K-STABILITY AND SPACE SEXTIC CURVES OF GENUS THREE

IVAN CHELTSOV, OLIVER LI, SIONE MA'U, ANTOINE PINARDIN

Abstract. We study Fano threefolds that can be obtained by blowing up the three-dimensional projective space along a smooth curve of degree six and genus three. We produce many new K-stable examples of such threefolds, and we describe all finite groups that can act faithfully on them.

Contents

1. Introduction

Let C be a smooth quartic curve in $\mathbb{P}^2$, let D be a divisor of degree 2 on the curve C such that

$$(\diamondsuit)\qquad h^0\big(\mathcal{O}_C(D)\big)=0.$$

Then K_C+D is very ample [27], and the linear system $|K_C+D|$ gives an embedding $\phi\colon C\hookrightarrow\mathbb{P}^3$. We set $C_6=\phi(C)$. Then C_6 is a smooth curve of degree 6 and genus 3.

Let $\pi\colon X\to\mathbb{P}^3$ be the blow up of the curve C_6. Then X is a Fano threefold in the deformation family №2.12 in the Mori–Mukai list, and every smooth member of this family can be obtained in this way. Moreover, the Fano threefold X can be given in $\mathbb{P}^3\times\mathbb{P}^3$ by

$$(\clubsuit)\qquad (x_0,x_1,x_2,x_3)M_1\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}=(x_0,x_1,x_2,x_3)M_2\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}=(x_0,x_1,x_2,x_3)M_3\begin{pmatrix}y_0\\y_1\\y_2\\y_3\end{pmatrix}=0$$

for appropriate 4×4 matrices M_1, M_2, M_3 such that π is induced by the projection to the first factor, where $([x_0:x_1:x_2:x_3],[y_0:y_1:y_2:y_3])$ are coordinates on $\mathbb{P}^3\times\mathbb{P}^3$.

Let $\pi'\colon X\to\mathbb{P}^3$ be the morphism induced by the projection $\mathbb{P}^3\times\mathbb{P}^3\to\mathbb{P}^3$ to the second factor. Then π' is a blow up of $\mathbb{P}^3$ along a smooth curve C_6' of degree 6 and genus 3, and the π'-exceptional surface is spanned by the strict transforms of the trisecants of the curve C_6. Furthermore, we have

Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

the following commutative diagram:

$$(\bigstar)\qquad \begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\chi"] & & \mathbb{P}^3 \end{tikzcd}$$

where χ is the birational map given by the linear system consisting of all cubic surfaces containing C_6. Note that the curves C_6 and C_6' are isomorphic, but they are not necessarily projectively isomorphic.

We can find the equations of the curves C_6 and C_6' as follows. Rewrite ($\clubsuit$) as

$$\begin{cases} L_{10}y_0 + L_{11}y_1 + L_{12}y_2 + L_{13}y_3 = 0, \\ L_{20}y_0 + L_{21}y_1 + L_{22}y_2 + L_{23}y_3 = 0, \\ L_{30}y_0 + L_{31}y_1 + L_{32}y_2 + L_{33}y_3 = 0, \end{cases},$$

where the L_{ij}'s are linear functions in x_0, x_1, x_2, x_3. Set

$$M = \begin{pmatrix} L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{pmatrix}.$$

Let f_0, f_1, f_2, f_3 be the determinants of the 3×3 matrices obtained from the matrix M by removing its first, second, third, fourth columns, respectively. Then $C_6 = \{f_0 = 0, f_1 = 0, f_2 = 0, f_3 = 0\}$, and the birational map $\chi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ in the diagram ($\bigstar$) is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto \big[f_0 : f_1 : f_2 : f_3\big]$$

up to a composition with an automorphism of the projective space $\mathbb{P}^3$. Similarly, one can also describe the defining equations of the sextic curve C_6'.

Example 1 ([19, 2])**.** Let

$$X = \Big\{x_0y_1 + x_1y_0 - \sqrt{2}x_2y_2 = 0, x_0y_2 + x_2y_0 - \sqrt{2}x_3y_3 = 0, x_0y_3 + x_3y_0 - \sqrt{2}x_1y_1 = 0\Big\} \subset \mathbb{P}^3 \times \mathbb{P}^3.$$

Then X is a smooth Fano threefold in the family №2.12, the curve C_6 is given by

$$\begin{cases} 2\sqrt{2}x_1x_2x_3 - x_0^3 = 0, \\ x_0^2x_1 + \sqrt{2}x_0x_2^2 + 2x_2x_3^2 = 0, \\ x_0^2x_2 + \sqrt{2}x_0x_3^2 + 2x_1^2x_3 = 0, \\ x_0^2x_3 + \sqrt{2}x_0x_1^2 + 2x_1x_2^2 = 0, \end{cases}$$

and C_6' is given by the same equations replacing each x_i by y_i. One has $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$, and X is the only smooth Fano threefold in the deformation family №2.12 that admits a faithful action of the Klein simple group $\mathrm{PSL}_2(\mathbb{F}_7)$. The map χ in ($\bigstar$) can be chosen to be an involution.

The following result has been proven in [2].

Theorem 2 ([2, § 5.4])**.** *Let X be the Fano threefold from Example 1. Then X is K-stable.*

Hence, a general member of the family №2.12 is K-stable, since K-stability is an open condition. We expect that every smooth Fano threefold in this family is K-stable. To show this, it is enough to prove that

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) > 0$$

for every prime divisor $\mathbf{F}$ over X [22, 32], where $A_X(\mathbf{F})$ is the log discrepancy of the divisor $\mathbf{F}$, and

$$S_X\big(\mathbf{F}\big) = \frac{1}{(-K_X)^3}\int\limits_0^\infty \mathrm{vol}\big(-K_X - u\mathbf{F}\big)du.$$

Unfortunately, we are unable to prove this result at the moment. Instead, we prove a weaker result. To state it, let E be the π-exceptional surface, and let E' be the π'-exceptional surface.

Theorem A. *Let $\mathbf{F}$ be a prime divisor over X such that $\beta(\mathbf{F}) \leqslant 0$, and let Z be its center on X. Then Z is a point in the intersection $E \cap E'$.*

Let us present applications of this result. By [39, Corollary 4.14], Theorem A implies

Corollary 3. *If $\mathrm{Aut}(X)$ does not fix points in $E \cap E'$, then X is K-stable.*

Since the action of the group $\mathrm{Aut}(\mathbb{P}^3, C_6)$ lifts to X, Corollary 3 implies

Corollary 4. *If $\mathrm{Aut}(\mathbb{P}^3, C_6)$ does not fix a point in C_6, then X is K-stable.*

Since the group $\mathrm{Aut}(\mathbb{P}^3, C_6)$ acts faithfully on the curve C_6, Corollary 4 implies the following generalization of Theorem 2, which has more applications (see Section 2).

Corollary 5. *If $\mathrm{Aut}(\mathbb{P}^3, C_6)$ is not cyclic, then X is K-stable.*

Proof. If the group $\mathrm{Aut}(\mathbb{P}^3, C_6)$ fixes a point $P \in C_6$, it acts faithfully on the one-dimensional tangent space to the curve C_6 at the point P by [20, Lemma 2.7], so that $\mathrm{Aut}(\mathbb{P}^3, C_6)$ is cyclic. $\square$

What do we know about $\mathrm{Aut}(X)$? This group is finite [9], and we have the following exact sequence:

$$1 \to \mathrm{Aut}\big(\mathbb{P}^3, C_6\big) \to \mathrm{Aut}(X) \to \boldsymbol{\mu}_2,$$

where $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathrm{Aut}(C, [D])$, and the final homomorphism is surjective $\iff$ $\mathrm{Aut}(X)$ contains an element that swaps E and E'. For instance, if X is the smooth Fano threefold from Example 1, then the group $\mathrm{Aut}(X)$ contains such an element — it is the involution given by

$$\big([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]\big) \mapsto \big([y_0 : y_1 : y_2 : y_3], [x_0 : x_1 : x_2 : x_3]\big),$$

which implies that $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$ in this case. In Section 4, we will discuss the possibilities for the group $\mathrm{Aut}(X)$ in more details. In particular, we will present a criterion when $\mathrm{Aut}(X)$ contains an element that swaps E and E', and we will prove the following result (cf. [38, Theorem 1.1]).

Theorem B. *A finite group G has a faithful action on a smooth Fano threefold in the deformation family №2.12 if and only if G is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$ or $\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$.*

As we mentioned in Example 1, the family №2.12 contains a unique smooth Fano threefold that admits a faithful action of the group $\mathrm{PSL}_2(\mathbb{F}_7)$. Similarly, we prove in Section 4 that the deformation family №2.12 contains a unique smooth threefold that admits a faithful action of the group $\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$, and the full automorphism group of this threefold is $\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$.

Remark 6. Let G be a subgroup in $\mathrm{Aut}(X)$. If G has an element that swaps the surfaces E and E', then X is a G-Mori fiber space (over a point), and X is also known as a G-Fano threefold (see [37]). In this case, it is natural to ask the following three nested questions:

(1) Is there a G-equivariant birational map $X \dashrightarrow \mathbb{P}^3$? Cf. [12, 13, 29].
(2) Is X G-solid? Cf. [10, 36].
(3) Is X G-birationally rigid? Cf. [11].

Inspired by [29, Corollary 6.11], we conjecture that the answer to the first question is always negative. If $G \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$, then X is G-birationally rigid [2, Theorem 5.23], so, in particular, it is G-solid. We believe that X is also G-birationally rigid if $G \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$.

To consider more applications of Theorem A, let $\Bbbk$ be a subfield in $\mathbb{C}$ such that C_6 is defined over $\Bbbk$. Then X and the Sarkisov link (★) are defined over $\Bbbk$. In particular, the curve C_6' is defined over $\Bbbk$. Moreover, it follows from [4, 30] that C_6' and C_6 are isomorphic over $\Bbbk$, which can be shown directly. By [39, Corollary 4.14], Theorem A implies the following corollaries.

Corollary 7. *If $E \cap E'$ does not have $\Bbbk$-points, then X is K-stable.*

Corollary 8. *If C_6 does not have $\Bbbk$-points, then X is K-stable.*

Using [39, Corollary 4.14], we also obtain

Corollary 9. *Every smooth Fano threefold in the deformation family №2.12 which is defined over a subfield of the field $\mathbb{C}$ and does not have points in this subfield is K-stable.*

We will present applications of Corollaries 8 and Corollary 9 in Section 2.

Acknowledgements. We started this project in Auckland (New Zealand) back in December 2022 during Ivan Cheltsov and Antoine Pinardin's visit, which has been supported by the Leverhulme Trust grant RPG-2021-229. Oliver Li is supported by the Australian Commonwealth Government.

We would like to thank Harry Braden, Dougal Davis, Linden Disney-Hogg, Tim Dokchitser, Igor Dolgachev, Sasha Kuznetsov, Yuri Prokhorov, Giorgio Ottaviani, Costya Shramov and Yuyang Zhou for their help in this project.

2. Examples

2.1. **$\mathfrak{S}_4$-invariant curves.** Let us use notations introduced in Section 1. Suppose, in addition, that

$$M_1 = \begin{pmatrix} 0 & a & 1 & 0 \\ a & 0 & 0 & -1 \\ 1 & 0 & 0 & a \\ 0 & -1 & a & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 & 0 & a \\ 1 & 0 & a & 0 \\ 0 & a & 0 & -1 \\ a & 0 & -1 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & a & -1 \\ 0 & 0 & 1 & a \\ a & 1 & 0 & 0 \\ -1 & a & 0 & 0 \end{pmatrix},$$

where $a \in \mathbb{C}$ such that $a(a^6-1) \neq 0$. Then X is a smooth Fano threefold in the family №2.12, and

$$M = \begin{pmatrix} ax_1 + x_2 & ax_0 - x_3 & ax_3 + x_0 & ax_2 - x_1 \\ ax_3 + x_1 & ax_2 + x_0 & ax_1 - x_3 & ax_0 - x_2 \\ ax_2 - x_3 & ax_3 + x_2 & ax_0 + x_1 & ax_1 - x_0 \end{pmatrix},$$

so that $C_6 = \{f_0 = 0, f_1 = 0, f_2 = 0, f_3 = 0\}$ for

$$\begin{aligned} f_0 = (1-a^3)x_0^3 - (2a^2+2a)x_0^2x_1 + (2a^2+2a)x_0^2x_2 + \\ + (2a^2+2a)x_0^2x_3 + (a^3-1)x_0x_1^2 - (2a^2-2a)x_0x_1x_2 - (2a^2-2a)x_0x_1x_3 + \\ + (a^3-1)x_0x_2^2 + (2a^2-2a)x_0x_2x_3 + (a^3-1)x_0x_3^2 - (2a^3+2)x_1x_2x_3, \end{aligned}$$

$$\begin{aligned} f_1 = (1-a^3)x_0^2x_1 + (-2a^2-2a)x_0x_1^2 + (2a^2-2a)x_0x_1x_2 - \\ - (2a^2-2a)x_0x_1x_3 + (2a^3+2)x_0x_2x_3 + (a^3-1)x_1^3 + (2a^2+2a)x_1^2x_2 - \\ - (2a^2+2a)x_1^2x_3 + (-a^3+1)x_1x_2^2 + (2a^2-2a)x_1x_2x_3 + (1-a^3)x_1x_3^2, \end{aligned}$$

$$\begin{aligned} f_2 = (a^3-1)x_0^2x_2 - (2a^2-2a)x_0x_1x_2 - (2a^3+2)x_0x_1x_3 + \\ + (-2a^2-2a)x_0x_2^2 - (2a^2-2a)x_0x_2x_3 + (a^3-1)x_1^2x_2 + (2a^2+2a)x_1x_2^2 + \\ + (2a^2-2a)x_1x_2x_3 + (1-a^3)x_2^3 + (2a^2+2a)x_2^2x_3 + (a^3-1)x_2x_3^2, \end{aligned}$$

$$f_3 = (1-a^3)x_0^2x_3 + (2a^3+2)x_0x_1x_2 - (2a^2-2a)x_0x_1x_3 - \\ - (2a^2-2a)x_0x_2x_3 + (2a^2+2a)x_0x_3^2 + (1-a^3)x_1^2x_3 - (2a^2-2a)x_1x_2x_3 + \\ + (2a^2+2a)x_1x_3^2 + (1-a^3)x_2^2x_3 + (2a^2+2a)x_2x_3^2 + (a^3-1)x_3^3.$$

It follows from [34] that the curve C_6 is isomorphic to the plane quartic curve in $\mathbb{P}^2_{x,y,z}$ that is given by the equation $\det(xM_1 + yM_2 + zM_3) = 0$, which can be rewritten as

$$x^4 + y^4 + z^4 + \lambda(x^2y^2 + x^2z^2 + y^2z^2) = 0$$

for $\lambda = -\frac{2a^4+2}{(a^2+1)^2}$, cf. [18, § 14]. So, it follows from [17] that $\mathrm{Aut}(C_6) \simeq \mathfrak{S}_4$ if $\lambda \neq 0$ and $\lambda^2+3\lambda+18 \neq 0$. Moreover, if $\lambda = 0$, then $\mathrm{Aut}(C_6) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. Furthermore, if $\lambda^2+3\lambda+18 = 0$, then C_6 is isomorphic to the Klein quartic curve, and $\mathrm{Aut}(C_6) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$.

Lemma 10. *The group* $\mathrm{Aut}(\mathbb{P}^3, C_6)$ *contains a subgroup isomorphic to* $\mathfrak{S}_4$.

Proof. Let G be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ that is generated by the following transformations:

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 & -1 & 1 \\ -3 & -1 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} -3 & -1 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 1 & -3 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}.$$

Then, using Magma, one can check that $G \simeq \mathfrak{S}_4$. Moreover, the curve C_6 is G-invariant. □

Corollary 11. *If* $\lambda \neq 0$ *and* $\lambda^2+3\lambda+18 \neq 0$, *then* $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathfrak{S}_4$.

Similarly, we prove

Lemma 12. *The group* $\mathrm{Aut}(X)$ *contains a subgroup isomorphic to* $\mathfrak{S}_4 \times \boldsymbol{\mu}_2$.

Proof. Let G be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ that is defined in the proof of Lemma 10. Then $G \simeq \mathfrak{S}_4$, the group G acts diagonally on $\mathbb{P}^3 \times \mathbb{P}^3$, and X is G-invariant. This gives an embedding $\mathfrak{S}_4 \hookrightarrow \mathrm{Aut}(X)$. Moreover, since the matrices M_1, M_2, M_3 are symmetric, the involution

$$\big([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]\big) \mapsto \big([y_0 : y_1 : y_2 : y_3], [x_0 : x_1 : x_2 : x_3]\big)$$

leaves X invariant and commutes with the $\mathfrak{S}_4$-action, which implies the required assertion. □

Corollary 13. *If* $\lambda \neq 0$ *and* $\lambda^2+3\lambda+18 \neq 0$, *then* $\mathrm{Aut}(X) \simeq \mathfrak{S}_4 \times \boldsymbol{\mu}_2$.

Applying Corollary 5, we conclude that the Fano threefold X is K-stable.

2.2. **$\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$-invariant curve.** Let $\widehat{G}$ be the subgroup in $\mathrm{GL}_4(\mathbb{C})$ generated by the matrices

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

and let G be the image of the group $\widehat{G}$ in $\mathrm{PGL}_4(\mathbb{C})$ via the natural projection $\mathrm{GL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$. Then $\widehat{G} \simeq \boldsymbol{\mu}_4.(\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3)$ and $G \simeq \boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$, and their GAP ID's are [192,4] and [48,3], respectively. Using GAP [23], one can check that $H^2(G, \mathbb{C}^*) \simeq \boldsymbol{\mu}_4$, and $\widehat{G}$ is a covering group of the group G.

Lemma 14. *Let* G' *be a subgroup in* $\mathrm{PGL}_4(\mathbb{C})$ *such that* $G' \simeq G$ *and* G' *does not fix points in* $\mathbb{P}^3$. *Then* G' *is conjugate to* G *in* $\mathrm{PGL}_4(\mathbb{C})$.

Proof. The claim follows from [10, Lemma 2.7] and the classification of finite subgroups in $\mathrm{PGL}_4(\mathbb{C})$, which can be found in [5]. Alternatively, one can prove the required assertion analyzing irreducible representations of the group $\widehat{G}$, which can be found in [15]. □

The main goal of this subsection is to show that the projective space $\mathbb{P}^3$ contains a G-invariant irreducible smooth non-hyperelliptic curve of degree 6 and genus 3, and this curve is unique up to the action of the normalizer of the group G in $\mathrm{PGL}_4(\mathbb{C})$. First, let us describe the normalizer. Set

$$C_4^{\pm} = \big\{(1 \mp \sqrt{3}i)x_1^2 - (1 \pm \sqrt{3}i)x_2^2 + 2x_3^2 = 0, 2x_0^2 - (1 \pm \sqrt{3}i)x_1^2 - \big(1 \mp \sqrt{3}i\big)x_2^2 = 0\big\} \subset \mathbb{P}^3.$$

Then $C_4^{\pm}$ is a G-invariant elliptic curve, and $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm})$ is the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm}) \simeq \boldsymbol{\mu}_2^3.\mathfrak{A}_4$. Let $G_{192,185}$ be the subgroup in $\mathrm{PGL}_4(\mathbb{C})$ generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then its GAP ID is [192,185]. Note that $\mathrm{Aut}(\mathbb{P}^3, C_4^{\pm}) \lhd G_{192,185} \simeq \boldsymbol{\mu}_2^3.\mathfrak{S}_4$ and $G \lhd G_{192,185}$.

Lemma 15. *The normalizer in $\mathrm{PGL}_4(\mathbb{C})$ of the subgroup G is the subgroup $G_{192,185}$.*

Proof. This follows from the fact that the curve $C_4^+ + C_4^-$ is $G_{192,185}$-invariant. $\square$

Let us describe G-orbits in $\mathbb{P}^3$ of length less than 48. To do this, we let

$$\begin{aligned} \Sigma_4 &= \mathrm{Orb}_G\big([1:0:0:0]\big), \\ \Sigma_{12} &= \mathrm{Orb}_G\big([1+i:\sqrt{2}:0:0]\big), \\ \Sigma_{12}' &= \mathrm{Orb}_G\big([1-i:\sqrt{2}:0:0]\big), \\ \Sigma_{16} &= \mathrm{Orb}_G\big([-1+\sqrt{3}i:-1-\sqrt{3}i:2:0]\big), \\ \Sigma_{16}' &= \mathrm{Orb}_G\big([-1-\sqrt{3}i:-1+\sqrt{3}i:2:0]\big), \\ \Sigma_{16}^u &= \mathrm{Orb}_G\big([1:1:1:u]\big) \text{ for } u \in \mathbb{C}, \\ \Sigma_{24}^t &= \mathrm{Orb}_G\big([2:t:0:0]\big) \text{ for } t \in \mathbb{C} \text{ such that } t \neq 0 \text{ and } t \neq \pm\sqrt{2} \pm \sqrt{2}i. \end{aligned}$$

Then Σ_4, Σ_{12}, Σ_{12}', Σ_{16}, Σ_{16}', Σ_{16}^u, Σ_{24}^t are G-orbits of length 4, 12, 12, 16, 16, 16, 24, respectively.

Lemma 16. *Let Σ be a G-orbit in $\mathbb{P}^3$ such that $|\Sigma| < 48$. Then Σ is one of the G-orbits*

$$\Sigma_4,\ \Sigma_{12},\ \Sigma_{12}',\ \Sigma_{16},\ \Sigma_{16}',\ \Sigma_{16}^u,\ \Sigma_{24}^t,$$

where $u \in \mathbb{C}$ and $t \in \mathbb{C}$ such that $0 \neq t \neq \pm\sqrt{2} \pm \sqrt{2}i$.

Proof. Let us describe subgroups of the group G. To do this, identify the matrices M, N, A, B with their images in $\mathrm{PGL}_4(\mathbb{C})$. Set

$$C = ANBMA^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Then, using [15], we see that all proper subgroups of the group G can be described as follows:

(i) $\langle B, C\rangle \simeq \boldsymbol{\mu}_4^2$ is the unique (normal) subgroup of order 16,
(ii) $\langle A, M, N\rangle \simeq \mathfrak{A}_4$ is one of four conjugated subgroups of order 12,
(iii) $\langle B, M, N\rangle \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_4$ is one of three conjugated subgroups of order 8,

(iv) $\langle M, N\rangle \simeq \boldsymbol{\mu}_2^2$ is the unique (normal) subgroup isomorphic to $\boldsymbol{\mu}_2^2$,
(v) $\langle B\rangle \simeq \boldsymbol{\mu}_4$ and $\langle CB\rangle \simeq \boldsymbol{\mu}_4$ are non-conjugate subgroups, their conjugacy classes consist of three subgroups, which are all subgroups of the group G isomorphic to $\boldsymbol{\mu}_4$,
(vi) $\langle A\rangle \simeq \boldsymbol{\mu}_3$ is one of sixteen conjugated subgroups of order 3,
(vii) $\langle M\rangle \simeq \boldsymbol{\mu}_2$ is one of three conjugated subgroups of order 2.

Now, let Γ be the stabilizer in G of a point in Σ. Then Γ is a proper subgroup of the group G, since G fixes no points in $\mathbb{P}^3$. So, we may assume that Γ is one of the subgroups $\langle B, C\rangle$, $\langle A, M, N\rangle$, $\langle B, M, N\rangle$, $\langle M, N\rangle$, $\langle B\rangle$, $\langle CB\rangle$, $\langle A\rangle$, $\langle M\rangle$. On the other hand, one can check that

(i) $\langle B, C\rangle$ does not fix points in $\mathbb{P}^3$,
(ii) the only fixed point of $\langle A, M, N\rangle$ is the point $[1:0:0:0]\in\Sigma_4$,
(iii) $\langle B, M, N\rangle$ does not fix points in $\mathbb{P}^3$,
(iv) $\langle M, N\rangle$ does not fix points in $\mathbb{P}^3\setminus\Sigma_4$,
(v) the only fixed point of $\langle B\rangle$ are the points
$$[1+i:\sqrt{2}:0:0],[1+i:-\sqrt{2}:0:0],[0:0:\sqrt{2}:1+i],[0:0:-\sqrt{2}:1+i],$$
which are contained in Σ_{12}, and the only fixed point of $\langle CB\rangle$ are the points
$$[\sqrt{2}:0:1-i:0],[-\sqrt{2}:0:1-i:0],[0:\sqrt{2}:0:1-i],[0:-\sqrt{2}:0:1-i],$$
which are contained in the G-orbit Σ_{12}',
(vi) the only fixed points of $\langle A\rangle$ are the points
 - $[-1+\sqrt{3}i:-1-\sqrt{3}i:2:0]\in\Sigma_{16}$,
 - $[-1-\sqrt{3}i:-1+\sqrt{3}i:2:0]\in\Sigma_{16}'$,
 - $[1:1:1:t]\in\Sigma_{16}^t$ for any $t\in\mathbb{C}$,
 - $[0:0:0:1]\in\Sigma_4$,
(vii) all fixed points of $\langle M\rangle$ are contained in the lines $\{x_0=x_1=0\}$ and $\{x_2=x_3=0\}$.

This implies the required assertion. $\square$

Now, we are ready to present a G-invariant irreducible smooth curve in $\mathbb{P}^3$ of degree 6 and genus 3. For every $u\in\mathbb{C}$ such that $u\neq 0$, let $\mathcal{M}_3^u$ be the linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(3)|$ that consists of all cubic surfaces passing through the G-orbit Σ_{16}^u. If $u^4=-3$, then the linear system $\mathcal{M}_3^u$ is 7-dimensional, and its base locus consists of one of the two elliptic curves C_4^+ or C_4^-. One the other hand, if $u^4\neq -3$, then the linear subsystem $\mathcal{M}_3^u$ is 3-dimensional, and its base locus is given by
$$(\heartsuit)\qquad \begin{cases} (u^4-1)x_3x_0^2+(u^4+3)x_0x_1x_2u+(u^4-1)x_3x_1^2-4x_3^3u^2+(u^4-1)x_2^2x_3=0,\\ (u^4-1)x_1x_0^2-u(u^4+3)x_0x_2x_3+4u^2x_1^3-(u^4-1)x_1x_2^2+(u^4-1)x_3^2x_1=0,\\ 4u^2x_0^3-(u^4-1)x_0x_1^2+(u^4-1)x_0x_2^2+(u^4-1)x_3^2x_0-u(u^4+3)x_1x_2x_3=0,\\ (u^4-1)x_2x_0^2+u(u^4+3)x_0x_1x_3-(u^4-1)x_2x_1^2-4u^2x_2^3-(u^4-1)x_3^2x_2=0.\end{cases}$$
Using this, one can check that the base locus is zero-dimensional unless
$$u\in\left\{\frac{-1\pm\sqrt{3}}{2}+\frac{1\mp\sqrt{3}}{2}i,\frac{-1\pm\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i,\frac{1\pm\sqrt{3}}{2}+\frac{1\pm\sqrt{3}}{2}i,\frac{1\mp\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i\right\}.$$
On the other hand, if $u=\frac{-1\pm\sqrt{3}}{2}+\frac{1\mp\sqrt{3}}{2}i$, $u=\frac{-1\pm\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i$, $u=\frac{1\pm\sqrt{3}}{2}+\frac{1\pm\sqrt{3}}{2}i$ or $u=\frac{1\mp\sqrt{3}}{2}+\frac{-1\pm\sqrt{3}}{2}i$, then the equations $(\heartsuit)$ define an irreducible G-invariant smooth curve in $\mathbb{P}^3$ of degree 6 and genus 3.

We will denote these curves by C_6, C_6', C_6'', C_6''', respectively. To be precise, we have

$$C_6 = \begin{cases} (x_0^2+x_1^2+x_2^2)x_3 - ix_3^3 - (1-i)x_1x_0x_2 = 0, \\ (x_0^2-x_2^2+x_3^2)x_1 + ix_1^3 + (1-i)x_3x_0x_2 = 0, \\ (x_1^2-x_2^2-x_3^2)x_0 - ix_0^3 - (1-i)x_1x_3x_2 = 0, \\ (x_0^2-x_1^2-x_3^2)x_2 - ix_2^3 - (1-i)x_1x_3x_0 = 0, \end{cases}$$

and C_6', C_6'', C_6''' can be obtained from C_6 by applying elements of the normalizer $G_{192,185}$.

Fix $u = \frac{-1\pm\sqrt{3}}{2} + \frac{1\mp\sqrt{3}}{2}i$. Then (♡) defines C_6. Choosing a different basis of the linear system $\mathcal{M}_3^u$, we obtain a birational map $\iota\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ given by $[x_0 : x_1 : x_2 : x_3] \mapsto [h_0 : h_1 : h_2 : h_3]$ for

$$\begin{aligned} h_0 &= (1+i)x_3x_0x_2 - x_1^3 + ix_1(x_0^2 - x_2^2 + x_3^2), \\ h_1 &= (1+i)x_3x_1x_2 - x_0^3 - ix_0(x_1^2 - x_2^2 - x_3^2), \\ h_2 &= (1+i)x_3x_0x_1 - x_2^3 - ix_2(x_0^2 - x_1^2 - x_3^2), \\ h_3 &= (i-1)x_1x_0x_2 - ix_3^3 + x_3(x_0^2 + x_1^2 + x_2^2). \end{aligned}$$

One can check that ι is a birational involution, and we have the following G-commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^3 \arrow[r, dashed, "\iota"'] & \mathbb{P}^3 \end{tikzcd}$$

where π is the blow up of the curve C_6, and τ is an involution. Then X is smooth Fano threefold in the deformation family №2.12, which can be defined as complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ given by

$$\begin{cases} y_3x_0 - y_2x_0 + iy_2x_1 + y_3x_1 - y_0x_2 + iy_1x_2 + y_0x_3 + y_1x_3 = 0, \\ iy_0x_0 - y_1x_1 + y_3x_2 + y_2x_3 = 0, \\ y_2x_0 + y_3x_0 + iy_2x_1 - y_3x_1 - y_0x_2 - iy_1x_2 - y_0x_3 + y_1x_3 = 0, \end{cases}$$

and π is induced by the projection to the first factor, where $([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3])$ are coordinates on $\mathbb{P}^3 \times \mathbb{P}^3$. Thus, in the notations in Section 1, we have

$$M_1 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & i & 1 \\ -1 & i & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

Note that M_1 and M_2 are symmetric, M_3 is skew-symmetric, and the involution τ is given by

$$\big([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]\big) \mapsto \big([y_0 : y_1 : y_2 : y_3], [x_0 : x_1 : x_2 : x_3]\big),$$

Corollary 17. *One has* $\mathrm{Aut}(\mathbb{P}^3, C_6) = G$ *and* $\mathrm{Aut}(X) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$.

Proof. First, using the classification of automorphism groups of smooth curves of genus three [17, 3], we see that C_6 is isomorphic to the Fermat quartic curve in $\mathbb{P}^2$. This can also be shown directly. Namely, it follows from [34] that C_6 is isomorphic to the plane quartic curve

$$\big\{\det(xM_1 + yM_2 + zM_3) = 0\big\} \subset \mathbb{P}^2_{x,y,z},$$

which is projectively isomorphic to the Fermat plane quartic curve.

We conclude that $\mathrm{Aut}(C_6) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. Therefore, if $\mathrm{Aut}(\mathbb{P}^3, C_6) \neq G$, then $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$, and the subgroup $\mathrm{Aut}(\mathbb{P}^3, C_6) \subset \mathrm{PGL}_4(\mathbb{C})$ is contained in the normalizer of the group G in $\mathrm{PGL}_4(\mathbb{C})$, which is impossible since the normalizer is the group $G_{192,185}$ by Lemma 15, and $G_{192,185}$ does not contain subgroups isomorphic to $\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$.

Therefore, we conclude that $\mathrm{Aut}(\mathbb{P}^3, C_6) = G$. Now, one can explicitly check that $\langle G, \tau\rangle \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$, where we consider G as a subgroup in $\mathrm{Aut}(X)$. This gives $\mathrm{Aut}(X) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. $\square$

By Corollary 5, the smooth Fano threefold X is K-stable.

In Section 4, we will see that X is the unique smooth Fano threefold in the family №2.12 whose automorphism group is isomorphic to the group $\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. To do this, we need the following result:

Theorem 18. *The only G-invariant irreducible smooth curves in $\mathbb{P}^3$ of degree 6 are C_6, C_6', C_6'', C_6'''.*

Proof. Let C be a G-invariant irreducible smooth curve in $\mathbb{P}^3$ of degree 6, and let g be its genus. Then $g \leqslant 4$ by the Castelnuovo bound. Thus, it follows from [7, 35] that either $g = 1$, or $g = 3$.

Note that $\Sigma_4 \not\subset C$, because stabilizers in G of points in C are cyclic by [20, Lemma 2.7].

Let $\Pi = \{x_3 = 0\}$, and let Γ be the the stabilizer of this plane in G. Then $\Gamma = \langle M, N, A\rangle \simeq \mathfrak{A}_4$, and all Γ-orbits in Π of length less than 12 can be described as follows:

(i) $\Sigma_4 \cap \Pi$ is the unique Γ-orbit of length 3,
(ii) $\Sigma_{12} \cap \Pi$ is a Γ-orbit of length 6,
(iii) $\Sigma_{12}' \cap \Pi$ is a Γ-orbit of length 6,
(iv) $\Sigma_{24}^t \cap \Pi$ is a Γ-orbit of length 6, where $0 \neq t \neq \pm\sqrt{2} \pm \sqrt{2}i$,
(v) $\Sigma_{16} \cap \Pi$, $\Sigma_{16}' \cap \Pi$ and $\Sigma_{16}^0 \cap \Pi$ are Γ-orbits of length 4.

Thus, since $\Sigma_4 \not\subset C$, $C \not\subset \Pi$, and $\Pi \cdot C$ is a Γ-invariant effective one-cycle of degree 6, we conclude that C contains at least one of the orbits Σ_{12} or Σ_{12}', and C does not contain Σ_{16}, Σ_{16}' and Σ_{16}^0.

If $g = 1$, it follows from [7, 35] that C does not contain G-orbits of length 12, which gives $g = 3$. Then it follows from [7, 35] that C contains two G-orbits of length 16, so $\Sigma_{16}^t \subset C$ for some $t \neq 0$.

Using the classification of automorphism groups of smooth curves of genus three [17, 3], we see that the curve C is isomorphic to the Fermat quartic curve in $\mathbb{P}^2$. Hence, the curve C is not hyperelliptic.

Let $\mathcal{M}_3$ be the linear subsystem in $|\mathcal{O}_{\mathbb{P}^3}(3)|$ that consists of all cubic surfaces passing through C. Then $\mathcal{M}_3$ is three-dimensional, and the curve C is its base locus by [27], because C is not hyperelliptic. Therefore, using the notations introduced earlier, we see that $\mathcal{M}_3 = \mathcal{M}_3^t$ for an appropriate $t \in \mathbb{C}$. Now, arguing as above, we see that

$$t \in \left\{\frac{-1 \pm \sqrt{3}}{2} + \frac{-1 \pm \sqrt{3}}{2}i, \frac{-1 \pm \sqrt{3}}{2} + \frac{1 \mp \sqrt{3}}{2}i, \frac{1 \pm \sqrt{3}}{2} + \frac{1 \pm \sqrt{3}}{2}i, \frac{1 \mp \sqrt{3}}{2} + \frac{-1 \pm \sqrt{3}}{2}i\right\},$$

which implies that C is one of the curves C_6, C_6', C_6'', C_6''' as claimed. $\square$

2.3. **Curves over $\mathbb{Q}$ without rational points.** Let us use notations introduced in Section 1. Suppose, in addition, that

$$C = \{x^4 + xyz^2 + y^4 + y^3z - 31yz^3 + 4z^4 = 0\} \subset \mathbb{P}^2_{x,y,z}.$$

Then C is smooth. One can show that $C(\mathbb{Q}) = \varnothing$ using the reduction modulo 3. Set

$$P_1 = [1 - i : 0 : 1], P_2 = [1 + i : 0 : 1], P_3 = [-1 + i : 0 : 1], P_4 = [-1 - i : 0 : 1].$$

and $D = 3(P_3 + P_4) - K_C$. Then D is defined over $\mathbb{Q}$, and D satisfies ($\diamondsuit$). Then C_6 is defined over $\mathbb{Q}$, and it is isomorphic to C over $\mathbb{Q}$. In particular, the curve C_6 does not contains $\mathbb{Q}$-rational points. Hence, by Corollary 8, the smooth Fano threefold X is K-stable.

One can explicitly find defining equations of C_6 as follows. Let $\mathcal{M}$ be the linear system of cubic curves in $\mathbb{P}^2$ whose general member is tangent to C with multiplicity 3 at the points P_1 and P_2. Then

$$\mathcal{M}\big|_C = 3P_1 + 3P_2 + |3(P_3 + P_4)|.$$

Thus, to compute the embedding $C \hookrightarrow \mathbb{P}^3$, it is enough to find a basis of the linear system $\mathcal{M}$, which can be done using linear algebra. After this, it is easy to find defining equations of the curve C_6.

2.4. **Real pointless threefolds.** Now, we explain how to construct real smooth Fano threefolds in the deformation family №2.12 that do not have real points. By Corollary 9, all of them are K-stable. We start with

Example 19. Let U be a three-dimensional Severi–Brauer variety defined over $\mathbb{R}$ such that $U \not\simeq \mathbb{P}^3_{\mathbb{R}}$. Recall from [24, 28] that U exists, it is unique, and, in particular, it is isomorphic to its dual variety. Set $W = U \times U$. Then

$$W_{\mathbb{C}} \simeq \mathbb{P}^3 \times \mathbb{P}^3.$$

Since $U \simeq U^\vee$, the Picard group $\mathrm{Pic}_{\mathbb{R}}(W)$ contains a real line bundle L such that $L_{\mathbb{C}}$ has degree $(1,1)$. Let V be any smooth complete intersection of three divisors in $|L|$. Then V is a smooth Fano threefold in the family №2.12, and V does not have real points, because W does not have real points.

Let us present another, more explicit, construction of pointless real smooth Fano threefolds in the deformation family №2.12. For a point $P = ([x_0 : x_1 : x_2 : x_3], [y_0 : y_1 : y_2 : y_3]) \in \mathbb{P}^3 \times \mathbb{P}^3$, let us consider the symmetric matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}$$

and the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & b_{01} & b_{02} & b_{03} \\ -b_{01} & 0 & b_{12} & b_{13} \\ -b_{02} & -b_{12} & 0 & b_{23} \\ -b_{03} & -b_{13} & -b_{23} & 0 \end{pmatrix}$$

defined (up to a common scalar multiple) as follows:

$$a_{nm} = \frac{x_n y_m + x_m y_n}{2}$$

and

$$b_{nm} = \frac{x_n y_m - x_m y_n}{2i}$$

for every $n \in \{0,1,2,3\}$ and $m \in \{0,1,2,3\}$ such that $n \neq m$, and $a_{nn} = x_n y_n$ for each $n \in \{0,1,2,3\}$. Set $M = A + iB$. Then

$$M = \begin{pmatrix} x_0y_0 & x_0y_1 & x_0y_2 & x_0y_3 \\ x_1y_0 & x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_0 & x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_0 & x_3y_1 & x_3y_2 & x_3y_3. \end{pmatrix}$$

Therefore, we see that the constructed map $P \mapsto M$ gives us the Serge embedding $\mathbb{P}^3 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^{15}$, where we consider $\mathbb{P}^{15}$ as a projectivization of the vector space of all 4×4 matrices.

Now, we consider matrices A and B on their own, and we also assume that all a_{ij} and b_{ij} are real. Then M is a Hermitian 4×4 matrix. Projectivizing the vector space of Hermitian 4×4 matrices, we obtain $\mathbb{P}^{15}_{\mathbb{R}}$ with coordinates $[a_{00} : a_{01} : \cdots : b_{13} : b_{23}]$. Let us consider M as a point in $\mathbb{P}^{15}_{\mathbb{R}}$, and set

$$V = \left\{ M \in \mathbb{P}^{15}_{\mathbb{R}} \;\middle|\; \mathrm{rank}(M) \leqslant 1 \right\} \subset \mathbb{P}^{15}_{\mathbb{R}}.$$

Then V is a real projective subvariety in $\mathbb{P}^{15}_{\mathbb{R}}$. Moreover, over $\mathbb{C}$, the subvariety $V_{\mathbb{C}}$ is the image of the map $P \mapsto M$ constructed above, which implies that $V_{\mathbb{C}} \simeq \mathbb{P}^3 \times \mathbb{P}^3$, so V is a form of $\mathbb{P}^3_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$. But $V \not\simeq \mathbb{P}^3_{\mathbb{R}} \times \mathbb{P}^3_{\mathbb{R}}$ over $\mathbb{R}$, because V is the Weil restriction of $\mathbb{P}^3$ over the reals [25, Exercise 8.1.6], which implies that $V(\mathbb{R}) \neq \varnothing$, and $\mathrm{Pic}_{\mathbb{R}}(V)$ is generated by the class of a hyperplane section.

Now, let H_1, H_2, H_3 be three real hyperplane sections of $V \subset \mathbb{P}^{15}_{\mathbb{R}}$, and let $X = H_1 \cap H_2 \cap H_3$. Suppose that X is smooth and three-dimensional. Then X is a real form of a smooth Fano threefold in the deformation family №2.12 such that $\mathrm{Pic}_{\mathbb{R}}(X) = \mathbb{Z}[-K_X]$. Moreover, Corollary 9 gives

Corollary 20. *If X does not have real points, then X is K-stable*

Such smooth Fano threefolds without real points do exists:

Example 21. Suppose that H_1 is cut out by $a_{00} + a_{11} + a_{22} + a_{33} = 0$. Then H_1 is smooth, because its preimage in $\mathbb{P}^3 \times \mathbb{P}^3$ via the map constructed above is given by

$$x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

Moreover, the fivefold H_1 does not have real points. Indeed, if $M \in V$, then the corresponding real numbers a_{00}, a_{11}, a_{22}, a_{33} are either all non-negative or all non-positive, and they cannot be all zero. Similarly, set $H_2 = \{a_{03} + 2a_{12} = 0\} \cap V$ and $H_3 = \{a_{02} + a_{13} + a_{23} = 0\} \cap V$. Then $V_{\mathbb{C}}$ is isomorphic to the complete intersection in $\mathbb{P}^3 \times \mathbb{P}^3$ given by

$$\begin{cases} x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0, \\ x_0y_3 + x_3y_0 + 2x_1y_2 + 2x_2y_1 = 0, \\ x_0y_2 + x_2y_0 + x_3y_1 + x_1y_3 + x_2y_3 + x_3y_2 = 0. \end{cases}$$

This complete intersection is a smooth threefold, so X is smooth, and it has no real points, because the divisor H_1 does not have real points.

3. The proof of Theorem A

Let us use all notations and assumptions introduced in Section 1. To start with, let us present few results from [1, 2] that will be used in the proof of Theorem A. Let $\mathbf{F}$ be a prime divisor over X, and let Z be its center on X. Suppose that

- either Z is a point,
- or Z is an irreducible curve.

Let P be any point in Z. Choose an irreducible smooth surface $S \subset X$ such that $P \in S$. Set

$$\tau = \sup\Big\{u \in \mathbb{Q}_{\geqslant 0} \ \Big|\ \text{the divisor } -K_X - uS \text{ is pseudo-effective}\Big\}.$$

For $u \in [0, \tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X - uS$, and let $N(u)$ be its negative part. Then $\beta(S) = 1 - S_X(S)$, where

$$S_X(S) = \frac{1}{-K_X^3}\int\limits_0^\infty \mathrm{vol}\big(-K_X - uS\big)du = \frac{1}{20}\int\limits_0^\tau P(u)^3 du.$$

Let us show how to compute $P(u)$ and $N(u)$. Set $H = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $H' = (\pi')^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Then

$$H \sim 3H' - E', E \sim 8H' - 3E', H' \sim 3H - E, E' \sim 8H - 3E,$$

where E and E' are exceptional surfaces of the blow ups π and π', respectively.

Example 22. Suppose that $S \in |H|$. Then $\tau = \frac{4}{3}$. Moreover, we have

$$P(u) \sim_{\mathbb{R}} \begin{cases} (4-u)H - E \ \text{ for } 0 \leqslant u \leqslant 1, \\ (4-3u)H' \ \text{ for } 1 \leqslant u \leqslant \dfrac{4}{3}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \ \text{ for } 0 \leqslant u \leqslant 1, \\ (u-1)E' \ \text{ for } 1 \leqslant u \leqslant \dfrac{4}{3}, \end{cases}$$

which gives $S_X(S) = \frac{1}{20}\int\limits_0^{\frac{4}{3}} \big(P(u)\big)^3 du = \frac{1}{20}\int\limits_0^1 (2-u)(u^2-10u+10)du + \frac{1}{20}\int\limits_1^{\frac{4}{3}}(4-3u)^3 du = \frac{53}{120}$.

Example 23. Suppose that $S=E$. Then $\tau=\frac{1}{2}$,

$$P(u)\sim_{\mathbb{R}}\begin{cases}4H-(1+u)E & \text{for } 0\leqslant u\leqslant\dfrac{1}{3},\\ (4-8u)H' & \text{for } \dfrac{1}{3}\leqslant u\leqslant\dfrac{1}{2},\end{cases}$$

and

$$N(u)=\begin{cases}0 & \text{for } 0\leqslant u\leqslant\dfrac{1}{3},\\ (3u-1)E' & \text{for } \dfrac{1}{3}\leqslant u\leqslant\dfrac{1}{2},\end{cases}$$

which gives $S_X(S)=\frac{1}{20}\int\limits_0^{\frac{1}{2}}4(1-u)(5-7u^2-10u)du+\frac{1}{20}\int\limits_{\frac{1}{2}}^{\frac{1}{3}}64(1-2u)^3du=\frac{11}{60}$.

Now, we choose an irreducible curve $C\subset S$ that contains the point P. For instance, if Z is a curve, and S contains Z, then we can choose $C=Z$. Since $S\not\subset\mathrm{Supp}(N(u))$, we can write

$$N(u)\big|_S=d(u)C+N'(u),$$

where $d(u)=\mathrm{ord}_C(N(u)|_S)$, and $N'(u)$ is an effective $\mathbb{R}$-divisor on S such that $C\not\subset\mathrm{Supp}(N'(u))$. Now, for every $u\in[0,\tau]$, we set

$$t(u)=\sup\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } P(u)\big|_S-vC \text{ is pseudo-effective}\Big\}.$$

For $v\in[0,t(u)]$, we let $P(u,v)$ be the positive part of the Zariski decomposition of $P(u)|_S-vC$, and we let $N(u,v)$ be its negative part. Following [1, 2], we let

$$S\big(W^S_{\bullet,\bullet};C\big)=\frac{3}{(-K_X)^3}\int\limits_0^\tau d(u)\Big(P(u)\big|_S\Big)^2du+\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^\infty\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu,$$

which we can rewrite as

$$S\big(W^S_{\bullet,\bullet};C\big)=\frac{3}{(-K_X)^3}\int\limits_0^\tau d(u)\Big(P(u)\big|_S\Big)^2du+\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^{t(u)}\big(P(u,v)\big)^2dvdu.$$

If Z is a curve, $Z\subset S$ and $C=Z$, then it follows from [1, 2] that

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\frac{1}{S_X(S)},\frac{1}{S\big(W^S_{\bullet,\bullet};C\big)}\right\}.\tag{1}$$

Let $f\colon\widetilde{S}\to S$ be the blow up of the point P, let F be the f-exceptional curve, let $\widetilde{N}'(u)$ be the strict transform on $\widetilde{S}$ of the $\mathbb{R}$-divisor $N(u)|_S$, and let $\widetilde{d}(u)=\mathrm{mult}_P(N(u)|_S)$. Then

$$f^*\big(N(u)\big|_S\big)=\widetilde{d}(u)F+\widetilde{N}'(u).$$

For every $u\in[0,\tau]$, set

$$\widetilde{t}(u)=\sup\Big\{v\in\mathbb{R}_{\geqslant 0}\ \Big|\ \text{the divisor } f^*\big(P(u)\big|_S\big)-vF \text{ is pseudo-effective}\Big\}.$$

For $v\in[0,\widetilde{t}(u)]$, we let $\widetilde{P}(u,v)$ be the positive part of the Zariski decomposition of $f^*(P(u)|_S)-vF$, and we let $\widetilde{N}(u,v)$ be its negative part. Let

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{3}{(-K_X)^3}\int\limits_0^\tau\widetilde{d}(u)\Big(f^*\big(P(u)\big|_S\big)\Big)^2du+\frac{3}{(-K_X)^3}\int\limits_0^\tau\int\limits_0^\infty\mathrm{vol}\big(f^*\big(P(u)\big|_S\big)-vF\big)dvdu.$$

Then

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{3}{(-K_X)^3}\int\limits_0^{\tau}\widetilde{d}(u)\big(\widetilde{P}(u,0)\big)^2du+\frac{3}{20}\int\limits_0^{\tau}\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\big)^2dvdu.$$

For every point $O\in F$, we let

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)=\frac{3}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)^2dvdu+F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big)$$

for

$$F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big)=\frac{6}{(-K_X)^3}\int\limits_0^{\tau}\int\limits_0^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)\cdot\mathrm{ord}_O\big(\widetilde{N}'(u)\big|_F+\widetilde{N}(u,v)\big|_F\big)dvdu.$$

Then it follows from [1, 2] that

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})}\geqslant\min\left\{\frac{1}{S_X(S)},\frac{2}{S\big(W^S_{\bullet,\bullet};F\big)},\inf_{O\in F}\frac{1}{S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)}\right\}.\tag{2}$$

Thus, if $S_X(S)<1$, $S(W^S_{\bullet,\bullet};F)<2$ and $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)<1$ for every point $O\in F$, then $\beta(\mathbf{F})>0$.

Now, we are ready to prove Theorem A. We must show that $\beta(\mathbf{F})>0$ if Z is not a point in $E\cap E'$. If Z is a surface, it follows from [21] that $\beta(\mathbf{F})>0$. Hence, we may assume that Z is **not** a surface.

Lemma 24 (cf. [8])**.** *Suppose that Z is a curve, $Z\subset E$, and $\pi(Z)$ is not a point. Then $\beta(\mathbf{F})>0$.*

Proof. Let e be the invariant of the ruled surface E defined in Proposition 2.8 in [26, Chapter V]. Then $e\geqslant -3$ [33]. Moreover, there exists a section C_0 of the projection $E\to C_6$ such that $C_0^2=-e$. Let ℓ a fiber of this projection. Then $H|_E\equiv 6\ell$ and $E|_E\equiv -C_0+\lambda\ell$ for some integer λ. Since

$$-28=-c_1\big(N_{C_6/\mathbb{P}^3}\big)=E^3=(-C_0+\lambda\ell)^2=-e-2\lambda,$$

we get $\lambda=\frac{28-e}{2}$, so e is even and $e\geqslant -2$. Since H' is nef and $H'|_E\equiv C_0+(18-\lambda)\ell$, we get

$$0\leqslant H'\cdot C_0=\big(C_0+(18-\lambda)\ell\big)\cdot C_0=\frac{8-e}{2},$$

which implies that $e\leqslant 8$. Thus, we see that $e\in\{-2,0,2,4,6,8\}$.

Set $S=E$ and $C=Z$. Let us estimate $S(W^S_{\bullet,\bullet};C)$. It follows from Example 23 that $\tau=\frac{1}{2}$ and

$$P(u)\big|_S\equiv\begin{cases}(1+u)C_0+\dfrac{20+e+ue-28u}{2}\ell & \text{for } 0\leqslant u\leqslant\dfrac{1}{3},\\ (4-8u)C_0+2(1-2u)(8+e)\ell & \text{for } \dfrac{1}{3}\leqslant u\leqslant\dfrac{1}{2}.\end{cases}$$

If $0\leqslant u\leqslant\frac{1}{2}$, then $N(u)=0$. If $\frac{1}{2}\leqslant u\leqslant\frac{1}{3}$, then $N(u)\big|_S=(3u-1)E'|_S$, where $E'|_S\equiv 3C_0+\frac{12+3e}{2}\ell$. By Proposition 2.20 in [26, Chapter V], we have $Z\equiv aC_0+b\ell$ for integers a and b such that $a\geqslant 0$ and $b\geqslant ae$. Since $\pi(Z)$ is not a point, we have $a\geqslant 1$. Then $\mathrm{ord}_C(E'|_S)\leqslant 3$. Hence, if $\frac{1}{3}\leqslant u\leqslant\frac{1}{2}$, then $d(u)\leqslant 3(3u-1)$. This gives

$$S(W^S_{\bullet,\bullet};C)=\frac{3}{20}\int\limits_{\frac{1}{3}}^{\frac{1}{2}}128(2u-1)^2d(u)du+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu\leqslant$$

$$\leqslant\frac{3}{20}\int\limits_{\frac{1}{3}}^{\frac{1}{2}}384(3u-1)(2u-1)^2du+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu=$$

$$=\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vC\big)dvdu=\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-v(aC_0+b\ell)\big)dvdu.$$

Thus, we conclude that $S(W^S_{\bullet,\bullet};C)\leqslant\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-v(aC_0+b\ell)\big)dvdu$.

Suppose that $b\geqslant 0$. Then

$$\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-v(aC_0+b\ell)\big)dvdu\leqslant\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)|_S-vC_0\big)dvdu.$$

On the other hand, we have

$$P(u)\big|_S-vC_0\equiv\begin{cases}(1+u-v)C_0+\dfrac{20+e+ue-28u}{2}\ell\ \text{if}\ 0\leqslant u\leqslant\dfrac{1}{3},\\(4-8u-v)C_0+2(1-2u)(8+e)\ell\ \text{if}\ \dfrac{1}{3}\leqslant u\leqslant\dfrac{1}{2}.\end{cases}$$

Hence, if $0\leqslant u\leqslant\frac{1}{3}$, then the divisor $P(u)\big|_S-vC_0$ is pseudoeffective $\iff$ it is nef $\iff$ $v\leqslant 1+u$. Likewise, if $\frac{1}{3}\leqslant u\leqslant\frac{1}{2}$, then $P(u)\big|_S-vC_0$ is pseudoeffective $\iff$ it is nef $\iff$ $v\leqslant 4-8u$. Then

$$S(W^S_{\bullet,\bullet};C)\leqslant\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)|_S-vC_0\big)dvdu=$$

$$=\frac{2}{45}+\frac{3}{20}\int\limits_0^{\frac{1}{3}}\int\limits_0^{1+u}\left((1+u-v)C_0+\frac{20+e+ue-28u}{2}\ell\right)^2dvdu+$$

$$+\frac{3}{20}\int\limits_{\frac{1}{3}}^{\frac{1}{2}}\int\limits_0^{4-8u}\big((4-8u-v)C_0+2(1-2u)(8+e)\ell\big)^2dvdu=\frac{23e}{1440}+\frac{221}{360}<1,$$

because $e\leqslant 8$. Then $\beta(\mathbf{F})>0$ by (1), since we know from Example 23 that $S_X(S)<1$.

Thus, to complete the proof, we may assume that $b<0$. Then $e<0$, so that $e=-2$, since $b\geqslant ae$. Hence, it follows from Proposition 2.21 in [26, Chapter V] that $a\geqslant 2$ and $b\geqslant -a$. Then

$$\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)|_S-vC\big)dvdu\leqslant\frac{3}{20}\int\limits_0^{\frac{1}{2}}\int\limits_0^{\infty}\mathrm{vol}\big(P(u)|_S-v(2C_0-2\ell)\big)dvdu.$$

Moreover, arguing as above, we compute

$$\frac{3}{20}\int\limits_{0}^{\frac{1}{2}}\int\limits_{0}^{\infty}\operatorname{vol}\big(P(u)|_S-v(2C_0-2\ell)\big)dvdu=\frac{41}{144},$$

which gives $S(W^S_{\bullet,\bullet};C)\leqslant\frac{2}{45}+\frac{41}{144}=\frac{79}{240}<1$, so that $\beta(\mathbf{F})>0$ by (1). $\square$

Similarly, we prove that

Lemma 25. *Suppose that Z is a curve, $Z\subset E'$, and $\pi'(Z)$ is not a point. Then $\beta(\mathbf{F})>0$.*

Now, suppose that Z is **not** a point in $E\cap E'$. To prove Theorem A, we must show that $\beta(\mathbf{F})>0$. Let P be a general point in Z. By Lemmas 24 and 25, we may assume that either $P\not\in E$ or $P\not\in E'$. Hence, without loss of generality, we may assume that $P\not\in E$. Let us show that $\beta(\mathbf{F})>0$.

Let S be a sufficiently general surface in $|H|$ that contains P. Then it follows from the adjunction formula that $-K_S\sim H'|_S$. Set $\Pi=\pi(S)$. Then Π is a general plane in $\mathbb{P}^3$ that contains $\pi(P)$. Write

$$\Pi\cap C_6=\big\{P_1,P_2,P_3,P_4,P_5,P_6\big\},$$

where P_1, P_2, P_3, P_4, P_5, P_6 are distinct points. Then π induces a birational morphism $\varpi\colon S\to\Pi$, which is a blow up of the intersection points P_1, P_2, P_3, P_4, P_5, P_6.

Lemma 26. *The divisor $-K_S$ is ample.*

Proof. We must show that at most three points among P_1, P_2, P_3, P_4, P_5, P_6 are contained in a line, and not all of these six points are contained in an irreducible conic.

If there exists a line $\ell\subset\Pi$ such that ℓ contains at least three points among P_1, P_2, P_3, P_4, P_5, P_6, then ℓ is a trisecant of the curve C_6, so that the line ℓ is contained in $\pi(E')$, and its strict transform on the threefold X is a fiber of the projection $E'\to C_6'$. But the planes in $\mathbb{P}^3$ containing $\pi(P)$ and a trisecant of the curve C_6 form a one-dimensional family. Hence, a general plane in $\mathbb{P}^3$ that contains the point $\pi(P)$ does not contain trisecants of the curve C_6. Therefore, we conclude that at most two points among P_1, P_2, P_3, P_4, P_5, P_6 are contained in a line.

Similarly, if the points P_1, P_2, P_3, P_4, P_5, P_6 are contained in an irreducible conic in Π, then its strict transform on the threefold X has trivial intersection with $H'\sim 3H-E$, which implies that this conic is the image of a fiber of the projection $E'\to C_6'$, which is impossible, since these fibers are mapped to lines in $\mathbb{P}^3$. Therefore, the divisor $-K_S$ is ample. $\square$

Thus, we can identify S with a smooth cubic surface in $\mathbb{P}^3$. Recall that $P\not\in E$.

Lemma 27. *Suppose that there exists a line $\ell\subset S$ such that $P\in\ell$. Then $\pi(\ell)$ is a conic.*

Proof. If $\pi(\ell)$ is not a conic, then $\pi(\ell)$ is a secant of the curve C_6 that contains $\pi(P)$. Let us show that we can choose Π such that it does not contain any secant of the curve C_6.

Let $\phi\colon\mathbb{P}^3\dashrightarrow\mathbb{P}^2$ be the linear projection from $\pi(P)$. Since C_6 is not hyperelliptic and $\pi(P)\not\in C_6$, one of the following two possibilities holds:

(1) $\phi(C_6)$ is a singular curve of degree 6, and ϕ induces a birational morphism $C_6\to\phi(C_6)$,

(2) $\phi(C_6)$ is a smooth cubic, and ϕ induces a double cover $C_6\to\phi(C_6)$.

In the second case, the curve C_6 is contained in an irrational cubic cone in $\mathbb{P}^3$, which is impossible, because the composition $\pi'\circ\pi^{-1}$ birationally maps every cubic surface containing C_6 to a plane in $\mathbb{P}^3$. Thus, we see that $\phi(C_6)$ is a singular irreducible curve of degree 6.

All secants of the curve C_6 containing $\pi(P)$ are mapped by ϕ to singular points of the curve $\phi(C_6)$. Since this curve has finitely many singular points, there are finitely many secants of the curve C_6 that pass through $\pi(P)$. Hence, since Π is a general plane in $\mathbb{P}^3$ that contains $\pi(P)$, we may assume that it does not contain secants of the curve C_6 containing $\pi(P)$, so $\pi(\ell)$ is a conic. $\square$

Let T be the unique hyperplane section of the surface $S \subset \mathbb{P}^3$ that is singular at P. Then it follows from Lemma 27 that either P is not contained in any line in S, and one of the following cases holds:

(a) T is an irreducible cubic curve that has a node at P;
(b) T is an irreducible cubic curve that has a cusp at P;

or P is contained in a unique line $\ell \subset S$, $\pi(\ell)$ is a conic, and one of the following cases holds:

(c) $T = \ell + C_2$ for a smooth conic C_2 that intersect ℓ transversally at P;
(d) $T = \ell + C_2$ for a smooth conic C_2 that is tangent to ℓ at P.

Let us construct another curve in S that is also singular at P. Namely, for each $i \in \{1, 2, 3, 4, 5, 6\}$, let ℓ_i be the proper transform on S of the unique line in Π that passes through the points $\pi(P)$ and P_i. Set $L = \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + \ell_6$. Then it follows from Example 22 that

$$P(u)\big|_S \sim_{\mathbb{R}} \begin{cases} \dfrac{2+u}{3}T + \dfrac{1-u}{3}L \text{ if } 0 \leqslant u \leqslant 1, \\ (4-3u)T \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3}. \end{cases}$$

Recall from Example 22 that $\tau = \frac{4}{3}$ and $S_X(S) = \frac{53}{120}$.

Let $\widetilde{T}$ and $\widetilde{L}$ be the proper transforms on $\widetilde{S}$ of the curves T and L, respectively. If $0 \leqslant u \leqslant 1$, then

$$f^*\big(P(u)|_S\big) - vF \sim_{\mathbb{R}} \frac{2+u}{3}\widetilde{T} + \frac{1-u}{3}\widetilde{L} + \frac{10-4u-3v}{3}F,$$

which implies that $\widetilde{t}(u) = \frac{10-4u}{3}$. Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then

$$f^*\big(P(u)|_S\big) - vF \sim_{\mathbb{R}} (4-3u)\widetilde{T} + (8-6u-v)F,$$

which implies that $\widetilde{t}(u) = 8 - 6u$.

Finally, set $R = E'|_S$. Then R is a smooth curve. Let $\widetilde{R}$ be its strict transform on $\widetilde{S}$. Then

$$\widetilde{N}'(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)\widetilde{R} \text{ if } 1 \leqslant u \leqslant \dfrac{4}{3}. \end{cases}$$

So, if $0 \leqslant u \leqslant 1$ or $P \notin E'$, then $\widetilde{d}(u) = 0$. Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$ and $P \in E'$, then $\widetilde{d}(u) = u - 1$.

Lemma 28. *Suppose that P is not contained in any line in S. Then $\beta(\mathbf{F}) > 0$.*

Proof. The curve T is irreducible. If $0 \leqslant u \leqslant 1$, then

$$P(u,v) \sim_{\mathbb{R}} \begin{cases} \dfrac{2+u}{3}\widetilde{T} + \dfrac{1-u}{3}\widetilde{L} + \dfrac{10-4u-3v}{3}F \text{ if } 0 \leqslant v \leqslant \dfrac{6-3u}{2}, \\ \dfrac{20-8u-6v}{3}\widetilde{T} + \dfrac{1-u}{3}\widetilde{L} + \dfrac{10-4u-3v}{3}F \text{ if } \dfrac{6-3u}{2} \leqslant v \leqslant 3-u, \\ \dfrac{10-4u-3v}{3}\big(2\widetilde{T} + \widetilde{L} + F\big) \text{ if } 3-u \leqslant v \leqslant \dfrac{10-4u}{3}, \end{cases}$$

and

$$N(u,v) = \begin{cases} 0 \text{ if } 0 \leqslant v \leqslant \dfrac{6-3u}{2}, \\ (2v-6+3u)\widetilde{T} \text{ if } \dfrac{6-3u}{2} \leqslant v \leqslant 3-u, \\ (2v-6+3u)\widetilde{T} + (v+u-3)\widetilde{L} \text{ if } 3-u \leqslant v \leqslant \dfrac{10-4u}{3}. \end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases} u^2-v^2-8u+10 \text{ if } 0\leqslant v\leqslant \dfrac{6-3u}{2},\\ 10u^2+12uv+3v^2-44u-24v+46 \text{ if } \dfrac{6-3u}{2}\leqslant v\leqslant 3-u,\\ (10-4u-3v)^2 \text{ if } 3-u\leqslant v\leqslant \dfrac{10-4u}{3}\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases} v \text{ if } 0\leqslant v\leqslant \dfrac{6-3u}{2},\\ 12-6u-3v \text{ if } \dfrac{6-3u}{2}\leqslant v\leqslant 3-u,\\ 30-12u-9v \text{ if } 3-u\leqslant v\leqslant \dfrac{10-4u}{3}.\end{cases}$$

Similarly, if $1\leqslant u\leqslant \frac{4}{3}$, then

$$P(u,v)\sim_{\mathbb{R}}\begin{cases} (4-3u)\widetilde{T}+(8-6u-v)F \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ (8-6u-v)\big(2\widetilde{T}+F\big) \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u,\end{cases}$$

and

$$N(u,v)=\begin{cases} 0 \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ (2v+9u-12)\widetilde{T} \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u.\end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases} 27u^2-v^2-72u+48 \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ 3(8-6u-v)^2 \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u,\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases} v \text{ if } 0\leqslant v\leqslant \dfrac{12-9u}{2},\\ 24-18u-3v \text{ if } \dfrac{12-9u}{2}\leqslant v\leqslant 8-6u.\end{cases}$$

Thus, if $P\in E'$, then

$$S\big(W^S_{\bullet,\bullet};F\big)=\frac{3}{20}\int\limits_1^{\frac{4}{3}}(27u^2-72u+48)(u-1)du+\frac{3}{20}\int\limits_0^1\int\limits_0^{\frac{6-3u}{2}}u^2-v^2-8u+10dvdu+$$
$$+\frac{3}{20}\int\limits_0^1\int\limits_{\frac{6-3u}{2}}^{3-u}10u^2+12uv+3v^2-44u-24v+46dvdu+\frac{3}{20}\int\limits_0^1\int\limits_{3-u}^{\frac{10-4u}{3}}(4u+3v-10)^2dvdu+$$
$$+\frac{3}{20}\int\limits_1^{\frac{4}{3}}\int\limits_0^{\frac{12-9u}{2}}27u^2-v^2-72u+48dvdu+\frac{3}{20}\int\limits_1^{\frac{4}{3}}\int\limits_{\frac{12-9u}{2}}^{8-6u}3(6u+v-8)^2dvdu=\frac{41}{24}<2.$$

Similarly, if $P\not\in E'$, then $S(W^S_{\bullet,\bullet};F)=\frac{409}{240}<\frac{41}{24}<2$.

Now, let O be a point in F. Let us compute $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)$. We have

$$S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)=\frac{3}{20}\int\limits_0^1\int\limits_0^{\frac{6-3u}{2}}v^2dvdu+\frac{3}{20}\int\limits_0^1\int\limits_{\frac{6-3u}{2}}^{3-u}(12-6u-3v)^2dvdu+$$
$$+\frac{3}{20}\int\limits_0^1\int\limits_{3-u}^{\frac{10-4u}{3}}(30-12u-9v)^2dvdu+\frac{3}{20}\int\limits_1^{\frac{4}{3}}\int\limits_0^{\frac{12-9u}{2}}8v^2dvdu+\frac{3}{20}\int\limits_1^{\frac{4}{3}}\int\limits_{\frac{12-9u}{2}}^{8-6u}(24-18u-3v)^2dvdu+F_O\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet}\big),$$

so that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)=\frac{63}{80}+F_O(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet})$. In particular, if $P\not\in E'$ and $O\not\in\widetilde{T}\cup\widetilde{C}$, then $F_O(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet})=0$, which implies that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)=\frac{63}{80}$. Let us compute $F_O(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet})$ in the remaining cases.

First, we deal with the case $P\not\in E'$. If $P\not\in E'$, then we have $O\not\in\mathrm{Supp}(\widetilde{N}'(u))$ for every $u\in[0,\frac{4}{3}]$. Moreover, if $P\not\in E'$ and $O\in\widetilde{L}$, then $O\not\in\widetilde{T}$, and $\widetilde{L}$ intersects F transversally at O, which gives

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)=\frac{63}{80}+\frac{6}{20}\int\limits_0^1\int\limits_{3-u}^{\frac{10-4u}{3}}\big(P(u,v)\cdot F\big)(v+u-3)\big(\widetilde{L}\cdot F\big)_Odvdu=\frac{19}{24}.$$

Similarly, if $P\not\in E'$ and $O\in\widetilde{T}$, then $O\not\in\widetilde{L}$ and

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)=\frac{63}{80}+\frac{6}{20}\int\limits_0^1\int\limits_{\frac{6-3u}{2}}^{\frac{10-4u}{3}}\big(P(u,v)\cdot F\big)(2v-6+3u)\big(\widetilde{T}\cdot F\big)_Odvdu+$$
$$+\frac{6}{20}\int\limits_1^{\frac{4}{3}}\int\limits_{\frac{12-9u}{2}}^{8-6u}\big(P(u,v)\cdot F\big)(2v+9u-12)\big(\widetilde{T}\cdot F\big)_O=\frac{63}{80}+\frac{6}{20}\int\limits_0^1\int\limits_{\frac{6-3u}{2}}^{3-u}(12-6u-3v)(2v-6+3u)\big(\widetilde{T}\cdot F\big)_Odvdu+$$
$$+\frac{6}{20}\int\limits_0^1\int\limits_{3-u}^{\frac{10-4u}{3}}(30-12u-9v)(2v-6+3u)\big(\widetilde{T}\cdot F\big)_Odvdu+\frac{6}{20}\int\limits_1^{\frac{4}{3}}\int\limits_{\frac{12-9u}{2}}^{8-6u}(24-18u-3)(2v+9u-12)\big(\widetilde{T}\cdot F\big)_O,$$

so $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)=\frac{63}{80}+\frac{5}{96}\big(\widetilde{T}\cdot F\big)_O\leqslant\frac{63}{80}+\frac{5}{96}\widetilde{T}\cdot F=\frac{107}{120}$. Hence, if $P\not\in E'$, then $\beta(\mathbf{F})>0$ by (2).

Therefore, to complete the proof of the lemma, we may assume that $P\in E'$. Since R is smooth, the curve $\widetilde{R}$ intersects F transversally at one point, so that

$$\mathrm{ord}_O\big(\widetilde{N}'(u)\big|_F\big)=\begin{cases}0\text{ if }0\leqslant u\leqslant 1,\\ 0\text{ if }1\leqslant u\leqslant\dfrac{4}{3}\text{ and }O\ne\widetilde{R}\cap F,\\ u-1\text{ if }1\leqslant u\leqslant\dfrac{4}{3}\text{ and }O=\widetilde{R}\cap F.\end{cases}$$

Hence, if $O\neq\widetilde{R}\cap F$, then $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)$ can be computed as in the case $P\not\in E'$. Thus, we may also assume that $O=\widetilde{R}\cap F$. Moreover, if $O\in\widetilde{L}$, then our previous calculations give

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)=\frac{6}{20}\int\limits_{1}^{\frac{4}{3}}\int\limits_{0}^{\widetilde{t}(u)}\big(\widetilde{P}(u,v)\cdot F\big)(u-1)dvdu+\frac{19}{24}=$$
$$=\frac{6}{20}\int\limits_{1}^{\frac{4}{3}}\int\limits_{0}^{\frac{12-9u}{2}}v(u-1)dvdu+\frac{6}{20}\int\limits_{1}^{\frac{4}{3}}\int\limits_{\frac{12-9u}{2}}^{8-6u}(24-18u-3v)(u-1)dvdu+\frac{19}{24}=\frac{191}{240}.$$

Similarly, if $O\in\widetilde{T}$, then, using our previous computations, we get

$$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)=\frac{1}{241}+\frac{63}{80}+\frac{5}{96}\big(\widetilde{T}\cdot F\big)_O\leqslant\frac{1}{241}+\frac{63}{80}+\frac{5}{96}\widetilde{T}\cdot F=\frac{43}{48}.$$

Thus, we see that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)<1$ for every point $O\in F$, so that $\beta(\mathbf{F})>0$ by (2). $\square$

To complete the proof of Theorem A, we may assume that $T=\ell+C_2$ and $P\in\ell\cap C_2$, where ℓ is a line such that $\pi(\ell)$ is a conic in $\mathbb{P}^2$, and C_2 is a smooth conic such that $\pi(C_2)$ is a line. Then C_2 is one of the curves ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, ℓ_6, so we may assume that $C_2=\ell_6$. Set $L'=\ell_1+\ell_2+\ell_3+\ell_4+\ell_5$. Let us denote by $\widetilde{\ell}$, $\widetilde{C}_2$, $\widetilde{L}'$ the strict transforms on the surface $\widetilde{S}$ of the curves ℓ, C_2, L', respectively. Then $\widetilde{\ell}\cap\widetilde{L}'=\varnothing$ and $\widetilde{C}_2\cap\widetilde{L}'=\varnothing$. Moreover, if $0\leqslant u\leqslant 1$, then

$$P(u,v)\sim_{\mathbb{R}}\begin{cases}\frac{2+u}{3}\widetilde{\ell}+\widetilde{C}_2+\frac{1-u}{3}\widetilde{L}'+\frac{10-4u-3v}{3}F\text{ if }0\leqslant v\leqslant 3-2u,\\ \frac{13-4u-3v}{6}\widetilde{\ell}+\widetilde{C}_2+\frac{1-u}{3}\widetilde{L}'+\frac{10-4u-3v}{3}F\text{ if }3-2u\leqslant v\leqslant\frac{9-4u}{3},\\ \frac{10-4u-3v}{3}\big(2\widetilde{\ell}+3\widetilde{C}_2+F\big)+\frac{1-u}{3}\widetilde{L}'\text{ if }\frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ \frac{10-4u-3v}{3}\big(2\widetilde{\ell}+\widetilde{L}'+3\widetilde{C}_2+F\big)\text{ if }3-u\leqslant v\leqslant\frac{10-4u}{3},\end{cases}$$

and

$$N(u,v)=\begin{cases}0\text{ if }0\leqslant v\leqslant 3-2u,\\ \frac{v+2u-3}{2}\widetilde{\ell}\text{ if }3-2u\leqslant v\leqslant\frac{9-4u}{3},\\ (2v+3u-6)\widetilde{\ell}+(3v+4u-9)\widetilde{C}_2\text{ if }\frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ (2v+3u-6)\widetilde{\ell}+(3v+4u-9)\widetilde{C}_2+(v+u-3)\widetilde{L}'\text{ if }3-u\leqslant v\leqslant\frac{10-4u}{3}.\end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases}u^2-v^2-8u+10\text{ if }0\leqslant v\leqslant 3-2u,\\ \frac{29}{2}-14u-3v+3u^2-\frac{v^2}{2}+2vu\text{ if }3-2u\leqslant v\leqslant\frac{9-4u}{3},\\ 11u^2+14uv+4v^2-50u-30v+55\text{ if }\frac{9-4u}{3}\leqslant v\leqslant 3-u,\\ (10-4u-3v)^2\text{ if }3-u\leqslant v\leqslant\frac{10-4u}{3},\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases} v \text{ if } 0\leqslant v\leqslant 3-2u,\\ \dfrac{3}{2}-u+\dfrac{v}{2} \text{ if } 3-2u\leqslant v\leqslant \dfrac{9-4u}{3},\\ 15-7u-4v \text{ if } \dfrac{9-4u}{3}\leqslant v\leqslant 3-u,\\ 30-12u-9v \text{ if } 3-u\leqslant v\leqslant \dfrac{10-4u}{3}.\end{cases}$$

Furthermore, if $1\leqslant u\leqslant \frac{4}{3}$, then

$$P(u,v)\sim_{\mathbb{R}}\begin{cases} (4-3u)(\widetilde{\ell}+\widetilde{C}_2)+(8-6u-v)F \text{ if } 0\leqslant v\leqslant 4-3u,\\ \dfrac{12-9u-v}{2}\widetilde{\ell}+(4-3u)\widetilde{C}_2+(8-6u-v)F \text{ if } 4-3u\leqslant v\leqslant \dfrac{20-15u}{3},\\ (8-6u-v)\big(2\widetilde{\ell}+3\widetilde{C}_2+F\big) \text{ if } \dfrac{20-15u}{3}\leqslant v\leqslant 8-6u,\end{cases}$$

and

$$N(u,v)=\begin{cases} 0 \text{ if } 0\leqslant v\leqslant 4-3u,\\ \dfrac{v+3u-4}{2}\widetilde{\ell} \text{ if } 4-3u\leqslant v\leqslant \dfrac{20-15u}{3},\\ (9u+2v-12)\widetilde{\ell}+(15u+3v-20)\widetilde{C}_2 \text{ if } \dfrac{20-15u}{3}\leqslant v\leqslant 8-6u.\end{cases}$$

This gives

$$\big(P(u,v)\big)^2=\begin{cases} 27u^2-v^2-72u+48 \text{ if } 0\leqslant v\leqslant 4-3u,\\ 56-84u-4v+\dfrac{63}{2}u^2-\dfrac{v^2}{2}+3vu \text{ if } 4-3u\leqslant v\leqslant \dfrac{20-15u}{3},\\ 4(8-6u-v)^2 \text{ if } \dfrac{20-15u}{3}\leqslant v\leqslant 8-6u,\end{cases}$$

and

$$P(u,v)\cdot F=\begin{cases} v \text{ if } 0\leqslant v\leqslant 4-3u,\\ 2-\dfrac{3u}{2}+\dfrac{v}{2} \text{ if } 4-3u\leqslant v\leqslant \dfrac{20-15u}{3},\\ 32-24u-4v \text{ if } \dfrac{20-15u}{3}\leqslant v\leqslant 8-6u.\end{cases}$$

Now, as in the proof of Lemma 28, we compute

$$S\big(W^S_{\bullet,\bullet};F\big)=\begin{cases} \dfrac{77}{45} \text{ if } P\in E',\\ \dfrac{1229}{720} \text{ if } P\not\in E'.\end{cases}$$

Similarly, if O is a point in F, we can compute $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)$ as we did this in the proof of Lemma 28. The results of these computations are presented in the following two tables:

condition	$O\in\widetilde{\ell}\cap\widetilde{C}_2\cap\widetilde{R}$	$\widetilde{\ell}\cap\widetilde{C}_2\ni O\not\in\widetilde{R}$	$\widetilde{\ell}\cap\widetilde{R}\ni O\not\in\widetilde{C}_2$	$\widetilde{\ell}\ni O\not\in\widetilde{R}\cup\widetilde{C}_2$	$\widetilde{C}_2\cap\widetilde{R}\ni O\not\in\widetilde{\ell}$
$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)$	$\frac{163}{180}$	$\frac{649}{720}$	$\frac{1859}{2160}$	$\frac{185}{216}$	$\frac{1801}{2160}$

condition	$\widetilde{C}_2\ni O\not\in\widetilde{R}\cup\widetilde{\ell}$	$O\in\widetilde{L}'\cap\widetilde{R}$	$\widetilde{L}'\ni O\not\in\widetilde{R}$	$\widetilde{R}\ni O\not\in\widetilde{\ell}\cup\widetilde{C}'\cap\widetilde{L}'$	$O\not\in\widetilde{\ell}\cup\widetilde{C}'\cap\widetilde{L}'\cup\widetilde{R}$
$S\big(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O\big)$	$\frac{112}{135}$	$\frac{571}{720}$	$\frac{71}{90}$	$\frac{71}{90}$	$\frac{113}{144}$

Thus, we proved that $S(W^S_{\bullet,\bullet};F)<2$, and we proved that $S(W^{\widetilde{S},F}_{\bullet,\bullet,\bullet};O)<1$ for every point $O\in F$. Therefore, using (2), we get $\beta(\mathbf{F})>0$. This completes the proof of Theorem A.

4. The proof of Theorem B

Let us use all assumptions and notations introduced in Section 1. Recall that

$$\mathrm{Aut}\big(\mathbb{P}^3,C_6\big)\simeq \mathrm{Aut}\big(C,[D]\big)\subset \mathrm{Aut}(C),$$

and all possibilities for the group $\mathrm{Aut}(C)$ are listed in [3, 17], where the two lists disagree a little bit. Moreover, since π is $\mathrm{Aut}(\mathbb{P}^3,C_6)$-equivariant, we can identify $\mathrm{Aut}(\mathbb{P}^3,C_6)$ with a subgroup in $\mathrm{Aut}(X)$. Then the action of the group $\mathrm{Aut}(X)$ on the set $\{E,E'\}$ gives a monomorphism

$$\mathrm{Aut}(X)/\mathrm{Aut}\big(\mathbb{P}^3,C_6\big)\hookrightarrow \boldsymbol{\mu}_2,$$

which is surjective if and only if $\mathrm{Aut}(X)$ has an element that swaps the surfaces E and E'.

Remark 29 ([17, Example 7.2.6]). We can choose M_1, M_2, M_3 in (♣) to be symmetric $\iff 2D\sim K_C$. Moreover, if M_1, M_2, M_3 are symmetric, then X admits the involution

$$\big([x_0:x_1:x_2:x_3],[y_0:y_1:y_2:y_3]\big)\mapsto\big([y_0:y_1:y_2:y_3],[x_0:x_1:x_2:x_3]\big).$$

In this case, we have $\mathrm{Aut}(X)\simeq\mathrm{Aut}(\mathbb{P}^3,C_6)\times\boldsymbol{\mu}_2$. For more details, see [34].

Remark 30 (Kuznetsov). Set $V=H^0(\mathcal{O}_C(K_C+D))$, $W=H^0(\mathcal{O}_C(2K_C-D))$ and $G=\mathrm{Aut}(C,[D])$. Let $\widehat{G}$ be a central extension of the group G such that D (considered as a line bundle) is $\widehat{G}$-linearizable. Then the sheaf $\mathcal{O}_C(D)$ admits a $\widehat{G}$-equivariant resolvent

$$0\to W^*\otimes\mathcal{O}_{\mathbb{P}^2}(-2)\to V\otimes\mathcal{O}_{\mathbb{P}^2}(-1)\to\mathcal{O}_C(D)\to 0,$$

which is known as the Beilinson resolvent. Since $W^*\otimes\mathcal{O}_{\mathbb{P}^2}(-2)\to V\otimes\mathcal{O}_{\mathbb{P}^2}(-1)$ is $\widehat{G}$-equivariant, the corresponding map $\rho\colon V^*\otimes W^*\to H^0\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)$ is equivariant, where $H^0\big(\mathcal{O}_{\mathbb{P}^2}(1)\big)\simeq H^0(\mathcal{O}_C(K_C))$ as $\widehat{G}$-representations. On the other hand, the embedding $X\hookrightarrow\mathbb{P}^3\times\mathbb{P}^3$ given by (♣) can be realized as

$$X=\big(\mathbb{P}(V^*)\times\mathbb{P}(W^*)\big)\cap\mathbb{P}\big(\ker(\rho)\big),$$

and the $\widehat{G}$-action on X factors through G, which is the natural G-action.

This remark gives

Lemma 31. *There exists a group homomorphism* $\eta\colon\mathrm{Aut}(X)\to\mathrm{Aut}(C)$ *such that its restriction to the subgroup* $\mathrm{Aut}(\mathbb{P}^3,C_6)\simeq\mathrm{Aut}(C,[D])$ *gives a natural embedding* $\mathrm{Aut}(\mathbb{P}^3,C_6)\hookrightarrow\mathrm{Aut}(C)$.

Proof. Let $\mathcal{M}$ be the two-dimensional linear system of divisors of degree $(1,1)$ on $\mathbb{P}^3\times\mathbb{P}^3$ that contains the threefold X. Then $\mathcal{M}$ can be identified with the projectivization of the three-dimensional vector space spanned by the matrices M_1, M_2, M_3, which we will identify with $\mathbb{P}^2_{x,y,z}$. Then $\mathrm{Aut}(X)$ naturally acts on this $\mathbb{P}^2_{x,y,z}$, because the action of the group $\mathrm{Aut}(X)$ on X lifts to its action on $\mathbb{P}^3\times\mathbb{P}^3$.

Moreover, the $\mathrm{Aut}(X)$-action on $\mathbb{P}^2_{x,y,z}$ preserves the quartic curve in $\mathbb{P}^2_{x,y,z}$ given by

$$\det\big(xM_1+yM_2+zM_2\big)=0,$$

which parametrizes singular divisors in $\mathcal{M}$. This curve is isomorphic to the curve C, which gives us the required homomorphism of groups $\eta\colon\mathrm{Aut}(X)\to\mathrm{Aut}(C)$. It follows from Remark 30 that this group homomorphism is functorial, so it gives a natural embedding $\mathrm{Aut}(\mathbb{P}^3,C_6)\hookrightarrow\mathrm{Aut}(C)$. □

Corollary 32. *Either* $\mathrm{Aut}(X)\simeq\mathrm{Aut}(\mathbb{P}^3,C_6)\times\boldsymbol{\mu}_2$ *or* $\mathrm{Aut}(X)$ *is isomorphic to a subgroup* $\mathrm{Aut}(C)$.

Now, we are ready to state a criterion when $\mathrm{Aut}(X)\neq\mathrm{Aut}(\mathbb{P}^3,C_6)$.

Lemma 33. $\mathrm{Aut}(X)\neq\mathrm{Aut}(\mathbb{P}^3,C_6)\iff$ *there is* $g\in\mathrm{Aut}(C)$ *such that* $g^*(D)\sim K_C-D$.

Proof. By Remark 30, the left copy of $\mathbb{P}^3$ in ($\bigstar$) can be be identified with $\mathbb{P}(H^0(\mathcal{O}_C(K_C+D))^\vee)$, while the right copy of $\mathbb{P}^3$ can be be identified with $\mathbb{P}(H^0(\mathcal{O}_C(2K_C-D))^\vee)$. Thus, if $\mathrm{Aut}(C)$ contains an automorphism g such that $g^*(D)\sim K_C-D$, we can use it to identify both copies of $\mathbb{P}^3$ in ($\bigstar$), which will give us an automorphism of X that swaps exceptional surfaces of the blow ups π and π'.

Vice versa, if the group $\mathrm{Aut}(X)$ is larger than $\mathrm{Aut}(\mathbb{P}^3,C_6)$, it follows from the proof of Lemma 31 that there exists $g\in\mathrm{Aut}(C)$ such that $g^*(D)\sim K_C-D$. $\square$

Recall that $\mathrm{Aut}(\mathbb{P}^3,C_6)\simeq\mathrm{Aut}(C,[D])$, where D is a divisor on C of degree 2 that satisfies ($\diamondsuit$). Using Remark 29, Lemma 33 and its proof, we obtain

Corollary 34. *One of the following three cases holds:*

- $2D\sim K_C$ *and* $\mathrm{Aut}(X)\simeq\mathrm{Aut}(C,[D])\times\boldsymbol{\mu}_2$,
- $2D\not\sim K_C$, *there is* $g\in\mathrm{Aut}(C)$ *such that* $g^*(D)\sim K_C-D$, *and*
$$\mathrm{Aut}(X)\simeq\big\langle\mathrm{Aut}(C,[D]),g\big\rangle.$$
- $\mathrm{Aut}(X)\simeq\mathrm{Aut}(C,[D])$, *and* $g^*(D)\not\sim K_C-D$ *for every* $g\in\mathrm{Aut}(C)$.

Corollary 35. *If* $\mathrm{Aut}(X)$ *is not isomorphic to any subgroup of* $\mathrm{Aut}(C)$, *then* $2D\sim K_C$.

Using Corollary 34, we can find all possibilities for $\mathrm{Aut}(X)$, but this requires a lot of work, because we have to analyze $\mathrm{Pic}^G(C)$ for every subgroup $G\subset\mathrm{Aut}(C)$. This can be done using

Proposition 36 ([16])**.** *Let G be a subgroup in* $\mathrm{Aut}(C)$. *Then there exists exact sequence*
$$1\to\mathrm{Hom}\big(G,\mathbb{C}^*\big)\to\mathrm{Pic}\big(G,C\big)\to\mathrm{Pic}^G\big(C\big)\to H^2\big(G,\mathbb{C}^*\big)\to 1,$$
where $\mathrm{Pic}(G,C)$ *is the group of G-linearized line bundles on C modulo G-equivariant isomorphisms.*

and

Remark 37. Let G be a subgroup in $\mathrm{Aut}(C)$, let $\Sigma_1,\ldots,\Sigma_n$ be all G-orbits in C of length less that $|G|$. We may assume that $|\Sigma_i|\geqslant|\Sigma_j|$ for $i\geqslant j$. For every $i\in\{1,\ldots,n\}$, set
$$e_i=\frac{|G|}{|\Sigma_i|}=\text{the order of the stabilizer in } G \text{ of a point in } \Sigma_i.$$
The *signature* of the G-action on C is the tuple $\big[g;e_1,\ldots,e_n\big]$, where g is the genus of the curve C/G. If $C/G\simeq\mathbb{P}^1$, then it follows from [16] that
$$\mathrm{Pic}\big(G,C\big)\simeq\mathbb{Z}\oplus\boldsymbol{\mu}_{a_1}\oplus\boldsymbol{\mu}_{a_2}\oplus\cdots\oplus\boldsymbol{\mu}_{a_{n-1}}$$
for $a_1=d_1,a_2=\frac{d_2}{d_1},\ldots,a_{n-1}=\frac{d_{n-1}}{d_{n-2}}$, where
$$\begin{aligned}d_1&=\gcd(e_1,\ldots,e_n),\\ d_2&=\gcd(e_1e_2,e_1e_3,\ldots,e_ie_j,\ldots,e_{n-1}e_n),\\ &\vdots\\ d_{n-1}&=\gcd(e_1e_2\cdots e_{n-1},\ldots,e_2\cdots e_{n-1}e_n).\end{aligned}$$
Moreover, if γ is a generator of the free part of $\mathrm{Pic}^G(C)$ in this case, then we have
$$4=\deg(K_C)=\mathrm{lcm}(e_1,\ldots,e_n)\left(n-2-\sum_{i=1}^{n}\frac{1}{e_i}\right)\deg(\gamma).$$

Let us show how to compute $\mathrm{Pic}^G(C)$ in some cases.

Example 38. Suppose that $\mathrm{Aut}(C)$ contains a subgroup $G \simeq \mathfrak{S}_4$. Then C is given in $\mathbb{P}^2_{x,y,z}$ by

$$x^4 + y^4 + z^4 + \lambda(x^2y^2 + x^2z^2 + y^2z^2) = 0$$

for some $\lambda \in \mathbb{C}$ such that $\lambda \notin \{-1, 2, -2\}$. One can show that

$$\mathrm{Aut}(C) \simeq \begin{cases} \mathfrak{S}_4 \text{ if } \lambda \neq 0 \text{ and } \lambda \neq \dfrac{-3 \pm 3\sqrt{7}i}{2}, \\ \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3 \text{ if } \lambda = 0, \\ \mathrm{PSL}_2(\mathbb{F}_7) \text{ if } \lambda = \dfrac{-3 \pm 3\sqrt{7}i}{2}. \end{cases}$$

We have $C/G \simeq \mathbb{P}^1$, and it follows from [31] that the signature is $[0; 2, 2, 2, 3]$. Thus, using Remark 37, we see that $\mathrm{Pic}(G, C) \simeq \mathbb{Z} \times \boldsymbol{\mu}_2^2$, and the free part of the group $\mathrm{Pic}(G, C)$ is generated by K_C. Moreover, using GAP, we compute $\mathrm{Hom}(G, \mathbb{C}^*) \simeq H^2(G, \mathbb{C}^*) \simeq \boldsymbol{\mu}_2$. Therefore, using Proposition 36, we get the following exact sequence of group homomorphisms:

$$0 \to \mathbb{Z} \times \boldsymbol{\mu}_2 \to \mathrm{Pic}^G(C) \to \boldsymbol{\mu}_2 \to 0.$$

We also know from [14] that $\mathrm{Pic}(C)$ contains two G-invariant even theta-characteristics θ_1 and θ_2. This immediately implies that $\mathrm{Pic}^G(C) = \langle \theta_1, \theta_2 \rangle \simeq \mathbb{Z} \times \boldsymbol{\mu}_2$.

Example 39 ([16])**.** Suppose that $\mathrm{Aut}(C) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$. Then C is given in $\mathbb{P}^2_{x,y,z}$ by

$$xy^3 + yz^3 + zx^3 = 0.$$

Set $G = \mathrm{Aut}(C)$. Using Example 1, we conclude that $\mathrm{Pic}^G(C)$ contains an even theta-characteristic θ. Now, arguing as in Example 38, we get $\mathrm{Pic}^G(C) = \langle \theta \rangle \simeq \mathbb{Z}$.

Example 40. Suppose that $\mathrm{Aut}(C) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. Then C is given in $\mathbb{P}^2_{x,y,z}$ by

$$x^4 + z^4 + z^4 = 0,$$

the group $\mathrm{Aut}(C)$ contains a unique subgroup isomorphic to $\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$, and C is the unique plane quartic curve admiting a faithful $\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$-action. Let G be this subgroup. Then the signature is $[0; 3, 3, 4]$. Therefore, using Remark 37, we get $\mathrm{Pic}(G, C) \simeq \mathbb{Z} \times \boldsymbol{\mu}_3$, where the free part is generated by K_C. Since $\mathrm{Hom}(G, \mathbb{C}^*) \simeq \boldsymbol{\mu}_3$ and $H^2(G, \mathbb{C}^*) \simeq \boldsymbol{\mu}_4$, it follows from Proposition 36 that

$$\mathrm{Pic}^G(C)/\langle K_C \rangle \simeq \boldsymbol{\mu}_4.$$

Moreover, we know from Section 2.2 that $\mathrm{Pic}^G(C)$ contains a divisor D of degree 2. Thus, we conclude that $\mathrm{Pic}^G(C) = \langle K_C, D \rangle \simeq \mathbb{Z} \times \boldsymbol{\mu}_2$, and $K_C - 2D$ is a two-torsion divisor.

Example 41. Let C be the Fermat quartic curve from Example 40, and let $G = \mathrm{Aut}(C) \simeq \boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. Then the signature is $[0; 2, 3, 8]$, so it follows from Remark 37 that

$$\mathrm{Pic}(G, C) \simeq \mathbb{Z} \times \boldsymbol{\mu}_2,$$

where the free part is generated by K_C. On can check that $\mathrm{Hom}(G, \mathbb{C}^*) \simeq \boldsymbol{\mu}_2$ and $H^2(G, \mathbb{C}^*) \simeq \boldsymbol{\mu}_2$. We claim that $\mathrm{Pic}^G(C)$ contains no divisors of degree 2. Indeed, if $\mathrm{Pic}^G(C)$ has a divisor D of degree 2, then $|K_C + D|$ gives a G-equivariant embedding $\phi\colon C \hookrightarrow \mathbb{P}^3$, which contradicts to Lemmas 14 and 15, because $\boldsymbol{\mu}_2^3.\mathfrak{S}_4$ does not contain subgroups isomorphic to G. Therefore, arguing as in Example 40, we see that $\mathrm{Pic}^G(C) = \langle K_C, \delta \rangle \simeq \mathbb{Z} \times \boldsymbol{\mu}_2$, where δ is a two-torsion divisor.

Using results described in Examples 38, 39, 40, 41, we get the following corollaries:

Corollary 42. *If* $\mathrm{Aut}(\mathbb{P}^3, C_6)$ *has a subgroup isomorphic to* $\mathfrak{S}_4$*, then one of the following holds:*

- $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathfrak{S}_4$ *and* $\mathrm{Aut}(X) \simeq \mathfrak{S}_4 \times \boldsymbol{\mu}_2$*,*
- $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$ *and* $\mathrm{Aut}(X) \simeq \mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$*.*

Corollary 43. *The smooth Fano threefold described in Example 1 is the unique smooth Fano threefold in the deformation family №2.12 that admits a faithful action of the group* $\mathrm{PSL}_2(\mathbb{F}_7)$*.*

Corollary 44. *The smooth Fano threefold described in Section 2.2 is the only smooth Fano threefold in the family №2.12 that admits a faithful action of the group* $\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$*.*

Proof. Suppose that $\mathrm{Aut}(X)$ has a subgroup isomorphic to $\boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$. Then arguing as in Example 41, we see that $\mathrm{Aut}(\mathbb{P}^3, C_6) \simeq \boldsymbol{\mu}_4^2 \rtimes \boldsymbol{\mu}_3$, and $\mathrm{Aut}(\mathbb{P}^3, C_6)$ is conjugate to the subgroup G that has been described in Section 2.2. Thus, the required assertion follows from Theorem 18. ☐

Now, we are ready to prove Theorem B.

Proof of Theorem B. It is enough to show that the automorphism group $\mathrm{Aut}(X)$ is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$ or $\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. Suppose this is not true. Let us seek for a contradiction.

Let $G = \mathrm{Aut}(C, [D])$. Then G is also not isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{F}_7) \times \boldsymbol{\mu}_2$ or $\boldsymbol{\mu}_4^2 \rtimes \mathfrak{S}_3$. Therefore, using [14] and the classification of automorphism groups of smooth plane quartic curves, we see that D is not an even theta-characteristic. So, by Corollary 35, the group $\mathrm{Aut}(X)$ is isomorphic to a subgroup of the group $\mathrm{Aut}(C)$.

Hence, using the classification of automorphism groups of smooth plane quartic curves again, we conclude that the group $\mathrm{Aut}(X)$ is isomorphic to one of the following groups:

$$\boldsymbol{\mu}_9,\ \boldsymbol{\mu}_{12},\ \mathrm{SL}_2(\mathbb{F}_3)\ (\text{GAP ID is [24,3]}),\ \boldsymbol{\mu}_4.\mathfrak{A}_4\ (\text{GAP ID is [48,33]}),$$

and it follows from Corollary 34 that either $G = \mathrm{Aut}(X)$ or G is a subgroup in $\mathrm{Aut}(X)$ of index 2. Thus, we have the following possibilities:

$\mathrm{Aut}(X)$	$\boldsymbol{\mu}_9$	$\boldsymbol{\mu}_{12}$	$\boldsymbol{\mu}_{12}$	$\mathrm{SL}_2(\mathbb{F}_3)$	$\boldsymbol{\mu}_4.\mathfrak{A}_4$	$\boldsymbol{\mu}_4.\mathfrak{A}_4$
G	$\boldsymbol{\mu}_9$	$\boldsymbol{\mu}_6$	$\boldsymbol{\mu}_{12}$	$\mathrm{SL}_2(\mathbb{F}_3)$	$\mathrm{SL}_2(\mathbb{F}_3)$	$\boldsymbol{\mu}_4.\mathfrak{A}_4$

Recall that D is a divisor on the quartic curve C such that $\deg(D) = 2$, the divisor D satisfies $\diamondsuit$, and its class $[D] \in \mathrm{Pic}(C)$ is G-invariant. Let us show that in each of our cases, such D does not exist.

First, using [17, 3], Proposition 36 and Remark 37, we can describe the equation of the curve C, the signature of the action of the group G on the curve C, the structure of the group $\mathrm{Pic}^G(S)$, and the degree of a generator γ of the free part of the group $\mathrm{Pic}^G(C)$. This gives the following possibilities:

G	Equation of C	Signature	Structure of $\mathrm{Pic}^G(S)$	$\deg(\gamma)$
$\boldsymbol{\mu}_6$	$y^4 - x^3z + z^4 = 0$	$[0; 2, 3, 3, 6]$	$\mathbb{Z} \oplus \mathbb{Z}_3$	1
$\boldsymbol{\mu}_9$	$y^3z - x(x^3 - z^3) = 0$	$[0; 3, 9, 9]$	$\mathbb{Z} \oplus \mathbb{Z}_3$	1
$\boldsymbol{\mu}_{12}$	$y^4 - x^3z + z^4 = 0$	$[0; 3, 4, 12]$	$\mathbb{Z}$	1
$\mathrm{SL}_2(3)$	$y^4 - x^3z + z^4 = 0$	$[0; 2, 3, 6]$	$\mathbb{Z} \oplus \mathbb{Z}_6$	4
$\boldsymbol{\mu}_4.\mathfrak{A}_4$	$y^4 - x^3z + z^4 = 0$	$[0; 2, 3, 12]$	$\mathbb{Z}$	4

In particular, if $G \cong \mathrm{SL}_2(3)$ or $G \simeq \boldsymbol{\mu}_4.\mathfrak{A}_4$, then C does not have G-invariant divisors of degree 2. Hence, we see that G is isomorphic to one of the following groups: $\boldsymbol{\mu}_6$, $\boldsymbol{\mu}_9$, $\boldsymbol{\mu}_{12}$.

Suppose that $G \simeq \boldsymbol{\mu}_{12}$. Then the action of G on C is generated by

$$[x : y : z] \mapsto \big[\omega_3 x : iy : z\big],$$

where ω_3 is a primitive cube root of the unity. Then G fixes the point $P = [1 : 0 : 0]$, which implies that $\mathrm{Pic}^G(S) = \mathbb{Z}[P]$, so that $D \sim 2P$, which contradicts to our assumption that D satisfies $\diamondsuit$.

Assume now that $G \simeq \boldsymbol{\mu}_9$. Then the G-action on the curve is given by

$$[x : y : z] \mapsto \big[\omega_9 x : \omega_9^{-3} y : z\big],$$

where ω_9 is a primitive ninth root of the unity. Set $P_1 = [1:0:0]$ and $P_2 = [0:1:0]$. Then

$$\mathrm{Pic}^G(S) = \langle P_1, P_2 \rangle,$$

because P_1 and P_2 are fixed by the action of the group G, and the divisor $P_1 - P_2$ is a 3-torsion. Then D is linearly equivalent to $2P_1$, $2P_2$ or $P_1 + P_2$, which contradicts $\diamondsuit$.

Finally, consider the case where G is isomorphic to $\boldsymbol{\mu}_6$. Then the G-action is given by

$$[x:y:z] \mapsto \big[x:-y:\omega_3 z\big],$$

Set $P = [0:0:1]$, $\Sigma_2 = [1:i:0] + [1:-i:0]$, and $\Sigma_2' = [1:1:0] + [1:-1:0]$. Then

$$K_C \sim 4P \sim \Sigma_2 + \Sigma_2',$$

and the divisors P, Σ_2, Σ' are G-invariant. This gives $\mathrm{Pic}^G(S) = \langle P, \Sigma_2 \rangle$, and $2P - \Sigma_2$ is a 3-torsion. Then D is linearly equivalent to $2P$, Σ_2, Σ_2', which contradicts $\diamondsuit$. □

References

[1] H. Abban, Z. Zhuang, *K-stability of Fano varieties via admissible flags*, Forum of Mathematics, Pi **10** (2022), e15.

[2] C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süß, N. Viswanathan, *The Calabi problem for Fano threefolds*, LMS Lecture Notes in Mathematics **485**, Cambridge University Press, 2023.

[3] F. Bars, *Automorphisms groups of genus 3 curves*, unpulished manuscript, 2006.

[4] O. Benoist, O. Wittenberg, *The Clemens-Griffiths method over non-closed fields*, Algebrebraic Geometry **7** (2020), 696–721.

[5] H. Blichfeldt, *Finite collineation groups*, University of Chicago Press, Chicago, 1917.

[6] W. Bosma, J. Cannon, C. Fieker, A. Steel, *Handbook of Magma functions*, (2010), 5017 pages.

[7] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, LMS Lecture Note Series **280**, Cambridge University Press, 2000.

[8] I. Cheltsov, P. Pokora, *On K-stability of $\mathbb{P}^3$ blown up along a quintic elliptic curve*, preprint, arXiv:2309.12528, 2023.

[9] I. Cheltsov, V. Przyjalkowski, C. Shramov, *Fano threefolds with infinite automorphism groups*, Izv. Math. **83** (2019), 860–907.

[10] I. Cheltsov, A. Sarikyan, *Equivariant pliability of the projective space*, Selecta Math. **29** (2023), paper 71, 84 pages.

[11] I. Cheltsov, C. Shramov, *Cremona groups and the icosahedron*, CRC Press, Boca Raton, FL, 2016.

[12] I. Cheltsov, Yu. Tschinkel, Z. Zhang, *Equivariant geometry of singular cubic threefolds*, preprint, arXiv:2401.10974, 2024.

[13] T. Ciurca, S. Tanimoto, Yu. Tschinkel, *Intermediate Jacobians and linearizability*, preprint, arXiv:2403.06047, 2024.

[14] L. Disney-Hogg, *Symmetries of Riemann surfaces and magnetic monopoles*, Ph.D. Thesis, University of Edinburgh, 2023.

[15] T. Dokchitser, `GroupNames.org`, 2021.

[16] I. Dolgachev, *Invariant stable bundles over modular curves $X(p)$*, in: Recent Progress in Algebra (Taejon/Seoul, 1997), Contemporary Mathematics **224**, Amer. Math. Soc., Providence, RI, 1999, 65–99.

[17] I. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge University Press, 2012.

[18] W. Edge, *Determinantal representations of $x^4 + y^4 + z^4$*, Proc. of the Cambridge Phil. Soc. **34** (1938), 6–21.

[19] W. Edge, *The Klein group in three dimensions*, Acta Math. **79** (1947), 153–223.

[20] H. Flenner, M. Zaidenberg, *Locally nilpotent derivations on affine surfaces with a $\mathbb{G}_m$-action*, Osaka J. Math. **42** (2005), 931–974.

[21] K. Fujita, *On K-stability and the volume functions of $\mathbb{Q}$-Fano varieties*, Proc. Lond. Math. Soc. **113** (2016), 541–582.

[22] K. Fujita, *A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.

[23] The GAP Group, *GAP – Groups, Algorithms, and Programming*, version 4.11.1, 2021.

[24] P. Gille, T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, 2006.

[25] S. Gorchinskiy, C. Shramov, *Unramified Brauer group and its applications*, Translations of Mathematical Monographs **246**, American Mathematical Society, 2018.
[26] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer–Verlag, 1983.
[27] M. Homma, *On projective normality and defining equations of a projective curve of genus three embedded by a complete linear system*, Tsukuba Jour. of Math. **4** (1980), 269–279.
[28] J. Kollár, *Severi-Brauer varieties; a geometric treatment*, preprint, arXiv:1606.04368, 2016.
[29] A. Kuznetsov, Yu Prokhorov, *Rationality over non-closed fields of Fano threefolds with higher geometric Picard rank*, to appear in Journal of the Institute of Mathematics of Jussieu.
[30] K. Lauter, J. P. Serre, *Geometric methods for improving the upper bounds on the number of rational points on algebraic curves over finite fields*, Jour. of Alg. Geom. **10** (2001), 19–36.
[31] The LMFDB collaboration, *The L-functions and modular forms database*, `https://www.lmfdb.org`, 2023.
[32] C. Li, *K-semistability is equivariant volume minimization*, Duke Math. Jour. **166** (2017), 3147–3218.
[33] M. Nagata, *On self-intersection number of a section on a ruled surface*, Nagoya Math. J. **37** (1970), 191–196.
[34] G. Ottaviani, *Effective methods for plane quartics, their theta characteristics and the Scorza map*, to appear in Annali dell' Universita di Ferrara.
[35] J. Paulhus, *A database of group actions on Riemann surfaces*, Springer Proceedings in Mathematics & Statistics **409** (2023), 693–708.
[36] A. Pinardin, *G-solid rational surfaces*, preprint, arXiv:2309.13757, 2023.
[37] Yu. Prokhorov, *G-Fano threefolds. II*, Adv. Geom. **13** (2013), 419–434.
[38] L. Wei, X. Yu, *Automorphism groups of smooth cubic threefolds*, Journal of the Math. Soc. of Japan **72** (2020), 1327–1343.
[39] Z. Zhuang, *Optimal destabilizing centers and equivariant K-stability*, Invent. Math. **226** (2021), 195–223.

Ivan Cheltsov
University of Edinburgh, Edinburgh, Scotland
`I.Cheltsov@ed.ac.uk`

Sione Ma'u
University of Auckland, Auckland, New Zealand
`s.mau@auckland.ac.nz`

Oliver Li
University of Melbourne, Melbourne, Australia
`oli@unimelb.edu.au`

Antoine Pinardin
University of Edinburgh, Edinburgh, Scotland
`antoine.pinardin@ed.ac.uk`