DIAGRAM METHOD FOR 3-FOLDS AND ITS APPLICATION TO KÄHLER CONE AND PICARD NUMBER OF CALABI-YAU 3-FOLDS. I

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Abstract. We prove the general diagram method theorem valid for the quite large class of 3-folds with $\mathbb{Q}$-factorial singularities (see Basic Theorems 1.3.2 and 3.2 and also Theorem 2.2.6). This gives the generalization of our results about Fano 3-folds with $\mathbb{Q}$-factorial terminal singularities (Preprint alg-geom/9311007).

As an application, we get the following result about 3-dimensional Calabi-Yau manifolds X: Assume that the Picard number $\rho(X) > 40$. Then one of two cases (i) or (ii) holds: (i) There exists a small extremal ray on X. (ii) There exists a nef element h such that $h^3 = 0$ (thus, the nef cone $NEF(X)$ and the cubic intersection hypersurface $\mathcal{W}_X$ have a common point; here, we don't claim that h is rational!).

As a corollary, we get: Let X be a 3-dimensional Calabi-Yau manifold. Assume that the nef cone $NEF(X)$ is finite polyhedral and X does not have a small extremal ray. Then there exists a rational nef element h with $h^3 = 0$ if $\rho(X) > 40$.

To prove these results on Calabi-Yau manifolds, we also use one result by V.V. Shokurov on the length of divisorial extremal rays (see Appendix by V.V. Shokurov). Thus, one should consider the results about Calabi-Yau 3-folds above as our common results with V.V. Shokurov.

We also discuss generalization of results above to so called $\mathbb{Q}$-factorial models of Calabi-Yau 3-folds, which sometimes permits to involve non-polyhedral case and small extremal rays to the game.

With Appendix by Vyacheslav V. Shokurov: Anticanonical Boundedness for Curves.

0. Introduction

We consider algebraic projective varieties over the field $\mathbb{C}$ of complex numbers.

In our paper [N8], we developed for Fano 3-folds X with terminal $\mathbb{Q}$-factorial singularities so called Diagram Method (for divisorial case). As an application, we proved that if Picard number $\rho(X) > 7$, then X either has a small extremal ray or a nef rational element h with $h^3 = 0$.

Here, we generalize this method for arbitrary 3-folds (also for divisorial case). It was very surprising for us that this is possible. See Basic Theorems 1.3.2 and 3.2.

As an application, we use this method for Calabi-Yau manifolds. We refer to very important papers of P.M.H. Wilson [W1], [W2] about terminology and basic

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results on Calabi-Yau 3-folds we have to use (we cite these results in Section 4.1). Using one result of V.V. Shokurov about length of divisorial extremal rays for log-terminal situation (see Appendix by V.V. Shokurov), we get the following result which one should consider as a common result of the author and V.V. Shokurov:

Theorem 0.1 (by the author and V.V. Shokurov). *Let X be a 3-dimensional Calabi-Yau manifold and the Picard number $\rho(X) > 40$.*

Then one of two cases (i) or (ii) below holds:

(i) There exists a small extremal ray on X.

(ii) There exists a nef element h such that $h^3 = 0$ (thus, the nef cone $NEF(X)$ and the cubic intersection hypersurface $\mathcal{W}_X$ have a common point; here, we don't claim that h is rational!).

As a corollary, we get

Theorem 0.1' (by the author and V.V. Shokurov). *Let X be a 3-dimensional Calabi-Yau manifold. Assume that the nef cone $NEF(X)$ (equivalently, Mori cone $\overline{NE}(X)$) is finite polyhedral and X does not have a small extremal ray.*

Then there exists a rational nef element h with $h^3 = 0$ if $\rho(X) > 40$.

Proof. See Theorems 4.3.1 and 4.3.1' for more exact statements and the proof.

It seems that existence of a *nef* rational element h with $h^3 = 0$ is very important for Calabi-Yau 3-folds. See [D-Gro], [Gro], [Gra], [Hu], [O], [W1], [W2], [W3]. By I.I. Piatetsky-Shapiro and I.R. Shafarevich [PŠ-Š], an algebraic $K3$ surface has a *nef* rational element with the square zero if its Picard number ≥ 5. See Sect. 6 where we discuss these results and their connection with our results.

In Sect. 5 we consider one possibility to extend our results for cases when either Mori cone $\overline{NE}(X)$ is not finite polyhedral or X has small extremal rays. It is connected with considering of so called $\mathbb{Q}$*-factorial models Y of a Calabi-Yau manifold X* which one gets as a sequence of contractions of divisorial extremal rays and flops in small extremal rays. These models have $\mathbb{Q}$-factorial canonical singularities, and we can apply Diagram Method to these models obtaining results similar to Theorems 0.1 and 0.1' (with replacing of the constant 40 by another constant). For some very special class of $\mathbb{Q}$-factorial models (we name them *very good*) we prove results similar to Theorems 0.1 and 0.1' (with the constant 163 instead 40). See Theorem 5.5 and Corollaries 5.6 and 5.9. We conjecture that analogs of Theorems 0.1 and 0.1' are valid for arbitrary $\mathbb{Q}$-factorial models (one should replace 40 by another absolute constant). See Conjecture 5.3.

In particular, this preprint contains results we announced in [N10].

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1. One effective variant of the diagram method for the divisorial case

1.1. Reminding. First, we recall one combinatorial result of [N8].

Let X be a projective algebraic variety with $\mathbb{Q}$-factorial singularities over an algebraically closed field. Let $N_1(X)$ be the $\mathbb{R}$-linear space generated by all algebraic curves on X by the numerical equivalence, and let $N^1(X)$ be the $\mathbb{R}$-linear space generated by all Cartier (or Weil) divisors on X by the numerical equivalence. Linear spaces $N_1(X)$ and $N^1(X)$ are dual to one another by the intersection pairing. Let $NE(X)$ be a convex cone in $N_1(X)$ generated by all effective curves on X. Let $\overline{NE}(X)$ be the closure of the cone $NE(X)$ in $N_1(X)$. It is called *Mori cone (*or *polyhedron)* of X. A non-zero element $x \in N^1(X)$ is called *nef* if $x \cdot \overline{NE}(X) \geq 0$. Let $NEF(X)$ be the set of all *nef* elements of X and the zero. It is the convex cone in $N^1(X)$ dual to Mori cone $\overline{NE}(X)$. A ray $R \subset \overline{NE}(X)$ with origin 0 is called *extremal* if from $C_1 \in \overline{NE}(X)$, $C_2 \in \overline{NE}(X)$ and $C_1 + C_2 \in R$ it follows that $C_1 \in R$ and $C_2 \in R$.

We consider a condition (i) for a set $\mathcal{R}$ of extremal rays on X.

(i) *If $R \in \mathcal{R}$, then all curves $C \in R$ fill out an irreducible divisor $D(R)$ on X. We call this extremal ray divisorial.*

In this case, we can correspond to $\mathcal{R}$ (and subsets of $\mathcal{R}$) an oriented graph $G(\mathcal{R})$ in the following way: Two different rays R_1 and R_2 are joined by an arrow R_1R_2 with the beginning in R_1 and the end in R_2 if $R_1 \cdot D(R_2) > 0$. Here and in what follows, for an extremal ray R and a divisor D we write $R \cdot D > 0$ if $r \cdot D > 0$ for $r \in R$ and $r \neq 0$. (The same for the symbols $\leq$, $\geq$ and $<$.)

A set $\mathcal{E}$ of extremal rays is called *extremal* if it is contained in a face of $\overline{NE}(X)$. Equivalently, there exists a nef element $H \in N^1(X)$ such that $\mathcal{E} \cdot H = 0$. Evidently, a subset of an extremal set is extremal too.

We consider the following condition (ii) for extremal sets $\mathcal{E}$ of extremal rays.

(ii) *An extremal set $\mathcal{E} = \{R_1, ..., R_n\}$ satisfies the condition (i), and for any real numbers $m_1 \geq 0, ..., m_n \geq 0$ which are not all equal to 0, there exists a ray $R_j \in \mathcal{E}$ such that $R_j \cdot (m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)) < 0$. In particular, the effective divisor $m_1D(R_1) + m_2D(R_2) + ... + m_nD(R_n)$ is not nef.*

A set $\mathcal{L}$ of extremal rays is called *E-set* (extremal in a different sense) if the $\mathcal{L}$ is not extremal but every proper subset of $\mathcal{L}$ is extremal. Thus, $\mathcal{L}$ is a minimal non-extremal set of extremal rays. Evidently, an E-set $\mathcal{L}$ contains at least two elements.

We consider the following condition (iii) for E-sets $\mathcal{L}$.

(iii) *Any proper subset of an E-set $\mathcal{L} = \{Q_1, ..., Q_m\}$ satisfies the condition (ii), and there exists a non-zero effective nef divisor $D(\mathcal{L}) = a_1D(Q_1) + a_2D(Q_2) + ... + a_mD(Q_m)$.*

We have the following statement:

Lemma 1.1.1. *An E-set $\mathcal{L}$ satisfying the condition (iii) is connected in the following sense: For any decomposition $\mathcal{L} = \mathcal{L}_1 \coprod \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are non-empty, there exists an arrow Q_1Q_2 such that $Q_1 \in \mathcal{L}_1$ and $Q_2 \in \mathcal{L}_2$.*

If $\mathcal{L}$ and $\mathcal{M}$ are two different E-sets satisfying the condition (iii), then there exists an arrow LM where $L \in \mathcal{L}$ and $M \in \mathcal{M}$.

Proof. See [N8, Lemma 1.1].

Let $NEF(X) = \overline{NE}(X)^* \subset N^1(X)$ be the cone of nef elements of X and $\mathcal{M}(X) = NEF(X)/\mathbb{R}^+$ its projectivization. We use usual relations of orthogonality between subsets of $\mathcal{M}(X)$ and $\overline{NE}(X)$. So, for $U \subset \mathcal{M}(X)$ and $V \subset \overline{NE}(X)$ we write $U \perp V$ if $x \cdot y = 0$ for any $\mathbb{R}^+x \in U$ and any $y \in V$. Thus, for $U \subset \mathcal{M}(X)$, $V \subset \overline{NE}(X)$ we denote

$$U^\perp = \{y \in \overline{NE}(X) \mid U \perp y\}, \quad V^\perp = \{x \in \mathcal{M}(X) \mid x \perp V\}$$

A subset $\gamma \subset \mathcal{M}(X)$ is called a *face* of $\mathcal{M}(X)$ if there exists a non-zero element $r \in \overline{NE}(X)$ such that $\gamma = r^\perp$. Similarly, a subset $\alpha \subset \overline{NE}(X)$ is called a *face* of $\overline{NE}(X)$ if there exists a non-zero element $h \in \mathcal{M}(X)$ such that $\alpha = h^\perp$.

A convex set is called *a closed polyhedron* (equivalently, *finite polyhedral*) if it is a convex hull of a finite set of points. A convex closed polyhedron is called *simplicial* if all its proper faces are simplexes. A convex closed polyhedron is called *simple* (equivalently, it has *simplicial angles*) if it is dual to a simplicial one. In other words, any its face of codimension k is contained exactly in k faces of the highest dimension. Evidently, $\mathcal{M}(X)$ is simple if the Mori cone $\overline{NE}(X)$ is polyhedral (has a finite set of extremal rays) and is simplicial (all proper faces of $\overline{NE}(X)$ are cones over simplexes). Evidently, the last property is equivalent to the fact that any extremal set of extremal rays on X is linear independent.

Let A, B are two vertices of an oriented graph G. The *distance* $\rho(A, B)$ in G is a length (the number of links) of a shortest oriented path of the graph G with the beginning in A and the end in B. The distance is $+\infty$ if this path does not exist. The *diameter* diam G of an oriented graph G is the maximum distance between ordered pairs of its vertices. By the Lemma 1.1.1, the diameter of an E-set is a finite number if this set satisfies the condition (iii).

We have the following

Theorem 1.1.2. *Let X be a projective algebraic variety with $\mathbb{Q}$-factorial singularities and* $\dim\ X \geq 2$. *Let us suppose that $\mathcal{M}(X)$ is closed and simple (equivalently, Mori cone $\overline{NE}(X)$ is a finite polyhedral simplicial cone).*

Assume that all extremal ray on X are divisorial (satisfies the condition (i) above), each extremal subset of extremal rays satisfies the condition (ii), and each E-set of extremal rays satisfies the condition (iii). Assume that there are some constants d, C_1, C_2 such that the conditions (a) and (b) below hold:

(a)

$$diam\ G(\mathcal{L}) \leq d.$$

for any E-set of extremal rays on X.

(b)

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1 \sharp\mathcal{E};$$

and

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid d+1 \leq \rho(R_1, R_2) \leq 2d+1\} \leq C_2 \sharp\mathcal{E}.$$

for any extremal set $\mathcal{E}$ of extremal rays on X.

Then $\dim N_1(X) = \dim \overline{NE}(X) \leq (16/3)C_1 + 4C_2 + 6$.

Proof. This is a particular case of [N8, Theorem 1.2].

1.2. General results on divisorial extremal rays for 3-folds.

In fact, the most part of results here was contained in [N8].

We restrict considering normal projective 3-folds X with $\mathbf{Q}$-factorial singularities.

Let R be an extremal ray of Mori polyhedron $\overline{NE}(X)$ of X. A morphism $f : X \to Y$ on a normal projective variety Y is called the *contraction* of the ray R if for an irreducible curve C of X the image $f(C)$ is a point iff $C \in R$. The contraction f is defined by a linear system H on X (H gives the nef element of $N^1(X)$, which we denote by H also). It follows that an irreducible curve C is contracted iff $C \cdot H = 0$. We assume that the contraction f has properties: $f_*\mathcal{O}_X = \mathcal{O}_Y$ and the sequence

$$(1.2.1) \qquad 0 \to \mathbf{R}R \to N_1(X) \to N_1(Y) \to 0$$

is exact where the arrow $N_1(X) \to N_1(Y)$ is f_*. An extremal ray R is called *contractible* if there exists its contraction f with these properties.

The number $\kappa(R) = \dim Y$ is called *Kodaira dimension* of the contractible extremal ray R.

We recall (see above) that a subset γ of $\overline{NE}(X)$ is called a *face* if there exists a non-zero *nef* element $r \in \overline{NE}(X)$ such that $\gamma = r^\perp$. A face γ of $\overline{NE}(X)$ is called *contractible* if there exists a morphism $f : X \to Y$ on a normal projective variety Y such that $f_*\gamma = 0$, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and f contracts curves lying in γ only. The $\kappa(\gamma) = \dim Y$ is called *Kodaira dimension of* γ.

Let H be a general nef element orthogonal to a face γ of Mori polyhedron. *Numerical Kodaira dimension of* γ is defined by the formula

$$\kappa_{num}(\gamma) = \begin{cases} 3, & \text{if } H^3 > 0; \\ 2, & \text{if } H^3 = 0 \text{ and } H^2 \not\equiv 0; \\ 1, & \text{if } H^2 \equiv 0. \end{cases}$$

It is obvious that for a contractible face γ we have $\kappa_{num}(\gamma) \geq \kappa(\gamma)$. In particular, $\kappa_{num}(\gamma) = \kappa(\gamma)$ for a contractible face γ of Kodaira dimension $\kappa(\gamma) = 3$.

We will use the following statement which (in different variants) is standard:

Proposition 1.2.1. *Let X be a projective 3-fold with* **Q***-factorial singularities, $D_1, ..., D_m$ irreducible divisors on X and $f : X \to Y$ a surjective morphism such that $\dim X = \dim Y$ and $\dim f(D_i) < \dim D_i$. Let $y \in f(D_1) \cap ... \cap f(D_m)$.*

Then there are $a_1 > 0$, ..., $a_m > 0$ and an open U, $y \in U \subset f(D_1) \cup ... \cup f(D_m)$, such that

$$C \cdot (a_1 D_1 + ... + a_m D_m) < 0$$

if a curve $C \subset D_1 \cup ... \cup D_m$ belongs to a non-trivial algebraic family of curves on $D_1 \cup ... \cup D_m$ and $f(C) = point \ \in U$.

Proof. See Proposition 2.2.6 in [N8]

By this Proposition, we have

Lemma 1.2.2. *Let R be a contractible extremal ray of Kodaira dimension 3 and $f : X \to Y$ its contraction.*

Then there are three possibilities:

(I) All curves $C \in R$ fill an irreducible Weil divisor $D(R)$, the contraction f contracts $D(R)$ in a point and $R \cdot D(R) < 0$.

(II) All curves $C \in R$ fill an irreducible Weil divisor $D(R)$, the contraction f contracts $D(R)$ on an irreducible curve and $R \cdot D(R) < 0$.

(III) (small extremal ray) All curves $C \in R$ give a finite set of irreducible curves and the contraction f contracts these curves in points.

Proof. Assume that some curves of R fill an irreducible divisor D. Then $R \cdot D < 0$, by Proposition 1.2.1. Suppose that $C \in R$ and D does not contain C. It follows that $R \cdot D \geq 0$. We get a contradiction. It follows the Lemma.

According to the Lemma 1.2.2, we say that an extremal ray R has the *type (I), (II) or (III) (small)* if it is contractible of Kodaira dimension 3 and the statements (I), (II) or (III) respectively hold. Extremal rays of the type (I) and (II) we also call *divisorial*.

For a divisor D on X let

$$\overline{NE}(X,D) = (\text{image } \overline{NE}(D)) \subset \overline{NE}(X).$$

Lemma 1.2.3. *Let R be a divisorial extremal ray of the type (I).*

Then $\overline{NE}(X,D(R))$ is the ray $\overline{NE}(X,D(R)) = R$.

Let R be a divisorial extremal ray of the type (II), and f its contraction.

Then $\overline{NE}(X,D(R))$ is an angle $\overline{NE}(X,D(R)) = R + \mathbf{R}^+S$ with edges R and S, where $f_*\mathbf{R}^+S = \mathbf{R}^+(f(D))$.

Proof. This follows at once from the exact sequence (1-2-1).

Using Lemma 1.2.3, we get Lemmas 1.2.4 and 1.2.5 below.

Lemma 1.2.4. *Let R_1 and R_2 are two different extremal rays of the type (II) such that the divisors $D(R_1) = D(R_2)$.*

Then for $D = D(R_1) = D(R_2)$ we have:

$$\overline{NE}(X,D) = R_1 + R_2.$$

In particular, do not exist three different extremal rays of the type (II) such that their divisors are coincided.

If for two different extremal rays R_1, R_2 of the type (II), $D(R_1) = D(R_2)$ (thus, we have the case of the Lemma 1.2.4 above), we say that the set $\{R_1, R_2\}$ of extremal rays has *the type* $\mathfrak{B}_2$.

Lemma 1.2.5. *The divisors $D(R_1)$ and $D(R_2)$ of two different extremal rays of the type (I) do not intersect one another.*

The divisors of an extremal ray of the type (I) and a pair of the type $\mathfrak{B}_2$ do not intersect one another.

The divisors of two different pairs of the type $\mathfrak{B}_2$ do not intersect one another.

The next Lemma was proved in [N8, Theorem 2.3.3], but we give the proof since this statement is very important.

Lemma 1.2.6. *Suppose that a pair $\{R_1, R_2\}$ has the type $\mathfrak{B}_2$. Then*

$$\overline{NE}(X,D(R_1)) = \overline{NE}(X,D(R_2)) = R_1 + R_2$$

is a 2-dimensional face of Mori polyhedron of the numerical Kodaira dimension 3 *and such that $(R_1+R_2)^\perp$ is a face of the $NEF(X)$ of the codimension* 2.

Proof. Since the rays R_1, R_2 are extremal of Kodaira dimension 3, there are *nef* elements H_1, H_2 such that $H_1 \cdot R_1 = H_2 \cdot R_2 = 0$, ${H_1}^3 > 0$, ${H_2}^3 > 0$. Let $0 \neq C_1 \in R_1$ and $0 \neq C_2 \in R_2$. Let D be a divisor of the rays R_1 and R_2. Let us consider a map

(1-2-2) $$(H_1, H_2) \to H =$$

$$= (-D\cdot C_2)(H_2\cdot C_1)H_1 + (-D\cdot C_1)(H_1\cdot C_2)H_2 + (H_2\cdot C_1)(H_1\cdot C_2)D.$$

For a fixed H_1, we get a linear map $H_2 \to H$ of the set of *nef* elements H_2 orthogonal to R_2 into the set of *nef* elements H orthogonal to R_1 and R_2. This map has a one dimensional kernel, generated by $(-D\cdot C_2)H_1+(H_1\cdot C_2)D$. It follows that R_1+R_2 is a 2-dimensional face of $\overline{NE}(X)$.

For a general *nef* element $H=a_1H_1+a_2H_2+bD$ orthogonal to this face, where $a_1,a_2,b>0$, we have $H^3=(a_1H_1+a_2H_2+bD)^3\geq(a_1H_1+a_2H_2+bD)^2\cdot(a_1H_1+a_2H_2)=(a_1H_1+a_2H_2+bD)\cdot(a_1H_1+a_2H_2+bD)\cdot(a_1H_1+a_2H_2)\geq(a_1H_1+a_2H_2)^2\cdot(a_1H_1+a_2H_2+bD)\geq(a_1H_1+a_2H_2)^3>0$, since $a_1H_1+a_2H_2+bD$ and $a_1H_1+a_2H_2$ are *nef*. It follows that the face R_1+R_2 has the numerical Kodaira dimension 3.

The last statement follows from construction.

By Proposition 1.2.1, we have

Lemma 1.2.7. *Let $\mathcal{E}=\{R_1,...,R_n\}$ be a set of divisorial extremal rays of the type (I) or (II) on X and the $\mathcal{E}$ is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3,*

Then there are real $a_1,...,a_n$ such that

$$R_i\cdot(a_1D(R_1)+\cdots a_nD(R_n))<0 \tag{1-2-3}$$

for all $R_i\in\mathcal{E}$.

It follows that $\mathcal{E}$ is linear independent if all divisors $D(R_1),...,D(R_n)$ are different.

Proof. We only need proving the last statement. Let us assume that we have a linear dependence condition $c_1R_1+\cdots c_sR_s+c_{s+1}R_{s+1}+\cdots+c_nR_n=0$. We can suppose that $c_1,...,c_s$ are positive and $c_{s+1},...,c_n$ are non-positive. By (1-2-3) and our condition,

$$(c_1R_1+\cdots+c_sR_s)\cdot(a_1D(R_1)+\cdot+a_sD(R_s))<0$$

and

$$(c_{s+1}R_{s+1}+\cdots+c_nR_s)\cdot(a_1D(R_1)+\cdot+a_sD(R_s))\leq 0.$$

We get the contradiction.

We have the following inverse statement based on standard arguments connected with Perron-Frobenius Theorem.

Lemma 1.2.8. *We have the following inverse statement to the previous one: Let $\mathcal{E}=\{R_1,...,R_n\}$ be a set of divisorial extremal rays of the type (I) or (II) on X and all divisors $D(R_1),...,D(R_n)$ are different. Assume that there are positive $a_1,...,a_n$ such that*

$$R_i\cdot(a_1D(R_1)+\cdots a_nD(R_n))<0$$

for all $R_i\in\mathcal{E}$.

Then for any $b_1\geq 0,...,b_n\geq 0$ which are not all equal to 0, there exists $1\leq j\leq n$ such that

$$R_j\cdot(b_1D(R_1)+\cdots+b_nD(R_n))<0.$$

For each $1\leq k\leq n$, there are non-negative $u_{1k},...,u_{nk}$ such that $R_k\cdot(u_{1k}D(R_1)+\cdots+u_{nk}D(R_n))<0$ and $R_j\cdot(u_{1k}D(R_1)+\cdots+u_{nk}D(R_n))=0$ if $j\neq k$ and $1\leq j\leq n$. It particular, the $\mathcal{E}$ is linear independent.

Proof. We can find $\lambda > 0$ such that $\lambda a_i \geq b_i$ for all $1 \leq i \leq n$ and one of these inequalities is equality. Suppose that $\lambda a_j = b_j$ for $1 \leq j \leq n$. Then

$$R_j \cdot (b_1 D(R_1) + \cdots + b_n D(R_n)) = R_j \cdot \lambda(a_1 D(R_1) + \cdots + a_n D(R_n)) \\ + R_j \cdot ((b_1 - \lambda a_1) D(R_1) + \cdots + (b_n - \lambda a_n) D(R_n)) < 0.$$

Evidently, $R_i \cdot (a_1 D(R_1) + \cdots a_{n-1} D(R_{n-1})) < 0$ for $1 \leq i \leq n-1$. Let us choose generators $r_i \in R_i$. Since $r_i \cdot D(R_n) \geq 0$ for $1 \leq i \leq n-1$, using induction, we can find $u_1 \geq 0, ..., u_{n-1} \geq 0$ such that $-r_i \cdot D(R_n) = r_i \cdot (u_1 D(R_1) + \cdots u_{n-1} D(R_{n-1}))$ for all $1 \leq i \leq n-1$. Thus, $u_1 D(R_1) + \cdots u_{n-1} D(R_{n-1}) + D(R_n)$ is orthogonal to $R_1, ..., R_{n-1}$. By the statement proved above, then $R_n \cdot (u_1 D(R_1) + \cdots u_{n-1} D(R_{n-1}) + D(R_n)) < 0$.

This finished the proof.

Using Lemma 1.2.8, we get

Lemma 1.2.9. *Let $\mathcal{E} = \{R_1, ..., R_n\}$ be a set of divisorial extremal rays of the type (I) or (II) on X and all divisors $D(R_1), ..., D(R_n)$ are different. Assume that there are positive $a_1, ..., a_n$ such that*

$$R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) < 0$$

for all $R_i \in \mathcal{E}$. Let us additionally suppose that $C \cdot D(R_i) \geq 0$ for any curve $C \subset D(R_i)$, $C \notin R_i$ and all $1 \leq i \leq n$.

Then $R_1 + \cdots + R_n$ is a face of the dimension n (thus, it is a cone over n-dimensional simplex) and of the numerical Kodaira dimension 3 of $\overline{NE}(X)$ and such that $(R_1 + \cdots + R_n)^\perp$ is a face of the cone $NEF(X)$ of the codimension n.

Proof. Let H be a nef element on X. By Lemma 1.2.8, there are non-negative linear functions $b_1(H), ..., b_n(H)$ such that $H' = H + \sum_{i=1}^{i=n} b_i(H) D(R_i)$ is orthogonal to $R_1, ..., R_n$. By additional condition, H' is nef. The map $H \to H'$ gives a linear map from the set of nef elements on X to the set of nef elements orthogonal to $R_1, ..., R_n$. This map has n-dimensional kernel generated by $D(R_1), ..., D(R_n)$. If follows that the extremal rays $R_1, ..., R_n$ belong to a face of $\overline{NE}(X)$ of dimension $\leq n$. By Lemma 1.2.8, $R_1, ..., R_n$ are linear independent. If follows that $R_1, ..., R_n$ belong to a face γ of $\overline{NE}(X)$ of the dimension n. By induction, we can suppose that any $n-1$-element subset of $\{R_1, ..., R_n\}$ generates a face of $\overline{NE}(X)$ which is a cone over $n-1$-dimensional simplex. It follows that the $\gamma = R_1 + \cdots + \mathcal{R}_n$ is a cone over n-dimensional simplex.

Like above, one can prove that $(H')^3 \geq H^3 > 0$ for an ample H. Thus, the face $R_1 + \cdots + \mathcal{R}_n$ has Kodaira dimension 3. We get the last property by the construction.

Using Perron-Frobenius Theorem, we get

Lemma 1.2.10. *Let $\{R_1, ..., R_n\}$ be a set of divisorial extremal rays of the type (I) or (II) and with different divisors $D(R_1), ..., D(R_n)$. Let us suppose that any its proper subset satisfies conditions of Lemma 1.2.8 but the set $\{R_1, ..., R_n\}$ itself does not.*

Then there exist positive $a_1, ..., a_n$ such that

$$(1.2.4) \qquad R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) \geq 0$$

for all $1 \le i \le n$. *Additionally, we have one of cases:*

(a) $R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) = 0$ *for all* $1 \le i \le n$. *Then the set of positive* $(a_1, ..., a_n)$ *with the property (1-2-4), is defined uniquely up to multiplication on* $\lambda > 0$.

(b) There exists $1 \le j \le n$ *such that the inequality (1-2-4) is strong for this* j*:*

$$R_j \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) > 0.$$

Proof. Let us suppose that after changing numeration, for some $1 \le m < n$, we have $R_i \cdot D(R_j) = 0$ for all $1 \le i \le m < j \le n$. By our condition, we can find $a_1 > 0, ..., a_m > 0$ and $a_{m+1} > 0, ..., a_n > 0$ such that

$$R_i \cdot (a_1 D(R_1) + \cdots + a_m D(R_m)) < 0$$

for all $1 \le i \le m$, and

$$R_j \cdot (a_{m+1} D(R_{m+1}) + \cdots + a_n D(R_n)) < 0$$

for all $m+1 \le j \le n$. Evidently, for small $\epsilon > 0$, we then have

$$R_k \cdot \epsilon(a_1 D(R_1) + \cdots + a_m D(R_m)) + (a_{m+1} D(R_{m+1}) + \cdots + a_n D(R_n)) < 0$$

for all $1 \le k \le n$. We get the contradiction.

Thus, the subdivision above is impossible. Then, by Perron–Frobenius Theorem, there are positive $a_1, ..., a_n$ with the property (1-2-4) above.

Assume that we have the case (a). Let us suppose that there are positive $b_1, ..., b_n$ such that $R_i \cdot (b_1 D(R_1) + \cdots + b_n D(R_n)) \ge 0$ for all $1 \le i \le n$. There exists $\lambda > 0$ such that $b_i - \lambda a_i \ge 0$ and one of these inequalities is equality. Then

$$R_i \cdot (b_1 D(R_1) + \cdots + b_n D(R_n)) =$$
$$\lambda R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) + R_i \cdot ((b_1 - \lambda a_1) D(R_1) + \cdots + (b_n - \lambda a_n) D(R_n)) \ge 0.$$

for all $1 \le i \le n$. If at least one $b_i - \lambda a_i > 0$, we then get a contradiction with Lemma 1.2.8.

From Lemma 1.2.10, we get

Lemma 1.2.11. *Let* $\{R_1, ..., R_n\}$ *be a set of divisorial extremal rays of the type (I) or (II) and with different divisors* $D(R_1), ..., D(R_n)$. *Let us suppose that any its proper subset satisfies the condition of Lemma 1.2.8 but the set* $\{R_1, ..., R_n\}$ *itself does not. Let us additionally suppose that* $C \cdot D(R_i) \ge 0$ *for any curve* $C \subset D(R_i)$, $C \notin R_i$ *and all* $1 \le i \le n$.

Then, in notation of Lemma 1.2.10, the element $H = a_1 D(R_1) + \cdots + a_n D(R_n)$ *is* nef. *The set* $\{R_1, ..., R_n\}$ *is not contained in a face of* $\overline{NE}(X)$ *of Kodaira dimension* 3. *For the case (a) of Lemma 1.2.10, the set* $\{R_1, ..., R_n\}$ *is extremal.*

Proof. By additional condition, the H is nef. The set $\{R_1, ..., R_n\}$ is not contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3 by Proposition 1.2.1 and Lemma 1.2.8.

If we have the case (a) of Lemma 1.2.10, then $\{R_1, ..., R_n\}$ is contained in the face of $\overline{NE}(X)$ orthogonal to H. This finishes the proof.

As a particular case, we get

Lemma 1.2.12. *Assume that all finite polyhedral faces γ of $\overline{NE}(X)$ with the property codim $\gamma^\perp = \dim \gamma$ are contractible and their numerical Kodaira dimension is equal to Kodaira dimension (here the $\gamma^\perp$ is the corresponding face of $NEF(X)$). Let $\mathcal{L} = \{R_1, ..., R_n\}$ be an E-set of divisorial extremal rays of the type (I) or (II). Let us additionally suppose that $C \cdot D(R_i) \geq 0$ for any curve $C \subset D(R_i),\ C \notin R_i$ and all $1 \leq i \leq n$.*

Then there are non-negative $a_1, ..., a_n$ which are not all equal to zero such that $a_1 D(R_1) + \cdots + a_n D(R_n)$ is nef

Proof. By Proposition 1.2.1 and Lemma 1.2.9, there exists a minimal subset $\mathcal{L}' \subset \mathcal{L}$ such that for $\mathcal{L}'$ conditions of the Lemma 1.2.11 hold. By Lemma 1.2.11, we get the statement.

Using Theorem 1.1.2 and Lemmas above, we get

Theorem 1.2.13. *Let X be a projective 3-fold with $\mathbb{Q}$-factorial singularities. Let us suppose that Mori cone $\overline{NE}(X)$ is finite polyhedral and any its face has Kodaira dimension 3. Assume that all extremal rays on X are divisorial of the type (I) or (II) and X does not have a pair of extremal rays of the type $\mathfrak{B}_2$.*

Then extremal and E-sets of extremal rays on X satisfy the conditions (i), (ii) and (iii) of Section 1.1, and any proper face of $\overline{NE}(X)$ is a cone over simplex.

Assume that there are some constants d, C_1, C_2 such that the conditions (a) and (b) below hold:

(a)

$$diam\ G(\mathcal{L}) \leq d$$

for any E-set $\mathcal{L}$ of extremal rays on X.

(b)

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1 \sharp \mathcal{E};$$

and

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid d+1 \leq \rho(R_1, R_2) \leq 2d+1\} \leq C_2 \sharp \mathcal{E}.$$

for any extremal set $\mathcal{E}$ of extremal rays on X (here we use distance in the graph $G(\mathcal{E})$).

Then we have the inequality:

$$\dim\ N_1(X) = \dim\ \overline{NE}(X) \leq (16/3) C_1 + 4 C_2 + 6.$$

1.3. Basic Theorem.

Definition 1.3.1. We say that an algebraic 3-fold X belongs to the class $\mathcal{LT}$ if X has $\mathbb{Q}$-factorial singularities; each face γ of $\overline{NE}(X)$ generated by a finite set of divisorial extremal rays of the type (I) or (II) and with the property $\dim \gamma =$ codim $\gamma^\perp$ for the face $\gamma^\perp$ of $NEF(X)$, and of numerical Kodaira dimension 3 is contractible and has Kodaira dimension 3; the contraction of any sequence of extremal rays of the type (I) or (II) starting from X gives a 3-fold with $\mathbb{Q}$-factorial singularities and with the properties above.

For a 3-fold X from the class $\mathcal{LT}$ we say that it has constants $q(X)$, $d(X)$, $C_1(X)$, $C_2(X)$ if we have:

$$\sharp \mathcal{E} \leq q(X)$$

for any extremal set $\mathcal{E}$ of Kodaira dimension 3 (i.e. $\mathcal{E}$ is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3) of extremal rays the type (II) such that the graph $G(\mathcal{E})$ is full (i.e. any two rays $R_1, R_2 \in \mathcal{E}$ are joint by both arrows R_1R_2 and R_2R_1); the diameter

$$\mathrm{diam}\ G(\mathcal{L}) \leq d(X)$$

for any E-set $\mathcal{L}$ of extremal rays of the type (I) or (II) such that any proper subset of $\mathcal{L}$ is extremal of Kodaira dimension 3 and $\mathcal{L}$ has the property (iii) of Sect. 1.1;

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid 1 \leq \rho(R_1, R_2) \leq d(X)\} \leq C_1(X)\sharp\mathcal{E};$$

and

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid d(X)+1 \leq \rho(R_1, R_2) \leq 2d(X)+1\} \leq C_2(X)\sharp\mathcal{E}.$$

for any extremal set $\mathcal{E}$ of Kodaira dimension 3 of extremal rays of the type (I) or (II) with different divisors.

We want to prove the following basic result:

Basic Theorem 1.3.2. *Let X be a 3-fold from the class $\mathcal{LT}$. We assume that Mori cone $\overline{NE}(X)$ is finite polyhedral and the conditions (A) and (B) below hold:*

(A) The $\overline{NE}(X)$ does not have a face of Kodaira dimension 1 or 2;

(B) The $\overline{NE}(X)$ does not have a small extremal ray (i.e. all extremal rays on X are divisorial of the type (I) or (II)).

Then we have the following statements about X with the constants $q(X)$, $d(X)$, $C_1(X)$, $C_2(X)$ above:

(1) The X does not have a pair of extremal rays of the type $\mathfrak{B}_2$ and Mori cone $\overline{NE}(X)$ is simplicial;

(2) The number of extremal rays of the type (I) on X is not greater than $q(X)$;

(3) The $\rho(X) = \dim N_1(X) \leq (16/3)C_1(X) + 4C_2(X) + 6$.

Proof. Let us prove (1). We need the following analog of [N8, Lemma 2.5.7].

Lemma 1.3.3. *Let X be a 3-fold from the class $\mathcal{LT}$ and Mori cone $\overline{NE}(X)$ is finite polyhedral.*

Let $\mathcal{E}$ be the set of all extremal rays of a proper face $[\mathcal{E}]$ of $\overline{NE}(X)$. Let

$$\{R_{11}, R_{12}\} \cup ... \cup \{R_{t1}, R_{t2}\}$$

be a set of different pairs of extremal rays of the type $\mathfrak{B}_2$. Assume that $R\cdot D(R_{i1}) = 0$ for all $R \in \mathcal{E}$ and all i, $1 \leq i \leq t$.

Then there are extremal rays $Q_1, ..., Q_r$ such that the following statements hold:

(a) $r \leq t$;

(b) For any i, $1 \leq i \leq r$, there exists j, $1 \leq j \leq t$, such that $Q_i \cdot D(R_{j1}) > 0$ (in particular, Q_i is different from extremal rays of pairs of extremal rays $\{R_{u1}, R_{u2}\}$ of the type $\mathfrak{B}_2$);

(c) For any j, $1 \leq j \leq t$, there exists an extremal ray Q_i, $1 \leq i \leq r$, such that

$$Q_i \cdot D(R_{j1}) > 0;$$

(d) The set $\mathcal{E}\cup\{Q_1, ..., Q_r\}$ is extremal, and extremal rays $\{Q_1, ..., Q_r\}$ are linearly independent.

Proof. If $t = 0$, we can take $r = 0$. Thus, we assume that $t \geq 1$.

Since $R_{ij}\cdot D(R_{ij}) < 0$, $1 \leq i \leq t, 1 \leq j \leq 2$, the set $\mathcal{E}$ does not contain the rays R_{ij}. Let H be a general nef element orthogonal to $[\mathcal{E}]$. Since $t \geq 1$, there exists $a > 0$ such that $H' = H + aD(R_{11})$ is nef and H' is orthogonal to $\mathcal{E}$ and one of the rays R_{11}, R_{12}. Let this ray be R_{11}. Then the set $E\cup\{R_{11}\}$ is extremal and is contained in a (proper) face of $\overline{NE}(X)$. It follows, $\dim[\mathcal{E}] < \dim[\mathcal{E}\cup\{R_{11}\}] < \dim\overline{NE}(X)$, and $\dim[\mathcal{E}] < \dim\overline{NE}(X) - 1$. Let us consider a linear subspace $V(\mathcal{E}) \subset N_1(X)$ generated by all extremal rays $\mathcal{E}$. By our condition, $V(\mathcal{E})$ is a linear envelope of the face $[\mathcal{E}]$ of $\overline{NE}(X)$.

Let us consider the factorization map $\pi : N_1(X) \to N_1(X)/V(\mathcal{E})$. Since the cone $\overline{NE}(X)$ is polyhedral, the cone $\pi(\overline{NE}(X))$ is generated by images of extremal rays T such that the set $\mathcal{E}\cup\{T\}$ is contained in a face $[\mathcal{E}\cup\{T\}]$ of $\overline{NE}(X)$ of the dimension $\dim[\mathcal{E}] + 1$. In particular, since $\dim[\mathcal{E}] < \dim N_1(X) - 1$, the face $[\mathcal{E}\cup\{T\}]$ is proper, and the set $\mathcal{E}\cup\{T\}$ is extremal.

There exists a curve C on X such that $C\cdot D(R_{11}) > 0$. This curve C (as any element $x \in \overline{NE}(X)$) is a linear combination of extremal rays T with non-negative coefficients and extremal rays from $\mathcal{E}$ with real coefficients. We have $R\cdot D(R_{11}) = 0$ for any extremal ray $R \in \mathcal{E}$. Thus, there exists an extremal ray T above such that $T\cdot D(R_{11}) > 0$. It follows that T is different from extremal rays of pairs of the type $\mathfrak{B}_2$. We take $Q_1 = T$. By our construction, the set $\mathcal{E}\cup\{Q_1\}$ is extremal. If $Q_1\cdot D(R_{j1}) > 0$ for any j such that $1 \leq j \leq t$, then $r = 1$, and the set $\{Q_1\}$ gives the set we were looking for. Otherwise, there exists a minimal j such that $2 \leq j \leq t$ and $Q_1\cdot D(R_{j1}) = 0$. Then we replace $\mathcal{E}$ by the set $\mathcal{E}_1$ of all extremal rays in the face $[\mathcal{E}\cup\{Q_1\}]$ of the dimension $\dim[\mathcal{E}_1] = \dim[\mathcal{E}] + 1$, and the set

$$\{R_{11}, R_{12}\}\cup ... \cup\{R_{t1}, R_{t2}\}$$

by

$$\{R_{j1}, R_{j2} \mid 1 \leq j \leq t,\ Q_1\cdot D(R_{j1}) = 0\},$$

and repeat this procedure.

Also, we need the following Lemma:

Lemma 1.3.4. *Let X be a 3-fold from the class LT and Mori cone $\overline{NE}(X)$ is polyhedral. Assume that extremal rays on X have the type (I) or (II). Let R_{11}, R_{12} be a pair of extremal rays of the type $\mathfrak{B}_2$ on X and $\rho(X) \geq 3$.*

Then there exists an extremal ray R on X such that R does not belong to a pair of extremal rays of the type $\mathfrak{B}_2$ and the sets $\{R_{11}, R\}$ and $\{R_{12}, R\}$ generate 2-dimensional faces of $\overline{NE}(X)$.

Proof. Let R_{21}, R_{22} be another pair of extremal rays of the type $\mathfrak{B}_2$. By Lemmas 1.2.5 and 1.2.6, the extremal rays R_{11}, R_{12} generate a 2-dimensional face of $\overline{NE}(X)$ and divisors $D(R_{11})$ and $D(R_{21})$ are disjoint (have empty intersection). Applying Lemma 1.3.3 to $\mathcal{E} = \{R_{11}, R_{12}\}$ and the set $\{\{R_{21}, R_{22}\}\}$ of pairs of the type $\mathfrak{B}_2$, we find an extremal ray Q such that $Q\cdot D(R_{21}) > 0$. It follows that $D(Q)\cap D(R_{21})$ is a non-empty curve C. By Lemma 1.2.5, the Q has the type (II). By Lemma 1.2.6, $\overline{NE}(X, D(R_{21})) = R_{21} + R_{22}$ is a 2-dimensional face of $\overline{NE}(X)$. By Lemma 1.2.3, then the 2-dimensional angle $\overline{NE}(X, D(Q)) = Q + \mathbf{R}^+C$. Since $\overline{NE}(X, D(R_{11})) = R_{11} + R_{12}$ is another 2-dimensional face of $\overline{NE}(X)$ which does not have a common ray with $R_{21} + R_{22}$, the angle $\overline{NE}(X, D(Q))$ does not have a common ray with the angle $\overline{NE}(X, D(R_{11})$. Thus, $D(Q) \cap D(R_{11}) = \emptyset$. By Lemma 1.2.5, the Q does not belong to a pair of the type $\mathfrak{B}_2$.

Let H be a general *nef* element orthogonal to the 2-dimensional face $R_{11} + R_{12}$. Then there exists $\alpha > 0$ such that the *nef* element $H' = H + \alpha D(Q)$ is orthogonal to the set of extremal rays R_{11}, R_{12}, Q. If an extremal ray R is different from these three extremal rays, evidently, $H' \cdot R > 0$ since all extremal rays on X are divisorial and Q does not belong to a pair of the type $\mathfrak{B}_2$. It follows that $R_{21} + R_{22} + Q$ is a simplicial 3-dimensional face of $\overline{NE}(X)$. It follows that Q is the extremal ray we are looking for.

Thus, we can suppose that the pair R_{11}, R_{12} is the only pair of extremal rays of the type $\mathfrak{B}_2$ on X.

If there exists an extremal ray Q such that $D(Q) \cap D(R_{11}) = \emptyset$, like above, Q is the required extremal ray. Thus, we can suppose that any extremal ray Q which is different from R_{11}, R_{12}, has the property $D(Q) \cap D(R_{11}) \neq \emptyset$. In particular, $D(R_{11}) \cap D(Q)$ is a non-empty curve.

Since $\overline{NE}(X)$ is polyhedral, there exists an extremal ray $Q_1 \notin \{R_{11}, R_{12}\}$ such that $R_{11} + Q_1$ is a 2-dimensional face of $\overline{NE}(X)$. If $Q_1 + R_{12}$ is a 2-dimensional face of $\overline{NE}(X)$, the Q_1 is the desired extremal ray. Thus, we can suppose that $Q_1 + R_{12}$ is not a 2-dimensional face of $\overline{NE}(X)$. Similarly, we can find an extremal ray $Q_2 \notin \{R_{11}, R_{12}\}$ such that $R_{12} + Q_2$ is a 2-dimensional face of $\overline{NE}(X)$ but $R_{11} + Q_2$ is not. Then $Q_1 \neq Q_2$. We recall that besides, we suppose that $C_1 = D(R_{11}) \cap D(Q_1)$ and $C_2 = D(R_{11}) \cap D(Q_2)$ are non-empty curves.

We will normalize the generator $C \in T$ of a divisorial extremal ray T by the condition $C \cdot D(T) = -2$.

Let r_{ij} be the generator of R_{ij} and q_i of Q_i. Let $t = q_1 \cdot D(R_{11})$ and $t_1 = r_{11} \cdot D(Q_1)$ and $t_2 = r_{12} \cdot D(Q_1)$.

Since $R_{11} + Q_1$ is a 2-dimensional face of $\overline{NE}(X)$ and all faces of $\overline{NE}(X)$ have Kodaira dimension 3, by Proposition 1.2.1, there are positive a_1, a_2 such that $r_{11} \cdot (a_1 D(R_{11}) + a_2 D(Q_1)) = -2a_1 + t_1 a_2 < 0$ and $q_1 \cdot (a_1 D(R_{11}) + a_2 D(Q_1)) = t a_1 - 2a_2 < 0$. Thus, $tt_1 < 4$. We claim that $t_1 < t_2$. Let us assume that $t_2 \leq t_1$. By inequality above, $tt_2 < 4$. Let H be a general *nef* element orthogonal to R_{12}. Then

$$H' = H + ((H \cdot q_1)/(2 - (tt_2)/2))((t_2/2)D(R_{11}) + D(Q_1))$$

is a *nef* element which is orthogonal to extremal rays R_{12}, Q_1 only. We very use here the inequality $t_2 \leq t_1$ to check that $R_{11} \cdot H' > 0$. Thus, $R_{12} + Q_1$ is a 2-dimensional face of $\overline{NE}(X)$. We get a contradiction. Thus, we have proved the claim: $t_1 < t_2$.

Let us consider the curve $C_1 = D(R_{11}) \cap D(Q_1)$. Then $C_1 = u_1 r_{11} + u_2 r_{12}$. Let $r_{11} \cdot r_{12} = m$ (equivalently, the m is the degree of the maps $f_{11} \mid r_{12} : r_{12} \to f_{11}(D(R_{11}))$ and $f_{12} \mid r_{11} : r_{11} \to f_{12}(D(R_{12}))$ where f_{11} and f_{12} are the contractions of R_{11} and R_{12} respectively). Then

$$(t_2 : t_1) = (r_{12} \cdot D(Q_1) : r_{11} \cdot D(Q_1)) = (r_{12} \cdot C_1 : r_{11} \cdot C_1) = (u_1 m : u_2 m).$$

Thus,

(1.2.5) $$C_1 = u_1 r_{11} + u_2 r_{12}, \quad (u_1 : u_2) = (t_2 : t_1), \quad t_2 > t_1$$

Similarly, for Q_2 we introduce $s_1 = r_{11} \cdot D(Q_2), s_2 = r_{12} \cdot D(Q_2)$ where $s_2 < s_1$. And we have for $C_2 = D(R_{11}) \cap D(Q_2)$:

(1-2-6) $$C_2 = v_1 r_{11} + v_2 r_{12},\ (v_1 : v_2) = (s_2 : s_1),\ s_2 < s_1.$$

By (1-2-5) and (1-2-6), $C_1 \cdot D(Q_2) = u_1 s_1 + u_2 s_{2 \geq} u_1 s_1 > 0$, Thus, divisors $D(Q_1)$ and $D(Q_2)$ have a common curve. It follows that 2-dimensional angles $\overline{NE}(X, D(Q_1)) = \mathbf{R}^+ C_1 + Q_1$ and $\overline{NE}(X, D(Q_2)) = \mathbf{R}^+ C_2 + Q_2$ should have a common ray. By (1-2-5) and (1-2-6), we then get that rays R_{11}, R_{12}, Q_1, Q_2 generate a 3-dimensional space V, and are extremal rays of the 3-dimensional polyhedral convex cone $\overline{NE}(X) \cap V$ with 2-dimensional faces $R_{11} + R_{12}$, $R_{11} + Q_1$, $R_{12} + Q_2$. Besides, angles $\mathbf{R}^+ C_1 + Q_1$ and $\mathbf{R}^+ C_2 + Q_2$ have a common ray. One should draw the picture to see a contradiction with (1-2-5) and (1-2-6). This finishes the proof of Lemma.

Now we can prove the statement (1). Thus, suppose that X satisfies the conditions of Theorem and has a pair $\{R_{11}, R_{12}\}$ of the type $\mathfrak{B}_2$. By Lemma 1.3.4, we can find an extremal ray Q which does not belong to a pair of extremal rays of the type $\mathfrak{B}_2$ and $Q + R_{11}$, $Q + R_{12}$ are 2-dimensional faces of $\overline{NE}(X)$. Let us contract the extremal ray Q. By our conditions, we get a 3-fold X' from the class LT, without small extremal rays, with polyhedral $\overline{NE}(X')$, and the image of the pair R_{11}, R_{12} of the type $\mathfrak{B}_2$ will be a pair of the type $\mathfrak{B}_2$ again. Thus, using the Lemma 1.3.4 $\rho(X) - 2$ times, we get a 3-fold Y which satisfies the condition of the Theorem, has $\rho(Y) = 2$, and has a pair of extremal rays $\{R_{11}, R_{12}\}$ of the type $\mathfrak{B}_2$. Then evidently $\overline{NE}(Y) = R_{11} + R_{12}$. We have $R_{11} \cdot D(R_{11}) < 0$ and $R_{12} \cdot D(R_{12}) < 0$. Thus, any curve of Y has negative intersection with the effective divisor $D(R_{11}) = D(R_{12})$. We get the contradiction.

This proves the statement (1).

Now let us prove (2): X does not have more than $q(X)$ extremal rays of the type (I).

By Proposition 1.2.1 and Lemma 1.2.5, divisors of different extremal rays of the type (I) do not have a common point and their number is finite. By Lemma 1.2.9, the set of extremal rays of the type (I) generates a simplicial face of $\overline{NE}(X)$ of Kodaira dimension 3. Let

$$\{R_1, ..., R_s\}$$

be the hole set of extremal rays of the type (I) on X. We should prove that $s \leq q(X)$.

We say that two divisorial extremal rays are joint (more formally: divisorially joint) if their divisors have a common point. It defines connected components of a set of divisorial extremal rays.

Let $\mathcal{E}$ be a maximal extremal set of extremal rays on X containing the set $\{R_1, ..., R_s\}$ and such that each connected component of $\mathcal{E}$ contains at least one of extremal rays $R_1, ..., R_s$. Let T be a connected component of the $\mathcal{E}$. Assume that T has two different extremal rays R_i, R_j from $R_1, ..., R_s$. After contracting all extremal rays of T different from R_i, R_j, we get a 3-fold Y from the class $\mathcal{L}T$, and images of extremal rays R_i, R_j give different extremal rays of the type (I) on Y such that their divisors are not disjoint. We get the contradiction with Lemma 1.2.5.

Thus, $\mathcal{E}$ has exactly s connected components $T_1, ..., T_s$ such that T_i contains the extremal ray R_i. Evidently, the maximal $\mathcal{E}$ does exist.

By Proposition 1.2.1, for $1 \le i \le s$, there exists an effective divisor $D(T_i)$ which is a linear combination of divisors of rays from T_i with positive coefficients and $R \cdot D(T_i) < 0$ for any $R \in T_i$. Evidently, the contraction of T_i contracts all T_i to a point. Thus, any curve of divisors of T_i belongs to the sum of extremal rays of T_i with positive coefficients. Thus, for this curve C we also have $C \cdot D(T_i) < 0$. Using the divisors $D(T_i)$, similarly to Lemma 1.3.4, we can find extremal rays

$$\{Q_1, ..., Q_r\}$$

with properties:

(a) $r \le s$;

(b) For any i, $1 \le i \le r$, there exists j, $1 \le j \le t$, such that $Q_i \cdot D(T_j) > 0$ (in particular, Q_i is different from extremal rays of $\mathcal{E}$ and does not have the type (I));

(c) For any j, $1 \le j \le s$, there exists an extremal ray Q_i, $1 \le i \le r$, such that

$$Q_i \cdot D(T_j) > 0;$$

(d) The set $\{Q_1, ..., Q_r\}$ of extremal rays is extremal.

By our conditions, all extremal rays on X are divisorial. Thus, by (b), the extremal rays $Q_1, ..., Q_r$ have the type (II).

Let us take the ray Q_i, and let $Q_i \cdot D(T_j) > 0$. By Lemma 1.2.9, the set T_j generates a simplicial face γ_j of $\overline{NE}(X)$. We have mentioned above that each curve of divisors of rays from T_j belongs to this face. It follows that $\overline{NE}(X, D(Q_i))$ is a 2-dimensional angle bounded by the ray Q_i and a ray from the face γ_j since the divisor $D(Q_i)$ evidently has a common curve with one of divisors $D(R)$, $R \in T_j$. Since any two sets from $T_1, ..., T_s$ do not have a common extremal ray, the faces $\gamma_1, ..., \gamma_s$ do not have a common ray (not necessarily extremal). It follows that the angle $\overline{NE}(X, D(Q_i))$ does not have a common ray with the face γ_k for $k \ne j$. Thus, the divisor $D(Q_i)$ does not have a common point with divisors of rays T_k. It follows that $r = s$ and we can choose numeration $Q_1, ..., Q_s$ such that $Q_i \cdot D(T_i) > 0$ but $D(Q_i)$ do not have a common point with divisors of extremal rays T_j if $j \ne i$.

Let us fix i, $1 \le i \le s$. By our construction, the set $\mathcal{E} \cup \{Q_i\}$ has connected components

$$T_1, ..., T_{i-1},\ T_i \cup \{Q_i\},\ T_{i+1}, ..., T_s.$$

By definition of $\mathcal{E}$, then the $\mathcal{E} \cup \{Q_i\}$ is not extremal. Thus, it contains an E-set (minimal non-extremal) $\mathcal{L}_i$ which contains Q_i. By Lemmas 1.2.12 and 1.1.1, the $\mathcal{L}_i$ is connected. Thus, $\{Q_i\} \subset \mathcal{L}_i \subset T_i \cup \{Q_i\}$. Let us consider the sets $\mathcal{L}_1, ..., \mathcal{L}_s$. By Lemma 1.1.1, the $\mathcal{L}_i, \mathcal{L}_j$ are joint by arrows in both directions. By our construction, it follows that Q_i, Q_j are joint by arrows Q_iQ_j and Q_jQ_i for any $1 \le i < j \le s$. Thus, $s \le q(X)$ because the set $\{Q_1, ..., Q_s\}$ is extremal.

This finishes the proof of the statement (2).

From the (1) and Theorem 1.2.13, the statement (3) follows. This finishes the proof of Basic Theorem 1.3.2.

In fact, Basic Theorem 1.3.2 is the generalization of similar theorem about Fano 3-folds with terminal $\mathbb{Q}$-factorial singularities proved in [N8]. We will apply Basic Theorem 1.3.2 to Calabi-Yau 3-fold in Section 3.

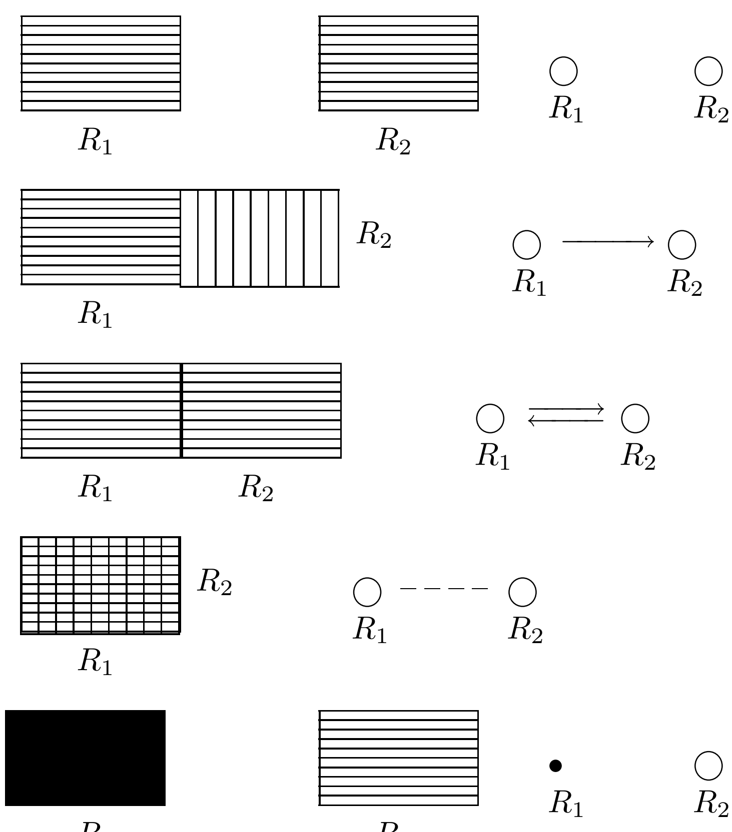
To apply Theorem 1.3.2, one should work with extremal and E-sets of extremal rays of the type (I) or (II). For Fano 3-folds this is done in [N8]. The next Section 2 is, in fact, devoted to generalization of this part of [N8] for 3-folds from the class $\mathcal{L}T$ (see Theorem 2.2.6). Roughly speaking, we should reduce the problem to sets of divisorial extremal rays which have non-single arrows only. The last case is very similar to surfaces.

2. Sets of divisorial extremal rays

2.1. Special divisorial extremal rays.

Here we continue geometrical study started in Section 1.2 of configurations of divisorial extremal rays of the types (I) or (II) for 3-folds.

Let T be a set of divisorial extremal rays of the types (I) or (II). We recall (see Section 1.1) that we correspond to T a graph $G(T)$ as follows: we draw an arrow R_1R_2 from R_1 to R_2 if $R_1 \cdot D(R_2) > 0$. We draw an extremal ray of the type (I) (respectively (II)) as a black (respectively white) vertex. A pair of the type $\mathfrak{B}_2$ we draw as a pair of white vertices connected by a dotted line. For this pair $\{R_1, R_2\}$ we have $D(R_1) = D(R_2)$ and $R_1 \cdot D(R_1) < 0, R_2 \cdot D(R_1) < 0$. Thus, two divisorial extremal rays R_1, R_2 give one of pictures of Figure 1. One should consider a single arrow R_1R_2 (thus, $R_1 \cdot D(R_2) > 0$, but $R_2 \cdot D(R_1) = 0$) as a "weak orthogonality" of extremal rays. They are completely orthogonal (are not joint by arrows) if their divisors do not have a common point: this follows from Lemma 1.2.2.



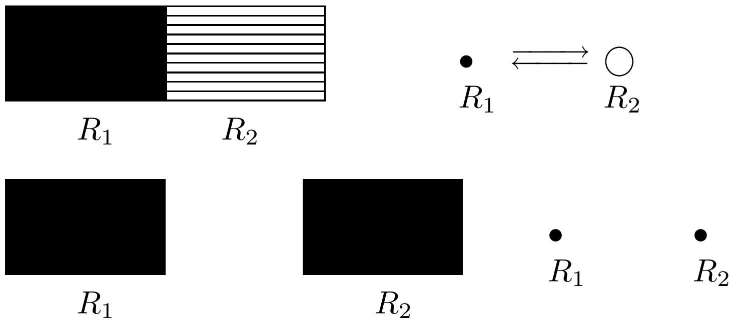


Figure 1.

Definition 2.1.1. We say that extremal rays $Q_1, Q_2, ..., Q_m,\ m \geq 2$, define the configuration $\mathfrak{C}_m$ if all these extremal rays have the type (II), extremal rays $Q_2, ..., Q_m$ are divisorially disjoint (intersection of any two divisors $D(Q_2), D(Q_3), ..., D(Q_m)$ is empty) and Q_iQ_1 is a single arrow (thus, $Q_i \cdot D(Q_1) > 0,\ Q_1 \cdot D(Q_i) = 0$) for all $i = 2, ..., m$.

The following Lemma was proved in [N8]. Since it is important, we give the proof.

Lemma 2.1.2. *Assume that divisorial extremal rays $Q_1, Q_2, ..., Q_m$ define the cofiguration of the type $\mathfrak{C}_m,\ m \geq 2$. Then Q_i does not belong to a pair of the type $\mathfrak{B}_2$ and $\overline{NE}(X, D(Q_i)) = Q_1 + Q_i$ is the 2-dimensional face of the numerical Kodaira dimension 3 of Mori polyhedron for any $2 \leq i \leq m$.*

Besides, $Q_1 + Q_2 + \cdots + Q_m$ is m-dimensional face of $\overline{NE}(X)$ of numerical Kodaira dimension 3 and such that the face $(Q_1 + Q_2 + \cdots + Q_m)^\perp$ of $NEF(X)$ has codimension m.

Proof. Let $2 \leq i \leq m$. Evidently, the curve $D(Q_i) \cap D(Q_1)$ belongs to Q_1. By Lemma 1.2.3, it follows that $\overline{NE}(X, D(Q_i)) = Q_1 + Q_i$. If Q_i belongs to a pair of the type $\mathfrak{B}_2$, the angle $Q_1 + Q_i$ contains another extremal ray (different from Q_1 and Q_i), which is impossible.

Let H be a *nef* element orthogonal to the ray Q_1. Let $0 \neq C_i \in Q_i$. Let us consider a map

$$H \to H' = H + \sum_{i=2}^{m} (-(H \cdot C_i)/(C_i \cdot D(Q_i)))D(Q_i). \qquad \text{(2-1-1)}$$

It is a linear map of the set of *nef* elements H orthogonal to Q_1 into the set of *nef* elements H' orthogonal to the rays $Q_1, Q_2, ..., Q_m$. Here the element H' is *nef* because $C \in Q_1 + Q_i$ for any curve $C \subset D(Q_i)$ if $2 \leq i \leq m$. The kernel of the map (2-1-1) has the dimension $m - 1$. It follows that the rays $Q_1, Q_2, ..., Q_m$ belong to a face of $\overline{NE}(X)$ of a dimension $\leq m$. On the other hand, multiplying rays $Q_1, ..., Q_m$ on the divisors $D(Q_1), ..., D(Q_m)$, one can see very easily that the rays $Q_1, ..., Q_m$ are linearly independent. Thus, they generate a m-dimensional face of $\overline{NE}(X)$. Let us show that this face is $Q_1 + Q_2 + ... + Q_m$. To prove this, we show that every $m - 1$ subset of $\mathcal{E}$ is contained in a face of $\overline{NE}(X)$ of a dimension $\leq m - 1$.

If this subset contains the ray Q_1, this subset has the type $\mathfrak{C}_{m-1}$. By induction, we can suppose that this subset belongs to a face of $\overline{NE}(X)$ of the dimension $m-1$. Let us consider the subset $\{Q_2, Q_3, ..., Q_m\}$. Let H be an ample element of X. For the element H, the map (2-1-1) gives an element H' which is orthogonal to the rays $Q_2, ..., Q_m$, but is not orthogonal to the ray Q_1. It follows that the set $\{Q_2, ..., Q_m\}$ belongs to a face of the Mori polyhedron of the dimension $< m$. Like above (see the proof of Lemma 1.2.6), one can see that for a general H orthogonal to Q_1 the element H' has $(H')^3 \geq H^3 > 0$.

The last property follows from the construction.

We use the next statement quite often:

Lemma 2.1.3. *Let X be a 3-fold from the class $\mathcal{LT}$ (see Definition 1.3.1). Let R be a divisorial extremal ray of the type (II), and D_1, D_2 are two different irreducible Weil divisors different from $D(R)$.*

Then the curve $D_1 \cap D_2$ does not belong to R.

Proof. Let $f : X \to X'$ be the contraction of the ray R. If $D_1 \cap D_2$ belongs to R, then intersection of Weil divisors $f(D_1)$ and $f(D_2)$ is zero-dimensional. It is impossible if X' is $\mathbb{Q}$-factorial.

We denote by $\mathbf{P}C$ the projectivization of a cone C with the beginning at zero. Let R be an extremal ray of the type (II). By Lemma 1.2.3, the projectivization $\mathbf{P}\overline{NE}(X, D(R))$ is an interval with one of its endpoints $\mathbf{P}R$.

Considering lines generated by these intervals we use the following well-known and elementary

Proposition 2.1.4. *Let S be a set of lines such that any two lines of S have a common point.*

Then there are two cases:

(a) There exists a 2-dimensional plane Π such that each line of S belongs to Π.

(b) There exists a point P which belongs to each line of S.

The Π and P are unique if S has at least two different lines.

Applying this statement, we get

Lemma 2.1.5. *Let $\mathcal{S}$ be a set of divisorial extremal rays of the type (I) or (II) such that divisors $D(R_i)$, $D(R_j)$ have a common point for any two extremal rays from $\mathcal{S}$ (equivalently, R_i and R_j are connected by non-single, single arrow or dotted line). Let us suppose that $\mathcal{S}$ contains at least 3 elements. Then S has one of the following types:*

(a) $\dim[\mathcal{S}] = 3$*;*

(b) Angles $\overline{NE}(X, D(R_i))$, $R_i \in \mathcal{S}$, have a common ray (not necessarily extremal) Q.

Let P be a set of divisorial extremal rays. We say that P is *divisorially connected* if there does not exist a decomposition $P = P_1 \cup P_2$ such that both P_1 and P_2 are non-empty and for any $R \in P_1$ and any $Q \in P_2$ divisors $D(R)$ and $D(Q)$ do not have a common point. It defines *divisorially connected components* of a set of extremal rays. Also, we can say what does it mean that two sets P_1 and P_2 of divisorial extremal rays are *divisorially joint*: this means that there exist extremal rays $Q_1 \in P_1$ and $Q_2 \in P_2$ such that divisors $D(Q_1)$ and $D(Q_2)$ have a common point (in particular, this divisors or even extremal rays Q_1, Q_2 may coincide).

Definition 2.1.6. A divisorial extremal ray R is called *special* if R satisfies one of conditions:

(1) R has the type (I);

(2) R belongs to a pair of extremal rays of the type $\mathfrak{B}_2$;

(3) There exists an extremal ray Q such that exactly one pair RQ or QR is an arrow: thus, either $R\cdot D(Q)>0$ and $Q\cdot D(R)=0$ or $Q\cdot D(R)>0$ and $R\cdot D(Q)=0$ (equivalently, the set $\{R,Q\}$ has the type $\mathfrak{C}_2$).

We have the following description of the special set:

Theorem 2.1.7. *Assume that a 3-fold X belongs to the class $\mathcal{LT}$.*

Then the set of special divisorial extremal rays is finite. Its divisorially connected component is one of the following :

$\mathfrak{A}_1$*: One extremal ray Q of the type (I).*

$\mathfrak{B}_2$*: Two different extremal rays Q_{11},Q_{12} of the type (II) with the same divisor $D(Q_{11})=D(Q_{12})$. Then $Q_{11}+Q_{12}$ is the 2-dimensional face of $\overline{NE}(X)$ of Kodaira dimension 3.*

$\mathfrak{C}_n$*, $n\ge 2$: n extremal rays $Q_1,...,Q_n$ of the type (II), such that divisors $D(Q_2),...,$ $D(Q_n)$ do not have a common point, and Q_i and Q_1 are joint by the single arrow Q_iQ_1 . For this case, $Q_1+\cdots+Q_n$ is the n-dimensional face of $\overline{NE}(X)$ of Kodaira dimension 3.*

$\mathfrak{B}_2\mathfrak{C}_n$*, $n\ge 1$: $n+1$ extremal rays $\{Q_{11},Q_2,...,Q_n,Q_{12}\}$ of the type (II), such that $Q_1=Q_{12},...,Q_n$ define the configuration $\mathfrak{C}_n$ (in notation above), and Q_{11},Q_{12} is the pair of the type $\mathfrak{B}_2$. The extremal rays Q_i,Q_{12} are connected by non-single arrows for $i\ge 2$. For this case, $Q_{11}+\cdots+Q_n$ is the face of $\overline{NE}(X)$ of dimension n and Kodaira dimension 3, and $Q_{11}+Q_{12}$ is the 2-dimensional face of $\overline{NE}(X)$ of Kodaira dimension 3.*

$\mathfrak{T}_3$ *(triangle): Three extremal rays Q_1,Q_2,Q_3 of the type (II) connected by single arrows Q_1Q_2, Q_2Q_3 and non-single arrows Q_1Q_3 and Q_3Q_1. For this case, $\overline{NE}(X,D(Q_1))=Q_1+Q_2$ and $\overline{NE}(X,D(Q_2))=Q_2+Q_3$ are 2-dimensional faces of $\overline{NE}(X)$ of Kodaira dimension 3. And $\overline{NE}(X,D(Q_3))=T+Q_3$ where $T\subset\overline{NE}(X,D(Q_1))=Q_1+Q_2$.*

$\mathfrak{T}_3'$ *(special triangle): Three extremal rays Q_1,Q_2,Q_3 of the type (II) connected by single arrows Q_1Q_2, Q_2Q_3 and Q_3Q_1. For this case, $\overline{NE}(X,D(Q_1))=Q_1+Q_2$, $\overline{NE}(X,D(Q_2))=Q_2+Q_3$ and $\overline{NE}(X,D(Q_3))=Q_3+Q_1$ are 2-dimensional faces of $\overline{NE}(X)$ of Kodaira dimension 3.*

Proof. Assume that extremal rays Q_1,Q_2 of the type (II) are joint by the single arrow Q_1Q_2, and extremal rays R_1,R_2 of the type (II) are joint by the single arrow R_1R_2. Let us suppose that the sets $\{Q_1,Q_2\}$ and $\{R_1,R_2\}$ are divisorially joint. We consider possible cases below.

Suppose that Q_1,R_1 are divisorially disjoint.

If $Q_2=R_2$, we get the configuration $\mathfrak{C}_3$. We prove that the only case is possible.

Assume that $Q_2\neq R_2$. Since the curve $D(R_1)\cap D(R_2)$ belongs to R_2, we have $R_2\cdot D(Q_1)=0$ (because $D(R_1)\cap D(Q_1)=\emptyset$). Similarly, $Q_2\cdot D(R_1)=0$.

Let us suppose that $D(Q_2)=D(R_2)=D$. Then any curve of the divisor D belongs to R_2+Q_2 by Lemma 1.2.4. Since $Q_2\cdot D(R_1)=R_2\cdot D(R_1)=0$, divisors $D(R_2)$ and $D(R_1)$ do not have a common curve. We get a contradiction, because, R_1R_2 is an arrow.

Thus, we can suppose that $D(Q_2)\neq D(R_2)$.

Let us suppose that R_1Q_2 is an arrow. We have proved that Q_2R_1 is not an arrow. It follows that the divisor $D(R_1)$ contains curves of 3 extremal rays: R_1, R_2, Q_2 which is impossible. Thus, R_1Q_2 and Q_2R_1 are not arrows. It follows that divisors $D(R_1)$ and $D(Q_2)$ are disjoint. Similarly, the divisors $D(R_2)$ and $D(Q_1)$ are disjoint. Since the curve $D(R_1) \cap D(R_2)$ belongs to R_2, we get that $R_2 \cdot D(Q_2) = 0$. Similarly, $Q_2 \cdot D(R_2) = 0$. It follows that divisors $D(R_2)$ and $D(Q_2)$ are divisorially disjoint. It follows that sets $\{R_1, R_2\}$ and $\{Q_1, Q_2\}$ are divisorially disjoint. We get a contradiction.

Thus, we proved that if the rays Q_1 and R_1 are divisorially disjoint, then $Q_2 = R_2$, and we have the configuration $\mathfrak{C}_3$.

Now, assume that Q_1 and R_1 are divisorially joint. Then, $D(Q_1) \cap D(R_1)$ is non-empty.

If $D(Q_1) = D(R_1)$, evidently, sets $\{Q_1, Q_2\}$ and $\{R_1, R_2\}$ are equal because the divisor $D(Q_1) = D(R_1)$ cannot contain 3 extremal rays by Lemma 1.2.3.

Thus, we assume that $D(Q_1) \cap D(R_1)$ is a non-empty curve. By Lemma 2.1.2, $\overline{NE}(X, D(Q_1)) = Q_1 + Q_2$ and $\overline{NE}(X, D(R_1)) = R_1 + R_2$ are 2-dimensional faces of $\overline{NE}(X)$. It follows that faces $Q_1 + Q_2$ and $R_1 + R_2$ have a common extremal ray.

Thus, sets $\{Q_1, Q_2\}$ and $\{R_1, R_2\}$ have a common extremal ray.

Assume that $Q_2 = R_1$ (the case $Q_1 = R_2$ is similar). We denote $Q_3 = R_2$. Thus, we have 3 extremal rays Q_1, Q_2, Q_3 where Q_1, Q_2 are joint by a single arrow Q_1Q_2 and Q_2, Q_3 are joint by a single arrow Q_2Q_3. The curve $D(Q_1) \cap D(Q_2)$ belongs to the ray Q_2 and $Q_2 \cdot D(Q_3) > 0$. it follows that divisors $D(Q_1)$ and $D(Q_3)$ have a non-empty common curve. The extremal rays Q_1 and Q_3 cannot be joint by a single arrow Q_1Q_3 since then $D(Q_1)$ contains curves of 3 different extremal rays Q_1, Q_2, Q_3. Thus, the extremal rays Q_1, Q_3 are joint either by non-single arrows Q_1Q_3 and Q_3Q_1 or by the single arrow Q_3Q_1. Thus, we get cases $\mathfrak{T}_3$ and $\mathfrak{T}_3'$.

Now suppose that $Q_2 = R_2$. We denote $R_1 = Q_3$. Thus, we have 3 extremal rays Q_1, Q_2, Q_3 such that Q_1, Q_2 are joint by a single arrow Q_1Q_2, and Q_3, Q_2 are joint by a single arrow Q_3Q_2. If Q_1 and Q_3 are divisorially disjoint, we get the case $\mathfrak{C}_3$.

Suppose that $D(Q_1) \cap D(Q_3)$ is a non-empty curve. By Lemma 2.1.2, $\overline{NE}(X, D(Q_1)) = Q_1 + Q_2$ and $\overline{NE}(X, D(Q_3)) = Q_3 + Q_2$ are 2-dimensional faces of $\overline{NE}(X)$. The set of common points of this faces is the extremal ray Q_2. It follows that the curve $D(Q_1) \cap D(Q_3)$ belongs to the ray Q_2. By Lemma 2.1.3, this is impossible.

Thus, we had proved that divisors $D(Q_1)$ and $D(Q_3)$ do not have a common point.

As a result, we have proved that any configuration of two single arrows which are divisorially joint satisfies the statement of Theorem. Thus this is either $\mathfrak{C}_3$ or $\mathfrak{T}_3$ or $\mathfrak{T}_3'$.

Suppose that there exists a single arrow R_1R_2 which is divisorially joint with a configuration $\mathfrak{T}_3$ or $\mathfrak{T}_3'$ of extremal rays and different from arrows of this configurations. By the result above, the set $\{R_1, R_2\}$ has a common extremal ray with both sets $\{Q_1, Q_2\}$ and $\{Q_2, Q_3\}$. Thus, either $Q_2 = R_1$ or $Q_2 = R_2$. If $Q_2 = R_1$, the divisor $D(R_1)$ contains 3 extremal rays: R_1, R_2, Q_3 which is impossible. If $Q_2 = R_2$, then R_1, R_2, Q_3 give a configuration $\mathfrak{T}_3$ or $\mathfrak{T}_3'$. Then the angle $\overline{NE}(X, D(Q_3))$ has a side which simultaneously is a ray of the $Q_1 + Q_2$ and $R_1 + Q_2$. Thus, this side is $Q_{[illegible]}$. Then $D(Q_{[illegible]}) = D(Q_{[illegible]})$ which is impossible.

From above, it follows that a divisorially connected component S of a finite non-empty configuration of single arrows of extremal rays of the type (II) has the type either $\mathfrak{C}_n, n \geq 2$ or $\mathfrak{T}_3$ of $\mathfrak{T}_3'$. Using Lemma 2.1.2, one can easily see that if an extremal ray Q of this connected component S belongs to a pair Q, R of the type $\mathfrak{B}_2$, then S has the type $\mathfrak{C}_n, n \geq 2$ and $Q = Q_1$ in notation of Theorem. Thus, $S \cup \{R\}$ gives rise a configuration of the type $\mathfrak{B}_2\mathfrak{C}_n$.

Now suppose that we have two sets $\{Q_1, Q_2\}$ and $\{R_{11}, R_{12}\}$ of extremal rays such that Q_1Q_2 is a single arrow and $R_{11}R_{12}$ is a dotted line, and this two sets are divisorially joint. If these two sets have a common extremal ray, then they give a configuration $\mathfrak{B}_2\mathfrak{C}_2$ since above considerations.

Thus, let us assume that these two sets do not have a common extremal ray. By Lemmas 2.1.2 and 1.2.6, $\overline{NE}(X, D(Q_1)) = Q_1 + Q_2$ and $\overline{NE}(X, D(R_{11})) = R_{11} + R_{12}$ are 2-dimensional faces of $\overline{NE}(X)$. It follows that if divisors $D(Q_1), D(R_{11})$ have a common point, then sets $\{Q_1, Q_2\}$ and $\{R_{11}, R_{12}\}$ have a common extremal ray. But we assume that this is not the case. Thus, we can suppose that divisors $D(Q_1)$ and $D(R_{11})$ do not have a common point. The curve $D(Q_1) \cap D(Q_2)$ belongs to the ray Q_2. Since $D(Q_1) \cap D(R_{11}) = \emptyset$, then $Q_2 \cdot D(R_{11}) = 0$. Thus, $R_{11}Q_2$ is a single arrow. This is impossible by Lemma 2.1.2. Thus, we get a contradiction.

Thus, we proved that a finite configuration of single arrows and dotted lines has divisorially connected components of the type $\mathfrak{B}_2, \mathfrak{C}_n, \mathfrak{B}_2\mathfrak{C}_n, \mathfrak{T}_3, \mathfrak{T}_3'$ of Theorem.

Now let R be an extremal ray of the type (I). Let Q_1Q_2 be a single arrow. If $D(Q_1) \cap D(R)$ is non-empty, then $D(Q_1)$ contains 3 extremal rays Q_1, Q_2 and R which is impossible. The curve $D(Q_1) \cap D(Q_2)$ belongs to Q_2. From above, $Q_2 \cdot D(R) = 0$. Thus, RQ_2 is a single arrow which is impossible for an extremal ray R of the type (I). Thus, $\{R\}$ and $\{Q_1, Q_2\}$ are divisorially disjoint.

By Lemma 1.2.5 and considerations above, we proved that any finite set of special extremal rays has connected components of the Theorem. Let us suppose that this set S has n extremal rays. We claim that then $n \leq 2\rho(X)$ where $\rho(X) = \dim N_1(X)$. From the description of connected components of S, for any connected component S_i of S with m_i elements there exists at least $\max\{1, m_i - 1\} \geq m_i/2$ extremal rays with disjoint divisors. Since any extremal ray $R \in S$ has $R \cdot D(R) < 0$, it follows the inequality above.

All other statements of Theorem follow from Lemmas 1.2.6 and 2.1.2.

This finishes the proof of Theorem.

Now let us consider a divisorial extremal ray S which is not a special one but the divisor $D(S)$ has a common point with the divisor of one of special divisorial extremal rays. Thus, S is joint by non-single arrows with a special divisorial extremal ray. These extremal rays also look very specially. We describe them below.

Theorem 2.1.8. *Let X be a 3-fold from the class $\mathcal{LT}$. Let S be a non-special divisorial extremal ray of X. Let Z_S be the set of special divisorial extremal rays which are divisorially joint with S (by definition, S has the type (II) and is joint by non-single arrows with all extremal rays from Z_S). Assume that $Z_S \neq \emptyset$.*

Then Z_S is contained in one of divisorially connected components $C = C_S$ of the set of special extremal rays, and depending to their types of Theorem 2.1.7, we have the following description for $Z_S \subset C_S$:

($\mathfrak{A}_1(S)$) C_S has the type $\mathfrak{A}_1$, $Z_S = Q$;

($\mathfrak{C}^{(1)}_n(S)$) C_S has the type $\mathfrak{C}_n$, $Z_S = \{Q_i\}$, for one of $i \geq 1$;

($\mathfrak{C}_2^{(2)}(S)$) C_S has the type $\mathfrak{C}_2$, $Z_S = \{Q_1, Q_2\}$;
($\mathfrak{B}_2\mathfrak{C}_n(S)$) C_S has the type $\mathfrak{B}_2\mathfrak{C}_n$, $n > 1$, $Z_S = \{Q_i\}$ for one of $i > 1$;
($\mathfrak{B}_2(S)$) C_S has the type $\mathfrak{B}_2$, $Z_S = \{Q_{11}, Q_{12}\}$.
We remark that it follows that C_S cannot have the type $\mathfrak{T}_3$ and $\mathfrak{T}_3'$.

Proof. We consider one case only. Similarly, one can consider all other cases.

Let us assume that S is divisorially joint with a divisorially connected component of the type $\mathfrak{C}_n$, $n \geq 2$, of the set of special extremal rays. Thus, $D(S)$ has a common curve with one of divisors $D(Q_i)$, $i = 1, ..., n$ (we use notation of Theorem 2.1.7).

Let us suppose that $D(S)$ has a common curve with the divisor $D(Q_1)$. Then S and Q_1 are joint by non-single arrows. Thus, $Q_1 \cdot D(S) > 0$. Since $D(Q_i) \cap D(Q_1)$ belongs to Q_1, we get that $D(S)$ has a common curve with all divisors $D(Q_i)$, $i = 2, ..., n$, too. By Lemma 2.1.2, $\overline{NE}(X, D(Q_i)) = Q_1 + Q_i$ is a 2-dimensional face of $\overline{NE}(X)$. We then get that the 2-dimensional angle $\overline{NE}(X, D(S))$ has the second side (different from S) which belongs to the face $Q_1 + Q_i$, $i = 2, ..., n$. If $n > 2$, we then get that the second side of $\overline{NE}(X, D(S))$ is generated by the curve $D(S) \cap D(Q_i)$ which belongs to Q_1. This is impossible by Lemma 2.1.3. Thus, $n = 2$, and we get the case $\mathfrak{C}_2^{(2)}(S)$.

Now, assume that S is not divisorially joint with Q_1, thus $D(S) \cap D(Q_1) = \emptyset$. Then $D(S) \cap D(Q_i)$ is not empty for one of $i > 1$. Then, like above, $\overline{NE}(X, D(S))$ is a 2-dimensional angle with the second side generated by the curve $D(S) \cap D(Q_i)$ which belongs to the 2-dimensional face $\overline{NE}(X, D(Q_i)) = Q_i + Q_1$ of $\overline{NE}(X)$. This curve cannot belong to Q_1 since $D(S) \cap D(Q_1) = \emptyset$. Thus, the second side of the angle $\overline{NE}(X, D(S))$ is the ray of the $Q_i + Q_1$ different from Q_1. It follows that the extremal ray Q_i, $i = 2, ..., n$, such that $D(S) \cap D(Q_i) \neq \emptyset$ is unique. Thus, we get the case $\mathfrak{C}_n^{(1)}(S)$ if we additionally prove that S is divisorially disjoint with all special extremal rays different from $Q_1, ..., Q_n$.

By Theorem 2.1.7, any special extremal ray Q of the type (II) together with another special extremal ray Q' of the same component of the set of special extremal rays generate a 2-dimensional face $Q + Q'$ of $\overline{NE}(X)$, and this 2-dimensional face is either $\overline{NE}(X, D(Q))$ or $\overline{NE}(X, D(Q'))$ (for the extremal ray Q of the type (I), one should use that $\overline{NE}(X, D(Q)) = Q$). Using this property, like above, one can see that $D(S) \cap D(Q)$ is empty for any special extremal ray Q different from extremal rays $Q_1, ..., Q_n$ above. One can similarly consider cases when S is divisorially joint with other types of connected components of the set of special extremal rays.

Now, we have the following description for cases when $S \neq S'$ but $Z_S \cap Z_{S'} \neq \emptyset$. These cases are very rare.

Theorem 2.1.9. *Let X be a 3-fold from the class $\mathcal{LT}$. Let S, S' are non-special divisorial extremal rays such that $Z_S \cap Z_{S'} \neq \emptyset$.*

Then $S = S'$ except cases below (in notation of Theorems 2.1.7 and 2.1.8):

(1) Case $C_S = C_{S'} = \mathfrak{C}_2$, $Z_S = Z_{S'} = \{Q_1, Q_2\}$, S, S' are joint by non-single arrows, $\dim\,[Q_1, Q_2, S, S'] = 3$.

(2) Case $C_S = C_{S'} = \mathfrak{C}_2$, $Z_S = \{Q_1, Q_2\}$, $Z_{S'} = \{Q_2\}$, S, S' are joint by non-single arrows, $\dim\,[Q_1, Q_2, S, S'] = 3$.

(3) Case $C_S = C_{S'} = \mathfrak{B}_2$, $Z_S = Z_{S'} = \{Q_{11}, Q_{12}\}$, S, S' are joint by non-single arrows, $\dim\,[Q_{11}, Q_{12}, S, S'] = 3$

(4) Case $C_S = C_{S'} = \mathfrak{B}_2$, $Z_S = Z_{S'} = \{Q_{11}, Q_{12}\}$, S, S' are joint by non-single arrows, there exists a ray $T \subset Q_{11} + Q_{12}$ such that $\overline{NE}(X, D(S)) = T + S$ and

$\overline{NE}(X, D(S')) = T + S'$.

Proof. Let $S' \neq S$ but $Z_{S'} \cap Z_S \neq \emptyset$. Using Theorem 2.1.8, we consider all possible cases (we use notation of Theorems 2.1.8 and 2.1.7).

The case $(\mathfrak{A}_1(S))$, $Z_S = Z_{S'} = \{Q\}$. First, remark that $D(S) \cap D(S') \neq \emptyset$ since $D(S), D(S')$ both have non-empty intersection with $D(Q)$ and $Q \cdot D(S) > 0$ and $Q \cdot D(S') > 0$. Besides, $\overline{NE}(X, D(S)) = S + Q$ and $\overline{NE}(X, D(S')) = S' + Q$ are 2-dimensional angles with the common edge Q. If the curve $D(S) \cap D(S')$ has a component which does not belong to Q, then one of this angles is contained in another one which is impossible since S and S' are different extremal rays. Thus, the curve $D(S) \cap D(S')$ belongs to the extremal ray Q. This is impossible by Lemma 2.1.3.

The case $(\mathfrak{C}_n^{(1)}(S))$, C_S has the type $\mathfrak{C}_n$, $Z_S = Z_{S'} = \{Q_i\}$, for one of $i > 1$.

Let $C = D(S) \cap D(Q_i)$. We have $C \cdot D(Q_1) = 0$ because $D(S) \cap D(Q_1) = \emptyset$. Besides, $C \in Q_i + Q_1$ and $Q_i \cdot D(Q_1) > 0$, $Q_1 \cdot D(Q_1) < 0$. Thus, the ray $\mathbf{R}^+ C \subset Q_i + Q_1$ is defined uniquely by the property $\mathbf{R}^+ C \cdot D(Q_1) = 0$.

Now assume that $S \neq S'$ and $Z_{S'} = Z_S = \{Q_i\}$. Like above, $D(S') \cap D(Q_i) = C'$ and $\mathbf{R}^+ C' = \mathbf{R}^+ C$. By Lemma 2.1.2, $\overline{NE}(X, D(Q_i)) = Q_i + Q_1$ is a 2-dimensional face of $\overline{NE}(X)$. It follows that $\overline{NE}(X, D(S)) = \mathbf{R}^+ C + S$ and $\overline{NE}(X, D(S')) = \mathbf{R}^+ C + S'$. If the curve $D(S) \cap D(S')$ has a component which does not belong to $D(Q_i)$, we then get that one of angles $\overline{NE}(X, D(S))$, $\overline{NE}(X, D(S'))$ contains another, which is impossible since S, S' are different extremal rays of $\overline{NE}(X)$. Let us consider the contraction $f : X \to X'$ of the extremal ray Q_i. Then the curve $C = f(D(S)) \cap f(D(S'))$ belongs to the extremal ray $f(Q_1)$ of X'. This is impossible by Lemma 2.1.3.

The case $(\mathfrak{C}_2^{(2)}(S))$, C_S has the type $\mathfrak{C}_2$, $Z_S = \{Q_1, Q_2\}$. For this case, $D(S) \cap D(Q_1)$ contains a component which does not belong to Q_1 (otherwise, SQ_1 is a single arrow). It follows that the 2-dimensional angles $\overline{NE}(X, D(S))$ and $\overline{NE}(X, D(Q_1))$ have a common ray different from the edge Q_1 of the $\overline{NE}(X, D(Q_1))$. Besides, the curve $D(S) \cap D(Q_2)$ is not empty (since SQ_2 is arrow). Thus, the 2-dimensional angle $\overline{NE}(X, D(S))$ has a common ray with the angle $\overline{NE}(X, D(Q_2))$. It follows that the angle $\overline{NE}(X, D(S))$ is contained in the 3-dimensional space V generated by angles $\overline{NE}(X, D(Q_1))$ and $\overline{NE}(X, D(Q_2))$ with the common edge Q_1. The same is valid for any S' with $Z_{S'} = Z_S = \{Q_1, Q_2\}$. This gives the case (1) of the Theorem.

By Theorem 2.3.8, $Z_{S'} = \{Q_2\}$ if $Z_{S'} \cap Z_S \neq \emptyset$ and $Z_{S'} \neq Z_S$. Let $C' = D(S') \cap D(Q_2)$. Since $\overline{NE}(X, D(Q_2)) = Q_2 + Q_1$, $C' \in Q_2 + Q_1$. Since $Q_2 \cdot D(S) > 0, Q_1 \cdot D(S) > 0$, it then follows that $C' \cdot D(S) > 0$ and $D(S') \cap D(S) \neq \emptyset$. Thus, angles $\overline{NE}(X, D(S))$ and $\overline{NE}(X, D(S'))$ have a common ray. If $\overline{NE}(X, D(S)) \cap \overline{NE}(X, D(S')) = \mathbf{R}^+ C'$, like above, considering the contraction of the extremal ray Q_2, we get the contradiction with Lemma 2.1.3. Thus, $\overline{NE}(X, D(S)) \cap \overline{NE}(X, D(S'))$ contains a ray which is different from $\mathbf{R}^+ C'$. It follows that the angle $\overline{NE}(X, D(S'))$ is contained in the 3-dimensional space V above containing extremal rays Q_1, Q_2, S. This gives the case (2) of the Theorem.

The case $(\mathfrak{B}_2\mathfrak{C}_n(S))$, C_S has the type $\mathfrak{B}_2\mathfrak{C}_n$, $n > 1$, $Z_S = Z_{S'} = \{Q_i\}$ for one of $i > 1$. Like above for the case $\mathfrak{C}_n$, this case is impossible for $S \neq S'$.

The case $(\mathfrak{B}_2(S))$, C_S has the type $\mathfrak{B}_2$, $Z_S = \{Q_{11}, Q_{12}\}$: By Lemma 1.2.4, $\overline{NE}(X,\ D(Q_{11})) = Q_{11} + Q_{12}$ is a face of $\overline{NE}(X)$. It follows that $\overline{NE}(X, D(S)) = T + S$ where T is a ray of the angle $Q_{11} + Q_{12}$. Let S' be another extremal ray with $Z_{S'} = \{Q_{11}, Q_{12}\}$. Since $Q_{11} \cdot D(S') > 0, Q_{12} \cdot D(S') > 0$, we have $T \cdot D(S') > 0$. It follows that S, S' are connected by non-single arrows. Applying Lemma 2.1.5, we get cases (3) and (4) of the Theorem. This finishes the proof.

2.2. Extremal and E-sets of divisorial extremal rays.

We want to apply results of Section 2.1 to describe extremal and E-sets of extremal rays of the type (I) and (II) for 3-folds of the class $\mathcal{LT}$. But it is important for future studies considering more general elliptic, connected parabolic and Lanner subsets of divisorial extremal rays which had in fact appeared in Section 1.2.

Definition 2.2.1. Let $\mathcal{E} = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (I) or (II). Then $\mathcal{E}$ is called *elliptic* if there are $a_1 > 0, ..., a_n > 0$ such that

$$R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) < 0$$

for all $1 \le i \le n$.

Definition 2.2.2. Let $\mathcal{P} = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (I) or (II). Then $\mathcal{P}$ is called *connected parabolic* if each proper subset of $\mathcal{P}$ is elliptic and there are $a_1 > 0, ..., a_n > 0$ such that

$$R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) = 0$$

for all $1 \le i \le n$.

A set Q of extremal rays of the type (I) or (II) is called *parabolic* if each divisorially connected component of Q is connected parabolic.

Definition 2.2.3. Let $\mathcal{L} = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (I) or (II). Then $\mathcal{L}$ is called *Lanner* (respectively, *quasi-Lanner*) if each proper subset of $\mathcal{L}$ is elliptic (respectively, either elliptic or parabolic) and there are $a_1 > 0, ..., a_n > 0$ such that

$$R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) \ge 0$$

for all $1 \le i \le n$ and there exists $j,\ 1 \le j \le n$, such that

$$R_j \cdot (a_1 D(R_1) + \cdots a_n D(R_n)) > 0.$$

The following statement will be useful.

Proposition 2.2.4. *Let $\mathcal{E} = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (I) or (II) with different divisors $D(R_1), ..., D(R_n)$. Then $\mathcal{E}$ is elliptic if and only if for any non-negative $b_1, ..., b_n$ not all equal to 0 there exists $R_i,\ 1 \le i \le n$, such that*

$$R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) < 0.$$

Proof. If $\mathcal{E}$ is elliptic, then $\mathcal{E}$ satisfies the condition above by Lemma 1.2.10.

Now assume that $\mathcal{E}$ satisfies the condition of Proposition. If $\mathcal{E}$ is not elliptic, there exists a minimal non-elliptic subset $\mathcal{E}' \subset \mathcal{E}$. The subset $\mathcal{E}'$ is not empty since any one element subset of $\mathcal{E}$ is elliptic. Let $\mathcal{E}' = \{R_1, ..., R_t\}$ where $1 \le t \le n$. By Lemma 1.2.10, there are positive $c_1, ..., c_t$ such that

$$R_i \cdot (c_1 D(R_1) + \cdots + c_t D(R_t)) \ge 0$$

for any $1 \le i \le t$. The same inequality is evidently true for $t+1 \le i \le n$. We get a contradiction with the condition of Proposition.

Definition 2.2.5. A sequence

$$C = \{R_1, ..., R_n\},$$

$n \geq 1$, of extremal rays of the type (I) or (II) is called *chain* if R_i, R_j are divisorially disjoint for $j - i > 1$, and R_iR_{i+1} is a non-single arrow (thus, $R_{i+1}R_i$ is an arrow too) for any $1 \leq i < n$. Extremal rays R_1, R_n are called terminal for the chain.

Theorem 2.2.6. *Let X be a 3-fold from the class $\mathcal{LT}$. Let T be a divisorially connected set of extremal rays of the type (I) or (II) such that any proper subset of T is elliptic. (In particular, this is valid if either T itself is elliptic or is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3 or T is connected parabolic or Lanner or T is an E-set such that each proper subset of T is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3). Let us assume that each two extremal rays of T are different from a pair of the type $\mathfrak{B}_2$. Then T has one of the types :*

(A) All extremal rays of T have the type (II) and T does not have a single arrow.

(B) $T = \{R\} \cup C_1 \cup ... \cup C_k$ has only extremal rays of the type (II), where

$$C_1 = \{R_{11}, ..., R_{1n_1}\},\ C_2 = \{R_{21}, ..., R_{2n_2}\}, ..., C_k = \{R_{k1}, ..., R_{kn_k}\}$$

are divisorially disjoint to one another chains, and all arrows between R and extremal rays of these chains are single arrows $R_{j1}R,\ j = 1, ..., k$; besides each extremal ray of the chains $C_1, ..., C_k$ does not belong to a pair of the type $\mathfrak{B}_2$.

(C) $T = \{R_1, R_2, R_3, ..., R_n\},\ n \geq 3$, where all extremal rays of T have the type (II) and R_2R_1 is a single arrow, R_1R_2, R_2R_1 and R_2R_3, R_3R_2 are non-single arrows, $R_3, ..., R_n$ is a chain such that $R_4, ..., R_n$ are divisorially disjoint with extremal rays R_1 and R_2; besides, each extremal ray $R_1, R_2, R_3, ..., R_n$ does not belong to a pair of the type $\mathfrak{B}_2$.

(D) $T = \{R_1, R_2, ..., R_i, ..., R_n\}$ is a chain such that R_1 has the type (I), and R_i has the type (II) for all $i > 1$; besides, each extremal ray $R_1, R_2, ..., R_n$ does not belong to a pair of the type $\mathfrak{B}_2$.

(E) $T = \{Q_1, Q_2, Q_3\}$ has type of triangle: thus, Q_1Q_2 and Q_2Q_3 are single arrows, and Q_1Q_3, Q_3Q_1 are non-single arrows; besides, each extremal ray Q_1, Q_2, Q_3 does not belong to a pair of the type $\mathfrak{B}_2$.

(E') $T = \{Q_1, Q_2, Q_3\}$ has type of special triangle: thus, $Q_1Q_2,\ Q_2Q_3,\ Q_3Q_1$ are single arrows; besides, each extremal ray Q_1, Q_2, Q_3 does not belong to a pair of the type $\mathfrak{B}_2$.

Moreover, for the case (B), the T is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3. Thus, the case (B) is impossible when T is not contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3.

Proof. If T does not have an extremal ray of the type (I) and a single arrow, we get the case (A).

Thus, we can suppose that T has either an extremal ray of the type (I) or a single arrow.

Let T_1 be the set which contains all extremal rays of the type (I) and all extremal rays of single arrows of T. If $T = T_1$, then T_1 is divisorially connected, and by Theorem 2.1.7, we get one of following cases: (B) with one element chains, (D) with one extremal ray R_1, (E) or (E').

Let us suppose that $T \neq T_1$. By Theorem 2.1.7, T_1 has divisorially connected components of the types $\mathfrak{A}_1$, $\mathfrak{C}_2$, $\mathfrak{T}$, or $\mathfrak{T}'$. Since T is divisorially connected, for each connected component of T_1 there exists an extremal rays $Q \in T - T_1$ such that Q is joined by non-single arrows with an extremal ray of this connected component. By Theorem 2.1.8, then this connected component should have the type $\mathfrak{A}_1$ or $\mathfrak{C}_n$. It follows that all connected components of T_1 have the type $\mathfrak{A}_1$ or $\mathfrak{C}_n$, and T_1 has a connected component of one of these types.

We consider induction on $\#T$. For $\#T = 1, 2$ the statement is clear. Let us suppose that $\#T = n > 2$. We consider several cases.

Let us suppose that there exists an extremal ray $R \in T$ of the type (I). Let $Q \in T$ and $Q \neq R$. We claim that then $Q + R$ is a 2-dimensional face of Kodaira dimension 3 of $\overline{NE}(X)$. At first, suppose that $D(R) \cap D(Q) = \emptyset$. Let H be a *nef* element orthogonal to Q (the set of these elements H defines a face of codimension one in $NEF(X)$ by the exact sequence (1-2-1)). There exists a linear function $\alpha(H) \geq 0$ such that the map $H \to H' = H + \alpha(H)D(R)$ is linear with one-dimensional kernel, and H' is orthogonal to extremal rays Q and R. The H' is evidently *nef* since R has the type (I) and $C \in R$ for any curve $C \subset D(R)$. Thus, $Q + R$ is a 2-dimensional face of $\overline{NE}(X)$, and, by construction, the face $(Q+R)^\perp$ of $NEF(X)$ has codimension 2. Evidently, $(H')^3 \geq H^3 > 0$ (see the proof of Lemma 1.2.6). Thus, by definition of the class $\mathcal{L}T$, this face has Kodaira dimension 3. (Further, we will not be so formal, and just show that the corresponding face α of $\overline{NE}(X)$ has numerical Kodaira dimension 3; automatically, by our construction, it will have the property $\operatorname{codim} \alpha^\perp = \dim \alpha$ and will have Kodaira dimension 3 by definition of the class $\mathcal{L}T$). Now, suppose that $D(Q) \cap D(R) \neq \emptyset$. Since $\#T = n > 2$, the set $\{Q, R\}$ is elliptic. Thus, there are positive a_1, a_2 such that $Q \cdot (a_1 D(Q) + a_2 D(R)) < 0$, $R \cdot (a_1 D(Q) + a_2 D(R)) < 0$. Let H be an ample element. By Lemma 2.1.8, there are positive c_1, c_2 such that $H' = H + c_1 D(Q) + c_2 D(R)$ is orthogonal to both Q, R. Since R has the type (I) and $D(Q) \cap D(R) \neq \emptyset$, we then get $\overline{NE}(X, D(Q)) = Q + R$. It follows that $H' \cdot C \geq 0$ for any curve C. Thus, H' is *nef*. Evidently, H' is orthogonal to extremal rays Q, R only and $H' \geq H^3 > 0$. It follows that $Q + R$ is a face of dimension 2 and Kodaira dimension 3 of $\overline{NE}(X)$. Thus, we proved that $Q + R$ is a 2-dimensional face of $\overline{NE}(X)$ for any $Q \in T$ and $Q \neq R$.

Let us consider the contraction $f : X \to X'$ of the extremal ray R. Then the image $f(Q)$ is an extremal ray of X' for $Q \neq R$ since the claim we proved above. Evidently, if $D(Q) \cap D(R) = \emptyset$, the extremal ray $f(Q)$ has the same type ((I) or (II)) as Q. If $D(Q) \cap D(R) \neq \emptyset$, the $f(Q)$ has the type (I). By Theorem 2.1.9, the extremal ray Q with this property is unique. Since T is connected, the Q does exist. By the projection formula and Proposition 2.2.4, any proper subset of the set $f(T) = f(T - \{R\})$ is elliptic. It follows that the set $f(T)$ has the same properties as T but has one element less. By induction, we then get that $f(T)$ has the type (D). Considering preimages, one can easily see that then T has the same type (D) and has desirable properties.

Now, assume that $R_2 R_1$ is a single arrow for $R_2, R_1 \in T$. At first, let us suppose that there does not exist $Q \in T$ such that Q is joined by non-single arrows with the terminal R_1 of the arrow $R_2 R_1$. Thus, $R_1 \cdot D(Q) = 0$ for all $Q \in T$, $Q \neq R_1$. By Lemma 2.1.2, $\overline{NE}(X, D(R_2)) = R_1 + R_2$ is a 2-dimensional face of $\overline{NE}(X)$ of Kodaira dimension 3. It follows that $\overline{NE}(X, D(Q)) = P + Q$, $P \subset R_1 + R_2$ for an extremal ray Q which is joined by non-single arrows with R_2. By Lemma 2.1.9, this extremal ray Q is unique if it does exist. Then, like above, we can prove that

R_2, Q generate a 2-dimensional face $R_2 + Q$ of Kodaira dimension 3 of $\overline{NE}(X)$ for each $Q \in T,\ Q \neq R_2$ (for $Q = R_1$, this follows from Lemma 2.1.2). Let us consider the contraction $f : X \to X'$ of the extremal ray R_2. Then $f(Q)$ is a divisorial extremal ray of X' for $Q \in T,\ Q \neq R_2$. If there exists Q such that QR_2 is a non-single arrow, then $f(Q)f(R_1)$ is evidently a single arrow because Q and R_1 are divisorially disjoint by Lemma 2.1.8. If there does not exist Q such that QR_2 is a non-single arrow then there exists an extremal ray $S \in T$ such that SR_1 is a single arrow and S is divisorially disjoint with R_2. This follows from Lemmas 2.1.7, 2.1.8 and conditions on T above (otherwise, $T = \{R_2, R_1\}$). Then $f(S)f(R_1)$ is a single arrow. Thus, the image $f(T)$ is a divisorially connected set of extremal rays of the type (I) and (II) which contains a single arrow and like for the case above any proper subset of $f(T)$ is elliptic. By induction, the $f(T)$ has the type (B) and only the extremal ray $f(R_1)$ may belong to a pair of the type $\mathfrak{B}_2$. One can see easily using our construction, that then T has also the type (B) and only R_1 may belong to a pair of the type $\mathfrak{B}_2$.

Now we consider the case when R_2R_1 is a single arrow for $R_2, R_1 \in T$ and there exists $R_3 \in T$ such that R_3R_1, R_1R_3 are non-single arrows. This is the case $\mathfrak{C}_2^{(2)}(R_3)$ of Theorem 2.1.8. By Theorem 2.1.8, then R_3R_2, R_2R_3 are non-single arrows and R_2R_1 is the only single arrow in T with the terminal R_1.

Since $\#T \geq 3$ and any proper subset of T is elliptic, there are $a_2 > 0, a_3 > 0$ such that $R_2 \cdot (a_2D(R_2) + a_3D(R_3)) < 0$ and $R_3 \cdot (a_2D(R_2) + a_3D(R_3)) < 0$. Let H be an ample element. By Lemma 1.2.8, there exist $b_2 > 0,\ b_3 > 0$ such that $H' = H + b_2D(R_2) + b_3D(R_3)$ is orthogonal to both R_2, R_3. By Lemma 2.1.2, $\overline{NE}(X, D(R_2)) = R_1 + R_2$. It follows that $C \cdot H' \geq 0$ for any curve $C \subset D(R_2)$. We have $D(R_2) \cap D(R_3)$ is a non-empty curve. By Lemma 2.1.2 $\overline{NE}(X, D(R_2)) = R_1 + R_2$ is a face of $\overline{NE}(X)$. It follows that $\overline{NE}(X, D(R_3)) = S + R_3$ where S is a ray of $R_1 + R_2$. If follows that $C \cdot H' \geq 0$ for any curve $C \in D(R_3)$. This implies that H' is nef and only extremal rays R_2, R_3 are orthogonal to H'. Besides, like above, $(H')^3 \geq H^3 > 0$. Thus, $R_2 + R_3$ is a 2-dimensional face of $\overline{NE}(X)$ of Kodaira dimension 3.

As we have mentioned above, $\overline{NE}(X, D(R_3)) = S + R_3$, where $S \in R_1 + R_2$. Here $S \neq R_1$ since otherwise, R_3R_1 is a single arrow. We have $D(R_3) \cap D(R_1) \neq \emptyset$. Since $R_2 + R_3$ is a face of $\overline{NE}(X)$, it follows that $\overline{NE}(X, D(R_1)) \subset R_1 + R_2 + R_3$. Then, like above, we can see that $R_1 + R_3$ is a 2-dimensional face of $\overline{NE}(X)$ of Kodaira dimension 3. By Theorems 2.1.7 and 2.1.8, each extremal ray R_1, R_2, R_3 does not belong to a pair of the type $\mathfrak{B}_2$.

If $T = \{R_1, R_2, R_3\}$, we get the case (C).

Suppose that T contains an extremal ray Q different from R_1, R_2, R_3. By condition, then $\{R_1, R_2, R_3\}$ is elliptic. As we have seen, $\overline{NE}(X, D(R_1)), \overline{NE}(X, D(R_2)), \overline{NE}(X, D(R_3))$ are contained in $R_1 + R_2 + R_3$. Then like above (using Proposition 1.2.1 and Lemmas 1.2.8 and 1.2.9), we can prove that $R_1 + R_2 + R_3$ is a 3-dimensional face of Kodaira dimension 3 of $\overline{NE}(X)$.

Let $Q \in T$ is different from R_1, R_2, R_3. Since $R_1 + R_2 + R_3$ is a face of $\overline{NE}(X)$, by statements (1) and (2) of Lemma 2.1.9, the Q is divisorially disjoint with R_1 and R_2. Then, like above, using that $\overline{NE}(X, D(R_1)) \subset R_1 + R_2 + R_3$, we can prove that $R_1 + Q$ is a 2-dimensional face of Kodaira dimension 3 of $\overline{NE}(X)$. For $Q = R_2, R_3$ we have proven the same above. Thus, this statement valid for any $Q \in T$ different from R_1. Let us consider the contraction $f : X \to X'$ of R_1. By the statement above, $f(Q)$ is a divisorial extremal ray for any $Q \in T,\ Q \neq R_1$. Evidently, $f(R_2)$ has the type (I). Thus, considering $T' = f(T)$ we get the case we had considered above which gives rise the type (D) of Theorem. Thus, T' has the type (D). One can easily see that then T has the type (C) and each extremal ray of T does not belong to a pair of the type $\mathfrak{B}_2$.

Let us prove the last statement. Thus, we consider T which has the type (B). We use induction on $\#T$. For $\#T = 2$, the statement follows from Lemma 2.1.2. If $\#T > 2$, let us consider the contraction $f : X \to X'$ of the extremal ray R_{11} which we had used above when we considered this case. Then $T' = f(T)$ has the type (B) again and has the same properties as T but $\#T' = n - 1$. By induction, $T' = f(T)$ is contained in a face of Kodaira dimension 3 of $\overline{NE}(X')$. Then, evidently, T is also contained in a face of Kodaira dimension 3 of $\overline{NE}(X)$. This finishes the proof.

In Sect. 5, we will need another property of divisorial extremal rays which shows importance of quasi-Lanner sets of divisorial extremal rays.

Definition 2.2.7. A set $S = \{Q_1, ..., Q_k\}$ of divisorial extremal rays is called *semi-elliptic* if for any non-negative $a_1, ..., a_k$ there exists $j,\ 1 \leq j \leq k$, such that

$$Q_j \cdot (a_1 D(Q_1) + \cdots + a_k D(Q_k)) \leq 0.$$

In particular, by Lemma 1.2.8, an elliptic set of divisorial extremal rays is semi-elliptic.

Proposition 2.2.8. *Let $\mathcal{L} = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (I) or (II) with different divisors $D(R_1), ..., D(R_n)$. Assume that $\mathcal{L}$ is not semi-elliptic, i. e. there are $a_1, ..., a_n$ such that $R_i \cdot (a_1 D(R_1) + \cdots + a_n D(R_n)) \geq 0$ for all $1 \leq i \leq n$, and one of these inequalities is strict. Assume that any proper subset of $\mathcal{L}$ is semi-elliptic (i. e. $\mathcal{L}$ is minimal non semi-elliptic). Then $\mathcal{L}$ is quasi-Lanner (see Definition 2.2.3). Besides, any proper subset of $\mathcal{L}$ either is elliptic or is connected parabolic with $\sharp\mathcal{L} - 1$ elements.*

Proof. We use Proposition 2.2.4 as the equivalent definition of an elliptic set of divisorial extremal rays.

If any proper subset $\mathcal{L}' \subset \mathcal{L}$ is elliptic, then the set $\mathcal{L}$ is Lanner and is then quasi-Lanner.

Assume that $\mathcal{L}$ contains a semi-elliptic subset $\mathcal{L}'$ which is not elliptic. We should prove that then $\mathcal{L}'$ is connected parabolic and $\sharp\mathcal{L}' = \sharp\mathcal{L} - 1$. Let $\mathcal{P}$ be a minimal semi-elliptic and not elliptic subset of $\mathcal{L}'$. Then any proper subset of $\mathcal{P}$ is elliptic (use Proposition 2.2.4). It follows that $\mathcal{P}$ is connected parabolic.

Let $R \in \mathcal{L} - \mathcal{P}$. Assume that there exists an arrow from R to (an element of) $\mathcal{P}$. Let $\mathcal{P} = \{Q_1, ..., Q_r\}$ and for some positive $c_1, ..., c_r$ we have $\mathcal{P} \cdot (c_1 D(Q_1) + \cdots + c_r D(Q_r)) = 0$. Then for sufficiently big $\lambda > 0$ we evidently have $R \cdot (\lambda(c_1 D(Q_1) + \cdots + c_r D(Q_r)) + D(R)) > 0$ and $\mathcal{P} \cdot (c_1 D(Q_1) + \cdots + c_r D(Q_r)) + D(R)) \geq 0$. Thus $\{R\} \cup \mathcal{P}$ is not semi-elliptic. Since $\mathcal{L}$ is minimal non semi-elliptic, we get $\mathcal{L} = \{R\} \cup \mathcal{P}$ and $\mathcal{L}' = \mathcal{P}$ is connected parabolic with $\sharp\mathcal{L} - 1$ elements.

Thus, we can suppose that there is not an arrow from $\mathcal{L} - \mathcal{P}$ to $\mathcal{P}$. If $\mathcal{L} - \mathcal{P}$ is not elliptic, arguing similarly, we can find a connected parabolic subset $\mathcal{P}' \subset \mathcal{L} - \mathcal{P}$. Moreover, $\mathcal{P}$, $\mathcal{P}'$ are divisorially disjoint to one another and there is not an arrow from $\mathcal{L} - (\mathcal{P} \cup \mathcal{P}')$ to $\mathcal{P} \cup \mathcal{P}'$. Continuing these considerations, we prove that $\mathcal{L}$

$\mathcal{E}$ such that connected parabolic subsets $\mathcal{P}_1, ..., \mathcal{P}_m$ are divisorially disjoint to one another and there does not exist an arrow from $\mathcal{E}$ to connected parabolic subsets $\mathcal{P}_1, ..., \mathcal{P}_m$. If follows that $\mathcal{L}$ is semi-elliptic. We get a contradiction. This finishes the proof.

From our proof we also get

Proposition 2.2.9. *Let S be a set of extremal rays of the type (I) or (II) with different divisors. Then S is semi-elliptic if and only if S is a disjoint union of connected parabolic subsets $\mathcal{P}_1, ..., \mathcal{P}_m$ and an elliptic subset $\mathcal{E}$ such that parabolic subsets $\mathcal{P}_1, ..., \mathcal{P}_m$ are divisorially disjoint to one another and there does not exist an arrow from any element of $\mathcal{E}$ to any parabolic subset $\mathcal{P}_1, ..., \mathcal{P}_m$.*

3. A refined variant of the Diagram Method.

Here we want to prove a more strong variant of the Diagram Method Theorem which uses results of both Sections 1 and 2. This variant will be useful below for Calabi-Yau 3-folds.

We study 3-folds X from the class $\mathcal{L}T$ with conditions (a), (b) and (c) below:
(a) X has a finite polyhedral Mori cone $\overline{NE}(X)$;
(b) X does not have a small extremal ray;
(c) $\overline{NE}(X)$ does not have a face of numerical Kodaira dimension 1 or 2.

From (b) and (c), it follows that all extremal rays of X are divisorial of the type (I) or (II).

Using Theorem 2.2.6, we define analogous constants to the constants $d(X)$, $C_1(X)$, $C_2(X)$ of Definition 1.3.1.

Definition 3.1. We introduce some invariants of X.

Invariants k, l, l_2 are numbers of divisorially connected components of the types $\mathfrak{A}_1$, $\mathfrak{C}_k,\ k \geq 2$, $\mathfrak{C}_2$ respectively of the set of all special divisorial extremal rays on X. See Theorem 2.1.7.

The invariants

$$n(X)_D = \max_{F\in(D)} \sharp F - 1; \quad n(X)_C = \max_{F\in(C)} \sharp F - 1;$$
$$n(X)_A = \max_{F\in(A)} \sharp F - 1; \quad d(X)_A = \max_{F\in(A)} \operatorname{diam} G(F).$$

Here $F \in (A)$ (respectively $F \in (C)$ or $F \in (D)$) means that F runs through all E-sets of the type (A) (respectively (C) or (D)) of Theorem 2.2.6 such that any proper subset of F is extremal of Kodaira dimension 3 and F satisfies the condition (iii) of Sect. 1.1 (in particular, F is Lanner). These invariants are analogous to the invariant $d(X)$ of Definition 1.3.1.

Let S be a set of divisorial extremal rays. We denote as S' the subset of S which one gets by removing all extremal rays of the type (I) and all extremal rays of the type (II) which are ends of single arrows of X (i.e. this extremal ray belongs to a component of the type $\mathfrak{C}_k$ of the set of all special extremal rays on X and is the end of a single arrow of this component). We define a symmetric distance $\rho_A(R_1, R_2)$ in S by the formula

$$\rho_A(R_1,R_2) = \begin{cases} 0, & \text{if } R_1 = R_2, \\ \rho(R_1,R_2)_{S'}, & \text{if } \{R_1,R_2\} \subset S', \\ +\infty, & \text{otherwise.} \end{cases}$$

Here $\rho(R_1, R_2)_{S'}$ denotes the ordinary distance in the graph $G(S')$ using oriented paths. The set S' does not have single arrows. Thus, $\rho_A(R_1, R_2)$ is symmetric.

The invariant $C(X)_A$ is defined by the condition:

$$\sharp\{\{R_1, R_2\} \subset \mathcal{E} - \mathcal{E}_0 \mid 1 \le \rho_A(R_1, R_2) \le 2d(X)_A + 1\} \le C(X)_A \sharp(\mathcal{E} - \mathcal{E}_0)$$

for any extremal set $\mathcal{E}$ of the Kodaira dimension 3 of divisorial extremal rays on X and any its divisorially connected subset $\mathcal{E}_0 \subset \mathcal{E}$ of extremal rays of the type (II).

We have the following refinement of Basic Theorem 1.3.2.

Basic Theorem 3.2. *Let X be a 3-fold from the class $\mathcal{LT}$ and X satisfies conditions (a), (b) and (c) above. Then we have assertions (1), (2) and (3) below:*

(1) X does not have a pair of extremal rays of the type $\mathfrak{B}_2$ and the Mori cone $\overline{NE}(X)$ is simplicial. Thus, by Theorem 2.1.7, each divisorially connected component of the set of all special divisorial extremal rays has the type $\mathfrak{A}_1$, $\mathfrak{C}_n$, $\mathfrak{T}_3$ or $\mathfrak{T}'_3$.

(2) If the set of all special divisorial extremal rays on X has k divisorially connected components of the type $\mathfrak{A}_1$ and l connected components of types $\mathfrak{C}_{n_1}, ..., \mathfrak{C}_{n_l}$ respectively, then

$$k + (n_1 - 1) + ... + (n_l - 1) \le q(X)$$

(see Definition 1.3.1 for the invariant $q(X)$). Besides, if there exists a connected component of the type $\mathfrak{T}_3$ or $\mathfrak{T}'_3$, then every extremal ray of X belongs to this connected component and the Picard number $\rho(X) = 3$ (in particular, $k = l = 0$).

(3) We have the inequality:

$$\begin{aligned}\rho(X) = \dim N_1(X) &\le kn(X)_D + l_2 \max\{n(X)_C, n(X)_A\} + 8C(X)_A + 6\\ &\le q(X) \max\{n(X)_D, n(X)_C, n(X)_A\} + 8C(X)_A + 6.\end{aligned}$$

Proof. (1) follows from the statement (1) of Basic Theorem 1.3.2 and Theorem 2.1.7.

To prove second part of (2), we use the following Lemmas which are important as itself (in fact, they are contained in Section 1) and follow from the statement (1), Lemma 1.2.8, Lemma 1.1.1 and Theorem 1.2.13.

Lemma 3.3. *Let X be a 3-fold from the class $\mathcal{LT}$ and X satisfies the conditions (a), (b) and (c) above. Let $T = \{R_1, ..., R_n\}$ be a set of extremal rays of X. Then*

(i) If T is extremal, there are positive $a_1, ..., a_n$ such that $R_i \cdot (a_1 D(R_1) + ... + a_n D(R_n)) < 0$ for all $1 \le i \le n$, and for any non-negative $b_1, ..., b_n$ which are not all equal to zero there exists i, $1 \le i \le n$, such that $R_i \cdot (b_1 D(R_1) + ... + b_n D(R_n)) < 0$.

(ii) If T is not extremal, then T contains an E-subset $\mathcal{L}$ and there are non-negative $c_1, ..., c_n$ which are not all equal to zero such that $c_1 D(R_1) + ... + c_n D(R_n)$ is nef.

Lemma 3.4. *Let X be a 3-fold from the class $\mathcal{LT}$ and X satisfies the conditions (a), (b) and (c) above. Let $T_1 = \{R_1, ..., R_n\}$ and $T_2 = \{Q_1, ..., Q_m\}$ are two sets of extremal rays on X such that $R_i \cdot D(Q_j) = 0$ for all $1 \le i \le n$ and all $1 \le j \le m$ (equivalently, there does not exist an arrow from T_1 to T_2). Then either T_1 or T_2 is extremal.*

If $T_1 \cup T_2$ is the set of all extremal rays on X and $T_2 \neq \emptyset$, then T_1 is extremal and T_2 is not extremal.

Proof. We explain only the last statement. Let us suppose that T_2 is extremal. By condition and Lemma 3.3, there are positive $a_1, ...a_m$ such that $R \cdot (a_1 D(Q_1) + ... + a_m D(Q_m)) \le 0$ for any extremal ray $R \in T_1 \cup T_2$. By condition, $T_1 \cup T_2$ is the set of all extremal rays on X. It follows that, $C \cdot (a_1 D(Q_1) + ... + a_m D(Q_m)) \le 0$ for any curve C on X. Evidently, this is absurd. Thus, T_2 is not extremal. By Lemma 3.3, there are non-negative $c_1, ..., c_m$ such that $D(Q) = c_1 D(Q_1) + ... + c_m D(Q_m)$ is nef. By condition, $T_1 \cdot D(Q) = 0$. It follows that T_1 is extremal.

From the last statement of Lemma 3.4, it follows

Lemma 3.5. *Let X be a 3-fold from the class $\mathcal{LT}$ and X satisfies the conditions (a), (b) and (c) above. Then the set of all extremal rays on X is divisorially connected and is not extremal.*

Now, let us prove the statement (2) of Theorem 3.2. Assume that the set of special extremal rays has a connected component P of the type $\mathfrak{T}_3$ or $\mathfrak{T}_3'$. Let R be an extremal ray and $R \notin P$. By Theorem 2.1.8, then $R \cdot D(Q) = 0$ for any extremal ray $Q \in P$. We get a contradiction with Lemma 3.5. Thus, any extremal ray of X belongs to P. Since P contains exactly 3 extremal rays, $\rho(X) \le 3$. By definition, P contains extremal rays Q_1, Q_2, Q_3 which define single arrows Q_1Q_2 and Q_2Q_3. By Lemma 2.1.2, $Q_1 + Q_2$ and $Q_2 + Q_3$ are 2-dimensional faces of $\overline{NE}(X)$ which are evidently different. It follows that $\rho(X) > 2$. Thus, $\rho(X) = 3$.

Let us prove first part of (2). This is similar to the proof of the statement (2) of Basic Theorem 1.3.2. Let $P_1 = \{R_1\}, ..., P_k = \{R_k\}$ are connected components of the type $\mathfrak{A}_1$ and $S_1, ..., S_l$ are connected components of the types $\mathfrak{C}_{n_1}, ..., \mathfrak{C}_{n_l}$ of the set of special extremal rays on X where $S_t = \{Q_{t1}, Q_{t2}, ..., Q_{tn_t}\}$ and $Q_{t2}Q_{t1}, ..., Q_{tn_t}Q_{t1}$ are single arrows.

The set of all special extremal rays on X evidently satisfies the condition (i) of Lemma 3.3 and is then extremal. Let $\mathcal{E}$ be a maximal extremal set of extremal rays on X containing the set of all special extremal rays and such that each divisorially connected component of $\mathcal{E}$ contains at least one special extremal ray.

By Theorem 2.2.6, each connected component of $\mathcal{E}$ contains exactly one connected component of the set of special extremal rays. Thus, $\mathcal{E}$ contains $k + l$ connected components $\mathcal{E}_1, ..., \mathcal{E}_k, \mathcal{E}_{k+1}, ..., \mathcal{E}_{k+l}$. Here $R_i \in \mathcal{E}_i$ and $\mathcal{E}_i$ has the type (D) for $1 \le i \le k$. And $S_j \subset \mathcal{E}_{k+j}$ and $\mathcal{E}_{k+j}$ either has the type (B) or (C) for $1 \le j \le l$. Changing numeration, we can suppose that $\mathcal{E}_{k+j}$ has the type (C) for $1 \le j \le r_1$ (in particular, $n_1 = ... = n_{r_1} = 2$) and $\mathcal{E}_{k+j}$ has the type (B) for $r_1 < j \le l$. For $r_1 < j \le l$, we additionally denote as $\mathcal{E}_{k+j,2}, ..., \mathcal{E}_{k+j,n_j}$ the corresponding chains containing $Q_{j2}, ..., Q_{jn_j}$ respectively. These chains give divisorially connected components of $\mathcal{E}_{k+j} - \{Q_{j1}\}$.

Since $\mathcal{E}$ is extremal, by Lemma 3.3, there are positive $a(R)$ for $R \in \mathcal{E}$ such that

$$R \cdot \sum_{R \in \mathcal{E}} a(R) D(R) < 0$$

for any $R \in \mathcal{E}$.

Let us consider divisorially disjoint to one another sets $F_v,\ v \in V$, of extremal rays which are equal to one of sets and the corresponding divisors

$$D(F_v) = \sum_{R\in F_v} a(R)D(R)$$

(here $\sharp V = k + (n_1 - 1) + ... + (n_l - 1)$). One can easily see that $C \cdot D(F_v) \le 0$ for any $R \in F_v$ and any curve $C \subset D(R)$.

Similarly to Lemma 1.3.4 and the proof of the statement (2) of Basic Theorem 1.3.2, we can find an extremal set $A = \{U_v \mid v \in V\}$ containing extremal rays U_v of the type (II) such that we have the property: $U_v \cdot D(F_v) > 0$ but $U_v \cdot D(F_{v'}) = 0$ for $v' \ne v$. Similarly to the proof of the statement (2) of Theorem 1.3.2, one proves that the graph $G(A)$ is full. Thus, $\sharp A = \sharp V = k + (n_1 - 1) + ... + (n_l - 1) \le q(X)$. This finishes the proof of the statement (2) of Theorem 3.2.

Proof of (3). It is based on the following

Lemma 3.6. *Let X be a 3-fold from the class $\mathcal{LT}$ and X satisfies conditions (a), (b) and (c) above. Assume that $\rho(X) > 3$. Then there exists an extremal divisorially connected set $\mathcal{E}_0$ containing only extremal rays of the type (II) and such that any E-set $\mathcal{L}$ does not contain extremal rays of the type (I) and extremal rays of the type (II) which are terminal vertices of single arrows on X (in particular, $\mathcal{L}$ has the type (A) of Theorem 2.2.6) if $\mathcal{L}$ has at least two elements which do not belong to $\mathcal{E}_0$ and $\mathcal{L}' \cup \mathcal{E}_0$ is extremal for any proper subset $\mathcal{L}' \subset \mathcal{L}$.*

Besides,

$$\sharp\mathcal{E}_0 \le kn(X)_D + l_2 \max\{n(X)_C, n(X)_A\}.$$

Proof. We numerate as $R_1, ..., R_k$ the whole set of divisorial extremal rays of the type (I) and as $R_{k+1}, ..., R_{k+l_2}$ the whole set of divisorial extremal rays of the type (II) such that $R_{k+i}, 1 \le i \le l_2$, belongs to a connected component of the type $\mathfrak{C}_2$ of the set of all special extremal rays and R_{k+i} is the terminal vertex of the single arrow of this component.

We construct $\mathcal{E}_0$ in $k + l_2$ steps as a sequence

$$\emptyset = (\mathcal{E}_0)_0 \subset ... \subset (\mathcal{E}_0)_{k+l_2} = \mathcal{E}_0.$$

Here $(\mathcal{E}_0)_t, 1 \le t \le k + l_2$, has the property that there does not exist an E-set $\mathcal{L}$ which contains one of extremal rays $R_1, ..., R_t$, and $\mathcal{L}$ contains at least two elements which do not belong to $(\mathcal{E}_0)_t$, and $(\mathcal{E}_0)_t \cup \mathcal{L}'$ is extremal for any proper subset $\mathcal{L}' \subset \mathcal{L}$.

Suppose that we have constructed $(\mathcal{E}_0)_t$ with properties above and $t < k + l_2$. Assume that there exists an E-set U which contains R_{t+1}, and U contains at least two elements which do not belong to $(\mathcal{E}_0)_t$, and $U' \cup (\mathcal{E}_0)_t$ is extremal for any proper subset $U' \subset U$. Then we set

$$(\mathcal{E}_0)_{t+1} = (\mathcal{E}_0)_t \cup (U - \{R_{t+1}\}).$$

The $(\mathcal{E}_0)_{t+1}$ is extremal by the conditions on $(\mathcal{E}_0)_t$ and U above. Let us suppose that there exists an E-set $\mathcal{L}$ which contains R_i, $1 \le i \le t + 1$, and $\mathcal{L}$ contains at least two elements which do not belong to $(\mathcal{E}_0)_{t+1}$, and $\mathcal{L}' \cup (\mathcal{E}_0)_{t+1}$ is extremal for any proper subset $\mathcal{L}' \subset \mathcal{L}$. If $1 \le i \le t$, we have similar properties for $(\mathcal{E}_0)_t$ and get the contradiction. Thus, $i = t + 1$. By our conditions, there exists $Q \in \mathcal{L} - \{R_{t+1}\}$ such that $Q \notin (\mathcal{E}_0)_{t+1}$ and $(\mathcal{E}_0)_{t+1} \cup (\mathcal{L} - \{Q\})$ is extremal. Since $R_{t+1} \in \mathcal{L} - \{Q\}$

and $U - \{R_{t+1}\} \subset (\mathcal{E}_0)_{t+1}$, it follows that $U \subset (\mathcal{E}_0)_{t+1} \cup (\mathcal{L} - \{Q\})$ is extremal because $(\mathcal{E}_0)_{t+1} \cup (\mathcal{L} - \{Q\})$ is extremal. We get a contradiction since U is an E-set. Thus, $(\mathcal{E}_0)_{t+1}$ has desirable properties. Here, by construction and Theorem 2.2.6, the set U has the type (D) if $1 \le t+1 \le k$, and the set U has the type (C) or (A) if $k < t+1 \le k+l_2$. If there does not exist the E-set U above, we just put $(\mathcal{E}_0)_{t+1} = (\mathcal{E}_0)_t$.

By our construction, $\sharp\mathcal{E}_0 \le kn(X)_D + l_2 \max\{n(X)_C, n(X)_A\}$. By our construction, $\mathcal{E}_0$ is divisorially connected since any two different E-sets are connected by arrows (by Lemma 3.4) and extremal rays $R_1, ..., R_{k+l_2}$ are divisorially disjoint.

Let $R_{k+l_2+1}, ..., R_{k+l}$ be terminal vertices of arrows of all components of the types $\mathfrak{C}_k, k \ge 3$. By our construction, any E-set $\mathcal{L}$ does not contain any of extremal rays $R_1, ..., R_{k+l_2}$ if $\mathcal{L}$ has at least two elements which do not belong to $\mathcal{E}_0$ and $\mathcal{L}' \cup \mathcal{E}_0$ is extremal for any proper subset $\mathcal{L}' \subset \mathcal{L}$. We claim that we even have more: the E-set $\mathcal{L}$ above does not contain any of extremal rays $R_1, ..., R_{k+l}$. Actually, if the $\mathcal{E}$-set $\mathcal{L}$ contains R_i, $k+l_2 < i \le k+l$, we get a contradiction with Theorems 2.1.8 and 2.2.6: the set $\mathcal{L}$ should be of the type (B), but then it is extremal and cannot be an E-set.

Since $\rho(X) > 3$, by the statement (2), any terminal vertex of a single arrow on X is one of extremal rays $R_{t+1}, ..., R_{t+l}$. This finishes the proof of Lemma.

We continue the proof of Theorem 3.2. Let us consider the face $\gamma \subset \mathcal{M}(X) = NEF(X)/\mathbb{R}^+$. which is orthogonal to $\mathcal{E}_0$. Let us consider constants $C_1(X)'$ and $C_2(X)'$ which are defined by the properties:

$$\sharp\{(R_1, R_2) \in (\mathcal{E} - \mathcal{E}_0) \times (\mathcal{E} - \mathcal{E}_0) \mid 1 \le \rho(R_1, R_2) \le d(X)_A\} \le C_1(X)'\sharp(\mathcal{E} - \mathcal{E}_0);$$

and

$$\sharp\{(R_1, R_2) \in (\mathcal{E} - \mathcal{E}_0) \times (\mathcal{E} - \mathcal{E}_0) \mid d(X)_A + 1 \le \rho(R_1, R_2) \le 2d(X)_A + 1\} \le C_2(X)'\sharp(\mathcal{E} - \mathcal{E}_0).$$

for any extremal set $\mathcal{E}$ which contains $\mathcal{E}_0$ and distance ρ in the graph $G(\mathcal{E})$.

Directly applying to γ Theorem 1.2 from [N8], we get the inequality

$$\dim \gamma < (16/3)C_1(X)' + 4C_2(X)' + 6 \tag{3-1}$$

which is less than we want.

To get the desirable estimate

$$\dim \gamma < 8C(X)_A + 6, \tag{3-2}$$

one needs to change a little the proof of Theorem 1.2 from [N8] for our case (evidently, $C(X)_A \le (C_1(X)' + C_2(X)')/2$ and (3-2) implies (3-1)).

Let $\angle$ be an oriented (plane) angle of γ. Let $\mathcal{R}(\angle)$ be the set of all extremal rays of $\overline{NE}(X)$ which are orthogonal to the vertex of $\angle$. The set $\mathcal{R}(\angle)$ is a disjoint union

$$\mathcal{R}(\angle) = \mathcal{R}(\angle^\perp) \cup \{R_1(\angle)\} \cup \{R_2(\angle)\}$$

where $\mathcal{R}(\angle^\perp)$ contains all rays orthogonal to the plane of the angle $\angle$, the rays $R_1(\angle)$ and $R_2(\angle)$ are orthogonal to the first and second side of the oriented angle $\angle$, respectively. Evidently, the set $\mathcal{R}(\angle)$ and the ordered pair of rays $(R_1(\angle), R_2(\angle))$ define the oriented angle $\angle$ uniquely. We define the weight $\sigma(\angle)$ by the formula (for the proof of Theorem 1.2 from [N8], the definition of the weight $\sigma(\angle)$ was different):

$$\sigma_A(\angle) = \begin{cases} 1/2, & \text{if } 1 \le \rho_A(R_1(\angle), R_2(\angle)) \le 2d(X)_A + 1, \\ 0, & \text{if } 2d(X)_A + 1 < \rho_A(R_1(\angle), R_2(\angle)). \end{cases} \tag{3-3}$$

Similarly to the proof of Theorem 1.2 from [N8], one can check conditions of Lemma 1.4 (Vinberg's Lemma) from [N8] for the polyhedron $\mathcal{M} = \gamma$ with the constants $C = C(X)_A$ and $D = 0$. The proof even is simpler because the weight $\sigma_A(\angle')$ of the angle with opposite orientation is equal to $\sigma_A(\angle)$. It follows (3-2). For convenience of a reader, we recall Vinberg's Lemma.

Lemma 3.8. *Let $\mathcal{M}$ be a convex simple polyhedron of a dimension n. Let C and D are some numbers. Suppose that oriented angles (2-dimensional, plane) of $\mathcal{M}$ are supplied with weights and the following conditions (1) and (2) hold:*

(1) The sum of weights of all oriented angles at any vertex of $\mathcal{M}$ is not greater than $Cn + D$.

(2) The sum of weights of all oriented angles of any 2-dimensional face of $\mathcal{M}$ is at least $5 - k$ where k is the number of vertices of the 2-dimensional face.

Then

$$n < 8C + 5 + \begin{cases} 1 + 8D/n & \text{if } n \text{ is even}, \\ (8C + 8D)/(n-1) & \text{if } n \text{ is odd} \end{cases}.$$

In particular, for $C \ge 0$ and $D = 0$, we have

$$n < 8C + 6.$$

Since $\dim \gamma = \dim N_1(X) - \sharp\mathcal{E}_0 - 1$ and $\sharp\mathcal{E}_0 \le kn(X)_D + l_2 \max\{n(X)_C, n(X)_A\}$, we get the estimate $\dim N_1(X) \le kn(X)_D + l_2 \max\{n(X)_C, n(X)_A\}$. By the statement (2), $k + l_2 \le q(X)$. Thus, $\dim N_1(X) \le q(X) \max\{n(X)_D, n(X)_C, n(X)_A\} + 8C(X)_A + 6$. This finishes the proof of Theorem 3.2.

Remark 3.8. Let us consider Fano 3-folds X with $\mathbb{Q}$-factorial terminal singularities and which satisfy the conditions (a), (b) and (c). Then the constants $q(X) \le 1$, $n(X)_D \le 1$, $n(X)_C = -1$, $n(X)_A \le 1$, $d(X)_A = 1$ and $C(X)_A = 0$ (see [N8]). Thus, by Theorem 3.2, we get the main result of [N8]: $k + (n_1 - 1) + ... + (n_l - 1) \le 1$ and $\rho(X) \le 7$.

In Sects. 4 and 5, we apply Basic Theorems 1.3.2 and 3.2 to Calabi-Yau 3-folds.

4. Application to Calabi-Yau 3-folds

4.1. Reminding.

Here we recall general facts about Calabi-Yau 3-folds we shall use.

A projective algebraic 3-dimensional manifold X over $\mathbb{C}$ is called Calabi-Yau if the canonical class $K_X = 0$ and $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$. Then $N^1(X) = H^2(X; \mathbf{R})$.

It was shown by Wilson [W1] and [W2] that the *nef*-cone $NEF(X)$ is locally rational polyhedral away from the cubic intersection hypersurface

$$\mathcal{W}' = \{h \in H^2(X; \mathbf{R}) \mid h^3 = 0\}$$

The main tool here is to use results of Kawamata [Ka1] and Shokurov [Sh] (which generalize results of Mori [Mo1]) about ”polyhedrality” of Mori cone of algebraic varieties with log-terminal singularities. Thus, we can speak about faces $\gamma \subset NEF(X)$ considering only faces of the locally polyhedral cone $NEF(X)$ away from the cone $\mathcal{W}$. Then the corresponding face $\gamma' = \gamma^{\perp}$ of Mori cone $\overline{NE}(X)$ is finite polyhedral and $\dim \gamma' = \operatorname{codim} (\gamma')^{\perp}$ where $(\gamma')^{\perp} = \gamma$. Evidently, these faces $\gamma\prime$ are exactly finite polyhedral faces of $\overline{NE}(X)$ of numerical Kodaira dimension 3 and with the property $\operatorname{codim} (\gamma')^{\perp} = \dim \gamma'$. It is known (see [W1], [W2], [O]) that these faces are contractible and have Kodaira dimension 3 (one should apply theory of log-terminal extremal contractions of [Ka1] and [Sh]; in fact, one need these results to prove the results of Wilson we mentioned above). Thus, we can speak about extremal rays of Kodaira dimension 3 (they are orthogonal to faces of $NEF(X)$ of highest dimension) and extremal rays of the types (I), (II) and (III) (small). Applying results of [Ka1] and [Sh], we get that a sequence $X \to X'$ of contractions of divisorial extremal rays gives rise a 3-fold with $\mathbb{Q}$-factorial (canonical) singularities and X' inherits all properties of X we mentioned above. In particular, we get that X and X' belong to the class $\mathcal{L}T$.

The following fact is very important for us (see [W2] and also appendix of Shokurov): For an extremal ray R of the type (II) of a 3-dimensional Calabi-Yau manifold X, there exists a curve $C \in R$ such that

$$C \cdot D(R) = -2. \tag{4-1-1}$$

In fact, this curve C is the general fiber of the $f : D(R) \to f(D(R))$ for the contraction $f : X \to X'$ of the extremal ray R. But, we don’t need the last fact in this paper.

Let Q be another divisorial extremal ray of X. Using this curve $C \in R$, we correspond to the arrow RQ the weight $C \cdot D(Q)$. Thus, from now on, arrows of the graph $G(T)$ of a set T of extremal rays of the type (II) have the corresponding weights. We don’t show the corresponding weight only if this weight is equal to 1.

For an extremal ray R of the type (I) we choose some curve $C \in R$. This gives rise the corresponding weights of arrows too. As was shown by Shokurov (see Appendix), there exists a curve $C \in R$ such that $C \cdot D(R) = -1, -2$ or -3. We shall use this result later. Now, we fix the choice of the C showing the weight $-k = C \cdot D(R)$ of the vertex corresponding to R. Thus, from now on, arrows and black vertices of the graph $G(T)$ of any set T of extremal rays of the type (I) or (II) are equipped by weights. For a manifold X these weights are integers.

Here we want to apply the theory we developed above for studying of 3-dimensional Calabi-Yau manifolds.

4.2. Sets of divisorial extremal rays on Calabi-Yau manifolds.

We describe here elliptic, parabolic and Lanner (some quasi-Lanner too) sets of extremal rays of the type (I) or (II) on Calabi–Yau 3-dimensional manifolds. By Theorem 2.2.6, they have types (A), (B), (C), (D) or (E), (E’). By results of Section 1.2 and Theorem 2.2.6, this is enough for the description of sets of extremal rays of the type (I) or (II) which are either extremal of Kodaira dimension 3 or are E-sets $\mathcal{L}$ such that each proper subset of $\mathcal{L}$ is extremal of Kodaira dimension 3 and $\mathcal{L}$ satisfies the condition (iii) of Sect. 1.1.

4.2.1. Sets ot the type (A).

We have the following results:

Theorem 4.2.1.1. *Let X be a 3-dimensional Calabi-Yau manifold. Let $T = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (II) on X such that all divisors $D(R_1), ..., D(R_n)$ are different and the graph $G(T)$ does not have a single arrow (thus, it has the type (A)) and is connected.*

Then:

(a) The T is elliptic if and only if $G(T)$ is a Dynkin diagram of the root system type

$$\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2$$

of the Table 4.1.

(b) The T is parabolic if and only if $G(T)$ is an extended Dynkin diagram of the root system type

$$\tilde{\mathbf{A}}_n, \tilde{\mathbf{B}}_n, \widetilde{\mathbf{BC}}_n, \tilde{\mathbf{C}}_n, \widetilde{\mathbf{BD}}_n, \widetilde{\mathbf{CD}}_n, \tilde{\mathbf{D}}_n, \tilde{\mathbf{E}}_6, \tilde{\mathbf{E}}_7, \tilde{\mathbf{E}}_8, \widetilde{\mathbf{BF}}_4, \widetilde{\mathbf{CF}}_4, \widetilde{\mathbf{AG}}_2, \widetilde{\mathbf{GA}}_2,$$

of the Table 4.2 where weights of vertices show the coefficients a_i of Definition 2.2.2.

(c) The T is Lanner if and only if $G(T)$ is one of diagrams of the Table 4.3.

Proof. Applying (4-1-1), this is standard and well-known in fact.

4.2.2. Sets ot the type (B).

Theorem 4.2.2.1. *Let X be a 3-dimensional Calabi-Yau manifold. Let $T = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (II) on X such that all divisors $D(R_1), ..., D(R_n)$ are different and the graph $G(T)$ has the type (B) of Theorem 2.2.2. Thus, $T = \{R\} \cup C_1 \cup ... \cup C_k$, $k \geq 1$, where*

$$C_1 = \{R_{11}, ..., R_{1n_1}\},\ C_2 = \{R_{21}, ..., R_{2n_2}\}, ..., C_k = \{R_{k1}, ..., R_{kn_k}\}$$

are divisorially disjoint to one another chains, and all arrows between R and extremal rays of these chains are single arrows $R_{j1}R$, $j = 1, ..., k$.

Then the T cannot be parabolic or Lanner. The T is elliptic if and only if each chain $C_1, ..., C_k$ is one of the chains

$$\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{F}_4, \mathbf{G}_2$$

of Theorem 4.2.1.1, weights of single arrows $R_{j1}R$, $j = 1, ..., k$, are arbitrary natural numbers.

Proof. If T is either elliptic, or parabolic or Lanner, the chains $C_1, ..., C_k$ are elliptic because they give proper subsets of T. By Theorem 4.2.1.1, they have the types $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{F}_4$ or $\mathbf{G}_2$. One can easily show (see the proof of Theorem 2.2.6) that then T is elliptic for arbitrary weights of arrows $R_{j1}R$ which should be natural numbers. This finishes the proof.

4.2.3. Sets of the type (C).

Theorem 4.2.3.1. *Let X be a 3-dimensional Calabi-Yau manifold. Let $T = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (II) on X such that all divisors $D(R_1), ..., D(R_n)$ are different and the graph $G(T)$ has the type (C) of Theorem 2.2.6.*

Then T is elliptic, parabolic, Lanner or quasi-Lanner if and only if $G(T)$ is elliptic, parabolic, Lanner or quasi-Lanner diagram respectively of the Table 4.4 below.

Proof. The corresponding calculations are very simple using (4-1-1).

4.2.4. Sets of the type (D).

Theorem 4.2.4.1. *Let X be a 3-dimensional Calabi-Yau manifold. Let $T = \{R_1, ..., R_n\}$ be a set of extremal rays of the type (I) or (II) on X such that the graph $G(T)$ has the type (D) of Theorem 2.2.6.*

Then T is elliptic, parabolic, Lanner or quasi-Lanner if and only if $G(T)$ is elliptic, parabolic, Lanner or quasi-Lanner diagram respectively of the Table 4.5 below (we recall that the black vertex corresponds to an extremal ray of the type (I)).

Proof. The corresponding calculations are very simple using (4-1-1). We remark that $G(T)$ without the black vertex should have one of the types $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{F}_4$ or $\mathbf{G}_2$ by Theorem 3.2.1.1 (a).

4.2.5. Sets of the type (E).

Theorem 4.2.5.1. *Let X be a 3-dimensional Calabi-Yau manifold. Let $T = \{R_1, R_2, R_3\}$ be a set of extremal rays of the type (II) on X such that the graph $G(T)$ has the type (E) (triangle or special triangle) of Theorem 2.2.6.*

Then T is elliptic, parabolic, Lanner or quasi-Lanner if and only if $G(T)$ is elliptic, parabolic, Lanner or quasi-Lanner diagram respectively of the Table 4.6 below.

Proof. The corresponding calculations are very simple using (4-1-1).

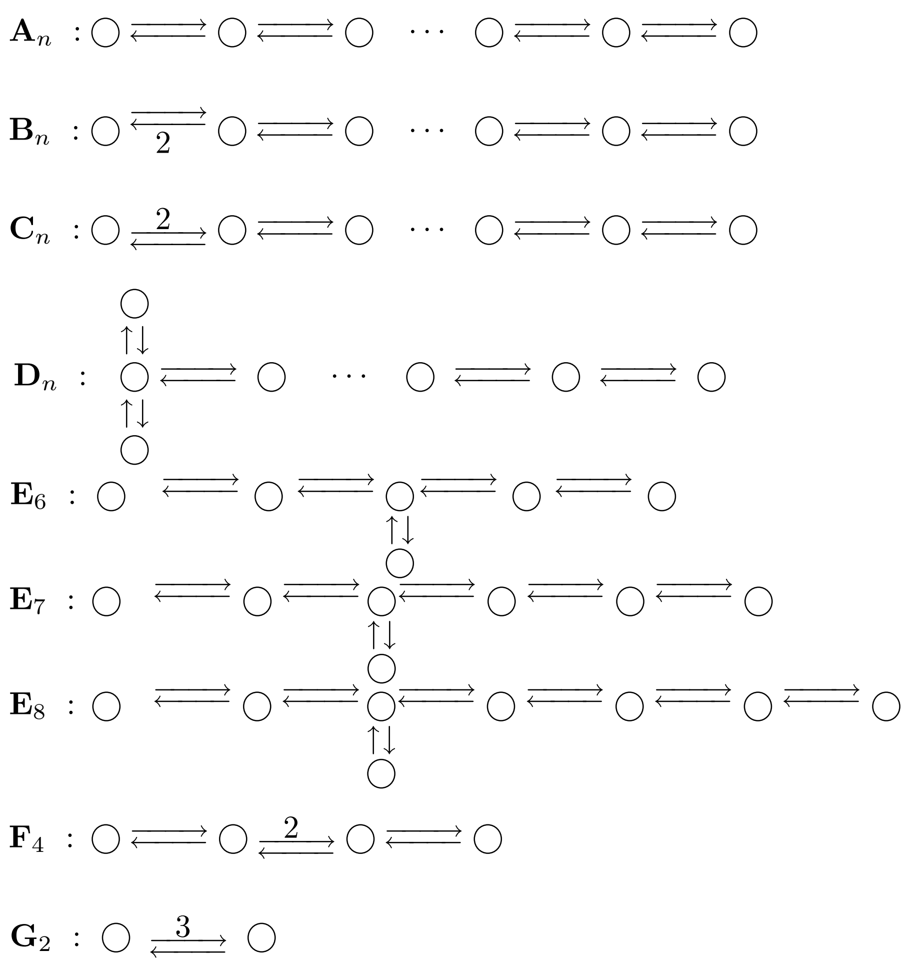


Table 4.1. Calabi–Yau elliptic diagrams without single arrows (classical Dynkin diagrams).

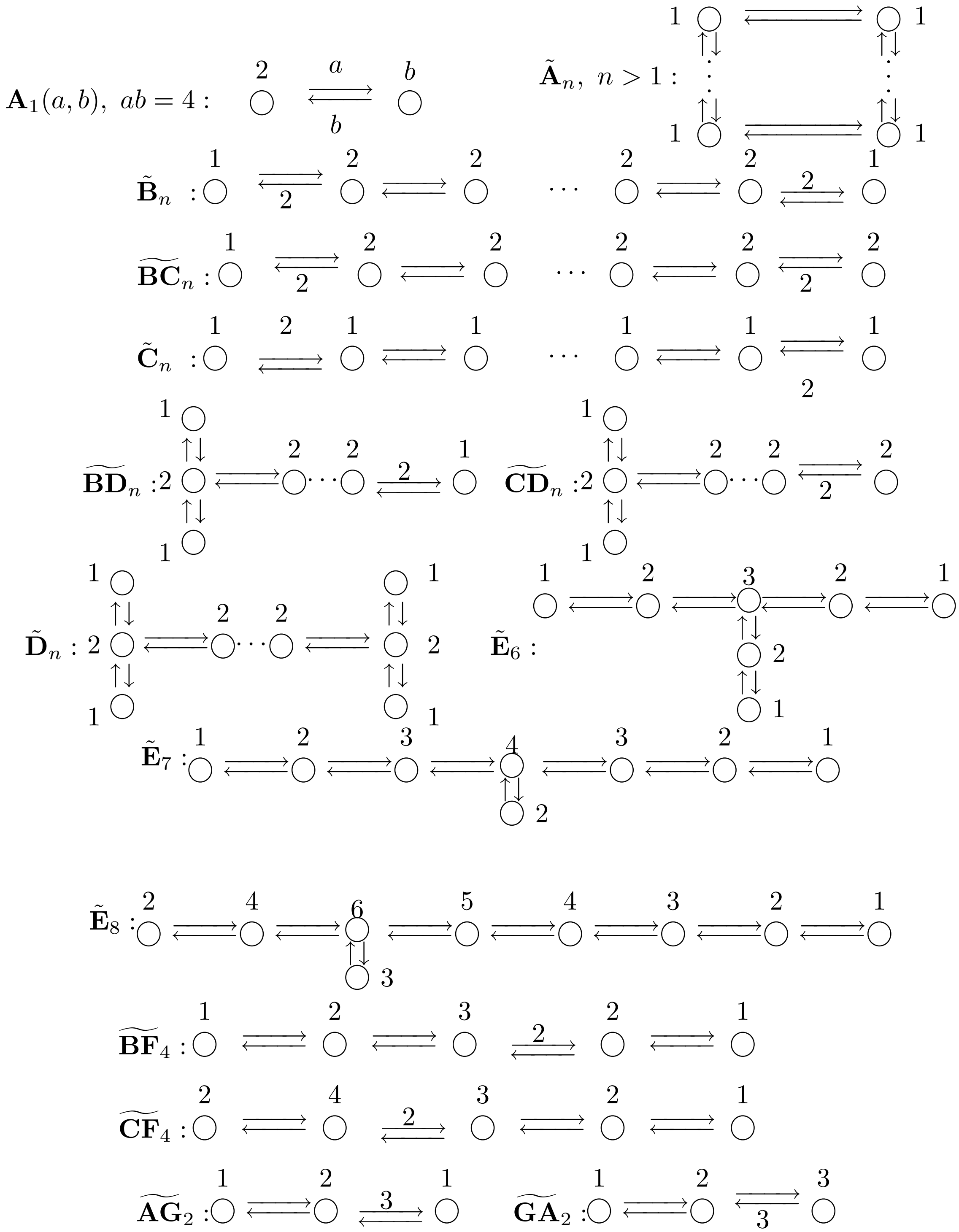
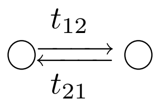
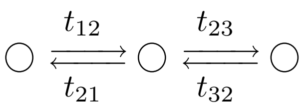


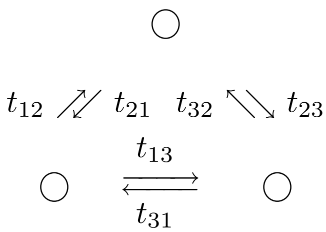
Table 4.2. Calabi–Yau parabolic diagrams without single arrows (classical extended Dynkin diagrams)



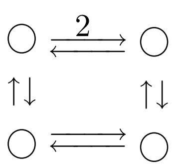
where $t_{12}t_{21} > 4$.

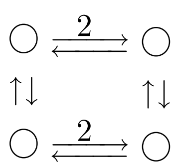


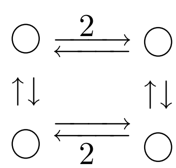
where $0 < t_{12}t_{21} < 4,\ 0 < t_{23}t_{32} < 4,\ t_{12}t_{21} + t_{23}t_{32} > 4$



where
$0 < t_{12}t_{21} < 4,\ 0 < t_{23}t_{32} < 4,\ 0 < t_{31}t_{13} < 4,$
$t_{12}t_{21} + t_{23}t_{32} + t_{31}t_{13} > 3$







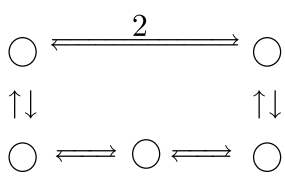
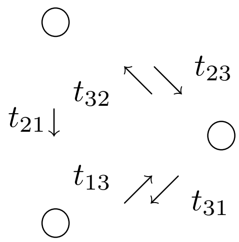
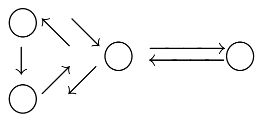


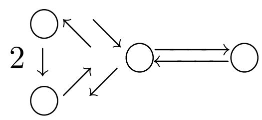
Table 4.3. Calabi–Yau Lanner diagrams without single arrows (classical Lanner diagrams).



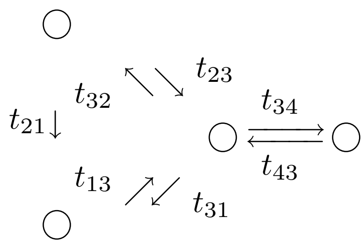
Elliptic diagram where
$t_{21}t_{13}t_{32} + 2t_{13}t_{31} + 2t_{23}t_{32} < 8$;
Parabolic diagram where
$t_{21}t_{13}t_{32} + 2t_{13}t_{31} + 2t_{23}t_{32} = 8$;
Quasi-Lanner diagram where
$t_{13}t_{31} \leq 4$, $t_{23}t_{32} \leq 4$
and $t_{21}t_{13}t_{32} + 2t_{13}t_{31} + 2t_{23}t_{32} > 8$.



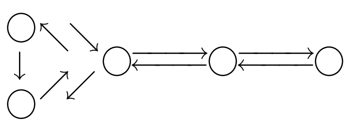
Elliptic diagram.



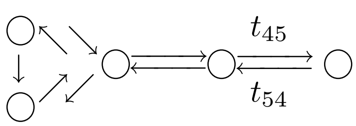
Parabolic diagram.



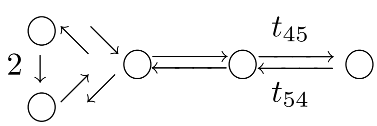
Quasi-Lanner diagram where
$t_{13}t_{31} + t_{34}t_{43} \leq 4$, $t_{23}t_{32} + t_{34}t_{43} \leq 4$,
$t_{21}t_{13}t_{32} + 2t_{13}t_{31} + 2t_{23}t_{32} \leq 8$,
$t_{21}t_{13}t_{32} + 2t_{13}t_{31} + 2t_{23}t_{32} + 2t_{34}t_{43} > 8$.



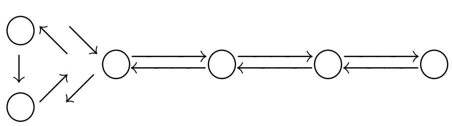
Elliptic diagram



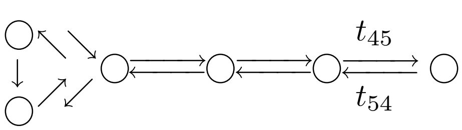
Lanner diagram
where $t_{45}t_{54} = 2$.



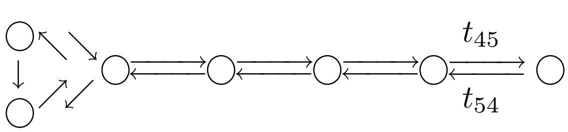
Quasi-Lanner diagram
where $1 \leq t_{45}t_{54} \leq 2$.



Elliptic diagram.

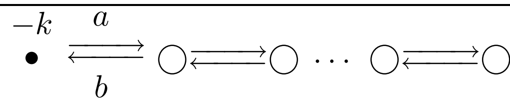


Lanner diagram
where $t_{45}t_{54} = 2$.

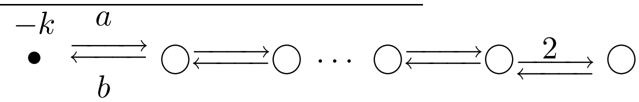


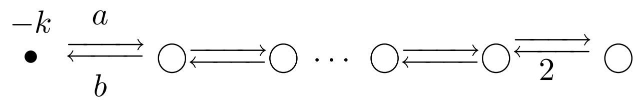
Quasi-Lanner diagram
where $t_{45}t_{54} = 2$.

Table 4.4. Calabi-Yau diagrams of the type (C).

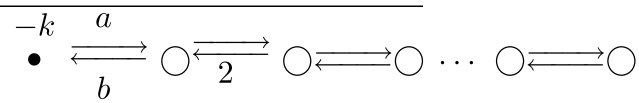
$\mathbf{A}^\bullet_n(k;a,b)$: 

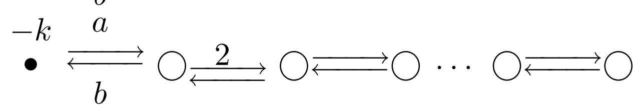
Elliptic diagram iff $ab < k(n+1)/n$. Parabolic diagram iff $ab = k(n+1)/n$. Quasi-Lanner diagram iff $kn/(n-1) \geq ab > k(n+1)/n$; in particular, $n \leq k+1$. Lanner diagram iff $kn/(n-1) > ab > k(n+1)/n$; in particular, $n \leq 2$ for $1 \leq k \leq 3$.

$\mathbf{B}^\bullet_n(k;a,b)_1$: 

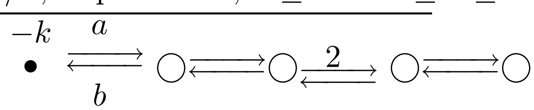
$\mathbf{C}^\bullet_n(k;a,b)_1$: 

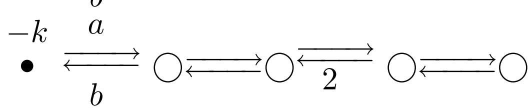
Elliptic diagram iff $ab < k$. Parabolic diagram iff $ab = k$. Quasi-Lanner diagram iff $kn/(n-1) \geq ab > k$; in particular, $n \leq k+1$. Lanner diagram iff $kn/(n-1) > ab > k$; in particular, $n < k+1$.

$\mathbf{B}^\bullet_n(k;a,b)_2$: 

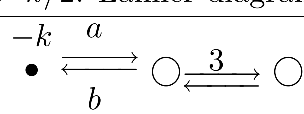
$\mathbf{C}^\bullet_n(k;a,b)_2$: 

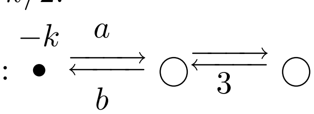
Elliptic diagram iff $ab < 2k/n$. Parabolic diagram iff $ab = 2k/n$. Quasi-Lanner diagram iff $2k/(n-1) \geq ab > 2k/n$; in particular, $n \leq 2k+1$. Lanner diagram iff $2k/(n-1) > ab > 2k/n$; in particular, $n \leq 2$ for $1 \leq k \leq 3$.

$\mathbf{F}^\bullet_4(k;a,b)_1$: 

$\mathbf{F}^\bullet_4(k;a,b)_2$: 

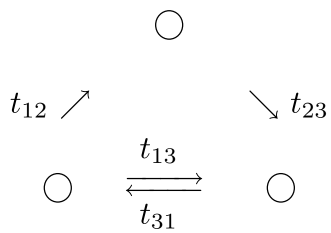
Elliptic diagram iff $ab < k/2$. Parabolic diagram iff $ab = k/2$. Quasi-Lanner diagram iff $k \geq ab > k/2$. Lanner diagram iff $k > ab > k/2$.

$\mathbf{G}^\bullet_2(k;a,b)_1$: 

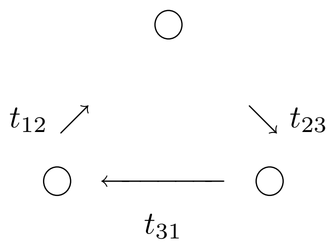
$\mathbf{G}^\bullet_2(k;a,b)_2$: 

Elliptic diagram iff $ab < k/2$. Parabolic diagram iff $ab = k/2$. Quasi-Lanner diagram iff $k \geq ab > k/2$. Lanner diagram iff $k > ab > k/2$.

Table 4.5. Calabi–Yau diagrams of the type (D) (n is equal to the number of white vertices).



Elliptic where
$t_{12}t_{23}t_{31} + 2t_{13}t_{31} < 8$
Parabolic where
$t_{12}t_{23}t_{31} + 2t_{13}t_{31} = 8$
Lanner where
$t_{13}t_{31} < 4$ and $t_{12}t_{23}t_{31} + 2t_{13}t_{31} > 8$
Quasi-Lanner where
$t_{13}t_{31} \le 4$ and $t_{12}t_{23}t_{31} + 2t_{13}t_{31} > 8$



Elliptic where
$t_{12}t_{23}t_{31} < 8$
Parabolic where
$t_{12}t_{23}t_{31} = 8$
Lanner where $t_{12}t_{23}t_{31} > 8$

Table 4.6: Calabi-Yau triangle and special triangle diagrams.

4.3. Basic results on Calabi-Yau 3-dimensional manifolds.

To prove our main results about Calabi-Yau 3-dimensional manifolds, we use our results above (especially, Basic Theorems 1.3.2 and 3.2 and Theorem 2.2.6) and the result of V.V. Shokurov about the length of extremal rays of the type (I) on Calabi-Yau manifolds (see Appendix of V.V. Shokurov).

Theorem 4.3.1 (by the author and V.V. Shokurov). *Let X be a 3-dimensional Calabi-Yau manifold and $\rho(X) > 40$.*

Then one of two cases (i) or (ii) below hold:

(i) There exists a small extremal ray on X.

(ii) There exists a nef element h such that $h^3 = 0$ (thus, the nef cone $NEF(X)$ and the cubic intersection hypersurface $\mathcal{W}_X$ have a common point; here, we don't claim that h is rational!).

Proof. Assume that X does not have a *nef* element h with $h^3 = 0$. Since, by Wilson [W1] and [W2], the $NEF(X)$ is locally rational polyhedral away $\mathcal{W}_X$, using compactness arguments, we get that $NEF(X)$ is rational finite polyhedral. It follows that the $\overline{NE}(X)$ is rational finite polyhedral. Thus, Theorem 4.3.1 follows from

Theorem 4.3.1'. *Let X be a 3-dimensional Calabi-Yau manifold with the finite rational polyhedral cone $NEF(X)$ (equivalently, $\overline{NE}(X)$). Let us assume that X does not have a small extremal ray and a nef rational element h such that $h^3 = 0$ (equivalently, all faces of $\overline{NE}(X)$ have Kodaira dimension 3). Then we have the following statements (1), (2) and (3) about X:*

(1) X does not have a pair of extremal rays of the type $\mathfrak{B}_2$ and $\overline{NE}(X)$ is simplicial;

(2) X has ≤ 2 extremal rays of the type (I) (more generally, $k + (n_1 - 1) + ... + (n_k - 1) \le 2$ with notations of Theorem 3.1);

(3) (by the author and V.V. Shokurov) The Picard number $\rho(X) = \dim N_1(X) \leq 4k + 5l_2 + 29 \leq 40$.

Proof. From the statement (1) of Theorem 1.3.2, the statement (1) follows.

Let $\mathcal{E}$ be an extremal set of Kodaira dimension 3 of extremal rays of the type (I) or (II) on X. By Proposition 1.2.1, the $\mathcal{E}$ is elliptic. By Theorem 2.2.6 and calculations of Section 4.2, the graph $G(\mathcal{E})$ is one of elliptic graphs of the Tables 4.1, 4.4, 4.5, 4.6 and Theorem 4.2.2.1.

In particular, $\#\mathcal{E} \leq 2$ if $\mathcal{E}$ contains extremal rays of the type (II) and the graph $G(\mathcal{E})$ is full (i.e. any two vertices are joined by non-single arrows). Thus, for the constant $q(X)$ of Basic Theorems 1.3.2 and 3.2 (see Definition 1.3.1), we have $q(X) \leq 2$. By Theorems 1.3.2, (2) and 3.2, (2) we get the statement (2).

Now let us estimate $\rho(X)$. To demonstrate how Basic Theorems 1.3.2, (3) and 3.2, (3) do work, we first give worse estimates for $\rho(X)$ which give these general Theorems.

First, let us apply Theorem 1.3.2, (3).

Considering Tables 4.1, 4.4, 4.5 and 4.6, one can easily see that for $d \geq 2$, we have (where we take the distance in the graph $G(\mathcal{E})$):

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1(d)\sharp\mathcal{E};$$

and

$$\sharp\{(R_1, R_2) \in \mathcal{E} \times \mathcal{E} \mid d+1 \leq \rho(R_1, R_2) \leq 2d+1\} \leq C_2(d)\sharp\mathcal{E}.$$

where

$$C_1(d) \leq 2d, \quad C_2(d) \leq 2(d+1). \tag{4-3-1}$$

Let $\mathcal{L}$ be an E-set on X. Since all extremal rays on X are of the type (I) or (II) and X does not have a pair of extremal rays of the type $\mathfrak{B}_2$, the $\mathcal{L}$ is Lanner by Lemma 1.2.11. Using calculations of Section 4.2, we then get that the diameter

$$\text{diam }(G(\mathcal{L})) \leq 4.$$

Here the maximum diam $(G(\mathcal{L})) = 4$ we get for the Lanner diagram with 6 vertices of the Table 4.4 and the Lanner diagrams $\mathbf{F}_4^\bullet(k; a, b)_1$ and $\mathbf{F}_4^\bullet(k; a, b)_2$ of the type (D) of the Table 4.5. Thus, the constant $d(X)$ of Basic Theorem 1.3.2 has the estimate $d(X) \leq 4$. Here, for Lanner diagrams of the type (D) (Table 4.5), we use the result of V.V. Shokurov (see Appendix): for an extremal ray R of the type (I) on X, there exists a curve $C \in R$ such that $C \cdot D(R) = -k$ where $1 \leq k \leq 3$. By (4-3-1), we can apply Theorem 1.3.2 with the constants $C_1(X) = 8$ and $C_2(X) = 10$. By Theorem 1.3.2, we get $\rho(X) = \dim N_1(X) \leq (16/3)C_1(X) + 4C_2(X) + 6 = (16/3) \cdot 8 + 4 \cdot 10 + 6 = 88 + 2/3 < 89$. Thus, Basic Theorem 1.3.2 gives the estimate

$$\rho(X) \leq 88.$$

Now let us apply Basic Theorem 3.2, (3). Similar considerations give constants:

$$n(X) \leq 4, n(X) \leq 5, n(X) \leq 4, d(X) \leq 2, C(X) \leq 5$$

Thus, by Basic Theorem 3.2, (3),

$$\rho(X) \leq 4k + 5l_2 + 8 \cdot 5 + 6 = 4k + 5l_2 + 46,$$

and

$$\rho(X) \leq 56.$$

(since $k + l \leq q(X) = 2$).

To get the strong estimate

$$\rho(X) \leq 4k + 5l_2 + 30 \leq 40,$$

we should change for this case the proof of the statement (3) of Basic Theorem 3.2. We should change the definition (3-3) of the weight $\sigma_A(\angle)$ of oriented angles of γ according to É.B. Vinberg [V1, §6, Sect. 1]. We denote this new weight as $\sigma_{AV}(\angle)$.

This is the following: We consider the extremal set $\mathcal{R}(\angle)'$ (see Definition 3.1) and connected components of the graph $G(R(\angle)')$. This graph contains non-single arrows only and this connected components have types

$$\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2$$

by Theorem 3.2.1.1, (a).

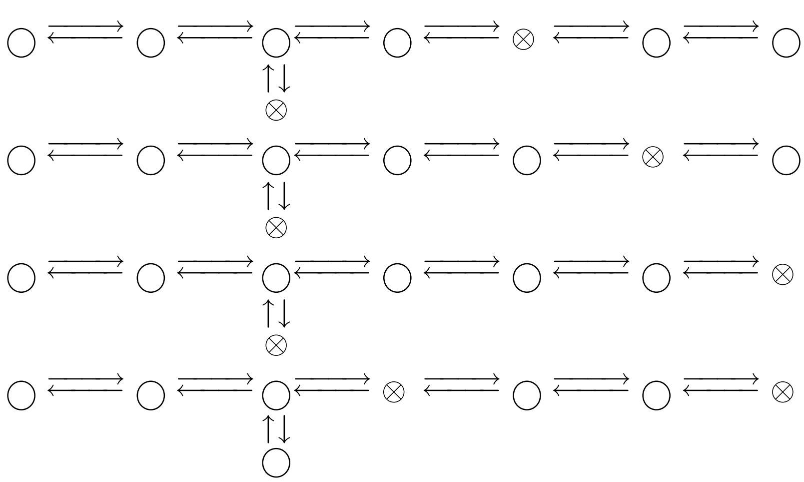
The weight $\sigma_{AV}(\angle)$ is non-negative. It is not equal to zero only if both extremal rays $R_1(\angle)$, $R_2(\angle)$ belong to one connected component S of the graph $G(\mathcal{R}(\angle)')$ and we have one of five cases below:

(I) $\rho_A(R_1(\angle), R_2(\angle)) = 1$ (i.e. $R_1(\angle)$ and $R_2(\angle)$ are adjoined in $G(S)$;

(II) the connected component S contains ≤ 7 vertices;

(III) the connected component S is classical (i.e. it has the type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$ or $\mathbf{D}_n$) and $R_1(\angle), R_2(\angle)$ belong to the terminal interval of the order ≤ 6 of $G(S)$ (for S of the type $\mathbf{D}_n$, $n \geq 5$, by definition, the terminal interval is a connected subgraph of $G(S)$ which contains a terminal vertex of S and is invariant relative to the involution automorphism of $\mathbf{D}_n$.

(IV) S has the type $\mathbf{E}_8$ and the pair $R_1(\angle)$, $R_2(\angle)$ is different from pairs of vertices of $\mathbf{E}_8$ below marked by $\otimes$:



The weight $\sigma_{AV}(\angle) = 1$ if

(V) S has ≤ 4 vertices.

In all other cases (I)–(IV) above $\sigma_{AV}(\angle) = 1/2$.

Now we should check conditions of Vinberg's Lemma 3.7 with the constants $C = 3$ and $D = 0$ which gives the desirable estimate: $\dim \gamma < 30$.

The proof of the condition (1) is very similar to Vinberg [V1, §6, Sect. 2] and is not difficult.

To prove the condition (2), one should also follow to Vinberg [V1, §6, Sects. 3—12]. For our case, elliptic diagrams are only "crystallographic", i. e. of the types

$$\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2,$$

and E-sets (or Lanner diagrams) have only types of Table 4.3. One should follow to Vinberg step by step (we should say that his considerations are long and very delicate, and it is a hard work), and check that in fact for our "crystallographic case" he uses only two arguments which work for our situation: There do not exist a configuration of extremal rays of the type (II) with one of extended Dynkin diagram of Table 4.2, because then some linear combination of divisors of these rays gives a *nef* element with zero cube. Besides, two non-extremal sets of extremal rays of the type (I) or (II) cannot be orthogonal (Lemma 3.4).

Only for the "non-crystallographic case", Vinberg uses *superhyperbolicity* arguments which probably do not work for our situation.

This finishes the proof of Theorem.

5. $\mathbb{Q}$-factorial models of Calabi-Yau 3-folds.

Applying Diagram Method to an arbitrary 3-dimensional Calabi-Yau manifold X, we have to avoid two problems: Mori cone $\overline{NE}(X)$ may not be finite polyhedral, and X may have small extremal rays. Here we want to discuss one possibility to avoid these problems and involve non-polyhedral case and small extremal rays to the game.

One can make several transformations (i) and (ii) below:

(i) Contraction of a divisorial extremal ray.

(ii) Flop in a small extremal ray.

Repeating this operations, we get some 3-fold Y. The 3-fold Y still has the most important for us properties: Y has $\mathbb{Q}$-factorial (canonical) singularities and belongs to the class $\mathcal{LT}$. See Kawamata [Ka2] and Shokurov [Sh2].

Definition 5.1. A 3-fold Y one can get starting from a 3-dimensional Calabi-Yau manifold X and repeating transformations (i) and (ii) is called a *$\mathbb{Q}$-factorial model of the Calabi-Yau manifold X.*

Considering $\mathbb{Q}$-factorial models, we get a chance to avoid cases when either Mori cone is not finite polyhedral or there exists a small extremal ray. If the $\mathbb{Q}$-factorial model Y has a finite polyhedral Mori cone and does not have a small extremal ray, we can apply Diagram Method to Y to find a rational *nef* element h on Y with cube 0. By the way, using the statement (1) of Basic Theorem 1.3.2, we can prove that transformations (i) and (ii) do not make situation worse from Diagram Method point of view.

Lemma 5.2. *Let Y be a $\mathbb{Q}$-factorial model such that we have properties (a), (b) and (c) below:*

(a) $\overline{NE}(Y)$ is finite polyhedral;

(b) Y does not have a small extremal ray;

(c) Y does not have a rational nef element h with $h^3 = 0$.

Then, starting from Y, repeating of transformations (i) and (ii) preserves properties (a), (b) and (c).

Proof. Let us consider a contraction $f : Y \to Y'$ of a divisorial extremal ray R on Y. One easily can see that Y' has properties (a) and (c) if Y has these properties. Besides, if Y does not have a small extremal ray, then Y' has a small extremal ray only if the divisor $D(R)$ contains another divisorial extremal ray Q. The image of Q then gives a small extremal ray on Y'. Thus, R and Q define a pair of the type $\mathfrak{B}_2$ on Y. This is impossible by Theorem 1.3.2, (1).

If Y does not have a small extremal ray, then Y does not have a flop and one does not have the transformation (ii).

We suggest the following

Conjecture 5.3. *There are absolute constants q, d, C_1, C_2, n_D, n_C, n_A, d_A, C_A such that for any 3-dimensional Calabi-Yau manifold X and any its $\mathbb{Q}$-factorial model Y we have estimates $q(Y) \le q$, $d(Y) \le d$, $C_1(Y) \le C_1$ and $C_2(Y) \le C_2$ for invariants of Definition 1.3.1, and $n(Y)_D \le n_D$, $n(Y)_C \le n_C$, $n(Y)_A \le n_A$, $d(X)_A \le d_A$ and $C(X)_A \le C_A$ for invariants of Definition 3.1.*

If this conjecture does hold, applying Theorems 1.3.2 and 3.2, we get an absolute estimate for $\rho(Y)$ if Y satisfies conditions (a), (b) and (c). Equivalently, if X has a $\mathbb{Q}$-factorial model Y with a finite polyhedral Mori cone, without small extremal rays and with big $\rho(Y)$, then Y has a rational *nef* element with cube zero. One can think that existence of rational *nef* element with cube 0 for a $\mathbb{Q}$-factorial model Y of X is of similar importance as for X.

Unfortunately, now we can prove Conjecture 5.3 only for very special $\mathbb{Q}$-factorial models Y.

We can consider part of results of Sect. 4, as the proof of Conjecture 5.3 for non-singular models (i.e. for 3-dimensional Calabi-Yau manifolds).

It is possible to extend these results for more large class of models which are still very special.

Definition 5.4. A $\mathbb{Q}$-factorial model Y of a 3-dimensional Calabi-Yau manifold is called *very good* if there exists a 3-dimensional Calabi-Yau manifold X and an extremal set $\mathcal{R} = \{R_1, ..., R_k\}$ of divisorial extremal rays on X with different divisors $D(R_1), ..., D(R_k)$ such that we have the following properties (i) and (ii):

(i) Y is a contraction $f : X \to Y$ of the face $\gamma = R_1 + ... + R_k$ of $\overline{NE}(X)$;

(ii) For any divisorial extremal ray R of Y there exists a divisorial extremal ray $\tilde{R}$ of X such that $R = f(\tilde{R})$. This means that for the contraction $f_R : Y \to Y'$ of R, the composition $f_R \cdot f : X \to Y'$ is the contraction of the face $R_1 + ... + R_k + \tilde{R}$.

We have

Theorem 5.5. *Let Y be a very good $\mathbb{Q}$-factorial model of a 3-dimensional Calabi-Yau manifold. Then Y has constants*

$$q(Y) \le 3, d(Y) \le 8, C_1(Y) \le 16, C_2(Y) \le 18,$$

and

$$n(Y)_D \le 8, n(Y)_C \le 9, n(Y)_A \le 9, d(Y)_A \le 8, C(Y)_A \le 17$$

In particular, by Theorem 1.2.3, Y has ≤ 3 extremal rays of the type (I) and

$$\rho(Y)\le (16/3)C_1(Y)+4C_2(Y)+6<164$$

if Y satisfies conditions (a), (b) and (c).

Considering preimage of the nef element on Y with cube 0, we get

Corollary 5.6. *Let X be a Calabi-Yau manifold X which has a very good $\mathbb{Q}$-factorial model Y such that Y has a finite polyhedral Mori cone $\overline{NE}(Y)$ and Y does not have a small extremal ray.*

Then X has a rational nef element h with $h^3=0$ if $\rho(Y)\ge 164$.

Sketch of the proof of Theorem 5.5. We use notation of Definition 5.4. For a set Q of divisorial extremal rays on Y, we denote $f^{-1}(Q)=\{R_1,...,R_k\}\cup\tilde{Q}$ where $\tilde{Q}=\{\tilde{R}\ \mid\ R\in Q\}$. The set $f^{-1}(Q)$ is called the preimage of Q and $\tilde{Q}$ is called the proper preimage of Q.

Next general statements will be very useful.

Lemma 5.7. *Let $\mathcal{E}$ be an elliptic set of divisorial extremal rays on Y. Then $f^{-1}(\mathcal{E})$ is elliptic.*

Proof. Let $\mathcal{E}=\{Q_1,...,Q_t\}$. Since $\mathcal{E}$ is elliptic, there are positive $a_1,...,a_t$ such that $Q_j\cdot(a_1D(Q_1)+\cdots+a_tD(Q_t))<0$ for all $1\le j\le t$. By Lemma 1.2.2, the set $\{R_1,...,R_k\}$ is elliptic. Hence, there are positive $b_1,...,b_k$ such that $R_i\cdot(b_1D(R_1)+\cdots+b_kD(R_k))<0$ for all $1\le i\le k$. Evidently,

$$f^*(a_1D(Q_1)+\cdots+a_tD(Q_t))=c_1D(R_1)+\cdots+c_kD(R_k)+a_1D(\tilde{Q}_1)+\cdots+a_tD(\tilde{Q}_t)$$

with non-negative $c_1,...,c_k$. By projection formula,

$$R_i\cdot(c_1D(R_1)+\cdots+c_kD(R_k)+a_1D(\tilde{Q}_1)+\cdots+a_tD(\tilde{Q}_t))=0$$

for all $1\le i\le k$, and

$$\begin{aligned}&\tilde{Q}_j\cdot(c_1D(R_1)+\cdots+c_kD(R_k)+a_1D(\tilde{Q}_1)+\cdots+a_tD(\tilde{Q}_t))=\\ =&Q_j\cdot(a_1D(Q_1)+\cdots+a_tD(Q_t)))<0.\end{aligned}$$

It follows that for a small $\epsilon>0$ the divisor

$$\begin{aligned}D(f^{-1}(Q))=&\epsilon(b_1D(R_1)+\cdots+b_kD(R_k))+\\ &+c_1D(R_1)+\cdots+c_kD(R_k)+a_1D(\tilde{Q}_1)+\cdots+a_tD(\tilde{Q}_t)\end{aligned}$$

has positive coefficients and has negative intersection with each element of $f^{-1}(Q)$. Thus, $f^{-1}(Q)$ is elliptic.

Lemma 5.8. *Let $\mathcal{L}$ be a Lanner set of divisorial extremal rays on Y. Then $f^{-1}(\mathcal{L})$ contains a quasi-Lanner subset $\mathcal{L}'$ such that $\tilde{\mathcal{L}}\subset\mathcal{L}'$ and such that for any non-empty subset $U\subset\tilde{\mathcal{L}}$ the set $\{R_1,...,R_k\}\cup(\mathcal{L}'-U)$ is elliptic.*

Proof. Let $\mathcal{L}=\{Q_1,...,Q_t\}$. Since $\mathcal{L}$ is Lanner, there are positive $a_1,...,a_t$ such that $R_i\cdot(a_1D(Q_1)+\cdots+a_tD(Q_t))\ge 0$ for all $1\le i\le t$ and one of these inequalities is strict. Evidently, $f^*(a_1D(Q_1)+\cdots+a_tD(Q_t)) = b_1D(R_1)+\cdots+b_kD(R_k)+a_1D(\tilde{Q}_1)+\cdots+a_tD(\tilde{Q}_t)$ where $b_j \geq 0$. By projection formula,

$$R\cdot(b_1D(R_1)+\cdots+b_kD(R_k)+a_1D(\tilde{Q_1})+\cdots+a_tD(\tilde{Q_t}))\geq 0$$

for any $R\in f^{-1}(\mathcal{L})$, and one of these inequalities is strict. Thus, the set $f^{-1}(\mathcal{L})$ is not semi-elliptic. By Proposition 2.2.8, $f^{-1}(\mathcal{L})$ contains a quasi-Lanner subset $\mathcal{L}'$. Since each proper subset of $\mathcal{L}$ is elliptic, by Lemma 5.7 any subset of $f^{-1}(\mathcal{L})$ is elliptic if it does not contain $\tilde{\mathcal{L}}$. It follows that $\tilde{\mathcal{L}}\subset\mathcal{L}'$ and $\{R_1,...,R_k\}\cup(\mathcal{L}'-U)$ is elliptic if U is a non-empty subset of $\tilde{\mathcal{L}}$.

Let us continue the proof of Theorem.

Let $\mathcal{L}$ be an E-set of divisorial extremal rays on Y such that any proper subset of $\mathcal{L}$ is extremal of Kodaira dimension 3 and $\mathcal{L}$ satisfies the condition (iii) of Sect. 1.1. By results of Sect. 1.2, the set $\mathcal{L}$ is Lanner. By Lemma 5.8, $\tilde{\mathcal{L}}\subset\mathcal{L}'$ where $\mathcal{L}'$ is a quasi-Lanner set of divisorial extremal rays on X. Considering image of $\mathcal{L}'$ by the morphism f, one sees that

$$\text{diam } \mathcal{L}\leq \text{diam } \mathcal{L}'.$$

Any proper subset of $\mathcal{L}'$ is either elliptic or connected parabolic (see Proposition 2.2.8). Using Theorem 2.2.6 and description in Sect. 4 of elliptic and connected parabolic sets of divisorial extremal rays on 3-dimensional Calabi-Yau manifolds, one can describe possible graphs of quasi-Lanner sets of divisorial extremal rays on X. In particular, one can see that

$$\sharp\mathcal{L}'\leq 10,\quad \text{diam } \mathcal{L}'\leq 8$$

for any quasi-Lanner set $\mathcal{L}'$ of divisorial extremal rays on a Calabi-Yau 3-dimensional manifold X (compare with [P, Fig. 1]). It follows that $n(Y)_A\leq 9$ and

$$d(Y)\leq 8.$$

By Theorem 2.2.6, we also get $n(Y)_D\leq 8$ and $n(Y)_C\leq 9$.

Now let us consider an extremal set $\mathcal{E}$ of Kodaira dimension 3 of divisorial extremal rays on Y. The set $f^{-1}(\mathcal{E})$ is also extremal of Kodaira dimension 3 and contains $\mathcal{R}=\{R_1,...,R_k\}$. For $Q_1,Q_2\in\mathcal{E}$ we have

$$\rho(Q_1,Q_2)=\rho_{\mathcal{R}}(\tilde{Q}_1,\tilde{Q}_2)$$

where $\rho(Q_1,Q_2)$ is the distance in the oriented graph $G(\mathcal{E})$ and

$$\rho_{\mathcal{R}}(\tilde{Q}_1,\tilde{Q}_2)=\min_{\gamma}\,(\rho(\gamma)-v(\gamma\cap\mathcal{R}))$$

where γ is an oriented path in $G(f^{-1}(\mathcal{E}))$ joining $\tilde{Q}_1$ and $\tilde{Q}_2$ and $v(\gamma\cap\mathcal{R})$ is the number of vertices of γ which belong to $\mathcal{R}$.

Using description in Sect. 4 of graphs of elliptic (in particular, extremal of Kodaira dimension 3) sets of divisorial extremal rays on X, one obtains that $q(Y)\leq 3$, $C_1(Y)\leq 16$, $C_2(Y)\leq 18$, $C(Y)\leq 17$. This finishes the proof.

We hope to describe more precisely quasi-Lanner sets of divisorial extremal rays on 3-dimensional Calabi-Yau manifolds and give better estimates for Theorem 5.5 and Corollary 5.6 in more advanced variant of this preprint.

We give another interesting application of Theorem 5.5.

Corollary 5.9. *Let X be a 3-dimensional Calabi-Yau manifold and Mori cone $\overline{NE}(X)$ is generated by a finite set of divisorial extremal rays. Let γ be a face of nef cone $NEF(X)$ and $R(\gamma^\perp)$ the set of all extremal rays orthogonal to γ. Assume that $R(\gamma^\perp)$ does not contain an extremal ray which belongs to a pair of the type $\mathfrak{B}_2$. Then γ contains a rational nef element h with $h^3 = 0$ if $\dim \gamma > 163$.*

Proof. Let $\mathcal{R} = R(\gamma^\perp)$ and $f : X \to Y$ the contraction of the face $\gamma^\perp \subset \overline{NE}(X)$ generated by $\mathcal{R}$. Since $\mathcal{R}$ does not have pairs of extremal rays of the type $\mathfrak{B}_2$, the morphism f is a sequence of contractions of divisorial extremal rays which are images of extremal rays from $\mathcal{R}$. Then Y has $\mathbb{Q}$-factorial singularities (canonical) and $\gamma = f^*(NEF(Y))$. Since $\overline{NE}(X)$ is generated by a finite set of divisorial extremal rays and $\mathcal{R}$ does not have an extremal ray which belongs to a pair of the type $\mathfrak{B}_2$, one sees that Y does not have a small extremal ray and is a very good $\mathbb{Q}$-factorial model of X with Mori cone $\overline{NE}(Y)$ generated by a finite set of divisorial extremal rays. If we additionally assume that $NEF(Y)$ (equivalently γ) does not have a *nef* element with cube 0, $\dim \gamma = \dim NEF(Y) < 164$ by Theorem 5.5.

At last, we mention the following conjecture by D. Morrison which is connected with the condition (a) above on Mori cone.

Conjecture 5.10. *(by D. Morrison,* [Mor2]*). For a Calabi-Yau manifold X, the nef cone $NEF(X)$ is rational finite polyhedral modulo the group Aut X of biregular automorphisms of X. In particular, $NEF(X)$ and Mori cone $\overline{NE}(X)$ are rational finite polyhedral if Aut X is finite.*

For example, the automorphism group Aut X of a 3-dimensional Calabi-Yau manifold X is finite if the cubic intersection hypersurface $\mathcal{W}_X$ is non-singular.

6. Concluding remarks.

We want to give several remarks about importance of existence of rational *nef* elements with cube 0 for Calabi-Yau 3-folds.

I.I. Piatetski-Shapiro and I.R. Shafarevich [PŠ-Š] proved that a K3 surface X has a rational *nef* element $h \in \text{Pic } X$ with $h^2 = 0$ if and only if the Picard lattice Pic X represents 0, i. e. there exists $0 \neq x \in \text{Pic } X$ such that $x^2 = 0$. In particular, this is true if $\rho(X) = \dim \text{Pic } X \geq 5$. Moreover, they proved that the linear system $|h|$ defines the elliptic fibration $|h| : X \to \mathbb{P}^1$ (i.e. the general fiber is an elliptic curve) if $h \in \text{Pic } X$ is *nef* and $h^2 = 0$.

One can ask about similar fact for Calabi-Yau 3-folds:

Question 6.1. *Does exist a rational nef element h with $h^3 = 0$ for a 3-dimensional Calabi-Yau manifold X if $\rho(X) = \dim N_1(X)$ is sufficiently big?*

Affirmative answer to this question is important because of two results below.

Theorem 6.2. *(P.M.H. Wilson,* [W1,(3.2)']*). Let X be a 3-dimensional Calabi-Yau manifold and h a rational nef element with $h^3 = 0$. Assume that $h^2 \not\equiv 0$ and $h \cdot c_2(X) > 0$. Then for a big N, the linear system $|Nh|$ defines an elliptic fibration $|Nh| : X \to S$ (i.e. S is a surface and the general fiber is an elliptic curve).*

Here the condition $h^2 \not\equiv 0$ is equivalent to that $\mathbb{C}h$ is not a singular point of the cubic intersection hypersurface $\mathbb{P}\mathcal{W}_X$. By Y. Miyaoka [Mi], we have the inequality $NEF(X) \cdot c_2(X) \geq 0$. Thus, for the "general case" when the cubic intersection hypersurface $\mathbb{P}\mathcal{W}_X$ is non-singular and $NEF(X) \cdot c_2(X) > 0$, existence of a rational *nef* element with cube 0 gives existence of an elliptic fibration. Thus, for this "general case", Theorems 4.3.1, 4.3.1' and Theorem 5.5 give existence of an elliptic fibration on Calabi-Yau manifolds under appropriate conditions. If either cubic intersection hypersurface $\mathbb{P}\mathcal{W}_X$ is singular or one only has the non-strict condition $NEF(X) \cdot c_2(X) \geq 0$, Corollary 5.9 is useful to satisfy conditions of Theorem 6.2, since Corollary 5.9 gives existence of rational *nef* elements h with cube 0 in "many" faces of the *nef* cone $NEF(X)$. We mention that it is conjectured that the condition $h \cdot c_2(X) > 0$ is not essential for the existence of the elliptic fibration $|Nh|$, see [W3]. By K. Oguiso [O], the linear system $|Nh|$ defines either elliptic or K3 or Abelian surface fibration if $|Nh|$ is not empty and N is big.

Existence of an elliptic fibration for a Calabi-Yau 3-fold (this Calabi-Yau 3-fold is called *elliptic*) is important because of the following result:

Theorem 6.3. *(M. Gross,* [Gro]*). There exists a finite set of families* $\mathcal{X}_1 \to \mathcal{M}_1, ..., \mathcal{X}_n \to \mathcal{M}_n$ *of elliptic Calabi-Yau* 3*-folds with* $\mathbb{Q}$*-factorial terminal singularities such that each elliptic Calabi-Yau* 3*-fold* X *with* $\mathbb{Q}$*-factorial terminal singularities is birationally isomorphic to a fiber of one of these families. (Here all families and isomorphisms preserve elliptic fiber structure.)*

See also connected results by I. Dolgachev and M. Gross [D-Gro] and A. Grassi [Gra].

Because of these Theorems 6.2. and 6.3 and results of this paper, one can think that may be classification of Calabi-Yau 3-folds is simpler for the high Picard number.

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Appendix by V.V. Shokurov: Anticanonical boundedness for curves

The purpose of this note is to generalize slightly results of [K].

Theorem. *Let $f: X \to S$ be a projective morphism of normal algebraic varieties over a field of characteristic zero, and D be an effective $\mathbb{R}$-divisor on X such that $K_X + D$ is $\mathbb{R}$-Cartier and divisorially log terminal near the generic points of a subvariety E, consisting of components of the* degenerate locus

$$\mathcal{E}xc(f) := \{x \in X \mid g \text{ is not finite at } x\}.$$

Then E is covered by a family (possibly disconnected) of effective 1-cycles $\{C_\lambda\}/S$ with $(-K_X - D.C_\lambda) \leq 2n$ where $n =$ dim E/S (and even $< 2n$ if $K_X + D$ is Kawamata log terminal in the generic points of E and $X \neq E$). Moreover, we could assume that the generic 1-cycles C_λ are curves, *i.e., reduced and irreducible, when $K_X + D$ is numerically definite, and the curves C_λ with $(K_X + D.C_\lambda) < 0$ (resp. ≤ 0) are rational.*

The Theorem implies the following results.

Corollary 1. *If, in addition, F is an $\mathbb{R}$-Cartier divisor such that $K_X + D + F$ is also divisorially log terminal near the generic points of E, $D + F$ is effective, and $K_X + D$ is numerically semi-negative with respect to f, then E is covered by a family (possibly disconnected) ofCKWARD E effective 1-cycles $\{C_\lambda\}/S$ (resp. with the curves as generic members when $K_X + D + F$ is numerically definite) with*

$$(F.C_\lambda) \geq -2n$$

(resp. $> -2n$ if $K_X + D + F$ is Kawamata log terminal in the generic points of E, and $X \neq E$).

For example, let f be an extremal divisorial contraction of a normal variety X with the exceptional $\mathbb{Q}$-Cartier divisor $F = E = \mathcal{E}xc(f)$, and K_X numerically semi-negative for f. Then we cover F by a family of rational curves $\{C_\lambda\}/S$ with

$$(F.C_\lambda) \geq -2 \dim X + 2.$$

In the low dimensional case, i.e., when $\dim X \leq 3$ we have a more sharp inequality $\geq -\dim X$. This is the best bound but it is known only in that case. (See Remark 3 below.)

Proof. According to the Theorem, take a family of 1-cycles $\{C_\lambda\}/S$ with respect to the second log divisor $K_X + D + F$. Then

$$(-K_X - D.C_\lambda) - (F.C_\lambda) = (-K_X - D - F.C_\lambda) \leq 2n$$

(resp. $< 2n$ if $K_X + D + F$ is Kawamata log terminal in the generic points of E, and $X \neq E$). But we assume that $(-K_X - D.C_\lambda) \geq 0$ which gives the required inequality.■

Corollary 2 ([Ko1], cf. also [K]). *Let $f: X \to S$ be a projective morphism of normal algebraic varieties over a field of characteristic zero, and D be an effective $\mathbb{R}$-divisor on X such that K_X+D is $\mathbb{R}$-Cartier, let H be an f-ample $\mathbb{R}$-Cartier divisor and $\varepsilon > 0$. Then the number of extremal contractions $cont_R$ and corresponding rays R such that $K_X + D$ is*

(*) divisorially log terminal near a generic point of the degenerate fibers of $cont_R$,

and such that $(K_X + D + \varepsilon H.R) < 0$, is finite.

Thus the half-cone $(K_X + D.R) < 0$ of the Kleiman-Mori cone $\overline{NE}(X/S)$ is locally polyhedral when (*) holds for all extremal rays in it, including the existence *of the extremal contractions $cont_R$.*

In particular, if X is normal projective with isolated singularities, and $\mathbb{Q}$-Gorenstein, we have at most finite number of extremal contractions which are negative for $K_X + \varepsilon H$ (cf. [Ko2, Th.]). Note that this does not mean that the half-cone $(K_X + D.R) < 0$ is always locally polyhedral. This means only, that whenever this fails, there exists an extremal *non-contracted* ray R with $(K_X + D.R) < 0$.

Proof. See [K].■

Conjecture. *In the Theorem we may replace the divisorially log terminal property near by the log canonical one in and always find a family $\{C_\lambda\}/S$ of curves.* Note that, even in the definite case either we consider a covering of the generic points of E, or we admit effective 1-cycles as curves C_λ for *degenerations*. Nevertheless in this case we abuse terminology and say on a covering by curves.

We should have even more.

Theorem′. *The Conjecture implies Corollaries 1-4 with the divisorially log terminal property near replaced by the log canonical one in and effective 1-cycles by curves.*

If $\dim X \leq 3$ *the Conjecture and the improved corollaries holds.*

Heuristic Arguments. Here we deduce the Conjecture and, in particular, the Theorem using the LMMP (the Log Minimal Model Program). Thus this proves the Theorem and Theorem′ for $\dim X \leq 3$ [Sh3]. For all dimensions, a proof of the Theorem refines [K] and [MM], and will be given later.

First, we discuss what means the log canonicity (or terminality) in a generic point of a subvariety $E \subseteq X$. A generic point P is one of its irreducible components. The log divisor $K_X + D$ is log canonical (resp. log terminal, purely log terminal, or Kawamata log terminal) in P if $K_X + D$ possesses that property in a generic point of P in a naive sense. A more rigorous point of view means that for the log discrepancies a_i of the divisors E_i such that the center $c(E_i) = P$, we have the log canonical property $a_i \geq 0$ (resp. in addition, $a_i > 0$ for such exceptional divisors E_i of one log nonsingular resolution over a neighborhood of the generic point P; $a_i > 0$ for all such exceptional divisors E_i; or $a_i > 0$ for all such divisors E_i) [Sh2]. Note that the purely log terminal property coincides with Kawamata one if P has codimension ≥ 2.

A property near means in a neighborhood.

We may assume that E and S are irreducible. For this add also that $\dim E/S = \dim E - \dim f(E)$ if E is irreducible, and the maximum of such dimensions for irreducible components of E in general.

Now we reduce the Theorem to the case when S is a point, i.e., X is a projective variety with a trivial morphism $f\colon X \to$ pt. . Taking a generic hyperplane section H on S and replacing X by its inverse image $f^{-1}H$, we could reduce the problem to the case when $f(E) = p$ is a point. Note that then $n =$ dim E.

If $S \neq p$ we can add a non-negative multiple of f^*H, where H is a generic hyperplane section on S through $f(E) = p$, in such a way that $K_X + D$ will be maximally log canonical in the generic point of E. This means that $K_X + D$ is log canonical in the generic point of E, and for any $\varepsilon > 0$, $K_X + D + \varepsilon f^*H$ is not so. In other words, there exists a divisor F with the log discrepancy 0 and with the center E.

In addition, if $S \neq p$, we may move and split D into a boundary preserving the log canonical property along E. So, $K_X + D$ will be log canonical in a neighborhood of the generic point of E.

If $F = E$ is a non-exceptional divisor on X, then D has the multiplicity 1 in E and we use the Adjunction Formula and Effectiveness [Sh2, 3.1 and 3.2.2]. This reduces to the case when $X = E^\nu$ and $S = p$.

Using the LMMP we can do the same in general. For this, we should replace X by a strict log minimal model $g\colon Y \to X$ of a neighborhood of E with respect to the log divisor $K_X + B$ where a boundary B has multiplicities $b_i =$ min $\{1, d_i\}$ (cf. [Sh2, 3.4]). According to the above perturbation of D, $D = B$ in a neighborhood of the generic point of E. Perhaps after an additional monoidal transform, we can also assume that $F = E$ is not exceptional on $X := Y$. Thereafter we replace $K_X + D$ by $f^*(K_X + D) = K_Y + D'$, i.e., D is replaced by an divisor D'. D' is effective by a construction of log models and by the Negativity of Birational Contractions [Sh2, 1.1]. The multiplicity of E in D is one by the construction. Here we meet one annoyance: dim E/S may be higher than $n =$ dim $g(E)/S$ when E is exceptional for a model g. However in that case we have an additional structure, namely, a projective *Iitaka* morphism $i = g|_E\colon E \to g(E)$. This means that i is a projective contraction, and $K_E + D_E = (K_X + D)|E$ is numerically trivial for i. Note that $n =$ dim $E/S =$ dim $i(E)$. The divisor E itself is a projective variety with a trivial morphism $f\colon E \to$ pt. . According to the Adjunction and Projection formula, the above properties of i follows from the assumption that $K_X + D$ is maximally log canonical in E, i.e., D has the multiplicity 1 in E. These imply also that a required family of curves in E induces that of on a subvariety $g(E)$ of the original X.

Take now $X = E$. By the construction X is again a normal, projective variety with a trivial morphism $f\colon X \to$ pt. . In addition, we have an Iitaka morphism $i\colon X \to E$ with respect to $K_X + D$. Also by the construction and the Adjunction formula $K_X + D$ is log canonical over the generic point of E. After additional blow ups, we can assume that X is $\mathbb{Q}$-factorial and $K_X + D$ is log terminal over the generic point of E. Since D has multiplicities $d_i > 1$ only for prime divisors D_i with $i(D_i) \neq E$, we can drop these components and suppose that $K_X + D$ is strictly log terminal, in particular, D is a boundary on X. However, this can spoil the Iitaka morphism i, namely, it may appear curves C in fibers of i such that $(K_X + D.C) \neq 0$. Nevertheless, i will be again the Iitaka morphism over the generic point of E. This is enough and coherent to the following arguments.

So, we should cover X by a family of curves $\{C_\lambda\}$ with $(-K_X - D.C_\lambda) \leq 2n$ where $n =$ dim E. Since $K_X + D$ is strictly log terminal and X is projective, we can apply the LMMP to X with the log divisor $K_X + D$. If $K_X + D$ is nef, then we take any family of curves $\{C_\lambda\}$ covering X. Otherwise we have an extremal contraction

$g: X \to Z$. If g has the fiber type, then the generic fiber F of g intersects the generic fiber of i in a finite set. Therefore, $\dim F \le n = \dim E$ and we can reduce the problem to fibers in this case. Note also that divisorial contractions and flips only decrease the intersection $(K_X + D.C_\lambda)$ for the generic curve of a covering family of X. Thus we can consider later only birational contractions.

If g is birational, we make a divisorial contraction or a flip of X. However, the exceptional locus F of g intersects the generic fiber of i at most in a finite set. If it does not intersect, we can make such transform and preserve birationally i. Otherwise $i(F) = E$ and F is finite over the generic point of E. In this case, we use above arguments for $E = F$ and the induction on the dimension of X, since the above arguments restrict the proof to a divisor. The correspondent Iitaka morphism is induced by i.

According to the termination of the LMMP, this completes the reduction to the case when $X = E$ and $S = p$ is a point. Thus X is normal and projective. After an additional blow ups, we can assume that X is also $\mathbb{Q}$-factorial. Now $n = dim X$, and we should cover X by a family of curves $\{C_\lambda\}$ with $(-K_X - D.C_\lambda) \le 2n$. Dropping D, and after an additional blow ups, we can assume that $D = 0$ and X has only log terminal or even terminal singularities. In the last case we use the MMP.

If K_X is nef, then we take any family of curves $\{C_\lambda\}$ covering X. Otherwise we apply the MMP. According to the termination, after a finite number of extremal transformation we reduce to the case when X possesses a Fano fibering. Taking the generic fiber, we get the case when X is a Fano variety having only terminal singularities. Then the required covering family exists according to Miyaoka and Mori [MM].

If $K_X + D$ is Kawamata log terminal in the generic points of E and $X \ne E$, we could replace f^*H by an effective Cartier divisor which is numerically negative on E, for instance, we could take a generic anti-hyperplane section of E over a neighborhood of $p = f(E)$. Then subsequent reductions give a stronger inequality $< 2n$. ■

Below in the proof of the Theorem, [MM] plays the same role. The previous heuristic arguments can be replaced by the following refinement of [K, Lemma]. We lose some properties stated in the conjecture. This is the price of homological methods (cf. [Sh3]).

Lemma. *Let $f: X \to Y$ be a projective morphism of normal algebraic varieties over a field of characteristic zero, H an $\mathbb{R}$-Cartier divisor on X, D an effective $\mathbb{R}$-divisor, E an irreducible component of the degenerate locus $\mathcal{E}xc(f)$ of f, $n = \dim E$, and $\nu: E^\nu \to E$ the normalization. Suppose that f is finite over the generic point of Y, H is nef with respect to f, $K_X + D$ is $\mathbb{R}$-Cartier and divisorially log terminal near the generic point of E and $f(E)$ is a point. Then*

$$(H^{n-1}.K_X + D.E) \ge ((\nu^*H)^{n-1}.K_{E^\nu})$$

(and even $>$ if H is ample and $K_X + D$ is Kawamata log terminal in the generic point of E).

Proof. For the non-strong inequality we can use an approximation of H by $\mathbb{Q}$-Cartier ample divisors. Moreover, we prove the following polylinear inequality

$$(H_1 \ldots H_{n-1}.K_X + D.E) \ge (\nu^*H_1 \ldots \nu^*H_{n-1}.K_{E^\nu})$$

where all H_i's are nef (and even $>$ if all H_i's are ample and $K_X + D$ is Kawamata log terminal in the generic points of E. Since any ample $\mathbb{R}$-divisor is a sum of an ample $\mathbb{R}$-divisor and an ample $\mathbb{Q}$-divisor, we may assume that all ample H_i's are $\mathbb{Q}$-Cartier and even Cartier.)

Thus as in [K], we suppose that Y is affine, H_i are very ample, and $n = 1$, i.e., we have no H_i's and E is a curve.

Since $K_X + D$ is divisorially log terminal near the generic point of E, then Supp D will be log nonsingular near the generic point of E unless $K_X + D$ is purely log terminal near the generic point of E.

In the log nonsingular case D is $\mathbb{R}$-divisor in the generic point of E. Then adding some small effective Cartier divisor trivial near the generic point of E we preserve the statement and may assume that $D - \varepsilon F \geq 0$ where $\varepsilon > 0$ and F is an effective Cartier divisor that coincides with D near the generic point of E. This makes $K_X + D$ Kawamata log terminal near the generic point of E and in particular purely log terminal near the generic point of E. As a limit for $\varepsilon \to 0$ we get the required inequality.

Thus perturbing D we may assume also that $K_X + D$ is purely and even Kawamata log terminal near the generic point of E except the case when E is a divisor, i.e. X is a surface. The above arguments also work for the last exception.

Since we are working over a field of characteristic zero, we can replace it by $\mathbb{C}$ and replace f by an analytic contraction over a small analytic neighborhood of $p = f(E)$ in Y.

Suppose that $(K_X + D.E) \leq$ deg K_{E^ν}. Then there exists a Cartier divisor A_0 on E such that deg $A_0 \geq (K_X + D.E)$ and $H^0(E^\nu, K_{E^\nu} - \nu^* A_0) \neq 0$. As in [K], we have $H^0(E, \omega(-A_0)) \neq 0$, and we can extend A_0 to a Cartier divisor A on X. Moreover, we may assume that $A - K_X - D$ is nef/Y and A is enough ample/Y on components of $\mathcal{E}xc(f)$ except E. Enough means enough for vanishings below.

According to our assumptions $K_X + D$ has nonpositive log discrepancies only for divisors F the centers of which intersects Y in finite sets.

Since $A - K_X - D$ is nef, we have $R^1 f_* \mathcal{I}_X(A) = 0$ where $\mathcal{I}_X$ is an ideal sheaf of a subscheme $S \subset X$ which intersects E in a finite set. This is by the proof of [KMM, 1.2.5] and the Kawamata log terminality of $K_X + D$ near the generic point of E. It is also important that D is effective,and X is normal. So, any function regular in codimension 2 on X will be regular everywhere.

According to the construction A is enough ample on S/Y, i.e., for $i \geq 1$, $R^i f_* \mathcal{O}_S(A) = 0$ by Serre. Therefore the exact sequence

$$0 \to \mathcal{I}_X(A) \to \mathcal{O}_X(A) \to \mathcal{O}_S(A) \to 0$$

implies the vanishing $R^1 f_* \mathcal{O}_X(A) = 0$.

Now the base change gives that $\mathrm{H}^1(f^{-1}p, A|_{f^{-1}p}) = 0$. Again the restriction sequence

$$0 \to \mathcal{J}(A|_{f^{-1}p}) \to \mathcal{O}_{f^{-1}p}(A|_{f^{-1}p}) \to \mathcal{O}_E(A_0) \to 0$$

and Serre vanishing imply the vanishing $\mathrm{H}^1(E, A_0) = 0$ and we obtain the required inequality as in [K]. For this note that the support of the ideal sheaf $\mathcal{J}$ intersects E at most in one-dimensional set E and we may choose A such that it is enough ample/Y, i.e., $\mathrm{H}^2(f^{-1}p, \mathcal{J}(A|_{f^{-1}p})) = 0$. ■

Proof of the Theorem. If $M = -\nu^*(K_X + D)$ is not f-nef on E^ν we have a curve C/S on E^ν with $(M.C) < 0$. Moreover, for 1-cycle $N\nu(C)/S$ with $N \gg$

0, $(-K_X - D.N\nu(C)) = (M.NC) \ll 0$. Then it is easy to add to $N\nu(C)$ an effective family of curves/S covering E and such that for the obtained 1-cycles C_λ/S, $(-K_X - D.C_\lambda) < 0 \leq 2n$.

Therefore, we suppose later that $M = -\nu^*(K_X + D)$ is f-nef on E. Then we can use the arguments of [K, Proof of the Theorem 1]. The only difference that now M is nef. Note that we can use [MM] even for a nef divisor M, because the required inequalities are not strong. Indeed, if M is a limit of ample $\mathbb{Q}$-Cartier divisors M_n for $n \to \infty$. Then the results of [MM] for M_n gives in a limit the same for M.■

Corollary 3 (cf. [K, Th. 2]). *For $f\colon X \to S$ and D as in the Theorem, let E be a subvariety, consisting of components of*

$$\begin{array}{c}\mathcal{E}xc(f){:=}\{x \in X \mid \text{an irreducible component of a fiber of } f \text{ through } x \\ \text{having the dimension greater than } d = \dim X/S\}\end{array}.$$

Then E is covered by a family (possibly disconnected) of effective 1-cycles $\{C_\lambda\}/S$ with $(-K_X - D.C_\lambda) \leq 2(n-d)$ where $n = \dim E/S$ (and even $< 2(n-d)$ if $K_X + D$ is Kawamata log terminal in the generic points of E). Moreover, we could assume that the generic 1-cycles C_λ are curves when $K_X + D$ is numerically definite, and these curves C_λ with $(K_X + D.C_\lambda) < 0$ (resp. ≤ 0) are rational.

Proof Take general hyperplane sections of S (cf. the Heuristic Arguments) and then use the Theorem.■

Corollary 4. *If, in addition to the Corollary 3, F is an $\mathbb{R}$-Cartier divisor such that $K_X + D + F$ is also divisorially log terminal near the generic points of E, and $K_X + D$ is numerically semi-negative with respect to f, then E is covered by a family (possibly disconnected) of effective 1-cycles (curves when $K_X + D + F$ numerically definite) $\{C_\lambda\}/S$ with*

$$(F.C_\lambda) \geq -2(n-d)$$

(resp. $> -2(n-d)$ if $K_X + D + F$ is Kawamata log terminal in the generic points of E).

Remarks. (1) Conjecturally, we my suppose that the generic members of $\{C_\lambda\}/S$ in the Corollaries 3 and 4 should always be curves. This is true when $n - d \leq 3$.

(2) If f is a projective morphism as in the Theorem we can introduce its D-length as the maximum of $(-K_X - D.C_\lambda)$ for families $\{C_\lambda\}/S$ covering X. Of course, the length is 0 when f is finite over the generic point of S, and by the Theorem the length is not higher than $2n$, if $K + D$ has in generic mild singularities, where $n = \dim X/S$.

According to [MM], a smooth projective variety X is uniruled if and only if its length (=0-length) is positive.

Here we could state problems for the length similar to that of for Fano indexes and minimal discrepancies [Sh1].

(3) The estimate $\leq 2n$ (respectively $< 2n$) is essentially derived from that of [MM] for Fano n-folds with terminal and even $\mathbb{Q}$-factorial singularities. For $n \leq 3$ and in the nonsingular case, according to the classification of such Fanos, we can improve the estimate up to $\leq n + 1$ (cf. [M]). Since terminal singularities are nonsingular for $n \leq 2$ this gives the same improvement in general for $n \leq 2$.

So, we have the following problems here. Does the estimate $\leq 2n$ sharp for $n \geq 3$? If not try to find the right one. Why not $n+1$?

(4) Perhaps, the proven results and the Conjecture hold in any characteristic.

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