STRONG RATIONAL CONNECTEDNESS OF TORIC VARIETIES

YIFEI CHEN AND VYACHESLAV SHOKUROV

Abstract. In this paper, we prove that: For any given finitely many distinct points $P_1, \dots, P_r$ and a closed subvariety S of codimension ≥ 2 in a complete toric variety over a uncountable (characteristic 0) algebraically closed field, there exists a rational curve $f : \mathbb{P}^1 \to X$ passing through $P_1, \dots, P_r$, disjoint from $S \setminus \{P_1, \dots, P_r\}$ (see Main Theorem). As a corollary, we prove that the smooth loci of complete toric varieties are strongly rationally connected.

Contents

1. Introduction

The concept of rationally connected varieties is independently invented by Kollár-Miyaoka-Mori ([KMM92b]) and Campana ([Ca92]). This kind of variety has interesting arithmetic and geometric properties.

A class of proper rationally connected varieties comes from the smooth Fano varieties ([Ca92], [KMM92a] or [Kol96]). Shokurov ([Sh00]), Zhang ([Zh06]), Hacon and McKernan ([HM07]) proved that FT (Fano type) varieties are rationally connected.

An interesting question is whether the smooth locus of a rationally connected variety is rationally connected. In general the answer of the question is NO. However, for the FT (or log del Pezzo) surface case, Keel and McKernan gave an affirmative answer, that is, if (S, Δ) is a log del Pezzo surface, then its smooth locus S^{sm} is rationally connected ([KM99]), but this does not imply the strong rational connectedness.

The concept of strongly rationally connected varieties (see Definition 5) was first introduced by Hassett and Tschinkel ([HT08]). A proper and smooth separably rationally connected variety X over an algebraically closed field is strongly rationally connected ([KMM92b] 2.1, or [Kol96] IV.3.9). Xu

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([Xu08]) announced that the smooth loci of log del Pezzo surfaces are not only rationally connected but also strongly rationally connected, which confirms a conjecture of Hassett and Tschinkel ([HT08], Conjecture 20). It is expected that the smooth locus of an FT variety is strongly rationally connected (cf. Example 2 and Main Theorem).

Throughout the paper, we are working over an uncountably algebraically closed field of characteristic 0. It is interesting that whether Main Theorem holds for any algebraically closed (or perfect) field.

Main Theorem. *Let X be a complete toric variety. Let $P_1, \dots, P_r$ be finitely many distinct points in X (P_i possibly singular). Then there is a geometrically free rational curve $f : \mathbb{P}^1 \to X$ over $P_i, 1 \leq i \leq r$ (see Definition 7). Moreover, f is free over P_i if all points P_i are smooth.*

Main Theorem can be rephrased as follows:

Let X be a complete toric variety. For any given distinct points $P_1, \dots, P_r \in X$ (possibly singular) and any given codimension ≥ 2 subvariety $S \subseteq X$, there is a rational curve $f : \mathbb{P}^1 \to X$ passing through $P_1, \dots, P_r$, disjoint from $S \setminus \{P_1, \dots, P_r\}$.

If all points P_i are smooth, then we get the following corollary.

Corollary 1. *The smooth locus of a complete toric variety is strongly rationally connected.*

2. Preliminaries

When we say that x is a point of a variety X, we mean that x is a closed point in X.

A *rational curve* is a nonconstant morphism $f : \mathbb{P}^1 \to X$.

A normal projective variety X is called *FT* (*Fano Type*) if there exists an effective $\mathbb{Q}$-divisor D, such that (X, D) is klt and $-(K_X + D)$ is ample. See [PSh09] Lemma-Definition 2.6 for other equivalent definitions.

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n. A *toric variety* $X(\Delta)$ is associated to a fan Δ, a finite collection of convex cones $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ (see [Fu93] or [Oda88]).

Example 2. Projective toric varieties are FT. Let K be the canonical divisor of the projective toric variety $X(\Delta)$, T be the torus of X, and $\Sigma = X \setminus T = \sum D_i$ be the complement of T in X. Then K is linearly equivalent to $-\Sigma$. Since X is projective, there is an ample invariant divisor L. Suppose that $L = \sum d_i D_i$. Let the polytope $\Box_L = \{m \in M | \langle m, e_i \rangle + d_i \geq 0, \forall e_i \in \Delta(1)\}$, where M is the dual lattice of N, and $\Delta(1)$ is the set consisting of 1-dimensional cones in Δ. Let u be an element in the interior of $\Box_L$. Let χ^u be the corresponding rational function of $u \in M$ (see [Fu93] section 1.3), and div χ^u be the divisor of χ^u. Then $D =$div $\chi^u + L$ is effective and ample and has support Σ. That is, $D = \sum d_i' D_i$ and all $d_i' > 0$.

Let ϵ be a positive rational number, such that all coefficients of prime divisors in ϵD are strictly less than 1. Then $\Sigma - \epsilon D$ is effective. It is easy to check that $(X, \Sigma - \epsilon D)$ is klt, and $-(K + \Sigma - \epsilon D) \sim \epsilon D$ is ample. Hence X is FT.

Definition 3. An *isogeny* of toric varieties is a finite surjective toric morphism. Toric varieties X and Y are said to be *isogenous* if there exists an isogeny $X \to Y$. The *isogeny class* of a toric variety X is a set consisting of all toric varieties Y such that X and Y are isogenous.

Theorem 4. *Let $f : X \to Y$ be a finite surjective toric morphism. Then there exists a finite surjective toric morphism $g : Y \to X$.*

Proof. Let $f : X \to Y$ be a finite surjective toric morphism of toric varieties and $\varphi : (N', \Delta') \to (N, \Delta)$ be the corresponding map of lattices and fans. Then we can identify N' as a sublattice of N and $\Delta' = \Delta$.

There is an positive integer r such that rN is a sublattice of N'. Let g be the corresponding toric morphism of $(rN, \Delta) \to (N', \Delta)$. Since (rN, Δ) and (N, Δ) induce an isomorphic toric variety, we get $g : Y \to X$ is a finite surjective toric morphism. □

The properties of isogeny:

1) Isogeny is an equivalence relation.

2) If a toric variety Y is in the isogeny class of X and $\mu : X \to Y$ is the isogeny, then there is a one-to-one correspondence between the set of orbits $\{O_i^X\}$ of X and the set of orbits $\{O_i^Y = \mu(O_i^X)\}$ of Y. Hence $\dim O_i^X = \dim O_i^Y$ for all i, and the number of orbits is independent of the choice of toric varieties in an isogeny class of X.

A variety X over a characteristic 0 field is rationally connected, if any two general points $x_1, x_2 \in X$ can be connected by a rational curve of X of a bounded family.

Definition 5. ([HT08] Definition 14.) A smooth rationally connected variety Y is *strongly rationally connected* if any of the following conditions hold:

(1) for each point $y \in Y$, there exists a rational curve $f : \mathbb{P}^1 \to Y$ joining y and a generic point in Y;

(2) for each point $y \in Y$, there exists a free rational curve containing y;

(3) for any finite collection of points $y_1, \ldots, y_m \in Y$, there exists a very free rational curve containing the y_j as smooth points;

(4) for any finite collection of jets

$$\mathrm{Spec}\ k[\epsilon]/\langle \epsilon^{N+1} \rangle \subset Y, \quad i = 1, \ldots, m$$

supported at distinct points $y_1, \ldots, y_m$, there exists a very free rational curve smooth at $y_1, \ldots, y_m$ and containing the prescribed jets.

Definition 6. Let X be a complete normal variety, B be a set of finitely many closed points in $\mathbb{P}^1$, and $g : B \to X$ be a morphism. A rational curve $f : \mathbb{P}^1 \to X$ is called *weakly free* over g if there exist an irreducible family of rational curves T and an evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ such that

1) $f = f_{t_0} = \mathrm{ev}|_{\mathbb{P}^1 \times t_0}$ for some $t_0 \in T$,

2) for any $t \in T$, $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t}$ is a rational curve and $f_t|_B = g$,

3) the evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ by $\mathrm{ev}(x,t) = f_t(x)$ is dominant.

We say that a *rational curve* $f' : \mathbb{P}^1 \to X$ is a *general deformation* of f, or f' is a *sufficiently general weakly free rational curve*, if there is an open dense subset U of T, such that $f' = f_t$ and $t \in U \subseteq T$. We say that a *weakly free rational curve* $g : \mathbb{P}^1 \to X$ is a *general deformation of* f, if there is an irreducible family T', such that $T \cap T'$ contains an open dense subset in T, $g = g_{t'}$ for some $t' \in T'$ and g is weakly free in its own family.

Definition 7. Let X be a complete normal variety, B be a set of finitely many closed points in $\mathbb{P}^1$, and $g : B \to X$ be a morphism. A rational curve $f : \mathbb{P}^1 \to X$ is called *geometrically free* over g if there exist an irreducible family of rational curves T and an evaluation morphism ev: $\mathbb{P}^1 \times T \to X$ such that

1) $f = f_{t_0} = \mathrm{ev}|_{\mathbb{P}^1 \times t_0}$ for some $t_0 \in T$,

2) for any $t \in T$, $f_t = \mathrm{ev}|_{\mathbb{P}^1 \times t}$ is a rational curve and $f_t|_B = g$,

3) for any codimension 2 subvariety Z in X, $f_t(\mathbb{P}^1) \cap Z \subseteq g(B)$ for general $t \in T$ (general meaning t belongs to a dense open subset in T, depending on Z).

If X is smooth over an uncountable field of characteristic 0, then weak freeness over g is equivalent to usual freeness over g if $|B| \leq 2$.

Remark. In our application, we usually assume g is one-to-one. Let $P_i = g(Q_i)$ where $B = \{Q_i\}$. Without confusion, we say f is geometrically free over $\{P_i\}$ (resp. weakly free over $\{P_i\}$) instead of saying that f is geometrically free over g (resp. weakly free over g).

Weak freeness and geometric freeness are generalizations of usual freeness (see [Kol96] II.3.1 Definition) if the curve passes through singularities. To consider weakly free rational curves or geometrically free rational curves, we think of them as general members in a certain family. In particular, we can suppose that the morphism ev is flat.

Example 8. Let X be a projective cone over a conic. Let T be the family of all lines through the vertex O. Then $l \in T$ is not free. However l is weakly free and geometrically free over O by construction.

We need a resolution as follows.

Theorem 9. *Let X be a toric variety. Let Σ be the invariant locus of X. Let $P_1, \ldots, P_r \in X$ be r points. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over $P_1, \ldots, P_r$. Then there exists a resolution $\pi : \tilde{X} \to X$, such that*

1) $\pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor with simple normal crossing;

2) $\pi^{-1}(P_j) \subseteq \pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor for each point P_j;

3) $\pi : \tilde{X} \to X$ is an isomorphism over $X \setminus (\mathrm{Sing}\, X \cup \{P_i\})$;

4) sufficiently general $\tilde{f}(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P_i\})$ over each P_j only in divisorial points of $\pi^{-1}(\Sigma \cup \{P_i\})$, where $\tilde{f} : \mathbb{P}^1 \to \tilde{X}$ is the proper birational transformation of a general deformation of f and is a (weakly) free rational curve.

More generally, let $f_j : \mathbb{P}^1 \to X$, $1 \leq j \leq m$ be finitely many sufficiently general weakly free rational curve over a subset of $\{P_i\}$, where $\{P_i\}$ is a set of finitely many distinct points in X. Then there exists a resolution $\pi : \tilde{X} \to X$ such that

1') $\pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor with simple normal crossing;

2') $\pi^{-1}(P_i) \subseteq \pi^{-1}(\Sigma \cup \{P_i\})$ is a divisor for each point P_i;

3') $\pi : \tilde{X} \to X$ is an isomorphism over $X \setminus (\mathrm{Sing}\ X \cup \{P_i\})$;

4') For each j, sufficiently general $\tilde{f}_j(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P_i\})$ over each P_i only in divisorial points of $\pi^{-1}(\Sigma \cup \{P_i\})$, where $\tilde{f}_j : \mathbb{P}^1 \to \tilde{X}$ is the proper birational transformation of a general deformation of f_j and is a (weakly) free rational curve.

Proof. When the ground field is of characteristic 0, 1)-3) follow from usual facts in the resolution theory, e.g. see [KM98] Theorem 0.2. However, in the toric or toroidal case, the same result holds for any field. More precisely, if all P_i are invariant, we can use a toric resolution. If some P_i are not invariant, they can be converted into toroidal invariant points P_i after a toroidalization.

We say that $\tilde{f}(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P_i\})$ over each P_i in a divisorial point x if x belongs to only one prime divisor of $\pi^{-1}(\Sigma \cup \{P_i\})$ for some i and the prime divisor is over P_i. To fulfill 4), we need extra resolution over intersections of the divisorial components of $\pi^{-1}(\Sigma \cup \{P_i\})$ through which general $\tilde{f}$ is passing over P_i. Termination of such resolution follows from an estimation by the multiplicities of intersection for $f(\mathbb{P}^1)$ with Σ. The last resolution is independent of the choice of a general rational curve by Lemma 12 below. However it depends on the choice of intersections of divisorial components. For more details, see the proof of Lemma 4.3.4 in [Ch09].

For the general statement, we can get 1')-3') in a similar manner above. To fulfill 4'), we just need extra resolutions over each point P_i. □

We discuss some examples of rational curves on projective spaces and quotient projective spaces.

Example 10. For any given subvariety S of codimension ≥ 2 in $\mathbb{P}^n$, any points $P_1, \ldots, P_r \in \mathbb{P}^n$, and any integer $d \geq r$, there exists a rational curve C of degree d, such that each $P_i \in C$ and $C \cap S = \emptyset$.

Indeed, we can construct a tree T with r branches, such that each P_i is a smooth points on a unique branch and disjoint from S. The tree can be smoothed into a rational curve C passing through $P_1, \ldots, P_r$, disjoint from S. The rational curve C has degree r. For $d \geq r$, we can attach $d - r$ rational curves to the tree T, and smooth it.

Applying Example 10, we get

Example 11. Let $\pi : \mathbb{P}^n \to X$ be a finite morphism, S be a codimension ≥ 2 subvariety in X, and $\{P_i\}_{i=1}^m$ be a set of m points outside S. Then there exists a rational curve C, such that each $P_i \in C$ and $C \cap S = \emptyset$.

In particular, the same result holds if X is a quotient space $\mathbb{P}^n/G$, where G is a finite group, for example, if X is a weighted projective space. It is well known that if X is a complete $\mathbb{Q}$-factorial toric variety with Picard number one, then there exist a weighted projective space Y and a finite toric morphism $\pi : Y \to X$. So the same result holds for rational curves on a complete $\mathbb{Q}$-factorial toric variety with Picard number one. It is a very special case of our Main Theorem.

3. Proof of Main Theorem

In this section we prove Main Theorem. Let us first prove Main Lemma, which is a special weak case of Main Theorem.

Main Lemma. *Let X be a complete toric variety. Let $P, Q \in X$ be two distinct points (P, Q possibly singular). Let $S \subseteq X$ be a closed subvariety of codimension ≥ 2. Then there exists a weakly free rational curve on X over P, Q, disjoint from $S \setminus \{P, Q\}$.*

To prove Main Lemma, we need some preliminaries.

Lemma 12. *Let f be a weakly free rational curve on X, and $F_1, \ldots, F_s \subseteq X$ be s proper irreducible subvarieties in X. Then there exist $s', 0 \leq s' \leq s$, subvarieties among $\{F_j\}$ (after renumbering we assume they are $F_1, \ldots, F_{s'}$) such that a general deformation of f intersects $F_1, \ldots, F_{s'}$, and is disjoint from $F_{s'+1}, \ldots, F_s$.*

The proof of this Lemma is a standard exercise in incidence relations. See [Ch09] Lemma 4.3.2 for a detailed proof.

Lemma 13. *Let X be a complete toric variety. Let $P, Q \in X$ be two points (possibly singular), and S be a closed subvariety of codimension ≥ 2. Let $F_1, \ldots, F_s$ be all the irreducible components of* Sing X. *Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over P, Q. Suppose $f(\mathbb{P}^1)$ intersects $F_1 \setminus \{P, Q\}, \ldots, F_{s'} \setminus \{P, Q\}$, and is disjoint from $F_{s'+1} \setminus \{P, Q\}, \ldots, F_s \setminus \{P, Q\}$. Then there exists a weakly free rational curve f' over $\{P, Q\}$, which is a general deformation of f, such that $f'(\mathbb{P}^1)$ is disjoint from $((S \setminus \operatorname{Sing} X) \cup F_{s'+1} \cup \ldots \cup F_s) \setminus \{P, Q\}$. Moreover, for any fixed closed subvariety Z of X, if $f(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$, then $f'(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$.*

Proof. Applying Theorem 9 to the toric variety X and two points $\{P, Q\}$, we get a resolution $\pi : \tilde{X} \to X$ satisfying 1)-3) in the theorem and a weakly free rational curve $\tilde{f} : \mathbb{P}^1 \to \tilde{X}$ satisfying 4) in the theorem. A general deformation $\tilde{f}'$ of $\tilde{f}$ is weakly free, so $\tilde{f}'$ is free by [Kol96] II.3.11 (Here we need the assumption that the ground field is uncountable and of characteristic 0.)

Moreover, we can assume that $\tilde{f}'$ is disjoint from $(S \setminus \mathrm{Sing}\ X) \setminus \pi^{-1}\{P, Q\}$ by [Kol96] II.3.7.

On the other hand, let Σ be the invariant locus of X. Notice that $\mathrm{Sing}\ X \subseteq \Sigma$. Then by Theorem 9, $\tilde{f}(\mathbb{P}^1)$ intersects $\pi^{-1}(\Sigma \cup \{P, Q\})$ divisorially over P, Q, and $\tilde{f}(\mathbb{P}^1)$ is disjoint from the closure of $\pi^{-1}(F_{s'+1} \setminus \{P, Q\}), \dots, \pi^{-1}(F_s \setminus \{P, Q\})$. So the general deformation $\tilde{f}'$ of $\tilde{f}$ intersects open subsets of divisors $\pi^{-1}(P)$ and $\pi^{-1}(Q)$, disjoint from the closure of $((S \setminus \mathrm{Sing}\ X) \setminus \pi^{-1}\{P, Q\}) \cup \pi^{-1}(F_{s'+1} \setminus \{P, Q\}) \cup \cdots \cup \pi^{-1}(F_s \setminus \{P, Q\})$. We apply Lemma 14 by replacing f' by $\tilde{f}'$, dominant morphism μ by $\pi : \tilde{X} \to X$, $\{P_i\}$ by $\{P, Q\}$, and S by $(S \setminus \mathrm{Sing}\ X) \cup F_{s'+1} \cup \cdots \cup F_s$. Then we get the weakly free rational curve $f' = \pi \tilde{f}' : \mathbb{P}^1 \to X$ is a general deformation of f (see Definition 6), passing through points P, Q and disjoint from $((S \setminus \mathrm{Sing}\ X) \cup F_{s'+1} \cup \cdots \cup F_s) \setminus \{P, Q\}$.

Moreover, we can assume that f' is a weakly free rational curve over P, Q, by a base change of the family to which f' belongs (For details, see the proof of Lemma 4.3.1 in [Ch09]).

The last statement can be proved similarly. □

Lemma 14. *Let X, X' be two complete varieties with $\dim X > 0$. Let $\mu : X' \to X$ be a dominant morphism. Then the image of a weakly free rational curve on X' is weakly free on X in the following sense:*

Let $P_1, P_2, \dots, P_r \in \mu(X)$ be r distinct points, and $S \subseteq X$ be a closed subvariety. Let $S' = \mu^{-1}S$, and $P'_1, P'_2, \dots, P'_r \in X'$ be points such that $\mu(P'_i) = P_i$ for $i = 1, \dots, r$. If $f' : \mathbb{P}^1 \to X'$ is a weakly free rational curve over $P'_1, P'_2, \dots, P'_r$, disjoint from $S' \setminus \{P'_1, P'_2, \dots, P'_r\}$, then $f = \mu \circ f''$ is a weakly free rational curve on X over $P_1, P_2, \dots, P_r$, disjoint from $S \setminus \{P_1, P_2, \dots, P_r\}$, where f'' is a general deformation of f'.

Proof. Since f' is weakly free, ev: $\mathbb{P}^1 \times T' \to X'$ is dominant, where T' is the family associated to f'. Since $\mu : X' \to X$ is dominant, ev: $\mathbb{P}^1 \times T' \to X' \to X$ is dominant. Hence for general deformation $f'' \in T'$ of f', $f = \mu \circ f''$ is a weakly free rational curve on X. □

Lemma 15. *Let X be a $\mathbb{Q}$-factorial toric variety, and O be a singular orbit of X. Then there exists an isogeny $\mu : Y \to X$, such that $\mu^{-1}(O)$ is smooth.*

Proof. Let (N, Δ) be the lattice and fan associated to X. Let N' be the sublattice generated by the primitive elements of the simplicial cone σ such that O is contained in the affine open subset σ corresponding to. Let Y be the toric variety corresponding to (N', Δ) and μ be the natural finite dominant morphism corresponding to $(N', \Delta) \to (N, \Delta)$. By construction of μ, $\mu^{-1}(O)$ is smooth. □

Proof of Main Lemma. Step 1. After $\mathbb{Q}$-factorization $q : X' \to X$, we can assume that X is a complete $\mathbb{Q}$-factorial toric variety ([Fj03] Corollary 3.6). Indeed, a weakly free rational curve on X' gives a weakly free rational curve on X by Lemma 14.

Step 2. A weakly free rational curve can be moved from any smooth variety of codimension ≥ 2 in the sense of Lemma 13. So we can reduce the proof of Main Lemma to the case $S = I(X)$, where $I(X)$ denotes the union of orbits of X of codimension ≥ 2. Since X is a toric variety, $\mathrm{Sing}\, X \subseteq I(X)$.

Indeed, for any subvariety $S \subseteq X$ of codimension ≥ 2, suppose there is a sufficiently general weakly free rational curve $f : \mathbb{P}^1 \to X$ over $P, Q \in X$, disjoint from $I(X) \setminus \{P, Q\}$. Apply Lemma 13 to the subvariety S, and the weakly free rational curve f. Since $\mathrm{Sing}\, X \subseteq I(X)$, $s' = 0$ in Lemma 13, that is, $f(\mathbb{P}^1)$ is disjoint from $F_1 \setminus \{P, Q\}, \dots, F_s \setminus \{P, Q\}$. Then there exists a weakly free rational curve f', which is a general deformation of f, such that $f'(\mathbb{P}^1)$ is disjoint from $((S \setminus \mathrm{Sing}\, X) \cup F_1 \cup \ldots \cup F_s) \setminus \{P, Q\} = ((S \setminus \mathrm{Sing}\, X) \cup \mathrm{Sing}\, X) \setminus \{P, Q\} = S \setminus \{P, Q\}$.

Step 3. Suppose that $I(X)$ consists of $\tilde{s}$ distinct orbits $O_1, \dots, O_{\tilde{s}}$. Let $f : \mathbb{P}^1 \to X$ be a sufficiently general weakly free rational curve over P, Q. By Lemma 12, we can assume that $f(\mathbb{P}^1)$ intersects with $O_1 \setminus \{P, Q\}, \dots, O_{s'} \setminus \{P, Q\}$, and is disjoint from $O_{s'+1} \setminus \{P, Q\}, \dots, O_{\tilde{s}} \setminus \{P, Q\}$ for some s'.

Notice that s' depends on the points P, Q and the variety X. However, since s' is bounded by $\tilde{s}$, and $\tilde{s}$ is independent of choice of X in an isogeny class, there exists an $\bar{s}$ such that for any toric variety Y in the isogeny class of X, and two distinct points $P', Q' \in Y$, there exists a weakly free rational curve $f'_{\bar{s}} : \mathbb{P}^1 \to Y$ over P', Q', such that $f'_{\bar{s}}(\mathbb{P}^1)$ intersects with at most $O_1^Y \setminus \{P', Q'\}, \dots, O_{\bar{s}}^Y \setminus \{P', Q'\}$, and is disjoint from $O_{\bar{s}+1}^Y \setminus \{P, Q\}, \dots, O_{\tilde{s}}^Y \setminus \{P, Q\}$, where O_i^Y are orbits of Y of codimension ≥ 2. Furthermore, we can assume that $\dim O_1^Y \geq \dim O_2^Y \geq \cdots \geq \dim O_{s'}^Y \geq \dim O_{s'+1}^Y \geq \cdots \geq \dim O_{\tilde{s}}^Y$. This order is good for us, because $\cup_{j \geq s} O_j^Y$ is closed for any s.

We fix a complete toric variety X, two points P, Q and a weakly free rational curve $f_{\bar{s}}$ over P, Q. By Lemma 14 and 15, we can suppose that the orbit $O_{\bar{s}}$ is smooth. Indeed, by Lemma 15, there is an isogeny $\mu : Y \to X$ such that $O_{\bar{s}}^Y = \mu^{-1}(O_{\bar{s}})$ is smooth. Let $P', Q' \in Y$ such that $\mu(P') = P, \mu(Q') = Q$. Then existence of a weakly free rational curve $f' : \mathbb{P}^1 \to Y$ over P', Q', disjoint from $O_{\bar{s}}^Y \cup \cdots \cup O_{\tilde{s}}^Y$, implies existence of a weakly free rational curve $f : \mathbb{P}^1 \to X$ over P, Q, disjoint from $O_{\bar{s}} \cup \cdots \cup O_{\tilde{s}}$, by Lemma 14 with $X' = Y, \{P_i\} = \{P, Q\}$ and $S = O_{\bar{s}}^Y \cup O_{\bar{s}+1}^Y \cup \cdots \cup O_{\tilde{s}}^Y$.

Step 4. Now, we prove that there is a weakly free rational curve $f_{\bar{s}-1}$ over P, Q such that $f_{\bar{s}-1}(\mathbb{P}^1)$ intersects at most $O_1 \setminus \{P, Q\}, \dots, O_{\bar{s}-1} \setminus \{P, Q\}$, and is disjoint from $O_{\bar{s}} \setminus \{P, Q\}, \dots, O_{\tilde{s}} \setminus \{P, Q\}$. Indeed, we have the following two cases:

1) If $f_{\bar{s}}(\mathbb{P}^1)$ is disjoint from $O_{\bar{s}} \setminus \{P, Q\}$, then let $f_{\bar{s}-1} = f_{\bar{s}}$.

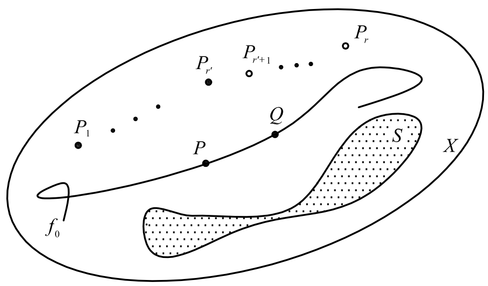
2) If $f_{\bar{s}}(\mathbb{P}^1)$ intersects $O_{\bar{s}} \setminus \{P, Q\}$, we apply Lemma 13 with $Z = O_{\bar{s}+1} \cup \cdots \cup O_{\tilde{s}}$ and $S = O_{\bar{s}} \cup Z$. Notice that S and Z are closed subvarieties of X of codimension ≥ 2, and $O_{\bar{s}}$ is smooth. In particular, $S \setminus \mathrm{Sing}\, X \supseteq O_{\bar{s}}$. By assumption, $f_{\bar{s}}(\mathbb{P}^1) \cap (Z \setminus \{P, Q\}) = \emptyset$. Therefore, by the Lemma, there exists a weakly free rational curve $f_{\bar{s}-1}$ on X, which is a general deformation of $f_{\bar{s}}$, such that $f_{\bar{s}-1}(\mathbb{P}^1)$ intersects at most $O_1 \setminus \{P, Q\}, \dots, O_{\bar{s}-1} \setminus \{P, Q\}$,

and is disjoint from $(O_{\bar{s}} \cup Z) \setminus \{P, Q\} = (O_{\bar{s}} \setminus \{P, Q\}) \cup (O_{\bar{s}+1} \setminus \{P, Q\}) \cup \cdots \cup (O_{\tilde{s}} \setminus \{P, Q\})$.

Step 5. By induction on $\bar{s}$, there is a weakly free rational curve f_0 over P, Q, disjoint from $I(X) \setminus \{P, Q\}$. $\square$

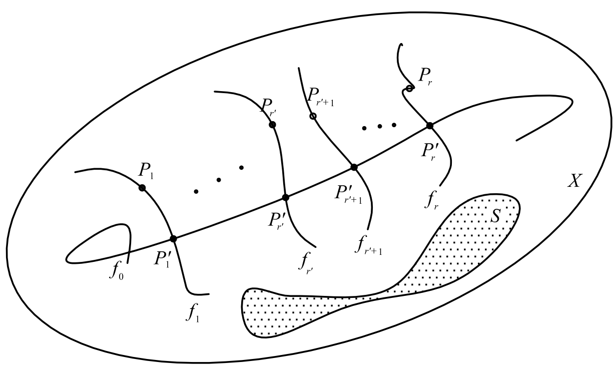
Proof of Main Theorem. Step 1. First, let us consider $S = \text{Sing } X$.

There is a free rational curve $f_0 : C_0 \cong \mathbb{P}^1 \to X$ disjoint from $\{P_i\} \cup S$. Indeed, we can apply Main Lemma to the subvariety $\{P_i\} \cup S$ and any two smooth points $P, Q \notin \{P_i\} \cup S$ in X. Since $f_0(\mathbb{P}^1)$ is in the smooth locus of X, f_0 is free and disjoint from $\{P_i\} \cup S$.



We construct a comb of smooth rational curves C and a morphism $f : C \to X'$ as follows.

I. Assume that $P_1, \ldots, P_{r'}$ are smooth points for some r', $1 \leq r' \leq r$, and $P_{r'+1}, \ldots, P_r$ are singular points of X. Choosing points $t_1, \ldots, t_r \in C_0$, such that $P'_i = f_0(t_i) \in X$ are distinct. For each j, applying the Main Lemma to $S = \text{Sing } X \cup \{P_i\}$ and points $P = P_j, Q = P'_j$, there is a weakly free rational curve $f_j : C_j \cong \mathbb{P}^1 \to X$ over P_j, P'_j for each $1 \leq j \leq r$, disjoint from $S \setminus \{P_j, P'_j\}$.



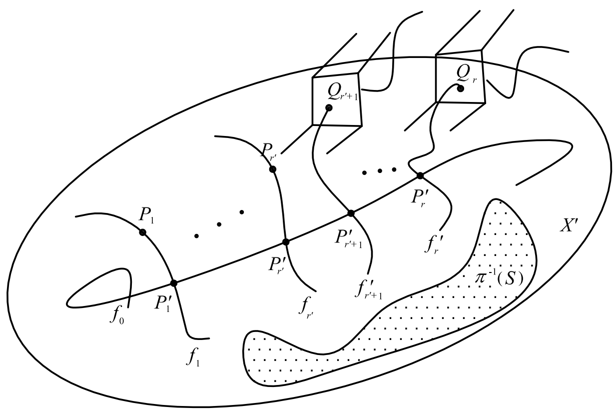
Applying the general statement of Theorem 9 to weakly free rational curves $f_0, f_1, \ldots, f_r$ and the set $\{P_i\} = \{P_i\}_{i \geq r'+1}$, we get a resolution $\pi : X' \to X$.

For each $1 \leq i \leq r'$, since P_i and P'_i are smooth points, $f_i(\mathbb{P}^1)$ is contained in the smooth locus of X. Therefore f_i is free for each $1 \leq i \leq r'$ by

[Kol96] II.3.11. We identify the curve $f_i : C_i \cong \mathbb{P}^1 \to X$ birationally with a free rational curve $f_i : C_i \cong \mathbb{P}^1 \to X'$. We also identify $P_i \in X$ with $P_i \in X'$ for $1 \le i \le r'$, and $P'_i \in X$ with $P'_i \in X'$ for $1 \le i \le r$. More precisely, $f_i(0_i) = P_i$, where $0_i \in C_i, 1 \le i \le r'$, and $f_i(\infty_i) = P'_i$ where $\infty_i \in C_i, 1 \le i \le r$.

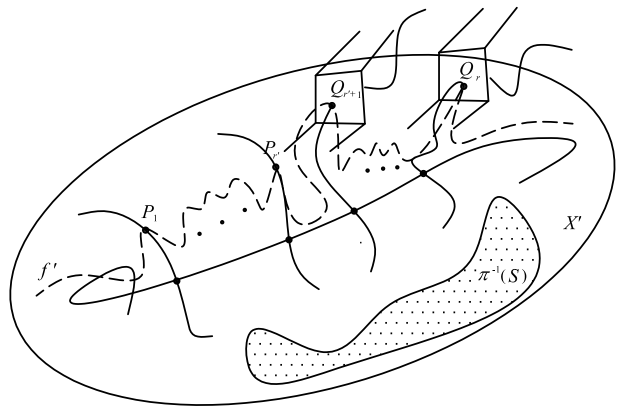
For each $r' + 1 \le j \le r$, P_j is singular. Let $f'_j : C_j \cong \mathbb{P}^1 \to X'$ be the proper birational transformation of a sufficiently general deformation of f_j. Since $\pi : X' \to X$ is a resolution in Theorem 9, $f'_j(C_j)$ intersects $\pi^{-1}P_j$ divisorially over P_j for $r' + 1 \le j \le r$, and is disjoint from the closure of $\pi^{-1}(S \setminus \{P_i\})$. Let Q_j be a point in $f'_j(C_j) \cap \pi^{-1}P_j$ over P_j for $r'+1 \le j \le r$. We can suppose that f_i is very free for $1 \le i \le r'$ and f'_j is very free for $r' + 1 \le j \le r$ by [KMM92a] 1.1. or [Kol96] II.3.11.

By construction of $f_i, 1 \le i \le r'$ and $f'_j, r' + 1 \le j \le r$, $f_i(C_i)$ and $f'_j(C_j)$ are disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \dots, P_r\}) = \pi^{-1}(S \setminus \{P_{r'+1}, \dots, P_r\})$.



II. Gluing $\cup_{i=0}^r C_i$, we get a comb of smooth rational curves $C = \sum_{i=0}^r C_i$ and a morphism $f : C \to X'$. Indeed, we identify points $\infty_i \in C_i$ with $t_i \in C_0$ for each $1 \le i \le r$. Then we have a comb of smooth rational curves $C = \sum_{i=0}^r C_i$ and a morphism $f : C \to X'$ because $f_0(t_i) = f_i(\infty_i) = P'_i$. Notice that $f(C)$ is disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \dots, P_r\})$.

In the end, $f : C \to X'$ can be smoothed into a rational curve $f' : \mathbb{P}^1 \to X'$ such that f' is free over $P_i, 1 \le i \le r'$ and $Q_j, r' + 1 \le j \le r$, and is disjoint from the closure of $\pi^{-1}(S \setminus \{P_1, \dots, P_r\})$ (We can generalize the proof of [Kol96] II.7.6 for comb to get f' is a free rational curve over $\{P_1, \dots, P_{r'}, Q_{r'+1}, \dots, Q_r\}$, not only with $\{P_1, \dots, P_{r'}, Q_{r'+1}, \dots, Q_r\}$ fixed, as stated in [Kol96] II.7.6. Or we can attach additional rational curves to enlarge of the family of f', such that f' is a free rational curve over $\{P_1, \dots, P_{r'}, Q_{r'+1}, \dots, Q_r\}$ after a base change).



Step 2. Now we consider any closed subvariety S of codimension ≥ 2.

By Step 1, there is a free rational curve $f' : \mathbb{P}^1 \to X'$ over $P_1, \dots, P_{r'}$, $Q_{r'+1}, \dots, Q_r$, disjoint from the closure of $\pi^{-1}(\text{Sing } X \setminus \{P_1, \dots, P_r\})$, where $\pi : X' \to X$ is the resolution in Step 1. On the other hand, $\pi^{-1}((S \setminus \text{Sing } X) \setminus \{P_1, \dots, P_r\})$ is a codimension ≥ 2 subvariety on X' by Theorem 9 3'). So a general deformation f'' of f' is free over $P_1, \dots, P_{r'}, Q_{r'+1}, \dots, Q_r$, disjoint from $\pi^{-1}((S \setminus \text{Sing } X) \setminus \{P_1, \dots, P_r\})$ by [Kol96] II.3.7. Since f' is disjoint from the closure of $\pi^{-1}(\text{Sing } X \setminus \{P_1, \dots, P_r\})$, f'' is disjoint from $\pi^{-1}(\text{Sing } X \setminus \{P_1, \dots, P_r\})$. Hence f'' is disjoint from $\pi^{-1}(\text{Sing } X \setminus \{P_1, \dots, P_r\}) \cup \pi^{-1}((S \setminus \text{Sing } X) \setminus \{P_1, \dots, P_r\}) = \pi^{-1}(S \setminus \{P_1, \dots, P_r\})$. Therefore, $\pi f''$ is a general deformation of $\pi f'$ over $P_1, \dots, P_r$, disjoint from $S \setminus \{P_1, \dots, P_r\}$, and thus $\pi f'$ is a geometrically free rational curve over $P_1, \dots, P_r$ on X. □

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Y. Chen: Department of Mathematics, The Johns Hopkins University. Baltimore, MD 21218, USA

E-mail address: `yichen@math.jhu.edu`

V. V. Shokurov: Department of Mathematics, The Johns Hopkins University. Baltimore, MD 21218, USA

Steklov Mathematics Institute, Russian Academy of Sciences, Cubkina Str. 8, 119991, Moscow, Russia

E-mail address: `shokurov@math.jhu.edu`