is not Kawamata log-terminal (see [23], Example 5.2.3). The singularity is non-exceptional if it is of type A_n or D_n. In these cases there is an n-complement that is not Kawamata log-terminal for $n \in N_0(\Phi_{\mathbf{sm}}) = \{1, 2\}$.

The key idea of the proof of Theorem 3.1 is simple: we find a special blow-up of X with an irreducible exceptional divisor S (Proposition 3.6) and then reduce the problem to a similar problem on a (possibly projective) variety S of smaller dimension using inductive properties of complements (Proposition 6.2).

Lemma 3.3. *Let $(X/Z, D)$ be a log variety such that the pair (X, D) is Kawamata log-terminal and the divisor $-(K_X + D)$ is numerically effective and big over Z. Then there is an effective $\mathbb{Q}$-divisor $D^{\mho}$ such that the pair $(X, D + D^{\mho})$ is Kawamata log-terminal and the divisor $-(K_X + D + D^{\mho})$ is ample over Z.*

The proof follows from Kodaira's lemma (see, for example, [8], Lemma 0-3-3).

Corollary 3.4. *Under the assumptions of Lemma* 3.3 (*and in the same notation*) *the Mori cone $\overline{\mathrm{NE}}(X/Z)$ is polyhedral and generated by contractible extremal rational curves.*

Definition 3.5. Let (X, Δ) be a log variety and let $g \colon Y \to X$ be a blow-up such that the exceptional locus of g contains precisely one irreducible divisor S. Assume that the pair $(Y, \Delta_Y + S)$ is purely log-terminal and the divisor $-(K_Y + \Delta_Y + S)$ is g-ample. Then $g \colon (Y \supset S) \to X$ is called a *purely log-terminal blow-up* of the pair (X, Δ).

Unlike log-terminal modifications ([25], Theorem 3.1), purely log-terminal blow-ups are not log-crepant.

Let $(X \ni o, D)$ be an exceptional singularity. By Corollary 2.6, there is at most one purely log-terminal blow-up (see [18], Proposition 6).

Proposition 3.6 (see [18], [20], § 3.1, cf. [26]). *Let $(X, \Delta + \Delta^0)$ be a log variety such that the variety X is $\mathbb{Q}$-factorial, $\Delta \geqslant 0$, $\Delta^0 \geqslant 0$, the divisor $K + \Delta + \Delta^0$ is log-canonical, but not purely log-terminal, and $K + \Delta$ is Kawamata log-terminal.* (*We do not require that Δ and Δ^0 have no common components.*) *Assume that the logMMP holds in dimension* $\dim(X)$. *Then there is a purely log-terminal blow-up $g \colon (Y \supset S) \to X$ of (X, Δ) such that*

(i) *the divisor $K_Y + \Delta_Y + S + \Delta^0_Y = g^*(K + \Delta + \Delta^0)$ is log-canonical,*

(ii) *the divisor $K_Y + \Delta_Y + S + (1 - \varepsilon)\Delta^0_Y$ is purely log-terminal and anti-ample over X for any $\varepsilon > 0$,*

(iii) *the variety Y is $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$.*

Such blow-ups are called *inductive blow-ups* of $(X, \Delta + \Delta^0)$. It should be noted that this definition depends on Δ and Δ^0 as well as on $\Delta + \Delta^0$. Such blow-ups are frequently used in the theory of complements. In the local case it is possible to construct a boundary Δ^0 such as that in Proposition 3.6 just by taking the pullback of some $\mathbb{Q}$-divisor on Z. In the global case the problem of constructing Δ^0 is much more difficult.

Proof. First we consider a log-terminal modification[2] $h\colon V\to X$ of the log pair $(X,\Delta+\Delta^0)$ (see [23], [13], Theorem 17.10). We write

$$h^*(K+\Delta+\Delta^0)=K_V+\Delta_V+\Delta_V^0+E,$$

where Δ_V and Δ_V^0 are the proper transforms of Δ and Δ^0, and E is an exceptional divisor. We choose h so that E is reduced and $E\neq 0$ (see [13], Theorem 17.10, [25], Theorem 3.1). We claim that the divisor $K_V+\Delta_V+E$ cannot be numerically effective over X. Indeed,

$$h^*(K+\Delta)=K_V+\Delta_V+\sum\alpha_iE_i,$$

where $\alpha_i<1$ for all i. Consequently,

$$h^*\Delta^0=\Delta_V^0+\sum(1-\alpha_i)E_i.$$

Therefore,

$$K_V+\Delta_V+E\equiv-\Delta_V^0\equiv\sum(1-\alpha_i)E_i\quad\text{over}\quad X,$$

where the divisor $\sum(1-\alpha_i)E_i$ is effective, exceptional and different from zero. This divisor cannot be h-numerically effective (see, for example, [23], § 1.1). Run the $(K_V+\Delta_V+E)$-MMP over X. At the final step we obtain a birational contraction $g\colon Y\to X$ such that (i)–(iii) in Proposition 3.6 hold.

We prove Theorem 3.1 by induction on d: assume that the assertion holds for $\dim(X)<d$. We replace the variety X by its $\mathbb{Q}$-factorialization (see [13], Theorem 6.11.1). All our assumptions remain valid. Consider the divisor $D^{\mho}$ (see Lemma 3.3) and put $D^{\triangledown}:=D^{\mho}+cf^*H$, where H is an effective Cartier divisor on Z passing through o and c is the log-canonical threshold:

$$c=c(X,D+D^{\mho},f^*H),$$

the largest number such that $K+D+D^{\mho}+cf^*H$ is log-canonical.

Then the following assertion holds.

Claim 3.7. *The divisor $K+D+D^{\triangledown}$ is anti-ample over Z and log-canonical but not Kawamata log-terminal.*

Note that D and $D^{\triangledown}$ can have common components. We consider the following two cases:

(a) the pair $(X,D+D^{\triangledown})$ is purely log-terminal (and $\lfloor D+D^{\triangledown}\rfloor\neq 0$),

(b) the pair $(X,D+D^{\triangledown})$ is not purely log-terminal.

In case (b) we consider an inductive blow-up $g\colon\widehat{X}\to X$ of $(X,D+D^{\triangledown})$. Let S be an (irreducible) exceptional divisor. According to [23], Lemma 5.4 (or [13], Lemma 19.2), it is sufficient to prove the existence of the desired complement on $\widehat{X}$. We write

$$\begin{gathered} g^*(K+D+D^{\triangledown})=K_{\widehat{X}}+\Delta+S+\widehat{D}^{\triangledown},\\ g^*(K+D)=K_{\widehat{X}}+\Delta+aS, \end{gathered}\tag{3.1}$$

[2]Sometimes log-terminal modifications are called *log-terminal resolutions*.

where $\widehat{D}^\triangledown$ and Δ are the proper transforms of $D^\triangledown$ and D, and $a<1$. Note that $\Delta+aS$ is not necessarily a boundary.

In case (a) we put $\widehat{X}=X$, $g=\mathrm{id}$ and $S=\lfloor D+D^\triangledown\rfloor$. In this case the variety S is irreducible by the connectedness lemma ([13], Theorem 17.4), since S is normal ([13], Corollary 17.5). We determine Δ from the equality $D=\Delta+aS$, where $0\leqslant a<1$ and S is not a component of Δ. We put

$$\widehat{D}^\triangledown := D+D^\triangledown-S-\Delta.$$

In both cases (a) and (b) Claim 3.7 and formula (3.1) imply that the following assertion holds (see [11], Proposition 3.10).

Claim 3.8. *The pair $(\widehat{X},\Delta+S+\widehat{D}^\triangledown)$ is log-canonical and is not Kawamata log-terminal, the pair $(\widehat{X},\Delta+aS)$ is Kawamata log-terminal, and the divisors $-(K_{\widehat{X}}+\Delta+S+\widehat{D}^\triangledown)$ and $-(K_{\widehat{X}}+\Delta+aS)$ are numerically effective and big over Z.*

Lemma 3.9. *One can find a $\delta_0>0$ and a boundary M on $\widehat{X}$ such that*

(i)
$$\Delta+aS\leqslant M\leqslant \Delta+S+(1-\delta_0)\widehat{D}^\triangledown, \tag{3.2}$$

(ii) *the pair $(\widehat{X},M)$ is Kawamata log-terminal,*

(iii) *the divisor $-(K+M)$ is numerically effective and big over Z.*

In particular, the Mori cone $\overline{\mathrm{NE}}(\widehat{X}/Z)$ is polyhedral.

Proof. By Claim 3.7, the divisor $K+D+(1-\delta_0)D^\triangledown$ is Kawamata log-terminal and anti-ample over Z for sufficiently small positive δ_0. We define M as a crepant pullback:

$$\begin{aligned} K_{\widehat{X}}+M &= g^*(K+D+(1-\delta_0)D^\triangledown)\\ &= g^*(K+D)+(1-\delta_0)\big(g^*(K+D+D^\triangledown)-g^*(K+D)\big)\\ &= K_{\widehat{X}}+\Delta+aS+(1-\delta_0)\big((K_{\widehat{X}}+\Delta+S+\widehat{D}^\triangledown)-(K_{\widehat{X}}+\Delta+aS)\big). \end{aligned} \tag{3.3}$$

Then

$$\begin{aligned} M &= \Delta+aS+(1-\delta_0)(S+\widehat{D}^\triangledown-aS)\\ &= \Delta+\big(1-\delta_0(1)\big)S+(1-\delta_0)\widehat{D}^\triangledown. \end{aligned}$$

Equality (3.3) implies that the pair $(\widehat{X},M)$ is Kawamata log-terminal ([11], Proposition 3.10), and the divisor $-(K+M)$ is numerically effective and big over Z. Inequality (3.2) holds if $a\leqslant 1-\delta_0(1)$, that is, if $0<\delta_0\ll 1$.

We shall define an auxiliary divisor $\widehat{D}^\lambda$.

A. Let $0<\lambda\ll\delta_0$. We put

$$\widehat{D}^\lambda := (1-\lambda)\widehat{D}^\triangledown.$$

We claim that the log divisor $K_{\widehat{X}}+\Delta+S+\widehat{D}^\lambda$ is purely log-terminal and anti-ample over Z.

Indeed, in case (b) the curves in the fibres of the morphism g generate an extremal ray R, since $\rho(\widehat{X}/X)=1$. We have

$$R\cdot(K_{\widehat{X}}+\Delta+S+\widehat{D}^{\triangledown})=0$$

(and the divisor $K_{\widehat{X}}+\Delta+S+\widehat{D}^{\triangledown}$ is strictly negative on all extremal rays different from R: see Claim 3.7 and equality (3.1)). Taking (3.1) into account, we obtain that $\widehat{D}^{\triangledown}\equiv-(1-a)S$ over X and the divisor $\widehat{D}^{\triangledown}$ is positive on R. Therefore, the log divisor $K_{\widehat{X}}+\Delta+S+\widehat{D}^{\lambda}$ is strictly negative on all extremal rays of the cone $\overline{\mathrm{NE}}(\widehat{X}/Z)$ for sufficiently small positive λ. By Kleiman's criterion, it is anti-ample. Finally, the pair $(\widehat{X},\Delta+S+\widehat{D}^{\lambda})$ is purely log-terminal, since $\widehat{D}^{\lambda}\leqslant\widehat{D}^{\triangledown}$. In case (a) our assertion is an obvious consequence of Claim 3.7.

Note that $M\leqslant\Delta+S+\widehat{D}^{\lambda}$ by (3.2).

B. We fix some set $F_1,\dots,F_r$ of principal divisors on $\widehat{X}$. For $n\gg0$ we consider a general element

$$F\in\left|-n(K_{\widehat{X}}+\Delta+S+\widehat{D}^{\lambda})-\sum F_i\right|$$

and put

$$B:=\widehat{D}^{\lambda}+\frac{1}{n}\left(F+\sum F_i\right).$$

It is possible to choose $F_1,\dots,F_r$ and n in such a way that

(i) the pair $(\widehat{X},\Delta+S+B)$ is purely log-terminal, and

(ii) the components of the divisor B generate $N^1(\widehat{X}/Z)$.

Our construction yields the numerical equivalence

$$K+\Delta+S+B\equiv0\quad\text{over}\quad Z.$$

Let $\varepsilon>0$ be such that the divisor $K+\Delta+S+(1+\varepsilon)B$ is purely log-terminal (see [13], Proposition 2.17) and

$$M\leqslant\Delta+S+(1-\varepsilon)B$$

(that is, $1-\delta_0\leqslant(1-\varepsilon)(1-\lambda)$: see the proof of Lemma 3.9). We run the $(K+\Delta+S+(1+\varepsilon)B)$-MMP over Z:

$$\begin{tikzcd}
& \widehat{X} \arrow[dl, "g"'] \arrow[dr, dashed] & \\
X \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "q"] \\
& Z &
\end{tikzcd}$$

In what follows a bar (for example, in $\bar{\Delta}$) stands for the proper transform on $\overline{X}$ of a divisor defined on $\widehat{X}$. For every extremal ray R we have $R\cdot B<0$ and $R\cdot(K+\Delta+S)>0$. Therefore, any contraction is either flipping or divisorial and contracts a component of B. In particular, no divisorial contraction can contract S. At the final step of the MMP we obtain that the divisor $(K+\Delta+S+(1+\varepsilon)B)$ is numerically effective over Z (we do not exclude the case when $\overline{X}=Z$). Since $K+\Delta+S+B\equiv0$, the divisor $-(K+\Delta+S)$ is also numerically effective over Z.

Lemma 3.10. *We can run the $(K+\Delta+S+(1+\varepsilon)B)$-MMP so that at every step there is a boundary $M \leqslant \Delta+S+(1-\varepsilon)B$ such that the divisor $K+M$ is Kawamata log-terminal and the divisor $-(K+M)$ is numerically effective and big over Z.*

Proof. By Lemma 3.9, there is such a boundary at the first step. If the divisor $K+\Delta+S+(1+\varepsilon)B \equiv \varepsilon B$ is not numerically effective over Z, then neither is $-(K+\Delta+S+(1-\varepsilon)B) \equiv \varepsilon B$. Put

$$t_0 := \sup\{t \mid -(K+M+t(\Delta+S+(1-\varepsilon)B-M)) \quad \text{is numerically effective}\}.$$

By Lemma 3.4 this supremum is a maximum that is attained at some extremal ray. Therefore, t_0 is rational and $0 < t_0 < 1$. Consider the boundary

$$M^0 := M + t_0(\Delta+S+(1-\varepsilon)B-M).$$

Then the divisor $-(K+M^0)$ is numerically effective over Z and $M^0 \leqslant \Delta+S+(1-\varepsilon)B$. We claim that the divisor $-(K+M^0)$ is big over Z. Assume the contrary. By the base-point-free theorem, the divisor $-(K+M^0)$ is semi-ample over Z and determines a contraction $\varphi\colon \widehat{X} \to W$ to a variety of smaller dimension. Let C be a general curve in a fibre. Then

$$C\cdot(K+M^0) = C\cdot(K+\Delta+S+B) = 0.$$

Therefore, $C\cdot(\Delta+S+B-M^0) = 0$. Since C is numerically effective, we have $\varepsilon C\cdot B \leqslant C\cdot(\Delta+S+B-M^0) = 0$ and $C\cdot B = 0$. By assertion (ii) in item B, we have $C \equiv 0$: a contradiction.

Further, the cone $\overline{\mathrm{NE}}(\widehat{X}/Z)$ is polyhedral. Consequently, there is an extremal ray R such that $R\cdot(K+M^0) = 0$ and

$$\varepsilon R\cdot B = -r\cdot(K+\Delta+S+(1-\varepsilon)B) < 0.$$

Therefore, $R\cdot(K+\Delta+S+(1+\varepsilon)B) < 0$. Let $h\colon \widehat{X} \to Y$ be a contraction of R. Put $M_Y^0 := h_*M^0$. Then

$$K+M^0 = h^*(K_Y+M_Y^0).$$

Therefore, the divisor $K_Y+M_Y^0$ is $\mathbb{Q}$-Cartier. The pair (Y, M_Y^0) is Kawamata log-terminal, and the divisor $-(K_Y+M_Y^0)$ is numerically effective and big over Z. If the contraction g is divisorial, then we can continue the process, replacing $\widehat{X}$ by Y and M by $M = M_Y^0$. Assume that g is a flipping contraction, and let

$$\begin{tikzcd} \widehat{X} \arrow[rr, dashed] \arrow[dr, "h"'] & & X^+ \arrow[dl, "h^+"] \\ & Y & \end{tikzcd}$$

be a flip. Put $M := {h^+}^{-1}(M_Y^0)$. Then the divisor $-(K_{X^+}+M^+) = -{h^+}^*(K_Y+M_Y^0)$ is numerically effective and big over Z. Hence, we can continue the process, replacing X by X^+.

C. Running the same MMP as in Lemma 3.10, we finally obtain a variety $\overline{X}$ such that

(i) the pair $(\overline{X}, \bar{\Delta}+\overline{S})$ is purely log-terminal,

(ii) the divisor $-(K+\bar{\Delta}+\overline{S})$ is numerically effective over Z.

Lemma 3.11. *The divisor $-(K+\bar{\Delta}+\overline{S})$ is semi-ample over Z. Moreover, if $-(K+\bar{\Delta}+\overline{S})$ is not ample, then it determines a birational contraction over Z whose exceptional locus is contained in* $\operatorname{Supp}(\overline{B})$. *In particular, the divisor*

$$-(K_{\overline{S}}+\operatorname{Diff}_{\overline{S}}(\bar{\Delta}))=-(K+\bar{\Delta}+\overline{S})|_{\overline{S}}$$

is big (and numerically effective) over $q(\overline{S})$.

Proof. Lemma 3.10 and the base-point-free theorem imply that the divisor $-(K+\bar{\Delta}+\overline{S})$ is semi-ample. Hence, for some $n\in\mathbb{N}$ the linear system $|-n(K+\bar{\Delta}+\overline{S})|$ determines a contraction $\overline{X}\to W$. For any curve C in a fibre we have $C\cdot\overline{B}=0$. Since the components of $\overline{B}$ generate $N^1(\overline{X}/Z)$ (see assertion (ii) in item B), we have $C\cdot\overline{B}_i<0$ for some component $\overline{B}_i$ of $\overline{B}$. Hence, $C\subset\operatorname{Supp}(\overline{B})$.

Note that $q\colon\overline{S}\to q(\overline{S})$ is also a contraction.

Lemma 3.12. $q_*\mathcal{O}_{\overline{S}}=\mathcal{O}_{q(\overline{S})}$ *and the variety $q(\overline{S})=f(g(S))$ is normal.*

Proof. See the proof of Lemma 3.6 in [23].

By Lemma 1.6, we have $\operatorname{Diff}_{\overline{S}}(\bar{\Delta})\in\Phi$ (recall that $\Phi=\Phi^d_{\mathbf{m}}$ or $\Phi_{\mathbf{sm}}$).

Lemma 3.13. *Assume that there is an n-complement $K_{\overline{S}}+\operatorname{Diff}_{\overline{S}}(\bar{\Delta})^+$ of the divisor $K_{\overline{S}}+\operatorname{Diff}_{\overline{S}}(\bar{\Delta})$ near $q^{-1}(o)$. Then there is an n-complement $K+D^+$ of $K+D$ near $q^{-1}(o)$. Moreover, if the pair $(\overline{S},\operatorname{Diff}_{\overline{S}}(\bar{\Delta})^+)$ is not Kawamata log-terminal, then the pair (X,D^+) is not exceptional.*

Proof. By Proposition 6.2 below, any n-complement of $K_{\overline{S}}+\operatorname{Diff}_{\overline{S}}(\bar{\Delta})$ can be extended to an n-complement of $K+\bar{\Delta}+\overline{S}$. By Proposition 6.1, we can pull back complements of $K+\Delta+S$ under divisorial contractions, since all these contractions are $(K+\Delta+S)$-positive. Finally, let us note that flips preserve complements, that is, the proper transform of an n-complement under a flip is an n-complement. Indeed, it is obvious that inequality (1.1) is invariant under any birational map that is an isomorphism in codimension one. The log-canonical property and linear equivalence (see Definition 1.1) are also invariant ([13], Proposition 2.28).

Lemma 3.14. *If* $\dim(q(\overline{S}))>0$, *then*

(i) *the pair $(X/Z\ni o,D)$ is non-exceptional, and*

(ii) *there is a non-exceptional n-complement of the divisor $K+D$ with $n\in N_{d-2}(\Phi)$.*

Proof. Assertion (i) follows from Corollary 2.7. Note that the assumptions of our theorem hold for the pair $(\overline{S}/q(\overline{S})\ni o,\operatorname{Diff}_{\overline{S}}(\bar{\Delta}))$ (see Lemma 1.6). By the induction hypothesis, we can assume that there is an n-complement $K_{\overline{S}}+\operatorname{Diff}_{\overline{S}}(\bar{\Delta})$ for $n\in N_{d-2}(\Phi)$ that is not Kawamata log-terminal. We complete the proof using Lemma 3.13.

We now resume the proof of Theorem 3.1.

D. Assume that the pair $(X/Z\ni o,D)$ is non-exceptional (that is, there is a non-exceptional complement $K+D+\Upsilon$) and $q(\overline{S})=o$. It is sufficient to prove that there is a non-exceptional n-complement of the divisor $K+D$ with $n\in N_{d-2}(\Phi)$.

By Lemma 3.14, we can assume that $q(\overline{S}) = o$, that is, the variety $\overline{S}$ is projective. By Corollary 2.5, the divisor Υ can be chosen so that $a(S, D+\Upsilon) = -1$ (and $a(E, D+\Upsilon) = -1$ for some $E \not\approx S$). Let $\widehat{\Upsilon}$ and $\overline{\Upsilon}$ be the proper transforms of Υ on $\widehat{X}$ and $\overline{X}$. Then

$$g^*(K+D+\Upsilon) = K_{\widehat{X}} + \Delta + S + \widehat{\Upsilon}.$$

Moreover,

$$a\big(E, \Delta + S + \widehat{\Upsilon}\big) = a\big(E, \bar{\Delta} + \overline{S} + \overline{\Upsilon}\big) = -1$$

(since $K_{\widehat{X}} + \Delta + S + \widehat{\Upsilon} \equiv 0$). Therefore, the divisor $K_{\overline{X}} + \bar{\Delta} + \overline{S} + \overline{\Upsilon}$ is not purely log-terminal (near the fibre $q^{-1}(o)$).

Lemma 3.15. *Under the assumptions of item* D *the divisor* $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\bar{\Delta} + \overline{\Upsilon})$ *is not Kawamata log-terminal.*

Proof. By the adjunction formula ([13], Theorem 17.6), it is sufficient to prove that the divisor $K + \bar{\Delta} + \overline{S} + \overline{\Upsilon}$ is not purely log-terminal near $\overline{S}$. By the above arguments, this follows from Lemma 3.17.

Lemma 3.15 and Conjecture 1.9 imply that there is an n-complement $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\bar{\Delta})$ with $n \in N_{d-2}(\Phi)$ that is not Kawamata log-terminal. We complete the proof of Theorem 3.1 using Lemma 3.13.

The following example illustrates the proof of Theorem 3.1.

Example 3.16. As in Example 3.2, let $(Z \ni o)$ be a two-dimensional Du Val singularity (rational double point), let $D = 0$, and let f be the identity map. In this case $g\colon \widehat{X} \to X$ is a weighted blow-up (with suitable weights) and $\widehat{X} \dashrightarrow \overline{X}$ is the identity map. Therefore, $S \simeq \mathbb{P}^1$. We write

$$\mathrm{Diff}_S(0) = \sum_{i=1}^{r} (1 - 1/m_i) P_i,$$

where $P_1, \dots, P_r$ are distinct points. We have the following correspondence between the types of $(Z \ni o)$ and the sets $(m_1, \dots, m_r)$ (see Examples 2.8 and 2.10):

$(Z \ni o)$	A_n	D_n	E_6	E_7	E_8
$(m_1, \dots, m_r)$	$r \leqslant 2$	$(2,2,m)$	$(2,3,3)$	$(2,3,4)$	$(2,3,5)$

Hence, the singularity $(Z \ni o)$ is exceptional if and only if it belongs to one of the types E_6, E_7, E_8.

Lemma 3.17 (see [19], cf. [23], Theorem 6.9, [5], Proposition 2.1). *Let* $(X/Z \ni o, D)$ *be a log variety and let* $f\colon X \to Z$ *be a structural morphism. Assume that*

(i) *the divisor* $K + D$ *is log-canonical and not purely log-terminal near* $f^{-1}(o)$,

(ii) $K + D \equiv 0$ *over* Z,

(iii) *there is an irreducible component* $S \subset \lfloor D \rfloor$ *such that* $f(S) \neq Z$.

Assume, moreover, that the logMMP can be run in dimension $\dim(X)$. *Then the pair* (X, D) *is not purely log-terminal near* $S \cap f^{-1}(o)$.

Corollary 3.18. *In the notation of Theorem* 3.1 *the following assertions are equivalent*:

(i) $(X/Z \ni o, D)$ *is an exceptional pair* (*of local type*),

(ii) $q(\overline{S}) = o$ *and* $(\overline{S}, \mathrm{Diff}_{\overline{S}}(\overline{D}))$ *is an exceptional pair* (*of global type*).

Proof. The implication i) $\Longrightarrow$ ii) follows from Lemmas 3.14 and 3.13. The reverse implication follows from Lemma 3.15.

Put

$$\mathrm{compl}'(X, D) := \min\left\{ m \,\middle|\, \begin{array}{l}\text{there is an } m\text{-complement of } K+D \\ \text{that is not Kawamata log-terminal}\end{array} \right\}.$$

Corollary 3.19. *Under the assumptions of Theorem* 3.1 (*and in the same notation*) *suppose*, *in addition*, *that the pair* $(X/Z \ni o, D)$ *is exceptional. Then*

$$\mathrm{compl}'(X, D) = \mathrm{compl}(\overline{S}, \mathrm{Diff}_{\overline{S}}(\overline{D})).$$

Proof. The inequality $\mathrm{compl}'(X, D) \leqslant \mathrm{compl}(\overline{S}, \mathrm{Diff}_{\overline{S}}(\overline{D}))$ follows from Lemma 3.13. Let us prove the reverse inequality. Let $K + D^+$ be an n-complement of $K + D$ that is not Kawamata log-terminal. Then $D^+ \geqslant D$. Corollary 2.6 implies that $a(S, D^+) = -1$. Consider the crepant pullback

$$g^*(K + D^+) = K_{\widehat{X}} + \Delta + S + \Upsilon$$

and let $\overline{\Upsilon}$ be the proper transform of Υ on $\overline{X}$. Then $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\bar{\Delta} + \overline{\Upsilon})$ is an n-complement of $K_{\overline{S}} + \mathrm{Diff}_{\overline{S}}(\bar{\Delta})$.

Note that for non-exceptional contractions we have only the inequality

$$\mathrm{compl}'(X, D) \leqslant \mathrm{compl}(\overline{S}, \mathrm{Diff}_{\overline{S}}(\overline{D})).$$

Example 3.20. Let $(X \ni o)$ be a terminal cE_8-singularity given by the equation

$$x_1^2 + x_2^3 + x_3^5 + x_4^r = 0,$$

where $\gcd(r, 30) = 1$, and let $g\colon (\widehat{X}, S) \to X$ be the weighted blow-up with weights $(15r, 10r, 6r, 30)$. Then $S = \mathbb{P}^2$ and

$$\mathrm{Diff}_S(0) = \frac{1}{2}L_1 + \frac{2}{3}L_2 + \frac{4}{5}L_3 + \frac{r-1}{r}L_4,$$

where $L_1, \dots, L_4$ are lines in $\mathbb{P}^2$ in general position. Then $\mathrm{compl}'(X, 0) = 1$ (since $(X \ni o)$ is a cDV-singularity). On the other hand, $\mathrm{compl}(\overline{S}, \mathrm{Diff}_{\overline{S}}(0)) = 6$.

§ 4. Exceptional Fano contractions

In this section we study exceptional Fano contractions for which assumptions of Theorem 3.1 hold.

Proposition 4.1. *We retain the notation and assumptions of Theorem* 3.1. *Assume*, *moreover*, *that Conjecture* 1.7 *is true in dimensions* $\leqslant d-1$ *and that* $(X/Z \ni o, D)$ *is an exceptional pair. Then*

$$a(E,D) \geqslant -1+\delta_d \quad \textit{for all} \quad E \not\approx S,$$

where $\delta_d > 0$ *is a constant that depends only on* d.

Proof. Let $K+D^+$ be an n-complement that is not Kawamata log-terminal (here $n \in N_{d-1}(\Phi)$). Then $D^+ \geqslant D$ (see Corollary 1.5). By the definition of exceptional contractions, we have $a(S,D^+)=-1$ and $a(E,D^+)>-1$ for all $E \not\approx S$. Therefore, $a(E,D^+) \geqslant -1+1/n$ (since $na(E,D^+)$ is an integer). Since $D^+ \geqslant D$, we have $a(E,D) \geqslant a(E,D^+)$. Hence, we can put

$$\delta_d := 1/\max(N_{d-1}(\Phi_{\mathbf{m}}^{d-1})).$$

Assuming that $D \in \Phi_{\mathbf{sm}}$, we obtain the following corollary.

Corollary 4.2. *We retain the notation and assumptions of Proposition* 4.1. *Let* D_i *be a component of* D *and let* $d_i = 1-1/m_i$ *be its coefficient. If* $D_i \not\approx S$, *then* $m_i \leqslant 1/\delta(d)$. *Hence, there are only finitely many values of* d_i.

Corollary 4.3 (cf. [10]). *Assume that the logMMP holds in dimensions* $\leqslant d$ *and Conjecture* 1.9 *is true in dimensions* $\leqslant d-1$. *Let* $(X \ni o)$ *be a* d*-dimensional Kawamata log-terminal singularity and* $F=\sum F_i$ *an effective reduced Weil divisor that is* $\mathbb{Q}$*-Cartier on* X *and passes through* o. *Then one of the following assertions holds*:

(i) $c_0(X,F)=1$,

(ii) $c_0(X,F) \leqslant 1-1/N_{d-1}$,

where $c_0(X,F)$ *is the log-canonical threshold* (X,F) (*see* [23] *and* [11]) *and* N_{d-1} *is the constant defined in* (1.3).

Note that this corollary is non-trivial only if Conjecture 1.7 is true in dimensions $\leqslant d-1$.

Proof. Put $c := c_0(X,F)$ and assume that $1-1/N_{d-1} < c < 1$. By Theorem 3.1 there is an n-complement $K+B$ of the divisor $K+cF$, where $n \leqslant N_{d-1}$. Let c_i^+ be the coefficient of the component F_i in B. Inequality (1.1) implies that $c_i^+ \geqslant 1$. Therefore, $F \leqslant B$ and the divisor $K+F$ is log-canonical: a contradiction.

In the case when

$$1-1/(N_{d-2}+1) \leqslant c = c_0(X,F) < 1,$$

the pair (X,cF) is exceptional. We expect that in any dimension there are only finitely many values of $c \in [1-1/(N_{d-2}+1,1)]$. This method enables us to prove (see, for example, [20], Corollary 6.0.9) that in dimension $d=2$ the range (the set of all values) of $c_0(X,F)$ in $[2/3,1]$ coincides with the set $\{2/3, 7/10, 3/4, 5/6, 1\}$.

Theorem 4.4. *Let $\varepsilon > 0$ be fixed. Let $(X/Z \ni o, D)$ be a d-dimensional log variety of local type such that*

(i) $D \in \Phi_{\mathbf{sm}}$ (*that is,* $d = \sum(1 - 1/m_i)D_i$*, where* $m_i \in \mathbb{N} \cup \{\infty\}$ *and the* D_i *are principal divisors*),

(ii)
$$\text{totaldiscr}\,(X, D) > -1 + \varepsilon, \tag{4.1}$$

(iii) *the divisor* $-(K + D)$ *is numerically effective and big over* Z,

(iv) $(X/Z \ni o, D)$ *is an exceptional pair.*

Further, let $\varphi\colon X' \to X$ *be a finite covering morphism such that*

(v) *the variety* X' *is normal and irreducible*,

(vi) *the morphism* φ *is etale in codimension one outside* $\operatorname{Supp}(D)$,

(vii) *the ramification index of* φ *at a general point of the components of the divisor* $\varphi^{-1}(D_i)$ *divides* m_i.

Assume also that the logMMP is true in dimensions $\leqslant d$ *and Conjectures* 1.8, 1.7 *and* 1.9 *hold for* $\Phi_{\mathbf{sm}}$ *in dimension* $d - 1$. *Then the degree of* φ *is bounded by a constant* $\operatorname{Const}(d, \varepsilon)$.

Proof. We retain the notation used in the proof of Theorem 3.1. Considering the fibred product of X and its $\mathbb{Q}$-factorialization, we can reduce the situation to the case when the variety X is $\mathbb{Q}$-factorial. Note that the fibre $\varphi^{-1} \circ f^{-1}(o)$ is connected (since X is regarded as a germ near the fibre $f^{-1}(o)$ and the variety X' is irreducible). Consider the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Z' & \xrightarrow{\pi} & Z \end{array}$$

where $X' \overset{f'}{\to} Z' \overset{\pi}{\to} Z$ is the Stein factorization. Then $f'\colon X' \to Z'$ is a contraction and $\pi\colon Z' \to Z$ is a finite morphism. We define divisors D' and $D^{\triangledown\prime}$ in the following way:

$$\begin{gathered} K_{X'} + D' = \varphi^*(K + D), \\ K_{X'} + D' + D^{\triangledown\prime} = \varphi^*(K + D + D^{\triangledown}) \end{gathered} \tag{4.2}$$

(see [23], § 2). This implies, for example, that the coefficient of the component $D'_{i,j}$ of the divisor $\varphi^{-1}(D_i)$ in D' has the following form:

$$d'_{i,j} = 1 - r_{i,j}(1 - (1 - 1/m_i)),$$

where $r_{i,j}$ is the ramification index at a general point of $D'_{i,j}$. Assumption (vii) of our theorem implies that $D' \in \Phi_{\mathbf{sm}}$ (and $D^{\triangledown\prime} \geqslant 0$). It is obvious that $K_{X'} + D' + D^{\triangledown\prime}$ is ample over Z'.

First we consider case (a) (see § 3), when the divisor $K + D + D^{\triangledown}$ is purely log-terminal, $S := \lfloor D + D^{\triangledown} \rfloor \neq 0$, $\widehat{X} = X$ and g is the identity map. Put $S' := \lfloor D' + D^{\triangledown\prime} \rfloor = \varphi^{-1}(S)$. By Lemma 3.14, the variety S is compact and $S \subset f^{-1}(o)$.

Using [23], § 2 (or [13], Proposition 20.3), we obtain that the pair $(X', D' + D^{\triangledown\prime})$ is purely log-terminal. The connectedness lemma ([13], Theorem 17.4) and the adjunction formula ([13], Theorem 17.6), imply that the variety S' is connected, irreducible and normal. We define a divisor Δ' by the equality $D' = \Delta' + a'S'$, where $0 \leqslant a' < 1$. Let $\overline{X}'$ be the normalization of the variety $\overline{X}$ in the function field of X'. There is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, dashed, "\psi'"'] & X \arrow[d, dashed, "\psi"] \\ \overline{X}' \arrow[r, "\overline{\varphi}"] & \overline{X} \end{tikzcd}$$

where $\overline{\varphi}\colon \overline{X}' \to \overline{X}$ is a finite morphism and $\psi\colon X \dashrightarrow \overline{X}$, $\psi'\colon X' \dashrightarrow \overline{X}'$ are birational maps such that ψ^{-1} and ψ'^{-1} do not contract divisors. Therefore, the covering morphism $\overline{\varphi}$ has a ramification divisor only over $\operatorname{Supp}(\psi_*(D)) \subset \overline{S} \cup \operatorname{Supp}(\bar{\Delta})$. The ramification index of $\overline{\varphi}$ at a general point of the component over $\psi_*(D_i)$ is equal to that of φ at a general point of the corresponding component over D_i. Using ψ_* and ψ'_*, we transform (4.2) into the relations

$$\begin{gathered} K_{\overline{X}'} + \overline{D}' = \overline{\varphi}^*(K_{\overline{X}} + \overline{D}), \\ K_{\overline{X}'} + \overline{D}' + \overline{D}^{\triangledown\prime} = \overline{\varphi}^*\bigl(K_{\overline{X}} + \overline{D} + \overline{D}^{\triangledown}\bigr), \end{gathered} \tag{4.3}$$

where $\overline{D}' := \psi'_* D'$ and $\overline{D}^{\triangledown\prime} := \psi'_* D^{\triangledown\prime}$. Let us recall that $\overline{S} = \lfloor \overline{D} + \overline{D}^{\triangledown} \rfloor$ is a prime divisor. Further, the first equality in (4.3) implies that

$$K_{\overline{X}'} + \bar{\Delta}' + \overline{S}' = \overline{\varphi}^*(K_{\overline{X}} + \bar{\Delta} + \overline{S}), \tag{4.4}$$

where $\bar{\Delta}' := \psi'_*\Delta'$ and $\overline{S}' := \psi'_* S'$. According to [23], § 2, and assertion (i) in item C (see also [13], Proposition 20.3), the divisor $K_{\overline{X}'} + \bar{\Delta}' + \overline{S}'$ is purely log-terminal. Moreover, assertion (ii) in item C and Lemma 3.11 imply that the divisor $-(K_{\overline{X}'} + \bar{\Delta}' + \overline{S})'$ is numerically effective and big over Z'. It is sufficient to prove that the degree of the restriction morphism $\overline{\phi} = \overline{\varphi}|_{\overline{S}'}\colon \overline{S}' \to \overline{S}$ is bounded. Indeed, $\deg\varphi = (\deg\overline{\phi})r$, where r is the ramification index over S. By (4.1) and assumption (vii) of Theorem 4.4, r is bounded. Consider the log pairs $(\overline{S}, \operatorname{Diff}_{\overline{S}}(\bar{\Delta}))$ and $(\overline{S}', \operatorname{Diff}_{\overline{S}'}(\bar{\Delta}'))$.

Restricting (4.4) to $\overline{S}$, we obtain the formula

$$K_{\overline{S}} + \operatorname{Diff}_{\overline{S}}(\bar{\Delta}) = \overline{\phi}^*\bigl(K_{\overline{S}'} + \operatorname{Diff}_{\overline{S}'}(\bar{\Delta}')\bigr).$$

In particular,

$$\bigl(K_{\overline{S}} + \operatorname{Diff}_{\overline{S}}(\bar{\Delta})\bigr)^{d-1} = (\deg\overline{\phi})\bigl(K_{\overline{S}'} + \operatorname{Diff}_{\overline{S}'}(\bar{\Delta}')\bigr)^{d-1}.$$

Both sides of this equality are positive by Lemma 3.11.

E. Consider the degree of $\overline{\phi}$ (in case (a)).

The proof of Theorem 3.1 shows that there is an n-complement $K_{\overline{X}}+\bar{\Delta}+\overline{S}+\overline{\Upsilon}$ of the log divisor $K_{\overline{X}}+\bar{\Delta}+\overline{S}$ with $n\leqslant \max N_{d-1}(\Phi_{\mathbf{sm}})<\infty$. We define a divisor $\overline{\Upsilon}'$ by the equality

$$K_{\overline{X}'}+\bar{\Delta}'+\overline{S}'+\overline{\Upsilon}'=\overline{\varphi}^*\bigl(K_{\overline{X}}+\bar{\Delta}+\overline{S}+\overline{\Upsilon}\bigr)$$

and put $\Theta:=\operatorname{Diff}_{\overline{S}}(\bar{\Delta}+\overline{\Upsilon})$, $\Theta':=\operatorname{Diff}_{\overline{S}'}(\bar{\Delta}'+\overline{\Upsilon}')$. Then $K_{\overline{S}}+\Theta$ and $K_{\overline{S}'}+\Theta'$ are n-complements. Since the divisor $K_{\overline{S}}+\Theta$ is Kawamata log-terminal (see Corollary 3.18), we have

$$\operatorname{totaldiscr}(\overline{S},\operatorname{Diff}_{\overline{S}}(\bar{\Delta}))\geqslant -1+\frac{1}{n}\geqslant -1+\beta,\qquad \beta=\frac{1}{\max N_{d-1}(\Phi_{\mathbf{sm}})}.$$

We obtain likewise that

$$\operatorname{totaldiscr}(\overline{S}',\operatorname{Diff}_{\overline{S}'}(\bar{\Delta}'))\geqslant -1+\beta.$$

According to Conjecture 1.8, the pairs $\bigl(\overline{S},\ \operatorname{Supp}(\operatorname{Diff}_{\overline{S}}(\bar{\Delta}))\bigr)$ and $\bigl(\overline{S}',\ \operatorname{Supp}(\operatorname{Diff}_{\overline{S}'}(\bar{\Delta}'))\bigr)$ belong to finitely many algebraic families. Taking into account that $\operatorname{Diff}_{\overline{S}}(\bar{\Delta})$, $\operatorname{Diff}_{\overline{S}'}(\bar{\Delta}')\in\Phi_{\mathbf{sm}}$ (see Lemma 1.6) and using the inequalities

$$\operatorname{Diff}_{\overline{S}}(\bar{\Delta})\leqslant\Theta,\qquad \operatorname{Diff}_{\overline{S}'}(\bar{\Delta}')\leqslant\Theta',$$

we obtain that the pairs $(\overline{S},\operatorname{Diff}_{\overline{S}}(\bar{\Delta}))$ and $(\overline{S}',\operatorname{Diff}_{\overline{S}'}(\bar{\Delta}'))$ also belong to finitely many algebraic families. Hence, $\deg\overline{\phi}$ is bounded.

Now consider case (b) (see § 3). Let $\widehat{X}'$ be the normalization of the dominant component of $\widehat{X}\times_X X'$ and S' the proper transform of S on $\widehat{X}'$. We claim that the morphism $g'\colon(\widehat{X}'\supset S')\to X'$ is a purely log-terminal blow-up of (X',D'). Consider the change of base

$$\begin{array}{ccc}\widehat{X}' & \xrightarrow{\widehat{\varphi}} & \widehat{X}\\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g}\\ X' & \xrightarrow{\varphi} & X\end{array}\tag{4.5}$$

It is clear that the morphism $\widehat{\varphi}\colon\widehat{X}'\to\widehat{X}$ is finite and its ramification divisor can be contained only in $S\cup\operatorname{Supp}(D)$. Then S' is an exceptional divisor of the blow-up $g'\colon\widehat{X}'\to X'$. Indeed,

$$K_{\widehat{X}'}+\Delta'+S'=\widehat{\varphi}^*(K_{\widehat{X}}+\Delta+S),\tag{4.6}$$

where Δ' is a boundary. This divisor is purely log-terminal (see [23], Corollary 2.2, and [13], Proposition 20.3) and anti-ample over X'. By the adjunction formula ([13], Theorem 17.6), the variety S' is normal. On the other hand, S' is connected near the fibre over $o'\in Z'$. Indeed, $-(K_{\widehat{X}'}+\widehat{D}'+\widehat{D}^{\nabla\prime})$ is numerically effective and big over Z' by (4.5) and Claim 3.8. Since $S'\subset\bigl\lfloor\widehat{D}'+\widehat{D}^{\nabla\prime}\bigr\rfloor$, the variety S' is connected by the connectedness lemma ([13], Theorem 17.4), which completes the proof of our assertion.

As in case (a) (see § 3), we consider the commutative diagram

$$\begin{tikzcd} \widehat{X}' \arrow[r, "\widehat{\varphi}"] \arrow[d, dashed, "\psi'"'] & \widehat{X} \arrow[d, dashed, "\psi"] \\ \overline{X}' \arrow[r, "\overline{\varphi}"] & \overline{X} \end{tikzcd}$$

and establish that the pairs $(\overline{S}, \mathrm{Diff}_{\overline{S}}(\bar{\Delta}))$ and $(\overline{S}', \mathrm{Diff}_{\overline{S}'}(\bar{\Delta}'))$ are bounded. Therefore, we can assume that $\deg\overline{\phi}$ is bounded, where $\overline{\phi}=\overline{\varphi}|_{\overline{S}'}\colon \overline{S}'\to\overline{S}$. It remains to show that the ramification index r of the morphism $\overline{\varphi}$ at a general point of S' is bounded. It is clear that r coincides with the ramification index $\widehat{\varphi}$ at a general point of $\widehat{S}'$. The following equality is similar to (3.1):

$$g^*(K_{X'}+D')=K_{\widehat{X}'}+\Delta'+a'S'. \tag{4.7}$$

Then

$$1-a'=r(1-a)\geqslant r(1+\mathrm{discr}\,(X,D))>r\varepsilon \tag{4.8}$$

(see [23], § 2 or [13], Proof 20.3). We claim that the pairs $\big(S', \mathrm{Diff}_{S'}(\Delta')\big)$ belong to finitely many algebraic families. Note that we cannot use Conjecture 1.8 directly, since it may happen that the divisor $-\big(K_{S'}+\mathrm{Diff}_{S'}(\Delta')\big)$ is not numerically effective.

As in case (a) we consider an n-complement $K_{\widehat{X}}+\Delta+S+\widehat{\Upsilon}$, where $n\leqslant \max N_{d-1}(\Phi_{\mathbf{sm}})$. We define divisors $\widehat{\Upsilon}'$ and $\widehat{D}^{\lambda\prime}$ in analogy with (4.2) (see item A):

$$\begin{aligned} K_{\widehat{X}'}+\Delta'+S'+\widehat{\Upsilon}'&=\widehat{\varphi}^*(K_{\widehat{X}}+\Delta+S+\widehat{\Upsilon}),\\ K_{\widehat{X}'}+\Delta'+S'+\widehat{D}^{\lambda\prime}&=\widehat{\varphi}^*(K_{\widehat{X}}+\Delta+S+\widehat{D}^{\lambda}). \end{aligned}$$

Then

$$K_{S'}+\mathrm{Diff}_{S'}(\Delta'+\widehat{\Upsilon}')\equiv 0.$$

According to item A, the divisor $K_{S'}+\mathrm{Diff}_{S'}(\Delta'+\widehat{D}^{\lambda\prime})$ is anti-ample. Therefore, the divisor $-\big(K_{S'}+\mathrm{Diff}_{S'}(\Delta'+\alpha\widehat{D}^{\lambda\prime}+(1-\alpha)\widehat{\Upsilon}')\big)$ is ample for any $\alpha>0$. Note that

$$\mathrm{totaldiscr}\,(S', \mathrm{Diff}_{S'}(\Delta'+\widehat{\Upsilon}'))\geqslant -1+1/n.$$

Hence, we can apply Conjecture 1.8 to the pair $\big(S', \mathrm{Diff}_{S'}(\Delta'+\alpha\widehat{D}^{\lambda\prime}+(1-\alpha)\widehat{\Upsilon}')\big)$ for small positive α. We obtain that the variety S' is bounded. As in item E, we see that the pair $(S', \mathrm{Diff}_{S'}(\Delta'))$ is bounded. Consider a sufficiently general curve l in a general fibre of $g'|_{S'}\colon S'\to g'(S)'$. Equality (4.7) implies that

$$-(K_{S'}+\mathrm{Diff}_{S'}(\Delta'))\cdot l=-(1-a')S'\cdot l. \tag{4.9}$$

It is clear that the intersection index $-(K_{S'}+\operatorname{Diff}_{S'}(\Delta'))\cdot l$ depends only on the pair $(S',\operatorname{Diff}_{S'}(\Delta'))$ and not on $\widehat{X}'$. So, we assume that the number $-(K_{S'}+\operatorname{Diff}_{S'}(\Delta'))\cdot l$ is fixed. Let us recall that the coefficients of $\operatorname{Diff}_{S'}(\Delta')$ are standard (see [23], Proposition 3.9, and [13], Proposition 16.6). Therefore,

$$\operatorname{Diff}_{S'}(\Delta')=\sum_{i=1}^{r}(1-1/m_i)\Xi_i',\qquad m_i\in\mathbb{N},\quad r\geqslant 0.$$

Put $m':=\operatorname{lcm}(m_1,\dots,m_r)$. According to [23], Proposition 3.9, $m'S'$ and $m'(K_{S'}+\operatorname{Diff}_{S'}(\Delta'))$ are Cartier divisors along l. Therefore, (4.9) can be written as $N=(1-a')k$, where $N=-m'l\cdot(K_{S'}+\operatorname{Diff}_{S'}(\Delta'))$ is a fixed positive integer and $k=-m'(l\cdot S)'$ also is a positive integer. Hence,

$$N=(1-a')k>kr\varepsilon\geqslant r\varepsilon$$

by (4.8). This shows that the ramification index r is bounded: $r<N/\varepsilon$, which completes the proof of the theorem.

In what follows we state some corollaries to Theorems 3.1 and 4.4. We consider mainly the three-dimensional case (it is well known that in this case all required conjectures are true; see [24] and [2]). Let us recall that in this case the non-exceptional contractions for which the assumptions of Theorem 3.1 hold have 1-, 2-, 3-, 4- or 6-complements. Putting $X=Z$ and $D=0$ in Theorem 4.4, we obtain the following corollary.

Corollary 4.5. *Let $(Z\ni o,D)$ be a three-dimensional exceptional Kawamata log-terminal singularity such that*

$$\operatorname{totaldiscr}(Z,D)>-1+\varepsilon$$

and $D\in\Phi_{\mathbf{sm}}$. Then

(i) *the order of the algebraic fundamental group $\pi_1^{\mathrm{alg}}(Z\setminus\operatorname{Sing}(Z))$ is bounded by a constant* $\operatorname{Const}(\varepsilon)$,

(ii) *the index of K_Z+D is bounded by a constant* $\operatorname{Const}(\varepsilon)$,

(iii) *for any exceptional divisor E over Z either $a(E)>0$ or $a(E)\in\mathfrak{M}(\varepsilon)$, where $\mathfrak{M}(\varepsilon)\subset(-1,0]$ depends only on ε.*

If we omit the assumption of exceptionality, then the order of $\pi_1^{\mathrm{alg}}(Z\setminus\operatorname{Sing}(Z))$ is not necessarily bounded. (Nevertheless, this set is finite; see [22], Theorem 3.6.) Assertion (i) of Corollary 4.5 holds for the topological fundamental group π_1 if this group is finite. M. Reid has communicated to the authors that the finiteness of $\pi_1(Z\setminus\operatorname{Sing}(Z))$ for three-dimensional log-terminal singularities was proved by Shepherd-Barron (unpublished).

Corollary 4.6 [19]. *Let $\varepsilon>0$ be fixed. Let $(X/Z\ni o,D)$ be a three-dimensional log variety of local type such that $K+D$ is a $\mathbb{Q}$-Cartier divisor and $-(K+D)$ is f-numerically effective and f-big. Assume that f is an exceptional contraction and*

$$\operatorname{totaldiscr}(X)>-1+\varepsilon.$$

(i) *If $\dim(Z)\geqslant 2$, then the order of the group $\pi_1^{\mathrm{alg}}(Z\setminus\operatorname{Sing}(Z))$ is bounded by a constant* $\operatorname{Const}(\varepsilon)$.

(ii) *If $\dim(Z)=1$, then the multiplicity of the central fibre $f^{-1}(o)$ is bounded by a constant* $\operatorname{Const}(\varepsilon)$.

Corollary 4.7 [24]. *Let $\varepsilon > 0$ be fixed. Let $(X/Z \ni o, D)$ be a three-dimensional exceptional log pair such that the structural morphism $f\colon X \to Z \ni o$ is a flipping contraction (that is, f contracts only finitely many curves),*

$$\operatorname{totaldiscr}(X, D) > -1 + \varepsilon,$$

$D \in \Phi^3_{\mathbf{m}}$, and assume that the divisor $-(K+D)$ is numerically effective and big over Z. Then

(i) *the relative Picard numbers $\rho(X/Z)$ and $\rho^{\mathrm{an}}(X/Z)$ are bounded by a* $\operatorname{Const}(\varepsilon)$,

(ii) *the number of components of the central fibre $f^{-1}(o)$ is bounded by a constant* $\operatorname{Const}'(\varepsilon)$.

Proof. We retain the notation used in the proof of Theorem 3.1. Take an n-complement $K_{\widehat{X}} + \Delta + S + \Upsilon$ with $n \leqslant N_2$ and run the $(K_{\widehat{X}} + \Delta + \Upsilon)$-MMP. We have $R \cdot S > 0$ for every extremal ray R. Therefore, S is not contracted. We finally obtain a model $p\colon \widetilde{X} \to Z$ with the p-numerically effective divisor $K_{\widetilde{X}} + \tilde{\Delta} + \widetilde{\Upsilon} \equiv -\widetilde{S}$. Since the divisor $K_{\widehat{X}} + \Delta + S + \Upsilon$ is numerically trivial, we have

$$a(E, \Delta + S + \Upsilon) = a\Big(E, \tilde{\Delta} + \widetilde{S} + \widetilde{\Upsilon}\Big)$$

for any divisor E of the field $K(X)$ (cf. [11], Proposition 3.10). This shows that $(\widetilde{X}, \tilde{\Delta} + \widetilde{S} + \widetilde{\Upsilon})$ is a purely log-terminal pair. Lemma 3.14 implies that $p(\widetilde{S}) = o$. Since the divisor $-\widetilde{S}$ is numerically effective over Z, it is obvious that $\widetilde{S}$ coincides with the fibre over o. We have

$$n\big(K_{\widetilde{S}} + \operatorname{Diff}_{\widetilde{S}}(\tilde{\Delta} + \widetilde{\Upsilon})\big) \sim 0,$$

and the divisor $K_{\widetilde{S}} + \operatorname{Diff}_{\widetilde{S}}(\tilde{\Delta} + \widetilde{\Upsilon})$ is Kawamata log-terminal (by the adjunction formula: [13], Theorem 17.6). Therefore,

$$\operatorname{totaldiscr}(\widetilde{S}, \operatorname{Diff}_{\widetilde{S}}(\tilde{\Delta} + \widetilde{\Upsilon})) \geqslant -1 + 1/n, \quad n \leqslant N_2.$$

It is obvious that $\operatorname{Diff}_{\widetilde{S}}(\tilde{\Delta} + \widetilde{\Upsilon}) \neq 0$. By [2], the variety $\widetilde{S}$ belongs to finitely many algebraic families. Hence, we can assume that the number $\rho(\widetilde{S})$ is bounded by a constant $\operatorname{Const}(\varepsilon)$.

Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}^{\mathrm{an}}_{\widetilde{X}} \xrightarrow{\exp} \mathcal{O}^{\mathrm{an}\,*}_{\widetilde{X}} \longrightarrow 0.$$

By the Kawamata–Viehweg vanishing theorem, we have $R^i f^* \mathcal{O}^{\mathrm{an}}_{\widetilde{X}} = 0, \;\; i > 0$. Therefore, $\operatorname{Pic}^{\mathrm{an}}(\widetilde{X}) = H^2(\widetilde{X}, \mathbb{Z})$. We obtain likewise that $H^2(\widetilde{S}, \mathbb{Z}) = \operatorname{Pic}(\widetilde{S})$. Since $\widetilde{S} = p^{-1}(o)$ is a topological retract of $\widetilde{X}$, we have $H^2(\widetilde{X}, \mathbb{Z}) = H^2(\widetilde{S}, \mathbb{Z})$. Therefore, $\rho^{\mathrm{an}}(\widetilde{X})$ is bounded and so is $\rho^{\mathrm{an}}(\widehat{X})$ (since $\widehat{X} \dashrightarrow \widetilde{X}$ is a sequence of flips). This completes the proof of assertion (i). We prove (ii) using the fact that $\rho^{\mathrm{an}}(X/Z)$ is equal to the number of components of the fibre $f^{-1}(o)$ (see the above arguments and [15], Corollary 1.3).

Corollary 4.8. *Let $\varepsilon > 0$ be fixed. Let $(Z \ni o, D)$ be a three-dimensional exceptional Kawamata log-terminal singularity such that*

$$\operatorname{totaldiscr}(X, D) > -1 + \varepsilon$$

and $D \in \Phi^3_{\mathbf{m}}$. Then the following assertions hold for a $\mathbb{Q}$-factorialization $f\colon X \to Z$:

(i) *the numbers $\rho(X/Z)$ and $\rho^{\mathrm{an}}(X/Z)$ are bounded by a* $\mathrm{Const}(\varepsilon)$,

(ii) *the number of components of $f^{-1}(o)$ is bounded by a* $\mathrm{Const}'(\varepsilon)$.

Note that for non-exceptional flipping contractions the number of components of a fibre is not bounded even in the terminal case (see [12], Example 13.7). Here we give an example of a flop-contraction such that the assumptions of Corollary 4.7 hold.

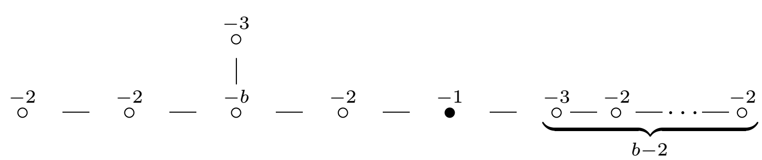
Example 4.9. Let $(Z \ni o)$ be the three-dimensional hypersurface singularity defined by the equation

$$x_1^3 + x_2^3 + x_3^5 + x_4^5 = 0$$

in $\mathbb{C}^4$. According to [7] this singularity is exceptional (and canonical); it is obviously not $\mathbb{Q}$-factorial. Let $f\colon X \to Z$ be the $\mathbb{Q}$-factorialization ([13], Theorem 6.11.1). By Lemma 2.3, $(X/Z \ni o, 0)$ is an exceptional pair. Hence, the assumptions of Corollary 4.7 hold (with $D = 0$).

Many examples of exceptional singularities can be found in [14] and [7]. Here we give an example of an exceptional Fano contraction $f\colon X \to Z$, $\dim(X) > \dim(Z)$.

Example 4.10 ([20], Theorem 7.1.12). We consider the surface $\mathbb{P}^1 \times \mathbb{C}^1$ and blow up the points in a fibre of the projection $\mathbb{P}^1 \times \mathbb{C}^1 \to \mathbb{C}^1$ to obtain a fibre bundle $f^{\min}\colon X^{\min} \to \mathbb{C}^1$ whose central fibre has the dual graph



with $b \geqslant 2$. Further, we contract the curves corresponding to the white nodes. We obtain an extremal contraction $f\colon X \to \mathbb{C}^1$ with two log-terminal points. The canonical divisor K_X is 3-complementary but not 1- or 2-complementary ([20], Theorem 7.1.12). Therefore, f is an exceptional contraction.

§ 5. The global case

In this section we modify Theorem 3.1 for the global case. As distinct from the local case, we must here assume that there is a boundary with rather "bad" singularities. Theorem 5.1 is a special case of Conjecture 1.9.

Theorem 5.1 (the global case). *Let (X, D) be a d-dimensional log variety of global type such that*

(i) *the pair (X, D) is Kawamata log-terminal,*

(ii) *the divisor $-(K + D)$ is numerically effective and big,*

(iii) *$D \in \Phi$, where $\Phi = \Phi^d_{\mathbf{m}}$ or $\Phi_{\mathbf{sm}}$.*

Assume that there is a boundary $D^\flat$ *such that*

(iv) *the pair* $K+D+D^\flat$ *is not Kawamata log-terminal*,

(v) *the divisor* $-(K+D+D^\flat)$ *is numerically effective and big.*

Assume that the logMMP is true in dimension d*. Then there is an* n*-complement of* $K+D$ *for* $n\in N_{d-1}(\Phi)$ *that is not Kawamata log-terminal.*

Proof. First we replace X by its $\mathbb{Q}$-factorialization. Then, as in Lemma 3.3, we consider a boundary $D^{\mho}\geqslant 0$ such that the divisor $-(K+D+D^\flat+D^{\mho})$ is ample (but $K+D+D^\flat+D^{\mho}$ is not necessarily log-canonical). Further, we find a $t\in\mathbb{Q}$ such that the pair $(X, D+t(D^\flat+D^{\mho}))$ is log-canonical but not Kawamata log-terminal (that is, t is the log-canonical threshold $c(X,D,D^\flat+D^{\mho})$). Put $D^\triangledown=t(D^\flat+D^{\mho})$.

The rest of the proof is similar to that of Theorem 3.1.

Corollary 5.2 (cf. [24], Corollary 2.8). *Let* (X,D) *be a* d*-dimensional log variety of global type such that*

(i) *the pair* (X,D) *is Kawamata log-terminal*,

(ii) *the divisor* $-(K+D)$ *is numerically effective and big*,

(iii) $D\in\Phi$, *where* $\Phi=\Phi^d_{\mathbf{m}}$ *or* $\Phi_{\mathbf{sm}}$,

(iv) $(K+D)^d>d^d$.

Assume that the logMMP is true in dimension d*. Then there is an* n*-complement* $K+D$ *with* $n\in N_{d-1}(\Phi)$ *that is not Kawamata log-terminal.*

Proof. By the Riemann–Roch theorem (see, for example, [11], Lemma 6.7.1), there is a boundary $D^\flat$ for which the assumptions of Theorem 5.1 hold.

Many examples of exceptional log del Pezzo surfaces can be found in [24], [1], [9] and [20].

§ 6. Appendix

In this section we state two properties of complements that can be useful in applications. We use Definition 1.1 in the case when D is a subboundary, that is, a $\mathbb{Q}$-divisor (not necessarily effective) with coefficients $d_i\leqslant 1$.

Proposition 6.1 ([24], Lemma 2.13). *Let* $n\in\mathbb{N}$ *be fixed. Let* $f\colon Y\to X$ *be a birational contraction and* D *a subboundary on* Y *such that*

(i) *the divisor* K_Y+D *is numerically effective over* X,

(ii) $f(D)=\sum d_i f(D_i)$ *is a boundary whose coefficients satisfy the inequality*

$$\lfloor (n+1)d_i\rfloor\geqslant nd_i.$$

If the divisor $K_X+f(D)$ *is* n*-complementary, then so is* K_Y+D.

Proof. Consider the crepant pullback

$$K_Y+D'=f^*(K_X+f(D)^+),\qquad f_*D'=D.$$

We write $D'=S'+B'$, where the divisor S' is reduced, the divisors S' and B' have no common components, and $\lfloor B'\rfloor\leqslant 0$. We claim that K_Y+D' is an n-complement of the divisor K_Y+D. It is sufficient to verify the inequality

$$nB'\geqslant\lfloor (n+1)\{D\}\rfloor.$$

By assumption (ii), we have $f(D)^+ \geqslant f(D)$. Therefore, $D' \geqslant D$ (since $D - D'$ is f-numerically effective; see [23], § 1.1). Finally, we have

$$nD' \geqslant nS' + \lfloor (n+1)B' \rfloor \geqslant n \lfloor D \rfloor + \lfloor (n+1) \{D\} \rfloor$$

since nD' is an integer divisor.

We obtain the next proposition by improving the technique in the proofs of Theorem 5.6 in [23] and Theorem 19.6 in [13].

Proposition 6.2 [19]. *Let $n \in \mathbb{N}$ be fixed. Let $(X/Z \ni o, D)$ be a log variety and let $S := \lfloor D \rfloor$ and $B := \{D\}$. Assume that*

(i) *the pair (X, D) is purely log-terminal,*

(ii) *the divisor $-(K_X + D)$ is numerically effective and big over Z,*

(iii) *$S \neq 0$ near the fibre $f^{-1}(o)$,*

(iv) *the coefficients of the divisor $D = \sum d_i D_i$ satisfy the inequality*

$$\lfloor (n+1)d_i \rfloor \geqslant nd_i. \tag{6.1}$$

Assume further that there is an n-complement $K_S + \operatorname{Diff}_S(B)^+$ of the log divisor $K_S + \operatorname{Diff}_S(B)$ near $f^{-1}(o) \cap S$. Then near $f^{-1}(o)$ there is an n-complement $K_X + S + B^+$ of the log divisor $K_X + S + B$ such that $\operatorname{Diff}_S(B)^+ = \operatorname{Diff}_S(B^+)$.

Proof. Let $g\colon Y \to X$ be a log resolution. We write

$$K_Y + S_Y + A = g^*(K_X + S + B),$$

where S_Y is the proper transform of S on Y and $\lfloor A \rfloor \leqslant 0$. By the adjunction formula ([13], Theorem 17.6), the divisor S is normal and the pair $(S, \operatorname{Diff}_S(B))$ is purely log-terminal. In particular, $g_S\colon S_Y \to S$ is a birational contraction. Therefore,

$$K_{S_Y} + \operatorname{Diff}_{S_Y}(A) = g_S^*(K_S + \operatorname{Diff}_S(B)).$$

Note that $\operatorname{Diff}_{S_Y}(A) = A|_{S_Y}$ (since Y is a non-singular variety). It is easy to show (see [20], § 4.2) that inequality (6.1) holds for the coefficients of $\operatorname{Diff}_S(B)$. Hence, we can apply Proposition 6.1 to g_S. We obtain the n-complement $K_{S_Y} + \operatorname{Diff}_{S_Y}(A)^+$ of $K_{S_Y} + \operatorname{Diff}_{S_Y}(A)$. In particular, by (1.1) there is a divisor

$$\Theta \in |-nK_{S_Y} - \lfloor (n+1) \operatorname{Diff}_{S_Y}(A) \rfloor|$$

such that

$$n \operatorname{Diff}_{S_Y}(A)^+ = \lfloor (n+1) \operatorname{Diff}_{S_Y}(A) \rfloor + \Theta.$$

By the Kawamata–Viehweg vanishing theorem, we have

$$\begin{aligned} &R^1 h_*(\mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor)) \\ &\qquad = R^1 h_*(\mathcal{O}_Y(K_Y + \lceil -(n+1)(K_Y + S_Y + A) \rceil)) = 0. \end{aligned}$$

The exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_Y(-nK_Y-(n+1)S_Y-\lfloor (n+1)A\rfloor) \\ &\longrightarrow \mathcal{O}_Y(-nK_Y-nS_Y-\lfloor (n+1)A\rfloor) \\ &\longrightarrow \mathcal{O}_{S_Y}(-nK_{S_Y}-\lfloor (n+1)A\rfloor\,|_{S_Y}) \longrightarrow 0 \end{aligned}$$

implies that the restriction

$$\begin{aligned} &H^0\bigl(Y,\mathcal{O}_Y(-nK_Y-nS_Y-\lfloor (n+1)A\rfloor)\bigr) \\ &\qquad\longrightarrow H^0\bigl(S_Y,\mathcal{O}_{S_Y}(-nK_{S_Y}-\lfloor (n+1)A\rfloor\,|_{S_Y})\bigr) \end{aligned}$$

is a surjection. Therefore, there is a divisor

$$\Xi \in |-nK_Y-nS_Y-\lfloor (n+1)A\rfloor|$$

such that $\Xi|_{S_Y}=\Theta$. Put

$$A^+ := \frac{1}{n}(\lfloor (n+1)A\rfloor+\Xi).$$

Then $n(K_Y+S_Y+A^+)\sim 0$ and $(K_Y+S_Y+A^+)|_{S_Y}=K_{S_Y}+\operatorname{Diff}_{S_Y}(A)^+$.

Note that we cannot use inversion of adjunction on Y, since A^+ can have negative coefficients. To ameliorate the situation, we put $B^+ := g_*A^+$. We have $n(K_X+S+B^+)\sim 0$ and

$$(K_X+S+B^+)|_S = K_S+\operatorname{Diff}_S(B)^+.$$

We have only to show that the divisor K_X+S+B^+ is log-canonical. Assume the contrary. Then the pair $(X,S+B+\alpha(B^+-B))$ is not log-canonical for some $\alpha<1$. It is clear that the divisor $-(K_X+S+B+\alpha(B^+-B))$ is numerically effective and big over Z. By inversion of adjunction ([13], Theorem 17.6), the pair $(X,S+B+\alpha(B^+-B))$ is purely log-terminal near $S\cap f^{-1}(o)$. Therefore, $LCS(X,B+\alpha(B^+-B))=S$ near $S\cap f^{-1}(o)$. On the other hand, by the connectedness lemma ([13], Theorem 17.4), $LCS(X,B+\alpha(B^+-B))$ is connected near $f^{-1}(o)$. Hence, the pair $(X,S+B+\alpha(B^+-B))$ is purely log-terminal. This contradiction completes the proof of the proposition.

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