HALPHEN PENCILS ON QUARTIC THREEFOLDS

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Abstract. For every smooth quartic threefold, we classify all pencils on it whose general
element is an irreducible surface birational to a smooth surface of Kodaira dimension zero.

1. Introduction

Let $C$ be a smooth curve in $P^2$ that is defined by a cubic homogeneous equation $f(x, y, z) = 0,$
and let $P_1, \cdots, P_9$ be nine distinct points on the curve $C$ such that the divisor

$$\sum_{i=1}^{9} P_i - O_{P^2}(3)|_C$$

is a torsion divisor of order $m \geq 1$ on the curve $C$. Then there is a curve $Z \subset P^2$ of degree $3m$
such that $\text{mult}_{P_i}(Z) = m$ for each point $P_i$. Let $P$ be the pencil given by the equation

$$\lambda f^m(x, y, z) + \mu g(x, y, z) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z]) \cong P^2,$$

where $g(x, y, z) = 0$ is a homogeneous equation of the curve $Z$ and $(\lambda, \mu) \in P^1$. Then a general
curve in $P$ is birational to an elliptic curve.

Remark 1.1. The construction of pencil $P$ can be generalized to the case when $C$ has at most
ordinary double points and the points $P_1, \cdots, P_9$ are not necessarily distinct (see [3]).

The pencil $P$ is called a standard Halphen pencil.

Definition 1.2. A Halphen pencil is a one-dimensional linear system whose general element is
an irreducible subvariety that is birational to a smooth variety of Kodaira dimension zero.

The following result is proved in [3].

Theorem 1.3. Every Halphen pencil on $P^2$ is birational to a standard Halphen pencil.

Let $X$ be a smooth quartic threefold in $P^4$. Then $X$ is not rational, because

$$\text{Bir}(X) = \text{Aut}(X)$$

due to [5]. The following threefold analogue of Theorem 1.3 is proved in [1].

Theorem 1.4. Suppose that $X$ is general. Then every Halphen pencil on $X$ is cut out by

$$\lambda f_1(x, y, z, t, w) + \mu g_1(x, y, z, t, w) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong P^4,$$

where $f_1(x, y, z, t, w)$ and $g_1(x, y, z, t, w)$ are linearly independent linear forms, and $(\lambda, \mu) \in P^1$.

The assertion of Theorem 1.4 may fail without the generality assumption (see [4]).

Example 1.5. Suppose that $X$ is given by an equation

$$w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong P^4,$$

where $q_i$ and $p_i$ are a homogeneous forms of degree $i$. Let $P$ be the pencil that is cut out by

$$\lambda x^2 + \mu (wx + q_2(x, y, z, t)) = 0$$

We assume that all varieties are projective, normal and defined over $\mathbb{C}$. 

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where \((\lambda, \mu) \in \mathbb{P}^1\). Then \(\mathcal{P}\) is a Halphen pencil if \(q_2(0, y, z, t, w) \neq 0\).

The purpose of the given paper is to prove the following result.

**Theorem 1.6.** Let \(\mathcal{M}\) be a Halphen pencil on \(X\). Then one of the following holds:

- the pencil \(\mathcal{M}\) is cut out on the hypersurface \(X\) by a pencil 
  \[
  \lambda f_1(x, y, z, t, w) + \mu g_1(x, y, z, t, w) = 0 \subset \text{Proj}\left(C[x, y, z, t, w]\right) \cong \mathbb{P}^4,
  \]
  where \(f_1\) and \(g_1\) are linearly independent linear forms, and \((\lambda, \mu) \in \mathbb{P}^1\),
- the hypersurface \(X \subset \mathbb{P}^4\) is given by an equation
  \[
  w^3 + w^2 q_2(x, y, z, t) + wx p_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}\left(C[x, y, z, t, w]\right) \cong \mathbb{P}^4,
  \]
  and the pencil \(\mathcal{M}\) is cut out on the hypersurface \(X\) by a pencil 
  \[
  \lambda x^2 + \mu \left(wx + q_2(x, y, z, t)\right) = 0
  \]
  where \(q_i\) and \(p_i\) are homogeneous forms of degree \(i\), \(q_2(0, y, z, t) \neq 0\), and \((\lambda, \mu) \in \mathbb{P}^1\).

For a point \(P \in X\), let us define a mobility threshold by

\[
\iota(P) = \sup \left\{ \lambda \in \mathbb{Q} \mid n \left(\pi^* (-K_X) - \lambda E\right) \right\}
\]

where \(\pi: Y \to X\) is a blow up of \(P\), and \(E\) is the exceptional divisor of \(\pi\). Then

- the inequalities \(2 \geq \iota(P) \geq 1\) holds,
- \(\iota(P) = 1 \iff \) the hyperplane section of \(X\) that is singular at \(P\) is a cone,
- \(\iota(P) = 3/2\) in the case when \(X\) contains no lines passing through \(P\).

**Remark 1.7.** The proof of Theorem 1.6 implies that \(\iota(P) = 2 \iff X\) can be given by

\[
w^3 + w^2 q_2(x, y, z, t) + wx p_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}\left(C[x, y, z, t, w]\right) \cong \mathbb{P}^4
\]

in a way such that \(P\) is given by \(x = y = z = t = 0\), where \(q_i(x, y, z, t)\) and \(p_i(x, y, z, t)\) are homogeneous polynomials of degree \(i \geq 2\) such that the inequality \(q_2(0, y, z, t) \neq 0\) holds.

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2. Preliminaries

Let \(X\) be a threefold with \(\mathbb{Q}\)-factorial singularities, and let \(\mathcal{M}\) be a pencil on \(X\) whose general element is irreducible. We consider the log pair \((X, \mu \mathcal{M})\) for some \(\mu \in \mathbb{Q}\) such that \(\mu \geq 0\).

**Remark 2.1.** Let \(\mathcal{H}\) be a linear system on the threefold \(X\) whose general element is an irreducible surface. Then \(\mathcal{M} = \mathcal{H}\) if there is a proper Zariski closed subset \(\Sigma \subseteq X\) such that

\[
\text{Supp}(M) \cap \text{Supp}(H) \subseteq \Sigma
\]

for every general divisors \(M \in \mathcal{M}\) and \(H \in \mathcal{H}\) that are chosen independently of the subset \(\Sigma\).

Let \(\rho: X \dashrightarrow \mathbb{P}^1\) be a map induced by the pencil \(\mathcal{M}\). Then there is a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\alpha} & X \\
\downarrow{\beta} & & \downarrow{\rho} \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}
\]

such that \(Y\) is smooth and \(\beta\) is a morphism. Let \(\mathcal{B}\) be the proper transform of \(\mathcal{M}\) on \(Y\). Then

\[
K_Y + \mu \mathcal{B} \equiv \alpha^* \left(K_X + \mu \mathcal{M}\right) + \sum_{i=1}^k a_i E_i,
\]
where $E_i$ is an exceptional divisor of the birational morphism $\alpha$ and $a_i$ is a rational number.

**Definition 2.2.** The singularities of the log pair $(X, \mu M)$ are terminal (canonical, respectively) in the case when each rational number $a_i$ is positive (nonnegative, respectively).

It is convenient to specify where the log pair $(X, \mu M)$ is not terminal.

**Definition 2.3.** A subvariety $Z \subset X$ is a center of canonical singularities of $(X, \mu M)$ if $\alpha(E_i) = Z$ for some $\alpha$-exceptional divisor $E_i$ such that $a_i \leq 0$.

The set of all centers of canonical singularities of $(X, \mu M)$ is denoted by $\text{CS}(X, \mu M)$.

**Remark 2.4.** The log pair $(X, \mu M)$ is terminal if and only if $\text{CS}(X, \mu M) = \emptyset$.

**Remark 2.5.** Let $C \subset X$ be a curve such that $C \not\subseteq \text{Sing}(X)$. Then

$$C \in \text{CS}(X, \mu M) \iff \text{mult}_C(M) \geq 1/\mu,$$

where $M$ is a general surface in the pencil $M$.

Suppose now that $M \equiv -\mu K_X$, and suppose that one of the following holds:

- either the divisor $-K_X$ is ample, or the divisor $-K_X$ is nef and big;
- the linear system $|-qK_X|$ is base point free for $q \gg 0$ and induces an elliptic fibration.

**Theorem 2.6.** Suppose that $M$ is a Halphen pencil. Then $\text{CS}(X, \mu M) \neq \emptyset$.

**Proof.** Suppose that $\text{CS}(X, \mu M) = \emptyset$. Then $\text{CS}(X, \epsilon M) = \emptyset$ as well for some positive rational number $\epsilon > \mu$. The divisor $K_X + \epsilon M$ is ample. We consider the numerical equivalence

$$K_Y + \epsilon B \equiv \alpha^*(K_X + \epsilon M) + \sum_{i=1}^k c_i E_i,$$

where $c_i$ is a rational number. Then each $c_i$ is positive, and $B$ is base point free.

Let $S$ be a general surface in $B$, and let $m$ be a big and divisible natural number. Then

$$\left|m(K_Y + \epsilon S)\right|$$

gives a dominant rational map $\xi: Y \dasharrow Z$ with $\text{dim}(Z) \geq 2$. The adjunction formula implies

$$m(K_Y + \epsilon S)_{|S} \sim mK_S,$$

which implies that $\text{dim}(V) \leq 1$, because $S$ has Kodaira dimension zero. It is a contradiction. □

How to decide whether the pencil $M$ is a Halphen pencil or not?

**Lemma 2.7.** Suppose that there is a nef and big divisor $D$ on the threefold $X$ such that $D \cdot C = 0$ for every base curve $C$ of the pencil $M$, and $(X, \mu M)$ is canonical. Then $M$ is a Halphen pencil.

**Proof.** It follows from [7] that for some rational number $\lambda > \mu$ there is a birational map

$$\xi: X \dasharrow W$$

such that the map $\xi$ is an isomorphism in codimension one, the log pair $(W, \lambda H)$ is log-terminal, and the divisor $K_W + \lambda H$ is nef, where $H$ is a proper transform of $M$ on the threefold $W$.

Let $H$ be a general surface in the pencil $H$. Since

$$H \equiv \frac{1}{\lambda - \mu} \left(K_W + \lambda H - (K_W + \mu H)\right),$$
the divisor $H$ is nef. Then $|mH|$ is base-point-free for some $m \gg 0$ by the abundance theorem.

Let $R$ be the proper transform of $|mH|$ on the threefold $X$. Then

$$D \cdot R \cdot M = 0$$

where $R$ and $M$ are general surfaces in $R$ and $M$, respectively.

It follows from Remark 2.7 that the linear system $R$ is composed from the pencil $M$, which implies that the pencil $H$ is base-point-free and induces a morphism $\pi: W \to \mathbb{P}^1$.

The log pair $(W, \mu H)$ is canonical, because the map $\xi: X \dashrightarrow W$ is a log flop with respect to the log pair $(X, \mu M)$. Hence, the surface $H$ has canonical singularities, and the equivalence

$$K_W + \mu H \equiv 0$$

implies that $K_H \equiv 0$. Consequently, the linear system $M$ is a Halphen pencil.

The proof of Lemma 2.7 implies the following corollary.

**Corollary 2.8.** Under the assumption and notation of Lemma 2.7, suppose that

$$M \sim -nK_X$$

for some $n \in \mathbb{N}$. Then general surface in $M$ is birational to a K3 surface or an abelian surface.

The following result is obvious but sometimes it is useful.

**Lemma 2.9.** Let $C$ be a curve such that $\text{Sing}(X) \not\supseteq C \in \mathcal{CS}(X, \mu M)$. Then $-K_X \cdot C \leq -K_X^3$.

**Proof.** Let $M_1$ and $M_2$ be general surfaces in $M$. Then the inequalities

$$\text{mult}_C(M_1 \cdot M_2) \geq \text{mult}_C(M_1)\text{mult}_C(M_2) \geq 1/\mu^2$$

holds. We can find $H \in |-mK_X|$ that does not contain components of $M_1 \cdot M_2$. Then

$$\frac{m}{\mu^2}(-K_X^3) = H \cdot M_1 \cdot M_2 \geq \left(-mK_X \cdot C\right)\text{mult}_C(M_1 \cdot M_2) \geq \frac{m}{\mu^2}(-K_X \cdot C),$$

which implies the required inequality $-K_X \cdot C \leq -K_X^3$.

Let us introduce the following objects:

- let $S$ be a normal irreducible surface;
- let $O$ be a smooth point of the surface $S$;
- let $R$ be an effective divisor on $S$ such that $O \in R$;
- let $\mathcal{D}$ be a linear system on the surface $S$ such that
  - the linear system $\mathcal{D}$ has no fix components,
  - the point $O$ is contained in the base locus of $\mathcal{D}$.

**Lemma 2.10.** Let $D_1$ and $D_2$ be general curves in $\mathcal{D}$. Then

$$\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) \leq \text{mult}_O(R)\text{mult}_O(D_1 \cdot D_1).$$

**Proof.** Put $S_0 = S$ and $O_0 = O$. Consider the sequence of blow ups

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0$$

such that $\pi_1$ is a blow up of the point $O_0$, and $\pi_i$ is a blow up of the point $O_{i-1}$ that is contained in the curve $E_{i-1}$, where $E_{i-1}$ is the exceptional curve of the blow up $\pi_{i-1}$, and $i = 2, \ldots, n$.

Let $D_j$ be the proper transform of $D_j$ on the surface $S_i$ for $i = 0, \ldots, n$ and $j = 1, 2$. Then

$$D_1^i \equiv D_2^i \equiv \pi_i^*(D_1^{i-1}) - \text{mult}_{O_{i-1}}(D_1^{i-1})E_i \equiv \pi_i^*(D_2^{i-1}) - \text{mult}_{O_{i-1}}(D_2^{i-1})E_i$$

for $i = 1, \ldots, n$. Put $d_i = \text{mult}_{O_{i-1}}(D_1^{i-1}) = \text{mult}_{O_{i-1}}(D_2^{i-1})$ for $i = 1, \ldots, n$. 


Let $R^i$ be the proper transform of $R$ on the surface $S_i$ for $i = 0, \ldots, n$. Then
\[ R^i \equiv \pi_i^* \left( R^{i-1} \right) - \text{mult}_{O_{i-1}} \left( R^{i-1} \right) E_i \]
for $i = 1, \ldots, n$. Put $r_i = \text{mult}_{O_{i-1}} \left( R^{i-1} \right)$ for $i = 1, \ldots, n$. Then $r_1 = \text{mult}_O(R)$.

We may chose blow ups $\pi_1, \ldots, \pi_n$ in a way such that the intersection $D^n_1 \cap D^n_2$ is empty in the neighborhood of the exclamational locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then
\[ \text{mult}_O \left( D_1 \cdot D_2 \right) = \sum_{i=1}^n d_i^2. \]

We may chose blow ups $\pi_1, \ldots, \pi_n$ in a way such that the intersections $D^n_1 \cap R^n$ and $D^n_2 \cap R^n$ are empty in the neighborhood of the exclamational locus of $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$. Then
\[ \text{mult}_O \left( D_1 \cdot R \right) = \text{mult}_O \left( D_2 \cdot R \right) = \sum_{i=1}^n d_i r_i, \]
where some numbers among $r_1, \ldots, r_n$ may be zero. Then
\[ \text{mult}_O \left( D_1 \cdot R \right) = \text{mult}_O \left( D_1 \cdot R \right) = \sum_{i=1}^n d_i r_i \leq \sum_{i=1}^n d_i r_1 \leq \sum_{i=1}^n d_i^2 r_1 = \text{mult}_O(R) \text{mult}_O \left( D_1 \cdot D_1 \right), \]
because $d_i \leq d_i^2$ and $r_1 \leq r_1 = \text{mult}_O(R)$ for every $i = 1, \ldots, n$.

The obvious assertion of Lemma 2.10 is a cornerstone of the proof of Theorem 1.6.

3. CURVES

Let $X$ be a smooth quartic hypersurface in $\mathbb{P}^4$, and let $\mathcal{M}$ be a Halphen pencil on it. Then
\[ \mathcal{M} \sim -nK_X, \]
because Pic($X$) = $\mathbb{Z}K_X$. Put $\mu = 1/n$. Then $(X, \mu \mathcal{M})$ is canonical (see [5], [6], [2]).

**Lemma 3.1.** Suppose that $\mathbb{C}S(X, \mu \mathcal{M})$ contain a point $P \in X$. Then
\[ \text{mult}_P(M) = n \text{mult}_P(T) = 2n, \]
where $M$ is any surface in $\mathcal{M}$, and $T$ is a surface in $| - K_X |$ that is singular at $P$.

**Proof.** Let $M_1$ and $M_2$ be two general surfaces in $\mathcal{M}$. Then the inequality
\[ \text{mult}_P \left( M_1 \cdot M_2 \right) \geq 4n^2 \]
holds (see [6], [2]). Let $H$ be a general surface in $| - K_X |$ that contains $P$. Then
\[ 4n^2 = H \cdot M_1 \cdot M_2 \geq \text{mult}_P \left( M_1 \cdot M_2 \right) \geq 4n^2, \]
which implies that $\text{mult}_P(M_1 \cdot M_2) = 4n^2$. Then the inequality
\[ \text{mult}_P(M_1) = \text{mult}_P(M_2) = 2n \]
holds (see [6], [2]). Similarly, we see that
\[ 4n = H \cdot T \cdot M_1 \geq \text{mult}_P(T) \text{mult}_P(M_1) = 2n \text{mult}_P(M_1) \geq 4n, \]
which implies that $\text{mult}_P(T) = 2$. Similarly, we see that
\[ 4n^2 = H \cdot M \cdot M_1 \geq \text{mult}_P(M) \text{mult}_P(M_1) = 2n \text{mult}_P(M) \geq 4n^2, \]
where $M$ is any surface in $\mathcal{M}$. The assertion is proved. \[ \Box \]

Let $M_1$ and $M_2$ be two general surfaces in $\mathcal{M}$.
Lemma 3.2. Suppose that $\mathcal{CS}(X, \mu M)$ contain a point $P \in X$. Then

$$M_1 \cap M_2 = \bigcup_{i=1}^r L_i,$$

where $L_1, \ldots, L_r$ are all lines on $X$ that pass through $P$, and $r < +\infty$.

Proof. It follows from Lemma 3.1 that there

$$4n^2 = H \cdot M_1 \cdot M_2 = \text{mult}_P(M_1 \cdot M_2) = 4n^2,$$

where $H$ is a general surface in $|-K_X|$ that passes through $P$. Thus, we see that the support of the cycle $M_1 \cdot M_2$ consists of all lines on $X$ that contains $P$, which completes the proof. □

Lemma 3.3. Suppose that $\mathcal{CS}(X, \mu M)$ contain a point $P \in X$. Then the inequalities

$$n/3 \leq \text{mult}_L (M) \leq n/2$$

hold for every line $L \subset X$ such that $P \in L$.

Proof. Let $D$ be a general hyperplane section of $X$ that contains $L$. Then

$$M \big|_D = \text{mult}_L(M) L + \Delta,$$

where $M$ is a general surface in $\mathcal{M}$, and $\Delta$ is an effective divisor such that

$$\text{mult}_P(\Delta) \geq 2n - \text{mult}_L(M).$$

We have $L \cdot L = -2$ on the surface $D$. Hence, we have

$$n + 2\text{mult}_L(\mathcal{M}) = L \cdot \Delta \geq \text{mult}_P(\Delta) \geq 2n - \text{mult}_L(M),$$

which implies $n/3 \leq \text{mult}_L(M)$.

Let $T$ be the surface in $-K_X$ that is singular at $P$. Then the cycle $T \cdot D$ is reduced and

$$T \cdot D = L + Z,$$

where $Z$ is plane cubic curve that passes through the point $P$. Thus, we have

$$3n = \left( \text{mult}_L(M) L + \Delta \right) \cdot Z = 3\text{mult}_L(M) + \Delta \cdot Z \geq 3\text{mult}_L(M) + 2n - \text{mult}_L(M),$$

which implies $\text{mult}_L(M) \leq n/2$. □

In particular, the set $\mathcal{CS}(X, \mu M)$ contains no curves and $n \neq 1$ if it contains a point.

Proposition 3.4. Suppose that $\mathcal{CS}(X, \mu M)$ contains a curve. Then $n = 1$.

Let us prove Proposition 3.4. Suppose that the set $\mathcal{CS}(X, \mu M)$ contains a curve $Z$. Then

$$\text{mult}_Z(M) = n,$$

the set $\mathcal{CS}(X, \mu M)$ contains no points, and $\deg(Z) \leq 4$ by Lemma 2.9.

Lemma 3.5. Suppose that $\deg(Z) = 1$. Then $n = 1$.

Proof. Let $\pi : V \to X$ be the blow up along the line $Z$. Then

$$B \equiv -nK_V,$$

where $B$ is a proper transform of $\mathcal{M}$ on the threefold $V$. There is a commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\eta} & \mathbb{P}^2 \\
\pi \downarrow & & \downarrow \\
X & \xrightarrow{\psi} & \mathbb{P}^2;
\end{array}$$

where $\psi$ is a projection from $Z$, and $\eta$ is a morphism induced by the linear system $|-K_V|$. 


It follows from the equivalence $B \equiv -nK_X$ implies that the pencil $B$ is contained in the fibers of the elliptic fibration $\eta$. So, the base locus of $B$ does not contain curves not contracted by $\eta$.

The set $\mathcal{CS}(V, \mu B)$ is not empty by the Theorem [2.9]. Then it does not contain points because the set $\mathcal{CS}(X, \mu M)$ contains no points. Hence, there is an irreducible curve $L \subset V$ such that $mult_L(B) = n$

and $\eta(L)$ is a point $Q \in \mathbb{P}^2$, where $B$ is a general surface in the pencil $B$.

Every hyperplane section of the hypersurface $X$ that contains the line $Z$ is smooth in general point of the line $Z$. But $B$ is the pull-back of a pencil $P$ on the plane $\mathbb{P}^2$ via $\eta$. Then

$$P \sim O_{\mathbb{P}^2}(n),$$

which implies that $mult_Q(P) = n$. Then $n = 1$, because general surface in $M$ is irreducible. $\square$

Thus, we may assume that the set $\mathcal{CS}(X, \mu M)$ does not contain lines.

**Lemma 3.6.** The curve $Z \subset \mathbb{P}^4$ is contained in a two-dimensional linear subspace.

**Proof.** Suppose that the curve $Z$ is not contained in any plane in $\mathbb{P}^4$. Then $deg(Z) \geq 3$, and $Z$ is smooth if $deg(Z) = 3$. If $deg(Z) = 4$, then $Z$ may have at most one double point.

Suppose that $Z$ is smooth. Let $\alpha: U \to X$ be the blow up at $Z$, and $F$ be the exceptional divisor of the birational morphism $\alpha$. Then the base locus of the linear system

$$\left|\alpha^*\left(-deg(Z)K_X\right) - F\right|$$

does not contain any curve. The later is impossible, because explicit calculations show that

$$\left(\alpha^*\left(-deg(Z)K_X\right) - F\right) \cdot D_1 \cdot D_2 < 0,$$

where $D_1$ and $D_2$ are proper transforms on $U$ of the surfaces $M_1$ and $M_2$, respectively.

We see that the curve $Z$ is a quartic curve with a double point $P$. Let $\beta: W \to X \subset \mathbb{P}^4$ be the composition of the blow up at the point $P$ with the blow up along the proper transform of the curve $Z$. Let $G$ and $E$ be $\beta$-exceptional divisors such that $\beta(E) = Z$ and $\beta(G) = P$. Then

$$\left|\beta^*\left(-4K_X\right) - E - 2G\right|$$

has no base curves. The later is impossible, because explicit calculations show that

$$\left(\beta^*\left(-4K_X\right) - E - 2G\right) \cdot R_1 \cdot R_2 < 0$$

where $R_1$ and $R_2$ are proper transforms on $W$ of the surfaces $M_1$ and $M_2$, respectively. $\square$

If $deg(Z) = 4$, then $n = 1$ by Remark [2.11]. Thus, we may assume that $2 \leq deg(Z) \leq 3$

**Lemma 3.7.** Suppose that $deg(Z) = 3$. Then $n = 1$.

**Proof.** Let $P$ be the pencil in $|-K_X|$ that contains all surfaces that pass through the irreducible reduced plane cubic curve $Z$, and let $D$ be a general surface in $P$. Then $D$ is a smooth surface, and the base locus of the pencil $P$ consists of the curve $Z$ and some line $L \subset X$. We have

$$\mathcal{M}|_D = nZ + mult_L(\mathcal{M})L + B \equiv nZ + nL,$$

where $B$ is a pencil on $D$ without fixed components. But on the surface $D$ we have

$$Z \cdot Z = 0, \quad Z \cdot L = 3, \quad L \cdot L = -2,$$

which implies that $L \in \mathcal{CS}(X, \mu M)$. But the set $\mathcal{CS}(X, \mu M)$ does not contain lines. $\square$
We may assume that $Z$ is a conic. Let $\alpha: U \to X$ be the blow up at $Z$, and let $D$ be a general surface in the pencil $|−K_U|$. Then $D$ is a smooth K3 surface and

$$-nK_U|_D \equiv B|_D \equiv nL,$$

where $B$ is a proper transform of the pencil $\mathcal{M}$ on the threefold $U$, and $L$ is the unique base curve of the pencil $|−K_U|$. The latter implies that

$$L \in \text{CS}(U, \mu B),$$

because $L^2 = −2$ on the surface $D$. Then $B = |−K_U|$ by Remark 2.1, which implies that $n = 1$. The assertion of Proposition 3.4 is proved.

4. General points

Let $X$ be a smooth quartic hypersurface in $\mathbb{P}^4$, and let $\mathcal{M}$ be a Halphen pencil on it. Then

$$\text{CS}\left(X, \frac{1}{n} \mathcal{M}\right) \neq \emptyset,$$

where $n \in \mathbb{N}$ such that $\mathcal{M} \sim −nK_X$. Suppose that $n \neq 1$.

**Remark 4.1.** To prove Theorem 1.6, we must show that $X$ can be given by an equation

$$w^3x + w^2q_2(x, y, z, t, w) + wxp_3(x, y, z, t, w) + q_4(x, y, z, t, w) = 0 \subset \text{Proj} \left(\mathbb{C}[x, y, z, t, w]\right) \cong \mathbb{P}^4,$$

where $q_i$ and $p_i$ are homogeneous polynomials of degree $i \geq 2$ such that $q_2(0, y, z, t, w) \neq 0$.

It follows from Lemmas 3.3, 3.2, 3.3 that there is a point $P \in X$ such that

- there are finitely many distinct lines $L_1, \ldots, L_r \subset X$ containing $P \in X$,
- the base locus of the pencil $\mathcal{M}$ consists of the lines $L_1, \ldots, L_r \subset X$,
- the equality $\text{mult}_P(M) = 2n$ and inequalities

$$n/3 \leq \text{mult}_{L_i}(M) \leq n/2$$

hold, where $M$ is a general surface in the pencil $\mathcal{M}$,
- for $T \in |−K_X|$ that $\text{mult}_P(T) \geq 2$, the equality $\text{mult}_P(T) = 2$ holds,
- for general surfaces $M_1$ and $M_2$ in the pencil $\mathcal{M}$, we have

$$\text{mult}_P\left(M_1 \cdot M_2\right) = 4n^2.$$

**Remark 4.2.** It follows from the proof of Lemma 2.7 that such point $P$ is unique.

The quartic threefold $X$ can be given by an equation

$$w^3x + w^2q_2(x, y, z, t, w) + wq_3(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj} \left(\mathbb{C}[x, y, z, t, w]\right) \cong \mathbb{P}^4,$$

where $q_i$ is a homogeneous polynomial of degree $i \geq 2$.

**Remark 4.3.** The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = q_2(x, y, z, t) = q_3(x, y, z, t) = q_4(x, y, z, t) = 0.$$

The surface $T \in |−K_X|$ that is singular at the point $P$ is cut out on $X$ by the equation $x = 0$.

**Remark 4.4.** The assertion $\text{mult}_P(T) = 2$ is equivalent to $q_2(0, y, z, t, w) \neq 0$.

Let $\pi: V \to X$ be the blow up at the point $P$ and $E$ be the $\pi$-exceptional divisor. Then

$$B \equiv \pi^*(-nK_X) - 2E \equiv -nK_V,$$

where $B$ is the proper transform of $\mathcal{M}$ on the threefold $V$.

**Remark 4.5.** The pencil $\mathcal{B}$ has no base curves in $E$, because

$$\text{mult}_P\left(M_1 \cdot M_2\right) = \text{mult}_P(M_1)\text{mult}_P(M_2).$$
Let $L_i$ be the proper transform of the line $L_i$ on $V$, for $i = 1, \ldots, r$. Then

$$B_1 \cdot B_2 = \sum_{i=1}^{r} \text{mult}_{L_i} (B_1 \cdot B_2) \bar{L}_i,$$

where $B_1$ and $B_2$ are proper transforms of $M_1$ and $M_2$ on the threefold $V$, respectively.

**Lemma 4.6.** Let $Z$ be an irreducible curve on $X$ such that $Z \not\in \{L_1, \ldots, Z_r\}$. Then

$$\deg(Z) \geq 2 \text{mult}_P(Z),$$

and the equality $\deg(Z) = 2 \text{mult}_P(Z)$ implies that

$$\mathcal{Z} \cap \left( \bigcup_{i=1}^{r} \bar{L}_i \right) = \emptyset,$$

where $\mathcal{Z}$ is a proper transform of the curve $Z$ on the threefold $V$.

**Proof.** The curve $\mathcal{Z}$ is not contained in the base locus of $\mathcal{B}$. Hence, we have

$$0 \leq B_i \cdot \mathcal{Z} \leq n \left( \deg(Z) - 2 \text{mult}_P(Z) \right),$$

which implies the required assertions. □

For every line $L \subset X$, it is known that $L$ has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ if and only if no two-dimensional linear subspace in $\mathbb{P}^4$ is tangent to the quartic $X$ along the line $L$.

**Lemma 4.7.** Suppose that $L_i$ has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$. Then $\text{mult}_{L_i}(\mathcal{M}) = n/2$.

**Proof.** Let $\alpha: W \to X$ be the blow up of the threefold $X$ at the line $L_i$, and let $F$ be the exceptional divisor of the blow up $\alpha$. Then the surface $F$ is the rational ruled surface $\mathbb{F}_1$.

Let $\Delta$ be the irreducible curve on the surface $F$ such that $\Delta^2 = -1$, and let $Z$ be the fiber of the morphism $\pi_F: F \to L_i$ over the point $P$. Then $F|_F \equiv -(\Delta + Z)$, which implies that

$$\mathcal{H}|_F \equiv nZ + \text{mult}_{L_i}(\mathcal{M})(\Delta + Z),$$

where $\mathcal{H}$ is the proper transform of the pencil $\mathcal{M}$ on the threefold $W$.

Let $\beta: U \to W$ be the blow up along the curve $Z$, and let $G$ be the exceptional divisor of the blow up $\beta$. The surface $E$ is the proper transform of $G$ on the threefold $V$. Then

$$n + \text{mult}_{L_i}(\mathcal{M}) \geq \text{mult}_Z(\mathcal{H}|_F) \geq 2n - \text{mult}_{L_i}(\mathcal{M}),$$

which implies that $\text{mult}_{L_i}(\mathcal{M}) \geq n/2$. But $\text{mult}_{L_i}(\mathcal{M}) \leq n/2$ by Lemma □

The surface $T \in |-K_X|$ has only isolated singularities, and $\text{mult}_P(T) = 2$.

**Lemma 4.8.** Suppose that $L_i$ has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ for $i = 1, \ldots, r$. Then $r \geq 3$.

**Proof.** Suppose that $P$ is an ordinary double point of $T$. Then the conic

$$q_2(0, y, z, t) = 0 \subset \text{Proj} \left( \mathbb{C}[y, z, t] \right) \cong \mathbb{P}^2$$

is irreducible. Let $H_i$ be a general hyperplane section of $X$ that passes through $L_i$. Then

$$H_i \cdot T = L_i + Z_i,$$

where $Z_i$ is an irreducible reduced cubic curve. The line $L_i$ contains at most three singular points of the surface $T$, because the line $L_i$ has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.

The curve $Z_i$ intersect $L_i$ at the point $P$ and at some smooth point of $T$. Then

$$L_i^2 = H_i \cdot L_i - Z_i \cdot L_i < -1/2.$$
Let $\bar{T}$ be the proper transform of $T$ on the threefold $V$. Then $\bar{T}$ is normal, and it follows from the inequality $L_1^2 < -1/2$ that $L_1^2 < -1$ on the surface $\bar{T}$. But

$$\text{Supp}(T \cdot M) = \bigcup_{i=1}^{3} L_i,$$

because $\text{mult}_P(T \cdot M) = 4n$. The equalities $\text{mult}_P(T) = 2n$ and $\text{mult}_P(M) = 2n$ imply that

$$\text{Supp}(\bar{T} \cdot M) = \bigcup_{i=1}^{r} L_i,$$

where $\bar{M}$ be the proper transform of $M \in \mathcal{M}$ on the threefold $V$. Hence, we have

$$\bar{M}|_{\bar{T}} = \sum_{i=1}^{r} m_i \bar{L}_i,$$

but $M \cdot L_1 = -n$ and $\bar{L}_i \cdot \bar{L}_j = 0$ for $i \neq j$ on the surface $\bar{T}$. Then

$$-n = \bar{M} \cdot \bar{L}_j = \sum_{i=1}^{r} m_i \bar{L}_i \cdot \bar{L}_j = m_j \bar{L}_j \cdot \bar{L}_j,$$

which implies that $m_j < n$. Let $H$ be a general hyperplane section of the quartic $X$. Then

$$4n = M \cdot T \cdot H = \sum_{i=1}^{r} m_i L_i \cdot H = \sum_{i=1}^{r} m_i = \sum_{i=1}^{r} m_i < rn,$$

which implies that $k > 4$. Thus, the inequality $r \geq 4$ holds.

Therefore, to complete the proof, we may assume that $r \leq 3$, and

$$q_2(x, y, z, t) = (\alpha_1 y + \beta_1 z + \gamma_1 t)(\alpha_2 y + \beta_2 z + \gamma_2 t) + xp_1(x, y, z, t)$$

where $p_1(x, y, z, t)$ is a linear form, and $(\alpha_1, \beta_1, \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2, \beta_2, \gamma_2)$.

Let $Z$ be the curve in $X$ that is cut out by the equations

$$x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0,$$

which implies that $\deg(Z) = 3$. Then $\text{Supp}(Z)$ contains a line among $L_1, \ldots, L_r$.

The curve $Z$ is reduced. It follows from Lemma 4.6 that $Z$ is none of the following curves:

- a union of an irreducible cubic curve and a line;
- a union of two lines and an irreducible conic.

Thus, the reducedness of $Z$ implies that $r = 3$ and $Z = L_1 + L_2 + L_3 + L$, where $L$ is a line on the quartic $X$ that does not contain $P$. Then $L$ intersects $M$ in at least three points

$$L_1 \cap L, \ L_2 \cap L, \ L_3 \cap L,$$

but $M \cdot L = n$, which implies that $L$ is contained in $M$ by Lemma 4.7, which is impossible. \qed

The following conditions are satisfied for general smooth quartic threefold:

- are at most 3 lines on pass though a given point;
- every line has normal bundle $O_{\mathbb{P}^3}(-1) \oplus O_{\mathbb{P}^1}$.

**Remark 4.9.** To prove Theorem 1.6 it is enough to show that

$$q_3(x, y, z, t) = xp_2(x, y, z, t) + q_2(x, y, z, t)p_1(x, y, z, t),$$

where $p_1$ and $p_2$ are some homogenous polynomials of degree 1 and 2, respectively.

The remaining part of the proof of Theorem 1.6 splits into the following cases:

- the conic $q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$ is irreducible (see Section 5);
- the conic $q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$ is reducible and reduced (see Section 6);
- the conic $q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$ is not reduced (see Section 7).
5. Good points

Let us use the assumptions and notation of Section 4. Suppose that the conic
\[ q_2(0, y, z, t) = 0 \subset \text{Proj} \left( \mathbb{C}[y, z, t] \right) \cong \mathbb{P}^2 \]
is reduced and irreducible. In this section we prove the following result.

**Proposition 5.1.** The polynomial \( q_3(0, y, z, t) \) is divisible by \( q_2(0, y, z, t) \).

Let us prove Proposition 5.1. Suppose that \( q_3(0, y, z, t) \) is not divisible by \( q_2(0, y, z, t) \).

**Lemma 5.2.** The surface \( T \) is singular outside of the point \( P \).

**Proof.** We suppose that the surface \( T \) is smooth outside of the point \( P \). Let \( \bar{T} \) be the proper transform of the surface \( T \) on the threefold \( V \). Then \( \bar{T} \) is smooth, and
\[ \bar{L}_i \cdot \bar{L}_i = -2, \quad \bar{L}_i \cdot \bar{L}_j = 0 \]
on the surface \( \bar{T} \) for every \( i = 1, \ldots, r \) and \( j = 1, \ldots, r \). On the other hand, we have
\[
\text{Supp} \left( T \cdot \bar{M} \right) = \bigcup_{i=1}^{3} L_i,
\]
because \( \text{mult}_P(T \cdot M) = 4n \). The equalities \( \text{mult}_P(T) = 2n \) and \( \text{mult}_P(M) = 2n \) imply that
\[
\text{Supp} \left( \bar{T} \cdot \bar{M} \right) = \bigcup_{i=1}^{r} \bar{L}_i,
\]
where \( \bar{M} \) be the proper transform of the surface \( M \) on the threefold \( V \). Hence, we have
\[
\bar{M} \big|_{\bar{T}} = \sum_{i=1}^{r} m_i \bar{L}_i,
\]
but \( M \cdot \bar{L}_t = -n \) and \( \bar{L}_i \cdot \bar{L}_j = 0 \) for \( i \neq j \) on the surface \( \bar{T} \). Then the equalities
\[
-n = \bar{M} \cdot \bar{L}_j = \sum_{i=1}^{r} m_i \bar{L}_i \cdot \bar{L}_j = m_j \bar{L}_j \cdot \bar{L}_j = -2m_j
\]
implicate that \( m_j = n/2 \). Let \( H \) be a general hyperplane section of the quartic \( X \). Then
\[
4n = M \cdot T \cdot H = \sum_{i=1}^{r} m_i L_i \cdot H = \sum_{i=1}^{r} m_i = rn/2,
\]
which implies that \( r = 8 \). But the lines \( L_1, \ldots, L_r \subset \mathbb{P}^4 \) are given by the equations
\[
x = q_2(x, y, z, t) = q_3(x, y, z, t) = q_4(x, y, z, t) = 0,
\]
which implies that \( r \leq 3 \), because
\[
r \leq \left| x = q_2(x, y, z, t) = q_3(x, y, z, t) = w = 0 \right| \leq 6
\]
due to our assumption that \( q_3(0, y, z, t) \) is not divisible by \( q_2(0, y, z, t) \).

Let \( \mathcal{R} \) be the linear system on the threefold \( X \) that is cut out by quadrics
\[
xh_1 + \lambda (wx + q_2) = 0,
\]
where \( h_1 = h_1(x, y, z, t) \) is an arbitrary linear form and \( \lambda \in \mathbb{C} \).

**Remark 5.3.** The linear system \( \mathcal{R} \) does not have fixed components.

The following result is crucial for the proof of Proposition 5.1.
Lemma 5.4. Let $R_1$ and $R_2$ be general surfaces in the linear system $\mathcal{R}$. Then

$$\sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2) \leq 6.$$ 

Proof. We may assume that $R_1$ is cut out by $wx + q_2(x, y, z, t) = 0$, and $R_2$ is cut out by the equation $xh_1(x, y, z, t) = 0$, where the polynomial $h_1(x, y, z, t) = 0$ is sufficiently general. Then

$$\text{mult}_{L_i}(R_1 \cdot R_2) = \text{mult}_{L_i}(R_1 \cdot T),$$

where $T$ is the hyperplane section of $X$ that is cut out by $x = 0$. Put

$$R_1 \cdot T = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i$ is a natural number, and $\Delta$ is an effective cycle, whose support does not contain lines that pass through $P$. Then the equality $m_i = \text{mult}_{L_i}(R_1 \cdot T)$ holds.

Let $\bar{R}_1$ and $\bar{T}$ be the proper transforms of $R_1$ and $T$ on the threefold $V$, respectively. Then

$$\bar{R}_1 \cdot \bar{T} = \sum_{i=1}^{r} m_i \bar{L}_i + \Omega,$$

where $\Omega$ is an effective cycle, whose support does not contain the curves $\bar{L}_1, \ldots, \bar{L}_r$.

The support of the cycle $\Omega$ does not contain curves that are contained in the exceptional divisor $E$, because $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$ by our assumption. We have

$$6 = E \cdot \bar{R}_1 \cdot \bar{T} = \sum_{i=1}^{r} m_i (E \cdot \bar{L}_i) + E \cdot \Omega \geq \sum_{i=1}^{r} m_i (E \cdot \bar{L}_i) = \sum_{i=1}^{r} m_i,$$

which is exactly what we want. \hfill $\square$

Let $M$ and $R$ be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and $\Delta$ is an effective cycle, whose support contains no lines among $L_1, \ldots, L_r$.

Lemma 5.5. The cycle $\Delta$ is not trivial.

Proof. Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by Remark 2.7. But $\mathcal{R}$ is not a pencil. \hfill $\square$

We have $\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i$. On the other hand, the inequality

$$\text{mult}_P(\Delta) \geq 6n - \sum_{i=1}^{r} m_i$$

holds, because $\text{mult}_P(M) = 2n$ and $\text{mult}_P(R) \geq 3$. It follows from Lemma 4.6 that

$$\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i \geq 2 \text{mult}_P(\Delta) \geq 2 \left(6n - \sum_{i=1}^{r} m_i\right),$$

which implies that $\sum_{i=1}^{r} m_i \geq 4n$. But it follows from Lemmas 2.10 and 3.3 that

$$m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2) \text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2$$

for every $i = 1, \ldots, r$, where $R_1$ and $R_2$ are general surfaces in $\mathcal{R}$. Then

$$\sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2)n/2 \leq 3n$$

by Lemma 5.4, which is a contradiction. The assertion of Proposition 5.1 is proved.
6. Bad points

Let us use the assumptions and notation of Section 4. Suppose that the conic
\[ q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2 \]
is reduced and reducible. Therefore, we have
\[ q_2(x, y, z, t) = (\alpha_1 y + \beta_1 z + \gamma_1 t)(\alpha_2 y + \beta_2 z + \gamma_2 t) + xp_1(x, y, z, t) \]
where \( p_1(x, y, z, t) \) is a linear form, and \( (\alpha_1, \beta_1, \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2, \beta_2, \gamma_2) \).

**Proposition 6.1.** The polynomial \( q_3(0, y, z, t) \) is divisible by \( q_2(0, y, z, t) \).

Let us prove Proposition 6.1. Suppose that \( q_3(0, y, z, t) \) is not divisible by \( q_2(0, y, z, t) \). Then
- either the polynomial \( q_3(0, y, z, t) \) is not divisible by \( \alpha_1 y + \beta_1 z + \gamma_1 t \),
- or the polynomial \( q_3(0, y, z, t) \) is not divisible by \( \alpha_2 y + \beta_2 z + \gamma_2 t \).

**Remark 6.2.** We may assume that \( q_3(0, y, z, t) \) is not divisible by \( \alpha_1 y + \beta_1 z + \gamma_1 t \).

Let \( Z \) be the curve in \( X \) that is cut out by the equations
\[ x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0. \]

**Remark 6.3.** The equality \( \text{mult}(Z) = 3 \) holds, but \( Z \) is not necessary reduced.

Hence, it follows from Lemma 4.6 that \( \text{Supp}(Z) \) contains a line among \( L_1, \ldots, L_r \).

**Lemma 6.4.** The support of the curve \( Z \) does not contain an irreducible conic.

**Proof.** Suppose that \( \text{Supp}(Z) \) contains an irreducible conic \( C \). Then
\[ Z = C + L_i + L_j \]
for some \( i \in \{1, \ldots, r\} \ni j \). Then \( i = j \), because otherwise the set \( (C \cap L_i) \cup (C \cap L_j) \) contains a point that is different from the point \( P \), which is impossible by Lemma 4.6. Thus, we see that
\[ Z = C + 2L_i, \]
and it follows from Lemma 4.6 that \( C \cap L_i = P \). Then \( C \) is tangent to \( L_i \) at the point \( P \).

Let \( \bar{C} \) be a proper transform of the curve \( C \) on the threefold \( V \). Then
\[ \bar{C} \cap \bar{L}_i \neq \emptyset, \]
which is impossible by Lemma 4.6. The assertion is proved. \( \square \)

**Lemma 6.5.** The support of the curve \( Z \) consists of lines.

**Proof.** Suppose that \( \text{Supp}(Z) \) does not consist of lines. Then it follows from Lemma 6.4 that
\[ Z = L_i + C, \]
where \( C \) is an irreducible plane cubic curve. But the equality \( \text{mult}(Z) = 3 \) implies that \( C \) must be singular at the point \( P \), which is impossible by Lemma 4.6. \( \square \)

Without loss of generality, we may assume that
\[ Z = a_1 L_1 + \cdots + a_k L_k + L, \]
where \( a_1, a_2, a_3 \in \mathbb{N} \) such that \( a_1 \geq a_2 \geq a_3 \) and \( \sum_{i=1}^{k} a_i = 3 \), and \( L \) is a line such that \( P \not\in P \).

**Remark 6.6.** We have \( L_i \neq L_j \) whenever \( i \neq j \).

Let \( H \) be a sufficiently general surface of \( X \) that is cut out by the equation
\[ \lambda x + \mu(\alpha_1 y + \beta_1 z + \gamma_1 t) = 0, \]
where \( (\lambda, \mu) \in \mathbb{P}^1 \). Then \( H \) has at most isolated singularities.
Remark 6.7. The surface \( H \) is smooth at the points \( P \) and \( L \cap L_i \), where \( i = 1, \ldots, k \).

Let \( \bar{H} \) and \( \bar{L} \) be the proper transforms of \( H \) and \( L \) on the threefold \( V \), respectively.

**Lemma 6.8.** The inequality \( k \neq 3 \) holds.

*Proof.* Suppose that the equality \( k = 3 \) holds. Then \( H \) is smooth. Put 
\[
B|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega,
\]
where \( B \) is a general surface in the pencil \( \mathcal{B} \), and \( \Omega \) is an effective divisor on the surface \( \bar{H} \) whose support does not contain any of the curves \( \bar{L}_1, \bar{L}_2 \) and \( \bar{L}_3 \). Then \( \bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E \), because the base locus of \( \mathcal{B} \) consists of the curves \( \bar{L}_1, \ldots, \bar{L}_r \). Then
\[
n = L \cdot \left( m_1 L_1 + m_2 L_2 + m_3 L_3 + \Omega \right) = \sum_{i=1}^{3} m_i + L \cdot \Omega \geq \sum_{i=1}^{3} m_i,
\]
which implies that \( \sum_{i=1}^{3} m_i \leq n \). On the other hand, we have
\[
-n = L_i \cdot \left( m_1 L_1 + m_2 L_2 + m_3 L_3 + \Omega \right) = -3m_i + L_i \cdot \Omega \geq -3m_i,
\]
which implies that \( m_i \geq n/3 \). Thus, we have \( m_1 = m_2 = m_3 = n/3 \) and
\[
\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = \Omega \cdot \bar{L}_3 = 0,
\]
which implies that \( \text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \text{Supp}(\Omega) \cap \bar{L}_3 = \emptyset \).

Let \( B' \) be another general surface in \( \mathcal{B} \). Arguing as above, we see that
\[
B'|_{\bar{H}} = \frac{n}{3} \left( \bar{L}_1 + \bar{L}_2 + \bar{L}_3 \right) + \Omega',
\]
where \( \Omega' \) is an effective divisor on \( \bar{H} \) whose support does not contain \( \bar{L}_1, \bar{L}_2 \) and \( \bar{L}_3 \) such that we have \( \text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \text{Supp}(\Omega') \cap \bar{L}_3 = \emptyset \). But \( \Omega \cdot \Omega' = n^2 \neq 0 \). Then
\[
\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,
\]
because \( |\text{Supp}(\Omega) \cap \text{Supp}(\Omega')| < +\infty \) due to generality of the surfaces \( B \) and \( B' \).

The base locus of the pencil \( \mathcal{B} \) consists of the curves \( \bar{L}_1, \ldots, \bar{L}_r \). Hence, we have
\[
\text{Supp}(B) \cap \text{Supp}(B') = \bigcup_{i=1}^{r} \bar{L}_i,
\]
but \( \bar{L}_i \cap \bar{H} = \emptyset \) whenever \( i \notin \{1, 2, 3\} \). Hence, we have
\[
\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \cup (\text{Supp}(\Omega) \cap \text{Supp}(\Omega')) = \text{Supp}(B) \cap \text{Supp}(B') \cap \bar{H} = \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3,
\]
which implies that \( \text{Supp}(\Omega) \cap \text{Supp}(\Omega') \subset \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \). In particular, we see that
\[
\text{Supp}(\Omega) \cap \left( \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \right) \neq \emptyset,
\]
because \( \text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset \). But \( \text{Supp}(\Omega) \cap \bar{L}_i = \emptyset \) for \( i = 1, 2, 3 \). \( \square \)

**Lemma 6.9.** The inequality \( k \neq 2 \) holds.

*Proof.* Suppose that the equality \( k = 2 \) holds. Then \( Z = 2L_1 + L_2 + L \). Put
\[
B|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega,
\]
where \( B \) is a general surface in the pencil \( \mathcal{B} \), and \( \Omega \) is an effective divisor on the surface \( \bar{H} \) whose support does not contain the curves \( \bar{L}_1 \) and \( \bar{L}_2 \). Then \( \bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E \) and
\[
n = L \cdot \left( m_1 L_1 + m_2 L_2 + \Omega \right) = m_1 + m_2 + L \cdot \Omega \geq m_1 + m_2,
\]
which implies that $m_1 + m_2 \leq n$. On the other hand, we have

$$\bar{T}|_{\bar{H}} = 2\bar{L}_1 + \bar{L}_2 + \bar{L} + E|_{\bar{H}} \equiv \left(\pi^*\left(-K_X\right) - 2E\right)|_{\bar{H}},$$

where $\bar{T}$ is the proper transform of the surface $T$ on the threefold $V$. Then

$$-1 = \bar{L}_1 \cdot \left(2\bar{L}_1 + \bar{L}_2 + \bar{L} + E|_{\bar{H}}\right) = 2\bar{L}_1 \cdot \bar{L}_1 + 2,$$

which implies that $\bar{L}_1 \cdot \bar{L}_1 = -3/2$ on the surface $\bar{H}$. Then

$$-n = \bar{L}_1 \cdot \left(m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega\right) = -3m_1/2 + L_1 \cdot \Omega \geq -3m_1/2,$$

which implies that $m_1 \geq 2n/3$. Similarly, we see that $\bar{L}_2 \cdot \bar{L}_2 = -3$ on the surface $\bar{H}$. Then

$$-n = \bar{L}_2 \cdot \left(m_1\bar{L}_1 + m_2\bar{L}_2 + \Omega\right) = -3m_2 + L_2 \cdot \Omega \geq -3m_2,$$

which implies that $m_2 \leq n/3$. Thus, we have $m_1 = 2m_2 = 2n/3$ and

$$\Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = 0,$$

which implies that $\text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \emptyset$.

Let $B'$ be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'|_{\bar{H}} = \frac{2n}{3}\bar{L}_1 + \frac{n}{3}\bar{L}_2 + \Omega',$$

where $\Omega'$ is an effective divisor on $\bar{H}$ whose support does not contain $\bar{L}_1$ and $\bar{L}_2$ such that

$$\text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \emptyset,$$

which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that

$$\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,$$

and arguing as in the proof of Lemma 6.8, we obtain a contradiction. \hfill \Box

It follows from Lemmas 6.8 and 6.9 that $Z = 3\bar{L}_1 + L$. Put

$$B|_{\bar{H}} = m_1\bar{L}_1 + \Omega,$$

where $B$ is a general surface $\mathcal{B}$, and $\Omega$ is an effective divisor such that $\bar{L}_1 \not\subseteq \text{Supp}(\Omega)$. Then

$$\bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E,$$

because the base locus of $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then

$$n = \bar{L} \cdot \left(m_1\bar{L}_1 + \Omega\right) = m_1 + \bar{L} \cdot \Omega \geq m_1,$$

which implies that $m_1 \leq n$. On the other hand, we have

$$\bar{T}|_{\bar{H}} = 3\bar{L}_1 + \bar{L} + E|_{\bar{H}} \equiv \left(\pi^*\left(-K_X\right) - 2E\right)|_{\bar{H}},$$

where $\bar{T}$ is the proper transform of the surface $T$ on the threefold $V$. Then

$$-1 = \bar{L}_1 \cdot \left(3\bar{L}_1 + \bar{L} + E|_{\bar{H}}\right) = 3\bar{L}_1 \cdot \bar{L}_1 + 2,$$

which implies that $\bar{L}_1 \cdot \bar{L}_1 = -1$ on the surface $\bar{H}$. Then

$$-n = \bar{L}_1 \cdot \left(m_1\bar{L}_1 + \Omega\right) = -m_1 + L_1 \cdot \Omega \geq -m_1,$$

which gives $m_1 \geq n$. Thus, we have $m_1 = n$ and $\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = 0$. Then $\text{Supp}(\Omega) \cap \bar{L}_1 = \emptyset$.

Let $B'$ be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'|_{\bar{H}} = n\bar{L}_1 + \Omega',$$
where \( \Omega' \) is an effective divisor on \( \bar{H} \) whose support does not contain \( \bar{L}_1 \) such that
\[
\text{Supp}(\Omega') \cap \bar{L}_1 = \emptyset,
\]
which implies that \( \Omega \cdot \Omega' = n^2 \). In particular, we see that \( \text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset \).

The base locus of the pencil \( \mathcal{B} \) consists of the curves \( \bar{L}_1, \ldots, \bar{L}_r \). Hence, we have
\[
\text{Supp}(\mathcal{B}) \cap \text{Supp}(\mathcal{B}') = \bigcup_{i=1}^{r} \bar{L}_i,
\]
but \( \bar{L}_i \cap \bar{H} = \emptyset \) whenever \( \bar{L}_i \neq \bar{L}_1 \). Then \( \text{Supp}(\Omega) \cap \bar{L}_1 \neq \emptyset \), because
\[
\bar{L}_1 \cup \left( \text{Supp}(\Omega) \cap \text{Supp}(\Omega') \right) = \text{Supp}(\mathcal{B}) \cap \text{Supp}(\mathcal{B}') \cap \bar{H} = \bar{L}_1,
\]
which is a contradiction. The assertion of Proposition 6.1 is proved.

7. Very bad points

Let us use the assumptions and notation of Section 4. Suppose that \( q_3 = y^2 + \alpha f_2(z, t) + x h_2(z, t) + x^2 a_1(x, y, z, t) + xy b_1(x, y, z, t) + y^2 c_1(y, z, t) \)
where \( a_1, b_1, c_1 \) are linear forms, \( f_2 \) and \( h_2 \) is a homogeneous polynomial of degree two.

**Proposition 7.1.** The equality \( f_2(z, t) = 0 \) holds.

Let us prove Proposition 7.1 by reductio ad absurdum. Suppose that \( f_2(z, t) \neq 0 \).

**Remark 7.2.** By choosing suitable coordinates, we may assume that \( f_2 = z t \) or \( f_2 = z^2 \).

We must use smoothness of the threefold \( X \) by analyzing the shape of \( q_4 \). We have
\[
q_4 = f_4(z, t) + x u_3(z, t) + y v_3(z, t) + x^2 a_2(x, y, z, t) + xy b_2(x, y, z, t) + y^2 c_2(y, z, t),
\]
where \( a_2, b_2, c_2 \) are homogeneous polynomial of degree two, \( u_3 \) and \( v_3 \) are homogeneous polynomial of degree two, and \( f_4 \) is a homogeneous polynomial of degree four.

**Lemma 7.3.** Suppose that \( f_2(z, t) = z t \) and \( f_4(z, t) = t^2 g_2(z, t) \) for some \( g_2(z, t) \in \mathbb{C}[z, t] \). Then \( v_3(z, 0) \neq 0 \).

**Proof.** Suppose that \( v_3(z, 0) = 0 \). The surface \( T \), which is cut out by \( x = 0 \), is given by
\[
w^2 y^2 + z t + y^2 c_1(x, y, z, t) + t^2 g_2(z, t) + y v_3(z, t) + y^2 c_2(x, y, z, t) \subset \text{Proj} \left( \mathbb{C}[y, z, t, w] \right) \cong \mathbb{P}^3
\]
which immediately implies that \( T \) has non-isolated singularity along the line \( x = y = t = 0 \), because we assume that \( v_3(z, 0) = 0 \). But the latter is impossible because \( X \) is smooth. \( \square \)

Arguing as in the proof of Lemma 7.3, we obtain the following corollary.

**Corollary 7.4.** Suppose that \( f_4(z, t) = z^2 g_2(z, t) \) for some \( g_2(z, t) \in \mathbb{C}[z, t] \). Then \( v_3(0, t) \neq 0 \).

The assertions of Lemma 7.3 and Corollary 7.4 are crucial for the proof of Proposition 7.1.

**Lemma 7.5.** Suppose that \( f_2 = z t \). Then \( f_4(0, t) = f_4(z, 0) = 0 \).

**Proof.** We may assume that \( f_4(0, t) \neq 0 \). Let \( \mathcal{H} \) be a linear system on \( X \) that is cut out by
\[
x \lambda + y \mu + z \nu = 0,
\]
where \( (\lambda, \mu, \nu) \in \mathbb{P}^2 \). Then the base locus of \( \mathcal{H} \) consists of the point \( P \).

Let \( \mathcal{R} \) be a proper transform of \( \mathcal{H} \) on the threefold \( V \). Then the base locus of \( \mathcal{R} \) consists of a single point that is not contained in any of the curves \( \bar{L}_1, \ldots, \bar{L}_r \).

The linear system \( \mathcal{R}|_B \) does not have base points, where \( B \) is a general surface in \( \mathcal{B} \). But
\[
R \cdot R \cdot B = 2n > 0,
\]
where $R$ is a general surface in the linear system $\mathcal{R}$. Thus, the linear system $\mathcal{R}|_B$ is not composed from a pencil. Hence, the curve $R \cdot B$ is irreducible and reduced by the Bertini theorem.

Let $H$ and $M$ be general surfaces in $\mathcal{H}$ and $\mathcal{M}$, respectively. Then $M \cdot H$ is irreducible and reduced. Thus, the linear system $\mathcal{M}|_H$ is a pencil.

The surface $H$ contains no lines passing through $P$, and $H_3$ can be given by

$$w^3x + w^2y^2 + w(y^2l_1(x, y, z, t) + xl_2(x, y, z, t)) + l_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, w]) \cong \mathbb{P}^3,$$

where $l_i(x, y, z, t)$ is a homogeneous polynomial of degree $i$.

Arguing as in Example 1.5 we see that there is a pencil $Q$ on the surface $H$ such that
- the equivalence $Q \sim \mathcal{O}_{\mathbb{P}^3}(2)|_H$ holds,
- the equality $\text{mult}_P(Q) = 4$ holds.

Arguing as in the proof of Lemma 3.1 we see that $\mathcal{M}|_H = Q$ by Remark 2.1 Then

$$\text{mult}_P(M) = 4$$

and $M \equiv -2K_X$, where $M$ is a general surface in the pencil $\mathcal{M}$.

The surface $M$ is cut out on $X$ by an equation

$$\lambda x^2 + x \left( A_0 + A_1(y, z, t) \right) + B_2(y, z, t) + B_1(y, z, t) + B_0 = 0,$$

where $A_i$ and $B_i$ are homogeneous polynomials of degree $i$, and $\lambda \in \mathbb{C}$.

It follows from $\text{mult}_P(M) = 4$ that $B_1(y, z, t) = B_0 = 0$.

The coordinated $(y, z, t)$ are also local coordinates on $X$ near the point $P$. Then

$$x = -y^2 - y \left( zt + yp_1(y, z, t) \right) + \text{higher order terms},$$

which is a Taylor power series for $x = x(y, z, t)$, where $p_1(y, z, t)$ is a linear form.

The surface $M$ is locally given by the analytic equation

$$\lambda y^4 + \left( -y^2 - yzt \right) + y^2p_1(y, z, t) \left( A_0 + A_1(y, z, t) \right) + B_2(y, z, t) + \text{higher order terms} = 0,$$

and $\text{mult}_P(M) = 4$. Hence, we see that $B_2(y, z, t) = \lambda A_0y^2$ and

$$A_1(y, z, t)y^2 + A_0y \left( zt + yp_1(y, z, t) \right) = 0$$

which implies that $A_0 = A_1(y, z, t) = 0$. Then $B_2(y, z, t) = 0$, which implies that a general surface of the pencil $\mathcal{M}$ is cut out on the hypersurface $X$ by the equation $x^2 = 0$, which is a absurd. □

Arguing as in the proof of Lemma 7.5 we obtain the following corollary.

**Corollary 7.6.** Suppose that $f_2 = z^2$. Then $f_4(0, t) = 0$.

Let $\mathcal{R}$ be the linear system on the threefold $X$ that is cut out by cubics

$$xh_2(x, y, z, t) + \lambda \left( w^2x + wy^2 + q_3(x, y, z, t) \right) = 0,$$

where $h_2 = h_2(x, y, z, t)$ is an arbitrary homogenous polynomial of degree 2 and $\lambda \in \mathbb{C}$.

**Remark 7.7.** The linear system $\mathcal{R}$ does not have fixed components.

Let $M$ and $R$ be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and $\Delta$ is an effective cycle, whose support contains no lines among $L_1, \ldots, L_r$.

**Lemma 7.8.** The cycle $\Delta$ is not trivial.

**Proof.** Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by Remark 2.1 But $\mathcal{R}$ is not a pencil. □
We have $\text{mult}_P(\Delta) \geq 8n - \sum_{i=1}^r m_i$, because $\text{mult}_P(M) = 2n$ and $\text{mult}_P(\mathcal{R}) \geq 4$. Then

$$\deg(\Delta) = 12n - \sum_{i=1}^r m_i \geq 2\text{mult}_P(\Delta) \geq 2\left(8n - \sum_{i=1}^r m_i\right)$$

by Lemma 4.6, because $\text{Supp}(\Delta)$ does not contain any of the lines $L_1, \ldots, L_r$.

**Corollary 7.9.** The inequality $\sum_{i=1}^r m_i \geq 4n$ holds.

Let $R_1$ and $R_2$ be general surfaces in the linear system $\mathcal{R}$. Then

$$m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2)\text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2$$

for every $1 \leq i \leq 4$ by Lemmas 2.10 and 3.3. Then

$$4n \leq \sum_{i=1}^r m_i \leq \sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2)n/2.$$

**Corollary 7.10.** The inequality $\sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8$ holds.

Now we suppose that $R_1$ is cut out on the quartic $X$ by the equation

$$w^2x + wy^2 + q_3(x, y, z, t) = 0,$$

and $R_2$ is cut out by $xh_2(x, y, z, t) = 0$, where $h_2(x, y, z, t) = 0$ is sufficiently general. Then

$$\text{mult}_{L_i}(R_1 \cdot T) = \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8n,$$

where $T$ is the hyperplane section of the hypersurface $X$ that is cut out by $x = 0$. But

$$R_1 \cdot T = Z_1 + Z_2,$$

where $Z_1$ and $Z_2$ are cycles on $X$ such that $Z_1$ is cut out by $x = y = 0$, and $Z_2$ is cut out by

$$x = wy + f_2(z, t) + yc_1(x, y, z, t) = 0.$$

**Lemma 7.11.** The equality $\sum_{i=1}^r \text{mult}_{L_i}(Z_1) = 4$ holds.

**Proof.** The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = y = q_4(x, y, z, t) = 0,$$

which implies that $\sum_{i=1}^r \text{mult}_{L_i}(Z_1) = 4$ holds. \qed

Hence, we see that $\sum_{i=1}^r \text{mult}_{L_i}(Z_2) \geq 4$. But $Z_2$ can be considered as a cycle

$$uy + f_2(z, t) + yc_1(y, z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}\left(\mathbb{C}[y, z, t, w]\right) \cong \mathbb{P}^3,$$

and, introducing new variable $u = w + c_1(y, z, t)$, we see that $Z_2$ can be considered as a cycle

$$uy + f_2(z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}\left(\mathbb{C}[y, z, t, u]\right) \cong \mathbb{P}^3,$$

and we can consider lines $L_1, \ldots, L_r$ as curves in $\mathbb{P}^3$ given by the equations $y = f_4(z, t) = 0$.

**Lemma 7.12.** The inequality $f_2(z, t) \neq zt$ holds.

**Proof.** Suppose that $f_2(z, t) = zt$. Then it follows from Lemma 7.3 and Corollary 7.4 that

$$f_4(z, t) = zt(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t)$$

for some $(\alpha_1, \beta_1) \in \mathbb{P}^1 \ni (\alpha_2, \beta_2)$. Then $Z_2$ can be given by

$$uy + zt = yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t) = 0 \subset \text{Proj}\left(\mathbb{C}[y, z, t, u]\right) \cong \mathbb{P}^3,$$

which implies that $Z_2 = Z_2^1 + Z_2^2$, where $Z_2^1$ and $Z_2^2$ are cycles on $\mathbb{P}^3$ such that $Z_2^2$ is given by

$$y = uy + zt = 0,$$
and $Z_2^2$ is given by $uy + zt = v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0$.

We may assume that $L_1$ is given by $y = z = 0$, and $L_2$ is given by $y = t = 0$. Then

$$Z_2^1 = L_1 + L_2,$$

which implies that $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) \geq 2$.

Suppose that $r = 4$. Then $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_2 \neq 0$. Hence, we see that

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_2,$$

because $v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on $L_1$ and $L_2$. But

$$L_3 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_4,$$

because $zt$ does not vanish on $L_3$ and $L_4$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Suppose that $r = 3$. We may assume that $(\alpha_1, \beta_1) = (1, 0)$, but $\alpha_2 \neq 0$ and $\beta_2 \neq 0$. Then

$$L_2 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z,t) + yc_2(y,z,t) - uz(\alpha_2 z + \beta_2 t)$ does not vanish on $L_2$. We have

$$f_4(z,t) = z^2 t(\alpha_2 z + \beta_2 t),$$

which implies that $v_3(0,t) \neq 0$ by Corollary 7.4. Hence, we see that

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_3,$$

because $v_3(z,t) + yc_2(y,z,t) - uz(\alpha_2 z + \beta_2 t)$ and $zt$ do not vanish on $L_1$ and $L_3$, respectively, which implies that $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$. The latter is a contradiction. Then $r \neq 3$.

We see that $r = 2$. Then we may assume that $(\alpha_1, \beta_1) = (1, 0)$, and either $\alpha_2 = 0$ or $\beta_2 = 0$.

Suppose that $\alpha_2 = 0$. Then $f_4(z,t) = \beta_2 z^2 t^2$. By Lemma 7.3 and Corollary 7.4 we have

$$v_3(0,t) \neq 0 \neq v_3(z,0),$$

which implies that $v_3(z,t) + yc_2(y,z,t) - \beta_2 z t$ does not vanish neither on $L_1$ nor on $L_2$. Then

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_2,$$

which implies that $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

We see that $\alpha_2 \neq 0$ and $\beta_2 = 0$. We have $f_4(z,t) = \alpha_2 z^3 t$. Then

$$v_3(0,t) \neq 0$$

by Corollary 7.4. Then $L_1 \not\subseteq \text{Supp}(Z_2^2)$ because $v_3(z,t) + yc_2(y,z,t) - \alpha_2 z^2$ does not vanish on $L_1$.

The line $L_2$ is given by the equations $y = t = 0$. But $Z_2$ is given by the equations

$$uy + zt = v_3(z,t) + yc_2(y,z,t) - \alpha_2 u z^2 = 0,$$

which implies that $L_2 \not\subseteq \text{Supp}(Z_2^2)$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Therefore, we see that $f_2(z,t) = z^2$. It follows from Lemma 7.6 that

$$f_4(z,t) = zg_3(z,t)$$

for some $g_3(z,t) \in \mathbb{C}[z,t]$. We may assume that $L_1$ is given by $y = z = 0$.

**Lemma 7.13.** The equality $g_3(0,t) = 0$ holds.

**Proof.** Suppose that $g_3(0,t) \neq 0$. Then $\text{Supp}(Z_2) = L_1$, because $Z_2$ is given by the equations

$$uy + z^2 = zg_3(z,t) + yv_3(z,t) + y^2 c_2(y,z,t) = 0,$$

and the lines $L_2, \ldots, L_r$ are given by the equations $y = g_3(z,t) = 0$.

The cycle $Z_2 + L_1$ is given by the equations

$$uy + z^2 = z^2 g_3(z,t) + zy v_3(z,t) + zy^2 c_2(y,z,t) = 0,$$

which implies that the cycle $Z_2 + L_1$ can be given by the equations

$$uy + z^2 = zy v_3(z,t) + zy^2 c_2(y,z,t) - uy g_3(z,t) = 0.$$
We have $Z_2 + L_1 = C_1 + C_2$, where $C_1$ and $C_2$ are cycles on $\mathbb{P}^3$ such that $C_1$ is given by

$$y = uy + z^2 = 0,$$

and the cycle $C_2$ is given by the equations

$$uy + z^2 = zv_3(z, t) + yc_2(y, z, t) - ug_3(z, t) = 0.$$

We have $C_1 = 2L_2$. But $L_1 \not\subseteq \text{Supp}(C_2)$ because the polynomial

$$zv_3(z, t) + yc_2(y, z, t) - ug_3(z, t)$$

does not vanish on $L_1$, because $g_3(0, t) \neq 0$. Then

$$Z_2 + L_1 = 2L_2,$$

which implies that $Z_2 = L_1$. Then $\sum_{i=1}^r \text{mult}_{Z_i}(Z_2) = 1$, which is a contradiction. □

Thus, we see that $r \leq 3$ and

$$f_2(z, t) = z^2(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$$

for some $(\alpha_1, \beta_1) \in \mathbb{P}^1 \ni (\alpha_2, \beta_2)$. Then $v_3(0, t) \neq 0$ by Corollary 7.14. But $Z_2$ can be given by

$$uy + z^2 = yv_3(z, t) + y^2c_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,$$

which implies that $Z_2 = Z_2^1 + Z_2^2$, where $Z_2^1$ and $Z_2^2$ are cycles on $\mathbb{P}^3$ such that $Z_2^1$ is given by

$$y = uy + z^2 = 0,$$

and the cycle $Z_2^2$ is given by the equations

$$uy + z^2 = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0,$$

which implies that $Z_2^2 = 2L_1$. Thus, we see that $\sum_{i=1}^r \text{mult}_{Z_i}(Z_2^2) \geq 2$.

**Lemma 7.14.** The inequality $r \neq 3$ holds.

**Proof.** Suppose that $r = 3$. Then $\beta_1 \neq 0 \neq \beta_2$, which implies that

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on $L_1$. But

$$L_2 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_3,$$

because $\beta_1 \neq 0 \neq \beta_2$. Then $\sum_{i=1}^r \text{mult}_{Z_i}(Z_2^2) = 0$, which is a contradiction. □

Thus, we see that either $r = 1$ or $r = 2$.

**Lemma 7.15.** The inequality $r \neq 2$ holds.

**Proof.** Suppose that $r = 2$. We may assume that either $\beta_1 \neq 0 = \beta_2$, or $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$.

Suppose that $\beta_2 = 0$. Then $f_1(z, t) = \alpha_2 z^3(\alpha_1 z + \beta_1 t)$ and

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on $L_1$. But $L_2$ is given by

$$y = \alpha_1 z + \beta_1 t = 0,$$

which implies that $z^2$ does not vanish on $L_2$, because $\beta_1 \neq 0$. Then $L_2 \not\subseteq \text{Supp}(Z_2^2)$, which immediately implies that $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Hence, we see that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$. Then

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)^2$ does not vanish on $L_1$. But $L_2 \not\subseteq \text{Supp}(Z_2^2)$, because the polynomial $z^2$ is not zero on $L_2$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. □
We see that $f_4(z, t) = z^2$ and $f_4(z, t) = \mu z^4$ for some $0 \neq \mu \in \mathbb{C}$. The cycle $Z_2^2$ is given by
\[ uy + z^2 = v_3(z, t) + yc_2(y, z, t) - \mu z^2 = 0, \]
where $v_3(0, t) \neq 0$ by Corollary [7.1]. Thus, we see that
\[ L_1 \not\subseteq \text{Supp}(Z_2^2), \]
because $v_3(z, t) + yc_2(y, z, t) - \mu z^2$ does not vanish on $L_1$. Then
\[ \sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0, \]
which is a contradiction. The assertion of Proposition [7.1] is completely proved.

The assertion of Theorem [1.6] follows from Propositions [3.4, 5.1, 6.1, 7.1].

References


