RICCI FLOW ON KÄHLER-EINSTEIN MANIFOLDS

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Abstract

This is the continuation of our earlier article [10]. For any Kähler-Einstein surfaces with positive scalar curvature, if the initial metric has positive bisectional curvature, then we have proved (see [10]) that the Kähler-Ricci flow converges exponentially to a unique Kähler-Einstein metric in the end. This partially answers a long-standing problem in Ricci flow: On a compact Kähler-Einstein manifold, does the Kähler-Ricci flow converge to a Kähler-Einstein metric if the initial metric has positive bisectional curvature? In this article we give a complete affirmative answer to this problem.

Contents

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1. Introduction

THEOREM 1.1
Let M be a Kähler-Einstein manifold with positive scalar curvature. If the initial metric has nonnegative bisectional curvature and positive bisectional curvature at least at one point, then the Kähler-Ricci flow converges exponentially fast to a Kähler-Einstein metric with constant bisectional curvature.

Remark 1.2
This problem was completely solved by R. S. Hamilton in the case of Riemann surfaces (see [20]). We also refer the reader to B. Chow's article [13] for more developments on this problem.

As a direct consequence, we have the following.

COROLLARY 1.3
The space of Kähler metrics with nonnegative bisectional curvature (and positive at least at one point) is path connected. The space of metrics with a nonnegative curvature operator (and positive at least at one point) is also path connected.

THEOREM 1.4
Let M be any Kähler-Einstein orbifold (see Definition 9.2) with positive scalar curvature. If the initial metric has nonnegative bisectional curvature and positive bisectional curvature at least at one point, then the Kähler-Ricci flow converges exponentially fast to a Kähler-Einstein metric with constant bisectional curvature. Moreover, M is a global quotient of $\mathbb{C}P^n$.

Clearly, Corollary 1.3 holds in the case of Kähler orbifolds. We want to mention here that there have been some earlier works in 1-dimensional complex orbifolds (see [14], [36]).

Remark 1.5
What we really need is for the Ricci curvature to be positive along the Kähler-Ricci flow. Since the positivity on Ricci curvature may not be preserved under the Ricci flow, we use the fact that the positivity of the bisectional curvature is preserved.

Remark 1.6
In view of the solution of the Frankel conjecture solved by S. Mori [27] and Siu and Yau [31], it suffices to study this problem on a Kähler manifold that is biholomorphic to $\mathbb{C}P^n$. However, we do not need to use the result of the Frankel conjecture. Moreover, we do not use explicitly the knowledge of the positive bisectional curvature. We use this condition only when we quote a result of Mok and Bando (see [10]) and a classification theorem by M. Berger [3].

Remark 1.7
We need the assumption on the existence of a Kähler-Einstein metric because we use a nonlinear inequality from [34]. Such an inequality is just the Moser-Trudinger-Onofri inequality if the underlying manifold is the Riemann sphere.

Remark 1.8
If we assume the existence of a lower bound for the functional $E_1 - E_0$,* then we are able to derive a convergence result similarly. Therefore, it is interesting to study the lower bound of $E_1 - E_0$ among metrics whose bisectional curvature is positive.

Remark 1.9
We learned from H. D. Cao (see [7]) that the holomorphic orthogonal bisectional curvature† is preserved under the Kähler-Ricci flow. (This follows from Mok's proof by a simple modification.) In this case, where the orthogonal bisectional curvature is positive, it is easy to see that positive Ricci curvature is preserved under the flow. Then our proof (of the convergence result) is extended to this case. Note that the bisectional curvature is not necessarily positive during the flow.

Now let us review briefly the history of Ricci flow. The Ricci flow was first introduced by R. S. Hamilton in [18], and it has been a subject of intense study ever since. The Ricci flow provides an indispensable tool for deforming Riemannian metrics toward canonical metrics, such as Einstein ones. It is hoped that by deforming a metric to a canonical metric, one can understand the geometric and topological structures of underlying manifolds. For instance, it was proved in [18] that any closed 3-manifold of positive Ricci curvature is diffeomorphic to a spherical space form. We refer the reader to [21] for more information.

If the underlying manifold is a Kähler manifold, the Ricci flow preserves the Kähler class. Following a similar idea of Yau [35], Cao [4] proved the global existence of the flow. If the first Chern class of the underlying Kähler manifold is zero or negative, the solution converges to a Kähler-Einstein metric. Consequently, Cao reproved the famous Calabi-Yau theorem (see [35]). On the other hand, if the first Chern class of the underlying Kähler manifold is positive, the solution of the Kähler-Ricci flow may not converge to any Kähler-Einstein metric. This is because there are compact Kähler manifolds with positive first Chern class which do not admit any Kähler-Einstein metrics (see [17], [33]). A natural and challenging problem is whether or not the Kähler-Ricci flow on a compact Kähler-Einstein manifold converges to a Kähler-Einstein metric. Our theorem settles this problem in the case of Kähler metrics of positive bisectional curvature or positive curvature operator. It was proved by S. Bando [1] for 3-dimensional Kähler manifolds and by N. Mok [25] for higher-dimensional Kähler manifolds that the positivity (or nonnegativity) of bisectional

*See Section 2.3 for definitions of E_0, E_1.
†It is the bisectional curvature between any two orthogonal complex plans.

curvature is preserved under the Kähler-Ricci flow. Further references to Kähler-Ricci flow can be found in [6] and our earlier work [10].

The typical method in studying the Ricci flow depends on pointwise bounds of the curvature tensor by using its evolution equation as well as the blow-up analysis. In order to prevent formation of singularities, one blows up the solution of the Ricci flow to obtain profiles of singular solutions. Those profiles involve Ricci solitons and possibly more complicated models for singularity. Then one tries to exclude formation of singularities by checking that these solitons or models do not exist under appropriate global geometric conditions. It is a common opinion that it is very difficult to detect how the global geometry affects those singularity models even for a very simple manifold like $\mathbb{C}P^2$. The first step is to classify those singularity models and hope to find their geometric information. Of course, it is already a very big task. There have been many exciting works on these (see [21]).

Our new contribution is to find a set of new functionals that are the Lagrangians of certain new curvature equations involving various symmetric functions of the Ricci curvature. We show that these functionals decrease essentially along the Kähler-Ricci flow and have uniform lower bounds. By computing their derivatives, we can obtain certain integral bounds on curvature of metrics along the flow.

For the reader's convenience, we recall what we studied in [10] regarding these new functionals. In [10] we proved that the derivative of each E_k along an orbit of automorphisms gives rise to a holomorphic invariant $\Im_k$, including the well-known Futaki invariant as a special one. When M admits a Kähler-Einstein metric, all these invariants $\Im_k$ vanish, and the functionals E_k are invariant under the action of automorphisms.

Next, we proved in [10] that these E_k are bounded from below. We then computed the derivatives of E_k along the Kähler-Ricci flow. Recall that the Kähler-Ricci flow is given by

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\omega + \partial\overline{\partial}\varphi)^n}{\omega^n} + \varphi - h_\omega, \tag{1.1}$$

where h_ω depends only on ω. The derivatives of these functionals are all bounded uniformly from above along the Kähler-Ricci flow. Furthermore, we found that E_0 and E_1 decrease along the Kähler-Ricci flow. These play a very important role in this article and in [10]. We can derive from these properties of E_k integral bounds on curvature; for example, for almost all Kähler metrics $\omega_{\varphi(t)}$ along the flow, we have

$$\int_M \left(R(\omega_{\varphi(t)}) - r\right)^2 {\omega_{\varphi(t)}}^n \to 0, \tag{1.2}$$

where $R(\omega_{\varphi(t)})$ denotes the scalar curvature and r is the average scalar curvature.

In complex dimension 2, using the above integral bounds on the curvature with Cao's Harnack inequality and the generalization of Klingenberg's estimate, we can bound the curvature uniformly along the Kähler-Ricci flow in the case of Kähler-Einstein surfaces. However, it is not enough in high dimension since formula (1.2) is not scaling invariant. We must find a new way of utilizing this inequality in higher-dimensional manifolds. First, we derive a uniform upper bound on the diameter by using the work of C. Sprouse [32] and J. Cheeger and T. H. Colding [8]. Next, we use a result of Li and Yau [23] and a theorem of C. B. Croke [15] to derive a uniform upper bound on both the Sobolev constant and the Poincaré constant on the evolved Kähler metric. Once these two important constants are bounded uniformly, we can use the Moser iteration to obtain C^0-estimates along the modified Kähler-Ricci flow. A priori, this curve of evolved Kähler-Einstein metrics is not even differentiable on the level of potentials in terms of the time parameter. This gives us a lot of trouble in deriving the desired C^0-estimates. What we need is to readjust this curve of automorphisms so that it is at least C^1-uniform on the level of Kähler potentials. The issue concerning modified Kähler-Ricci flow (see (4.3)) is discussed in detail in Section 4. Once the C^0-estimate is established, it is then possible to obtain the C^2-estimate (following a similar calculation of Yau [35]) and Calabi's C^3-estimates. Eventually, we can prove that the modified Kähler-Ricci flow converges exponentially to the unique Kähler-Einstein metric.

Unlike in [10], we do not use any pointwise estimate on curvature; in particular, we do not need to use the Harnack inequality. It appears to us that the fact that the set of functionals we found are essentially decreasing along the Kähler-Ricci flow and have a uniform lower bound at the same time has already excluded the possibilities of formation of singularities. In higher-dimensional manifolds this idea of having integral estimates on curvature terms may prove to be an effective and attractive alternative (in contrast to the usual pointwise estimates).

In this article we also extend our results to Kähler-Einstein orbifolds with positive bisectional curvature. Note that the limit metric of the Kähler-Ricci flow on orbifolds must be Einstein metric with positive bisectional curvature. M. Berger's theorem in [3] then implies that it must be of constant bisectional curvature. We then use the exponential map to explicitly prove that such an orbifold must be a global quotient of $\mathbb{C}P^n$.

The organization of our article is roughly as follows. In Section 2 we review briefly some basics in Kähler geometry and some results we obtained in [10]. In Section 3 we prove that for any Kähler metric in the canonical class with nonnegative Ricci curvature, if the scalar curvature is sufficiently close to the average in the L^2-sense, then it has uniform diameter bounds. Next, using the old results of Li and Yau and the result of C. B. Croke, we bound both the Sobolev constant and the Poincaré constant. In Section 4 we prove C^0-estimates for all time over the modified Kähler-Ricci flow. In Section 5 we prove that we can choose a uniform gauge. In Section 6 we obtain both C^2- and C^3-estimates. In Section 7 we prove the exponential convergence to the unique Kähler-Einstein metric with constant bisectional curvature. In Section 8 we prove that any orbifold that supports a Kähler metric with positive constant bisectional curvature is globally a quotient of $\mathbb{C}P^n$. In Section 9 we prove Theorem 1.5, make some concluding remarks, and propose some open questions.

Since this article was posted in arXiv as a preprint [11] in 2001, there has been recent progress on the Kähler-Ricci flow with positive first Chern class. In the spring of 2003 when Perelman was visiting the Massachusetts Institute of Technology, he announced uniform bounds on scalar curvature and diameter along the Kähler-Ricci flow with positive first Chern class (see [29]). He gave an outlined proof of these bounds. The proof used monotonicity for one of his new functionals. He also claimed that these bounds can be used in proving convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds with positive scalar curvature. This would give a new and more involved proof of Theorem 1.1. If the initial metric has nonnegative holomorphic bisectional curvature, then Perelman's scalar curvature bound implies the full curvature bound. An alternative proof of such a curvature bound in this nonnegatively curved case was announced in [5] by using Perelman's monotonicity and the Harnack estimate for the Kähler-Ricci flow. It remains to show that the Kähler-Ricci flow converges to a Kähler-Einstein metric as the time tends to infinity. This is still a nontrivial problem and is related to the Frankel conjecture. This problem was solved only in our recent work [12].

2. Setup and known results

2.1. Setup of notation

Let M be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on M. In local coordinates $z_1, \ldots, z_n$, this ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\overline{j}}\, dz^i \wedge dz^{\overline{j}} > 0,$$

where $\{g_{i\overline{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive (1,1)-form. In other words, the following holds:

$$\frac{\partial g_{i\overline{k}}}{\partial z^j} = \frac{\partial g_{j\overline{k}}}{\partial z^i} \qquad \text{and} \qquad \frac{\partial g_{k\overline{i}}}{\partial z^{\overline{j}}} = \frac{\partial g_{k\overline{j}}}{\partial z^{\overline{i}}}, \quad \forall\, i, j, k = 1, 2, \ldots, n.$$

The Kähler metric corresponding to ω is given by

$$\sqrt{-1} \sum_{1}^{n} g_{\alpha\overline{\beta}}\, dz^\alpha \otimes dz^{\overline{\beta}}.$$

For simplicity, in the following, we often denote by ω the corresponding Kähler metric. The Kähler class of ω is its cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$. By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$\omega_\varphi = \omega + \sqrt{-1}\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z^i \partial z^{\bar{j}}} > 0$$

for some real-valued function φ on M. The functional space in which we are interested (often referred to as the space of Kähler potentials) is

$$\mathscr{P}(M,\omega) = \left\{\varphi \,\middle|\, \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \,\text{on}\, M\right\}.$$

Given a Kähler metric ω, its volume form is

$$\omega^n = \frac{1}{n!}(\sqrt{-1})^n \det(g_{i\bar{j}})\, dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}}.$$

Its Christoffel symbols are given by

$$\Gamma^k_{ij} = \sum_{l=1}^{n} g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j} \qquad \text{and} \qquad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \sum_{l=1}^{n} g^{\bar{k}l} \frac{\partial g_{l\bar{i}}}{\partial z^{\bar{j}}}, \quad \forall\, i, j, k = 1, 2, \ldots, n.$$

The curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}, \quad \forall\, i, j, k, l = 1, 2, \ldots, n.$$

We say that ω is of nonnegative bisectional curvature if

$$R_{i\bar{j}k\bar{l}} v^i v^{\bar{j}} w^k w^{\bar{l}} \geq 0$$

for all nonzero vectors v and w in the holomorphic tangent bundle of M. The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature of ω is locally given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log\det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j}.$$

So its Ricci curvature form is

$$\operatorname{Ric}(\omega) = \sqrt{-1}\sum_{i,j=1}^{n} R_{i\bar{j}}(\omega)\, dz^i \wedge dz^{\bar{j}} = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{k\bar{l}}).$$

It is a real, closed (1, 1)-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to $\lambda\,\omega$ for some constant λ.

2.2. The Kähler-Ricci flow

Now we assume that the first Chern class $c_1(M)$ is positive. The normalized Ricci flow (see [18] and [19]) on a Kähler manifold M is of the form

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \quad \forall\, i,\ j = 1, 2, \ldots, n, \tag{2.1}$$

if we choose the initial Kähler metric ω with $c_1(M)$ as its Kähler class. The flow (2.1) preserves the Kähler class $[\omega]$. It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{{\omega_\varphi}^n}{\omega^n} + \varphi - h_\omega, \tag{2.2}$$

where h_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} h_\omega \qquad \text{and} \qquad \int_M (e^{h_\omega} - 1)\omega^n = 0.$$

As usual, the flow (2.2) is referred to as the Kähler-Ricci flow on M.

The following theorem was proved by S. Bando for 3-dimensional compact Kähler manifolds. This was later proved by N. Mok in [25] for all dimensional Kähler manifolds. Their proofs used Hamilton's maximum principle for tensors. The proof for higher dimensions is quite intriguing.

THEOREM 2.1 (see [1], [25])

Under the Kähler-Ricci flow, if the initial metric has nonnegative bisectional curvature, then the evolved metrics also have nonnegative bisectional curvature. Furthermore, if the bisectional curvature of the initial metric is positive at least at one point, then the evolved metric has positive bisectional curvature at all points.

Before Bando and Mok, R. S. Hamilton proved (by using his maximum principle for tensors) the following.

THEOREM 2.2

Under the Ricci flow, if the initial metric has nonnegative curvature operator, then the evolved metric also has nonnegative curvature operator. Furthermore, if the curvature operator of the initial metric is positive at least at one point, then the evolved metric has positive curvature operator at all points.

2.3. Results from our previous article [10]

In this section we collect a few results from our earlier paper [10]. First, we introduce the new functionals $E_k = E_k^0 - J_k$ ($k = 0, 1, 2, \ldots, n$), where E_k^0 and J_k are defined in Definitions 2.3 and 2.4.

Definition 2.3
For any $k = 0, 1, \ldots, n$, we define a functional E_k^0 on $\mathscr{P}(M, \omega)$ by

$$E^0_{k,\omega}(\varphi) = \frac{1}{V} \int_M \Big(\log \frac{{\omega_\varphi}^n}{\omega^n} - h_\omega\Big)\Big(\sum_{i=0}^{k} \mathrm{Ric}(\omega_\varphi)^i \wedge \omega^{k-i}\Big) \wedge {\omega_\varphi}^{n-k} + c_k,$$

where

$$c_k = \frac{1}{V} \int_M h_\omega\Big(\sum_{i=0}^{k} \mathrm{Ric}(\omega)^i \wedge \omega^{k-i}\Big) \wedge \omega^{n-k}.$$

Definition 2.4
For each $k = 0, 1, 2, \ldots, n-1$, we define $J_{k,\omega}$ as follows. Let $\varphi(t)\,(t \in [0, 1])$ be a path from 0 to φ in $\mathscr{P}(M, \omega)$; we define

$$J_{k,\omega}(\varphi) = -\frac{n-k}{V} \int_0^1 \int_M \frac{\partial \varphi}{\partial t}({\omega_\varphi}^{k+1} - \omega^{k+1}) \wedge {\omega_\varphi}^{n-k-1} \wedge dt.$$

Put $J_n = 0$ for convenience in notation.

It is straightforward to verify that the definition of J_k $(0 \le k \le n)$ is independent of the path chosen.

Remark 2.5
In a noncanonical Kähler class, we need to modify the definition slightly since h_ω is not defined. For any $k = 0, 1, \ldots, n,$ we define

$$E_{k,\omega}(\varphi) = \frac{1}{V} \int_M \log \frac{{\omega_\varphi}^n}{\omega^n} \Big(\sum_{i=0}^{k} \mathrm{Ric}(\omega_\varphi)^i \wedge \mathrm{Ric}(\omega)^{k-i}\Big) \wedge {\omega_\varphi}^{n-k}$$
$$- \frac{n-k}{V} \int_M \varphi\big(\mathrm{Ric}(\omega)^{k+1} - \omega^{k+1}\big) \wedge \omega^{n-k-1} - J_{k,\omega}(\varphi).$$

The second integral on the right-hand side is used to offset the change from ω to $\mathrm{Ric}(\omega)$ in the first term. The derivative of this functional is exactly the same as in the canonical Kähler class. In other words, the Euler-Lagrange equation is not changed.

If $\omega \in c_1(M)$, then we assume that $E_k = E_{k,\omega}$. Direct computations lead to the following.

THEOREM 2.6
For any $k = 0, 1, \ldots, n$, we have

$$\frac{dE_k}{dt} = \frac{k+1}{V}\int_M \Delta_\varphi\Big(\frac{\partial\varphi}{\partial t}\Big)\,\mathrm{Ric}(\omega_\varphi)^k \wedge {\omega_\varphi}^{n-k} - \frac{n-k}{V}\int_M \frac{\partial\varphi}{\partial t}\big(\mathrm{Ric}(\omega_\varphi)^{k+1} - {\omega_\varphi}^{k+1}\big)\wedge {\omega_\varphi}^{n-k-1}. \tag{2.3}$$

Here $\{\varphi(t)\}$ is any path in $\mathscr{P}(M,\omega)$.

PROPOSITION 2.7
Along the Kähler-Ricci flow, we have

$$\frac{dE_k}{dt} \le -\frac{k+1}{V}\int_M \big(R(\omega_\varphi) - r\big)\mathrm{Ric}(\omega_\varphi)^k \wedge {\omega_\varphi}^{n-k}. \tag{2.4}$$

When $k = 0, 1$*, we have*

$$\begin{aligned} \frac{dE_0}{dt} &= -\frac{n\sqrt{-1}}{V}\int_M \partial\frac{\partial\varphi}{\partial t}\wedge\bar\partial\frac{\partial\varphi}{\partial t}{\omega_\varphi}^{n-1} \le 0,\\ \frac{dE_1}{dt} &\le -\frac{2}{V}\int_M \big(R(\omega_\varphi) - r\big)^2 {\omega_\varphi}^n \le 0. \end{aligned} \tag{2.5}$$

In particular, both E_0 *and* E_1 *are decreasing along the Kähler-Ricci flow.*

We then prove that the derivatives of these functionals along holomorphic automorphisms give rise to holomorphic invariants. For any holomorphic vector field X, and for any Kähler metric ω, there exists a potential function θ_X such that

$$L_X\omega = \sqrt{-1}\partial\bar\partial\theta_X.$$

Here L_X denotes the Lie derivative along a vector field X and θ_X is defined up to the addition of any constant. Now we define $\Im_k(X,\omega)$ for each $k = 0, 1, \ldots, n$ by

$$\Im_k(X,\omega) = (n-k)\int_M \theta_X\,\omega^n + \int_M \big((k+1)\Delta\theta_X\,\mathrm{Ric}(\omega)^k \wedge \omega^{n-k} - (n-k)\,\theta_X\,\mathrm{Ric}(\omega)^{k+1}\wedge\omega^{n-k-1}\big).$$

Here and in the following, Δ denotes the Laplacian of ω. The integral is unchanged if we replace θ_X by $\theta_X + c$ for any constant c. This is because of the following identity in a canonical Kähler class (when $[\omega] = [\mathrm{Ric}(\omega)]$):

$$\int_M \omega^n - \int_M \mathrm{Ric}(\omega)^k \wedge \omega^{n-k} = 0, \quad \forall\, k = 0, 1, 2, \ldots, n.$$

The next theorem assures us that the above integral gives rise to a holomorphic invariant.

THEOREM 2.8
The integral $\Im_k(X, \omega)$ *is independent of choices of Kähler metrics in the Kähler class* $[\omega]$. *That is,* $\Im_k(X, \omega) = \Im_k(X, \omega')$ *so long as the Kähler forms* ω *and* ω' *represent the same Kähler class. Hence, the integral* $\Im_k(X, \omega)$ *is a holomorphic invariant, which is denoted by* $\Im_k(X, [\omega])$.

COROLLARY 2.9
The invariants $\Im_k(X, c_1(M))$ *all vanish for any holomorphic vector fields* X *on a compact Kähler-Einstein manifold. In particular, these invariants all vanish on* $\mathbb{C}P^n$.

COROLLARY 2.10
For any Kähler-Einstein manifold, E_k $(k = 0, 1, \ldots, n)$ *is invariant under actions of holomorphic automorphisms.*

Intuitively speaking, we should modify the Kähler-Ricci flow so that the new evolved metric in the new modified flow is "centrally positioned" with respect to a fixed Kähler metric ω. One crucial step in [10] is to keep the Kähler-Ricci flow fixed but to modify the Kähler-Einstein metric ω so that the evolved metrics in the Kähler-Ricci flow are centrally positioned with respect to the new family of evolved Kähler-Einstein metrics. For the convenience of the reader, we include the definition of *centrally positioned* here.

Definition 2.11
Any Kähler form ω_φ is called centrally positioned with respect to some Kähler-Einstein metric $\omega_\rho = \omega + \sqrt{-1}\partial\overline{\partial}\rho$ if it satisfies

$$\int_M (\varphi - \rho)\,\theta\,{\omega_\rho}^n = 0, \quad \forall\,\theta \in \Lambda_1(\omega_\rho). \tag{2.6}$$

PROPOSITION 2.12
Let $\varphi(t)$ *be the evolved Kähler potentials. For any* $t > 0$, *there always exists an automorphism* $\sigma(t) \in \mathrm{Aut}(M)$ *such that* $\omega_{\varphi(t)}$ *is centrally positioned with respect to* $\omega_{\rho(t)}$. *Here*

$$\sigma(t)^*\omega_1 = \omega_{\rho(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\rho(t),$$

where ω_1 *is a Kähler-Einstein metric.*

Remark 2.13
In [10], we proved the existence of at least one Kähler-Einstein metric $\omega_{\rho(t)}$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to $\omega_{\rho(t)}$. As a matter of fact, such a Kähler-Einstein metric is unique. However, a priori, we do not know if the curve $\rho(t)$ is differentiable or not.

PROPOSITION 2.14
On a Kähler-Einstein manifold, the K-energy ν_ω is uniformly bounded from above and below along the Kähler-Ricci flow. Moreover, there exists a uniform constant C such that

$$\begin{aligned} \left|J_{k,\omega_{\rho(t)}}\big(\varphi(t)-\rho(t)\big)\right| &\leq \left\{\nu_\omega\big(\varphi(t)\big)+C\right\}^{1/\delta}, \\ \log\frac{{\omega_\varphi}^n}{{\omega_{\rho(t)}}^n} &\geq -4C''\, e^{2(\nu_\omega(\varphi(t)+C)^{1/\delta}+C')}, \\ E_k\big(\varphi(t)\big) &\geq -e^{c(1+\max\{0,\nu_\omega(\varphi(t))\}+(\nu_\omega(\varphi(t))+C)^{1/\delta})}, \\ \int_M \log\left(\frac{\omega_\varphi^n}{\omega^n}\right) &\leq C, \quad \forall\,\varphi\perp\Lambda_1(M,g_\omega), \end{aligned}$$

where c, C, C', and C'' are some uniform constants and $\rho(t)$ is defined as in Proposition 2.12.

COROLLARY 2.15
The energy functional E_k $(k=0,1,\ldots,n)$ has a uniform lower bound from below along the Kähler-Ricci flow.

COROLLARY 2.16
For each $k=0,1,\ldots,n$, there exists a uniform constant C such that the following holds (for any $T\leq\infty$) along the Kähler-Ricci flow:

$$\int_0^T \frac{k+1}{V}\int_M \big(R(\omega_{\varphi(t)})-r\big)\ \mathrm{Ric}(\omega_{\varphi(t)})^k\wedge\omega_{\varphi(t)}^{n-k}\,dt\leq C.$$

When $k=1$, we have

$$\int_0^\infty \frac{1}{V}\int_M \big(R(\omega_{\varphi(t)})-r\big)^2\,\omega_{\varphi(t)}^n\,dt\leq C<\infty.$$

3. Estimates of Sobolev and Poincaré constants

In this section we prove that for any Kähler metric in the canonical Kähler class, if the scalar curvature is close enough to a constant in the L^2-sense and if the Ricci curvature is nonnegative, then there exists a uniform upper bound for both the Poincaré constant and the Sobolev constant. We first follow an approach taken by C. Sprouse [32] to obtain a uniform upper bound on the diameter.

In [8], J. Cheeger and T. H. Colding proved an interesting and useful inequality that converts integral estimates along geodesics to integral estimates on the whole manifold. In this section we assume that $m = \dim(M)$.

LEMMA 3.1 (see [8])
Let A_1, A_2, and W be open subsets of M such that $A_1, A_2 \subset W$ and all minimal geodesics $r_{x,y}$ from $x \in A_1$ to $y \in A_2$ lie in W. Let f be any nonnegative function. Then

$$\int_{A_1\times A_2}\int_{r_{x,y}} f\big(r(s)\big)\,ds\,d\mathrm{vol}_{A_1\times A_2}$$

$$\le\ C(m,k,\mathfrak{R})\big(\mathrm{diam}(A_2)\,\mathrm{vol}(A_1)+\mathrm{diam}(A_1)\,\mathrm{vol}(A_2)\big)\int_W f\,d\mathrm{vol},$$

where for $k \le 0$,

$$C(m,k,\mathfrak{R}) = \frac{\mathrm{area}(\partial B_k(x,\mathfrak{R}))}{\mathrm{area}(\partial B_k(x,\mathfrak{R}/2))}, \tag{3.1}$$

$$\mathfrak{R} \ge \sup\big\{d(x,y)\,\big|\,(x,y)\in(A_1\times A_2)\big\}, \tag{3.2}$$

and $B_k(x,r)$ denotes the ball of radius r in the simply connected space of constant sectional curvature k.

In this article, we always assume that Ric ≥ 0 on M, and thus $C(n,k,\mathfrak{R}) = C(n)$. Using this theorem of Cheeger and Colding, C. Sprouse [32] proved an interesting lemma.

LEMMA 3.2
Let (M,g) be a compact Riemannian manifold with Ric ≥ 0. *Then for any $\delta > 0$, there exists $\epsilon = \epsilon(n,\delta)$ such that if*

$$\frac{1}{V}\int_M \big((m-1)-\mathrm{Ric}_-\big)_+ < \epsilon(m,\delta), \tag{3.3}$$

then the $\mathrm{diam}(M) < \pi+\delta$. *Here* Ric_- *denotes the lowest eigenvalue of the Ricci tensor. For any function f on M, $f_+(x) = \max\{f(x),0\}$.*

Remark 3.3
Note that the right-hand side of equation (3.3) is not scaling correct. A scaling-correct version of this lemma should be the following: For any positive integer $a > 0$, if

$$\frac{1}{V}\int_M |\mathrm{Ric} - a\,|\,d\mathrm{vol} < \epsilon(m,\delta)\cdot a,$$

then the diameter has a uniform upper bound.

Remark 3.4
It is interesting to see what the optimal constant $\epsilon(m,\delta)$ is. Following this idea, the best constant should be

$$\epsilon(m,\delta) = \sup_{N>2} \frac{N-2}{8C(m)N^m}.$$

However, it is interesting to figure out definitively the best constant here.

Adopting C. Sprouse's arguments, we prove the similar lemma.

LEMMA 3.5
Let (M,ω) be a polarized Kähler manifold; $[\omega]$ is the canonical Kähler class. Then there exists a positive constant ϵ_0 which depends only on the dimension such that if the Ricci curvature of ω is nonnegative and if

$$\frac{1}{V}\int_M (R-n)^2\omega^n \le \epsilon_0^2,$$

then there exists a uniform upper bound on the diameter of the Kähler metric ω. Here r is the average of the scalar curvature.

Proof
We first prove that the Ricci form is close to its Kähler form in the L^1-sense (after proper rescaling). Note that

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar\partial f$$

for some real-valued function f. Thus

$$\int_M \big(\mathrm{Ric}(\omega)-\omega\big)^2\wedge\omega^{n-2} = \int_M (\sqrt{-1}\partial\bar\partial f)^2\wedge\omega^{n-2} = 0.$$

On the other hand, we have

$$\int_M \big(\mathrm{Ric}(\omega)-\omega\big)^2\wedge\omega^{n-2} = \frac{1}{n(n-1)}\int_M \big((R-n)^2 - |\mathrm{Ric}(\omega)-\omega|^2\big)\omega^n.$$

Here we already use the identity $\mathrm{tr}_\omega(\mathrm{Ric}(\omega)-\omega)=R-n$. Thus

$$\int_M |\mathrm{Ric}(\omega)-\omega|^2\,\omega^n=\int_M (R-n)^2\,\omega^n.$$

This implies that

$$\begin{aligned}\Big(\int_M |\mathrm{Ric}-1|\,\omega^n\Big)^2 &\le \int_M |\mathrm{Ric}(\omega)-\omega|^2\,\omega^n\cdot\int_M \omega^n\\ &=\int_M (R-n)^2\,\omega^n\cdot V\\ &\le \epsilon_0^2\cdot V\cdot V=\epsilon_0^2\cdot V^2,\end{aligned}$$

which gives

$$\frac{1}{V}\int_M |\mathrm{Ric}-1|\,\omega^n\le\epsilon_0. \tag{3.4}$$

The value of ϵ_0 is determined later.

Using this inequality, (3.4), we want to show that the diameter must be bounded from above. Note that in our setting, $m=\dim(M)=2n$. Unlike in [32], we are not interested in obtaining a sharp upper bound on the diameter.

Let A_1 and A_2 be two balls of small radius, and let $W=M$. Let $f=|\mathrm{Ric}-1|=\sum_{i=1}^m|\lambda_i-1|$, where λ_i is the eigenvalue of the Ricci tensor. We assume also that all geodesics are parameterized by arc length. By possibly removing a set of measure zero in $A_1\times A_2$, there is a unique minimal geodesic from x to y for all $(x,y)\in A_1\times A_2$. Let p,q be two points on M such that

$$d(p,q)=\mathrm{diam}(M)=D.$$

We also used $d\mathrm{vol}$ to denote the volume element in the Riemannian manifold M, and we used V to denote the total volume of M. For $r>0$, put $A_1=B(p,r)$ and $A_2=B(q,r)$. Then Lemma 3.1 implies that

$$\begin{aligned}&\int_{A_1\times A_2}\int_{r_{x,y}}|\mathrm{Ric}-1|\,ds\,d\mathrm{vol}_{A_1\times A_2}\\ &\quad\le C(n,k,R)\big(\mathrm{diam}(A_2)\,\mathrm{vol}(A_1)+\mathrm{diam}(A_1)\,\mathrm{vol}(A_2)\big)\int_W|\mathrm{Ric}-1|\,d\mathrm{vol}.\end{aligned}$$

Taking the infimum over both sides, we obtain

$$\begin{aligned}&\inf_{(x,y)\in A_1\times A_2}\int_{r_{x,y}}|\mathrm{Ric}-1|\,dt\\ &\quad\le 2rC(n)\Big(\frac{1}{\mathrm{vol}(A_1)}+\frac{1}{\mathrm{vol}(A_2)}\Big)\int_W|\mathrm{Ric}-1|\,d\mathrm{vol}\\ &\quad\le 4rC(n)\frac{D^n}{r^n}\frac{1}{V}\int_M|\mathrm{Ric}-1|\,d\mathrm{vol},\end{aligned}\tag{3.5}$$

where the last inequality follows from the relative volume comparison. We can then find a minimizing unit-speed geodesic γ from $x \in \overline{A_1}$ and $y \in \overline{A_2}$ which realizes the infimum and shows that for $L = d(x, y)$ much larger than π, γ cannot be minimizing if the right-hand side of (3.5) is small enough.

Let $E_1(t), E_2(t), \dots, E_m(t)$ be a parallel orthonormal basis along the geodesic γ such that $E_1(t) = \gamma'(t)$. Set now $Y_i(t) = \sin(\pi t/L)E_i(t)$, $i = 2, 3, \dots, m$. Denote by $L_i(s)$ the length functional of a fixed endpoint variation of curves through γ with variational direction Y_i; we have the second variation formula

$$\begin{aligned}\sum_{i=2}^{m} \frac{d^2 L_i(s)}{d s^2}\Big|_{s=0} &= \sum_{i=2}^{m} \int_0^L \big(g(\nabla_{\gamma'} Y_i, \nabla_{\gamma'} Y_i) - R(\gamma', Y_i, \gamma', Y_i)\big)\, dt \\ &= \int_0^L (m-1)\bigg(\frac{\pi^2}{L^2}\cos^2\Big(\frac{\pi t}{L}\Big)\bigg) - \sin^2\Big(\frac{\pi t}{L}\Big) \operatorname{Ric}(\gamma', \gamma')\, dt \\ &= \int_0^L \bigg((m-1)\frac{\pi^2}{L^2}\cos^2\Big(\frac{\pi t}{L}\Big) - \sin^2\Big(\frac{\pi t}{L}\Big)\bigg)\, dt \\ &\qquad + \int_0^L \sin^2\Big(\frac{\pi t}{L}\Big)\big(1 - \operatorname{Ric}(\gamma', \gamma')\big)\, dt \\ &= -\frac{L}{2}\Big(1 - (m-1)\frac{\pi^2}{L^2}\Big) + \int_0^L \sin^2\Big(\frac{\pi t}{L}\Big)\big(1 - \operatorname{Ric}(\gamma', \gamma')\big)\, dt.\end{aligned}$$

Note that

$$1 - \operatorname{Ric}(\gamma', \gamma') \le |\operatorname{Ric} - 1|.$$

Combining this calculation and inequality (3.5), we obtain

$$\begin{aligned}&\sum_{i=2}^{n} \frac{d^2 L_i(s)}{d s^2}\Big|_{s=0} \\ &\quad \le -\frac{L}{2}\Big(1 - (m-1)\frac{\pi^2}{L^2}\Big) + \int_0^L \sin^2\Big(\frac{\pi t}{L}\Big)|\operatorname{Ric} - 1|\, dt \\ &\quad \le -\frac{L}{2}\Big(1 - (m-1)\frac{\pi^2}{L^2}\Big) + 4rC(n)\frac{D^n}{r^n}\frac{1}{V}\int_M |\operatorname{Ric} - 1|\, d\mathrm{vol}. \qquad (3.6)\end{aligned}$$

Here in the last inequality, we have already used the fact that γ is a geodesic that realizes the infimum of the left-hand side of inequality (3.5). For any fixed positive larger number $N > 4$, let $D = N \cdot r$. Set $c = (1/V)\int_M |\operatorname{Ric} - 1|\, d\mathrm{vol}$. Note that

$$L = d(x, y) \ge d(p, q) - 2r = D\Big(1 - \frac{2}{N}\Big) \ge \frac{D}{2}.$$

Then inequality (3.6) leads to

$$\frac{1}{D}\sum_{i=2}^{n}\frac{d^2L_i(s)}{d\,s^2}\Big|_{s=0} \le -\frac{1-2/N}{2}\Big(1-(m-1)\frac{\pi^2}{L^2}\Big)+4C(n)\,\frac{N^{m-1}}{V}\cdot c\cdot V$$

$$=4C(n)N^{m-1}\Big(c-\frac{(N-2)}{2N}\frac{1}{4C(n)N^{m-1}}\Big)+\frac{1-2/N}{2}(m-1)\frac{\pi^2}{L^2}.$$

Note that the second term in the right-hand side can be ignored if $L \ge D/2$ is large enough. Set

$$\epsilon_0=\frac{(N-2)}{2N}\cdot\frac{1}{4C(n)N^{m-1}}=\frac{N-2}{8C(n)N^m}.$$

Then if

$$\frac{1}{V}\int_M (R-n)^2\omega^n \le \epsilon_0^2,$$

by the argument at the beginning of this proof we have inequality (3.4):

$$\frac{1}{V}\int_M |\mathrm{Ric}-1|\,d\mathrm{vol}<\epsilon_0,$$

which in turn implies

$$\frac{1}{D}\sum_{i=2}^{n}\frac{d^2L_i(s)}{d\,s^2}\Big|_{s=0}<0$$

for D large enough. Thus, if the diameter is too large, γ cannot be a length-minimizing geodesic. This contradicts our earlier assumption that γ is a minimizing geodesic. Therefore, the diameter must have a uniform upper bound. □

According to the work of C. B. Croke [15], Li and Yau [23], and Li [22], we state the following lemma on the upper bound of the Sobolev constant and Poincaré constant.

LEMMA 3.6
Let (M,ω) be any compact polarized Kähler manifold, where $[\omega]$ is the canonical class. If $\mathrm{Ric}(\omega)\ge 0$, $V=\int_M\omega^n \ge \nu > 0$, *and the diameter has a uniform upper bound, then there exists a constant $\sigma=\sigma(\epsilon_0,\nu)$ such that for every function $f\in C^\infty(M)$, we have*

$$\Big(\int_M |f|^{2n/(n-1)}\omega^n\Big)^{(n-1)/n}\le\sigma\Big(\int_M|\nabla f|^2\omega^n+\int_M f^2\omega^n\Big).$$

Furthermore, there exists a uniform Poincaré constant $c(\epsilon_0)$ such that the Poincaré inequality holds

$$\int_M \Big(f - \frac{1}{V}\int_M f\omega^n\Big)^2 \omega^n \le c(\epsilon_0)\int_M |\nabla f|^2\omega^n.$$

Here ϵ_0 is the constant that appeared in Lemma 3.5.

Proof

Note that (M,ω) has a uniform upper bound on the diameter. Moreover, it has a lower volume bound, and it has nonnegative Ricci curvature. Following a proof in [22] which is based on a result of C. B. Croke [15], we obtain a uniform upper bound on the Sobolev constant (independent of metric!).

Recall a theorem of Li and Yau [23] which gives a positive lower bound of the first eigenvalue in terms of the diameter when Ricci curvature is nonnegative:

$$\lambda_1(\omega) \ge \frac{\pi^2}{4D^2};$$

here λ_1, D denote the first eigenvalue and the diameter of the Kähler metric ω. Now D has a uniform upper bound according to Lemma 3.5. Thus the first eigenvalue of ω has a uniform positive lower bound, which, in turn, implies that there exists a uniform Poincaré constant. □

4. C^0-estimates

Let us first prove a general lemma on C^0-estimates.

LEMMA 4.1

Let ω_ψ be a Kähler metric such that

$$\sup_M \psi \le C_1$$

and

$$\int_M (-\psi)\omega_\psi^n \le C_2.$$

If the Sobolev constant and the Poincaré constant of ω_ψ are bounded from above by C_3, then there exists a uniform constant C_4 which depends only on the dimension and the constants C_1, C_2, and C_3 such that

$$|\psi| \le C_4.$$

We use this lemma several times, so we include a proof here for the convenience of the reader.

Proof
Denote by Δ_ψ the Laplacian of ω_ψ. Then, because $\omega + \partial\bar{\partial}\psi > 0$, we see that $\omega = \omega_\psi - \partial\bar{\partial}\psi > 0$. Taking the trace of this latter expression with respect to ω_ψ, we get

$$n - \Delta_\psi \psi = \mathrm{tr}_{\omega_\psi} \omega > 0.$$

Define now $\psi_-(x) = \max\{-\psi(x), 1\} \geq 1$. It is clear that

$$\psi_-{}^p(n - \Delta_\psi \psi) \geq 0.$$

Integrating this inequality, we get

$$\begin{aligned}
0 &\leq \frac{1}{V}\int_M \psi_-{}^p(n - \Delta_\psi \psi)\omega_\psi^n \\
&= \frac{n}{V}\int_M \psi_-{}^p \omega_\psi^n + \frac{1}{V}\int_M \nabla_\psi \psi_-{}^p \nabla_\psi \psi \omega_\psi^n \\
&= \frac{n}{V}\int_M \psi_-{}^p \omega_\psi^n + \frac{1}{V}\int_{\{\psi \leq -1\}} \nabla_\psi \psi_-{}^p \nabla_\psi \psi \omega_\psi^n \\
&= \frac{n}{V}\int_M \psi_-{}^p \omega_\psi^n + \frac{1}{V}\int_M \nabla_\psi \psi_-{}^p \nabla_\psi (-\psi_-) \omega_\psi^n \\
&= \frac{n}{V}\int_M \psi_-{}^p \omega_\psi^n - \frac{1}{V}\frac{4p}{(p+1)^2}\int_M |\nabla_\psi \psi_-{}^{(p+1)/2}|^2 \omega_\psi^n,
\end{aligned}$$

which yields, using the fact that $\psi_- \geq 1$ and hence that $\psi_-{}^p \leq \psi_-{}^{p+1}$,

$$\frac{1}{V}\int_M |\nabla_\psi \psi_-{}^{(p+1)/2}|^2 \omega_\psi^n \leq \frac{n(p+1)^2}{4pV}\int_M \psi_-{}^{p+1}\omega_\psi^n.$$

Since the Sobolev constant of ω_ψ is bounded from above, we can use the Sobolev inequality

$$\frac{1}{V}\left(\int_M |\psi_-|^{(p+1)n/(n-1)}\omega_\psi^n\right)^{(n-1)/n} \leq \frac{c(p+1)}{V}\int_M \psi_-{}^{p+1}\omega_\psi^n.$$

Moser's iteration shows us that

$$\sup_M \psi_- = \lim_{p\to\infty} \|\psi_-\|_{L^{p+1}(M,\omega_\psi)} \leq C\|\psi_-\|_{L^2(M,\omega_\psi)}.$$

Since the Poincaré constant is uniformly bounded from above, we can use the Poincaré inequality

$$\begin{aligned}
\frac{1}{V}\int_M \left(\psi_- - \frac{C}{V}\int_M \psi_- \omega_\psi^n\right)^2 \omega_\psi^n &\leq \frac{1}{V}\int_M |\nabla \psi_-|^2 \omega_\psi^n \\
&\leq \frac{C'}{V}\int_M \psi_- \omega_\psi^n,
\end{aligned}$$

where we have set $p = 1$ and used the same reasoning as before. This then implies that

$$\max\{-\inf_M \psi, 1\} = \sup_M \psi_- \leq \frac{C}{V} \int_M \psi_- \omega_\psi^n.$$

Since $\int_M e^{-h_\varphi + \psi} \omega_\psi^n = V$, we can easily deduce $\int_{\psi>0} \psi \omega_\psi^n \leq C$. Combining this together with the previous equation, we get

$$-\inf_M \psi \leq \frac{C}{V} \int_M (-\psi) \omega_\psi^n + C,$$

which proves the lemma. □

LEMMA 4.2
Along the Kähler-Ricci flow, the diameter of the evolving metric is uniformly bounded.

Proof
In our first work [10], we proved

$$\int_0^\infty dt \int_M (R - r)^2 \, \omega_\varphi^n \leq C.$$

Therefore, for any sequence $s_i \to \infty$, and for any fixed time period T, there exist $t_i \to \infty$ and $0 < s_i - t_i < T$ such that

$$\lim_{t_i \to \infty} \frac{1}{V} \int_M (R - r)^2 \, \omega_\varphi^n = 0. \tag{4.1}$$

Now for this sequence of t_i, applying Lemma 3.5, we show that there exists a uniform constant D such that the diameters of $\omega_{\varphi(t_i)}$ are uniformly bounded by $D/2$. Recall that the Ricci curvature is uniformly positive along the flow so that the diameter of the evolving metric has increased at most exponentially since

$$\frac{\partial}{\partial t} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} \leq g_{i\bar{j}}.$$

Now $t_{i+1} - t_i < 2T$ for all $i > 0$; this implies that the diameter of the evolving metric along the entire flow is controlled by $e^{2T}(D/2) \leq D$. (Choose T small enough in the first place.) □

Combining this with Lemma 3.6, we obtain the following.

THEOREM 4.3
Along the Kähler-Ricci flow, the evolving Kähler metric $\omega_{\varphi(t)}$ has a uniform upper bound on the Sobolev constant and Poincaré constant.

Before we go on any further, we want to review some results we obtained in our previous article [10].

Let $\varphi(t)$ be the global solution of the Kähler-Ricci flow. In the level of Kähler potentials, the evolution equation is

$$\frac{\partial \varphi(t)}{\partial t} = \log\left(\frac{\omega^n_{\varphi(t)}}{\omega^n}\right) + \varphi(t) - h_\omega.$$

According to [10, Lemma 6.5], there exists a one-parameter family of Kähler-Einstein metrics $\omega_{\rho(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\rho(t)$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to $\omega_{\rho(t)}$ for any $t \geq 0$. Suppose that $\omega_{\varphi(0)}$ is already centrally positioned with the Kähler-Einstein metric $\omega_1 = \omega + \sqrt{-1}\partial\bar{\partial}\rho(0)$. Normalize the value of $\rho(t)$ such that

$$\omega^n_{\rho(t)} = e^{-\rho(t)+h_\omega}\,\omega^n$$

or, equivalently,

$$\ln\left(\frac{\omega^n_{\varphi_{\rho(t)}}}{\omega^n}\right) = -\rho(t) + h_\omega. \tag{4.2}$$

Then the Kähler-Ricci flow equation can be rewritten as

$$\frac{\partial \varphi(t)}{\partial t} = \log\left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\right) + \varphi(t) - \rho(t). \tag{4.3}$$

Sometimes we may refer to this equation as the modified Kähler-Ricci flow. Next, we are ready to prove the C^0-estimates for both the Kähler potentials and the volume form when $t = t_i$.

THEOREM 4.4
There exists a uniform constant C such that

$$|\varphi(t) - \rho(t)| < C \qquad \text{and} \qquad \left|\frac{\partial \varphi}{\partial t}\right| \leq C.$$

In particular, we have

$$\left|\ln\det\left(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\right)\right| < C.$$

We need a lemma on the L^1-integral of the Kähler potentials.

LEMMA 4.5
Along the Kähler-Ricci flow on a Kähler-Einstein manifold, there exists a uniform bound C such that

$$-C \leq \int_M \big(\varphi(t) - \rho(t)\big)\omega^n_{\varphi(t)} \leq C.$$

Proof
As in [10, Section 10], we define

$$c(t) = \int_M \frac{\partial \varphi(t)}{\partial t}\, \omega^n_{\varphi(t)}.$$

In a Kähler-Einstein manifold, the K-energy has a uniform lower bound along the Kähler-Ricci flow. Thus

$$\int_0^\infty \int_M \left|\nabla \frac{\partial \varphi(t)}{\partial t}\right|^2_{\varphi(t)} \omega^n_{\varphi(t)}\, dt \le C.$$

Therefore, we can normalize the initial value of Kähler potential, so that

$$c(0) = \int_0^\infty e^{-t} \int_M \left|\nabla \frac{\partial \varphi(t)}{\partial t}\right|^2_{\varphi(t)} \omega^n_{\varphi(t)}\, dt \le C.$$

According to [10, Lemma 10.1], we have $c(t) > 0$ and

$$\lim_{t\to\infty} c(t) = 0.$$

In particular, this implies that there exists a constant C such that

$$\begin{aligned} C \ge c(t) &= \int_M \frac{\partial \varphi(t)}{\partial t}\, \omega^n_{\varphi(t)} \\ &= \int_M \left(\log\Big(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\Big) + \varphi(t) - \rho(t)\right) \omega^n_{\varphi(t)} > 0. \end{aligned}$$

In the last inequality we have used the fact that $c(t) > 0$. According to Proposition 2.14, we have

$$-C \le \int_M \log\Big(\frac{\omega^n_{\varphi}}{\omega^n_{\rho(t)}}\Big)\omega^n_{\varphi(t)} \le C.$$

Combining this with the previous inequality, we arrive at

$$-C \le \int_M \big(\varphi(t) - \rho(t)\big)\, \omega^n_{\varphi(t)} < C.$$

Here C is a constant that may be different from line to line. □

Next, we turn to the proof of Theorem 4.4.

Proof of Theorem 4.4
According to Proposition 2.12, we have

$$J_{\omega_{\rho(t)}}(\omega_{\varphi(t)}) < C.$$

Then

$$0 \le (I - J)(\omega_{\varphi(t)}, \omega_{\rho(t)}) \le (n+1) \cdot J_{\omega_{\rho(t)}}(\omega_{\varphi(t)}) < (n+1)C.$$

By definition, this implies that

$$0 \le \int_M \big(\varphi(t) - \rho(t)\big)(\omega^n_{\rho(t)} - \omega^n_{\varphi(t)}) \le C.$$

Combining this with Lemma 4.6, we obtain

$$-C \le \int_M \big(\varphi(t) - \rho(t)\big)\omega^n_{\rho(t)} \le C.$$

Since $\triangle_{\rho(t)}(\varphi(t) - \rho(t)) \ge -n$, by the Green formula we have

$$\begin{aligned}
&\sup_M\big(\varphi(t) - \rho(t)\big) \\
&\quad \le \frac{1}{V}\int_M \big(\varphi(t)-\rho(t)\big)\omega^n_{\rho(t)} - \max_{x\in M}\Big(\frac{1}{V}\int_M \big(G(x,\cdot)+C_4\big)\triangle_{\rho(t)}\big(\varphi(t) - \rho(t)\big)\omega^n_{\rho(t)}(y)\Big) \\
&\quad \le \frac{1}{V}\int_M \big(\varphi(t) - \rho(t)\big)\omega^n_{\rho(t)} + nC_4,
\end{aligned}$$

where $G(x, y)$ is the Green function associated to ω_ρ satisfying $G(x, \cdot) \ge 0$. Therefore, there exists a uniform constant C such that

$$\sup_M\big(\varphi(t) - \rho(t)\big) \le C.$$

By Lemma 4.5, we have

$$-C \le \int_M \big(\varphi(t) - \rho(t)\big)\omega^n_{\varphi(t)} \le C.$$

Furthermore, according to Theorem 4.3, the Kähler metrics $\omega_{\varphi(t)}$ have a uniform upper bound on both the Sobolev constant and the Poincaré constant. Now using Lemma 4.1, we conclude that there exists a uniform constant C such that

$$-C \le \big(\varphi(t) - \rho(t)\big) \le C.$$

Next, we consider

$$\begin{aligned}
\frac{\partial\varphi(t)}{\partial t} &= \log\Big(\frac{\omega^n_{\varphi(t)}}{\omega^n_{\rho(t)}}\Big) + \big(\varphi(t) - \rho(t)\big) \\
&\ge -C
\end{aligned}$$

for some uniform constant C. Recall that $|c(t)| = \big| \int_M \frac{\partial \varphi(t)}{\partial t}\, \omega_\varphi^n \big|$ is uniformly bounded. Therefore, there is some uniform constant C such that

$$\int_M \left|\frac{\partial \varphi}{\partial t}\right| \omega_\varphi^n \leq C.$$

In view of the fact that the K-energy is uniformly bounded below, we arrive at

$$\int_0^\infty dt \int_M \left|\nabla \frac{\partial \varphi}{\partial t}\right|^2 \omega_\varphi^n < \infty.$$

Since the Poincaré constant of the evolving Kähler metric is bounded, we have

$$\int_a^{a+1} dt \int_M \left(\frac{\partial \varphi}{\partial t}\right)^2 \omega_\varphi^n \leq C,$$

where $c > 0$ is a constant independent of $a > 0$.

Note that $\frac{\partial \varphi(t)}{\partial t}$ satisfies the evolution equation

$$\frac{\partial}{\partial t} \frac{\partial \varphi(t)}{\partial t} = \triangle_\varphi \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial t}$$

and the fact that both the Sobolev and the Poincaré constants of the evolving metrics are uniformly bounded. Applying Lemma 4.7, a parabolic version of Lemma 4.1, we prove that there exists a uniform constant C such that

$$-C \leq \frac{\partial \varphi(t)}{\partial t} = \log\left(\frac{\omega_{\varphi(t)}^n}{\omega_{\rho(t)}^n}\right) + \big(\varphi(t) + \rho(t)\big) < C.$$

It follows that

$$-C \leq \log\left(\frac{\omega_{\varphi(t)}^n}{\omega_{\rho(t)}^n}\right) < C.$$

□

By Proposition 2.12, there exists a one-parameter family of $\sigma(t) \in \mathrm{Aut}(M)$ such that $\omega_{\varphi(t)}$ is centrally positioned with respect to the Kähler-Einstein metric $\omega_{\rho(t)}$. Here

$$\sigma(t)^* \omega_1 = \omega_{\rho(t)} = \omega + \sqrt{-1}\partial\overline{\partial}\rho(t). \tag{4.4}$$

This condition "centrally positioned" plays an important role in deriving Proposition 2.14 there. However, it is no longer needed once we have Proposition 2.14.

LEMMA 4.6

There exists a uniform constant C such that for all integers $i = 1, 2, \ldots, \infty$, *we have*

$$|\rho(i) - \rho(i+1)| < C.$$

Moreover,

$$|\sigma(i+1)\sigma(i)^{-1}|_{\hbar} < C.$$

Here $\hbar$ *is the left-invariant metric in* Aut(M).

Proof

The modified Kähler-Ricci flow is

$$\frac{\partial}{\partial t}(\varphi - \rho) = \varphi - \rho + \log \frac{{\omega_{\varphi}}^n}{{\omega_{\rho(t)}}^n} - \frac{\partial \rho}{\partial t}.$$

Since $\frac{\partial \varphi}{\partial t}$ is uniformly bounded, we arrive at

$$|\rho(i) - \rho(i+1)| \leq |\rho(i) - \varphi(i)| + |\rho(i+1) - \varphi(i+1)| + |\varphi(i) - \varphi(i+1)| \leq C.$$

Since $\omega_{\rho(t)}$ is a Kähler-Einstein metric for any time t, we have (see (4.2))

$$\left| \log \frac{{\omega_{\rho(i+1)}}^n}{{\omega_{\rho(i)}}^n} \right| = |\rho(i) - \rho(i+1)| < C$$

and

$$|\sigma(i+1)\sigma(i)^{-1}|_{\hbar} < C.$$

□

This lemma allows us to do the following modification on the curve $\sigma(t) \in$ Aut(M). (We also modify the curve $\rho(t)$ by equation (4.4).) Fix all of the integer points $(\sigma(i), i = 1, 2, \ldots)$ of the curve $\sigma(t)$ first. At each unit interval, replace the original curve in Aut(M) by a straight line that connects the two endpoints in Aut(M). Such a new curve in Aut(M) satisfies all the estimates listed below. (For convenience, we still denote it as $\sigma(t)$, $\rho(t)$, respectively.)

(1) Theorem 4.4 still holds for this new curve $\rho(t)$ since we only change $\rho(t)$ by a uniformly controlled amount. (Fix at each integer points, and adapt linear interpolation between them.)

(2) The new curves $\sigma(t)$ and $\rho(t)$ are Lipschitz with a uniform Lipschitz constant for all the time $t \in [0, \infty)$. In fact, $\sigma(t)$ is an infinitely long piecewise linear in Aut$_r(M)$.

(3) There exists a uniform constant C such that

$$\left|\left(\frac{d}{dt}\sigma(t)\right)\cdot\sigma(t)^{-1}\right| < C \quad \text{for any } t \neq \text{integer}.$$

In the remaining part of this section, we give a technical lemma required by the proof of Theorem 4.4.

LEMMA 4.7
*If the Poincaré constant and the Sobolev constant of the evolving Kähler metrics are both uniformly bounded along the Kähler-Ricci flow, if $\frac{\partial\varphi}{\partial t}$ is bounded from below, and if $\int_a^{a+1} dt \int_M \left(\frac{\partial\varphi}{\partial t}\right)^2 \omega^n_{\varphi(t)}$ is uniformly bounded from above for any $a \geq 0$, then $\frac{\partial\varphi}{\partial t}$ is uniformly bounded from above and below.**

Proof
Since $\frac{\partial\varphi}{\partial t}$ has a uniform lower bound, there is a constant c such that $u = \frac{\partial\varphi}{\partial t} + c > 1$ holds all the time. Now u satisfies the equation

$$\frac{\partial}{\partial t} u = \triangle_\varphi u + u - c.$$

Set $d\,\mu(t) = \omega^n_{\varphi(t)}$ as the evolving volume element. Then

$$\frac{\partial}{\partial t} d\mu(t) = \triangle_\varphi u\, d\mu(t).$$

For any $a < b < \infty$, define η to be any positive increasing function that vanishes at a. Set

$$\psi(t,x) = \eta^2 u^{\beta-1}$$

for any $\beta > 2$. Then (here $\partial_t\, d\mu(t) = \triangle_\varphi\, u\, d\mu(t)$) we have

$$\begin{aligned}
&\int_a^b dt \int_M (\partial_t u)\psi\, d\mu(t) \\
&\quad = \int_a^b dt\Big(\partial_t \int_M u\psi\, d\mu(t) - \int_M u\frac{\partial\psi}{\partial t} - \int_M \psi u \triangle_\varphi u\Big) \\
&\quad = \eta(b)^2 \int_M u^\beta - \int_a^b dt \Big\{\int_M \big(u 2\eta\eta' u^{\beta-1} + u\eta^2(\beta-1)u^{\beta-2}\partial_t u\big) + \int_M \psi u \triangle_\varphi u\Big\}.
\end{aligned}$$

* This is a parabolic version of Moser iteration arguments. We give a detailed proof here for the convenience of the reader.

Thus

$$\begin{aligned}\int_a^b dt\Big(\int_M \beta(\partial_t u)\psi\, d\mu(t) + \int_M \psi u \triangle_\varphi u\Big)\\ &= \eta(b)^2 \int_M u^\beta - \int_a^b dt \int_M 2\eta\eta' u^\beta \\ &= \int_a^b dt\Big(\int_M \beta(\triangle_\varphi u + u - c)\eta^2 u^{\beta-1} + \int_M \eta^2 u^{\beta-1} u \triangle_\varphi u\Big) \\ &\le -\int_a^b dt\Big(\int_M \beta(\beta-1)u^{\beta-2}|\nabla u|^2\eta^2 - \int_M \beta\eta^2 u^\beta\Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\eta(b)^2 \int_M u^\beta + \int_a^b dt \int_M \beta(\beta-1)u^{\beta-2}|\nabla u|^2\eta^2\\ \le \int_a^b dt \int_M \beta\eta^2 u^\beta + \int_a^b dt \int_M 2\eta\eta' u^\beta.\end{aligned}$$

In other words,

$$\begin{aligned}\eta(b)^2 \int_M u^\beta + \int_a^b dt \int_M 4\Big(1-\frac{1}{\beta}\Big)|\nabla u^{\beta/2}|^2\eta^2\\ \le \int_a^b dt \int_M \beta(\eta^2 + 2\eta\eta')u^\beta\end{aligned}$$

or

$$\begin{aligned}\eta(b)^2 \int_M u^\beta + \int_a^b dt \int_M 4\Big(1-\frac{1}{\beta}\Big)(|\nabla u^{\beta/2}|^2\eta^2 + u^\beta\eta^2)\\ \le \int_a^b dt \int_M \beta(2\eta^2 + 2\eta\eta')u^\beta.\end{aligned}$$

In particular, this implies that

$$\max_{a\le t\le b} \int_M \eta(t)^2 u^\beta \le \int_a^b dt \int_M \beta(2\eta^2 + 2\eta\eta')u^\beta.$$

Let us first state a lemma.

LEMMA 4.8 (Sobolev inequality)
Assume that $0 \le a < b$, and assume that $v : M \times [a,b] \to \mathbf{R}$ is a measurable function such that

$$\sup_{a\le t\le b} |v(\cdot\,,t)|_{L^2(M,d\mu(t))} < \infty$$

and

$$\int_a^b \int_M |\nabla v|^2 \, d\mu \, dt < \infty;$$

then we have ($m = 2n = \dim(M)$)*:*

$$\int_a^b dt \int_M |v|^{2(m+2)/m} \, d\mu(t) \le \sigma \sup_{a \le t \le b} |v(\cdot\,,t)|^{4/m}_{L^2(M,d\mu(t))} \int_a^b dt \int_M (|\nabla v|^2 + v^2) \, d\mu(t).$$

Here σ is the Sobolev constant.

Proof
For any $a \le t \le b$, we have

$$\begin{aligned} |v(\cdot\,,t)|_{L^{2(m+2)/m}(M,d\mu(t))} &\le |v(\cdot\,,t)|^{2/(m+2)}_{L^2(M,d\mu(t))} \, |v(\cdot\,,t)|^{m/(m+2)}_{L^{2m/(m-2)}(M,d\mu(t))} \\ &\le |v(\cdot\,,t)^{2/(m+2)}_{L^2(M,d\mu(t))} \Big(\sigma \int_M (|\nabla v|^2 + v^2) \, d\mu(t) \Big)^{m/(2(m+2))}. \end{aligned}$$

The lemma follows by taking the $(2(m+2)/m)$th power on both sides and integrating over $[a,b]$. □

Now we return to the proof of Lemma 4.7. Let $v = \eta u^{\beta/2}$; we have

$$\begin{aligned} &\Big(\int_a^b dt \int_M |\eta^2 u^\beta|^{(m+2)/m} \Big)^{m/(m+2)} \\ &\quad \le \sigma^{m/(m+2)} \sup_{a \le t \le b} |v(\cdot\,,t)|^{4/(m+2)}_{L^2(M,d\mu(t))} \Big(\int_a^b dt \int_M (|\nabla v|^2 + v^2) \, d\mu(t) \Big)^{m/(m+2)} \\ &\quad \le C(m) \Big(\int_a^b dt \int_M \beta(2\eta^2 + 2\eta\eta') u^\beta \Big)^{2/(m+2)} \Big(\int_a^b dt \int_M \beta(2\eta^2 + 2\eta\eta') u^\beta \Big)^{m/(m+2)} \\ &\quad \le C(m)\beta \int_a^b dt \int_M (\eta^2 + \eta\eta') u^\beta. \end{aligned}$$

Here $C(m)$ is a constant depending only on the Sobolev constant of $(M, g(t))$ and dimension of manifold.

Now for any $a \le b_0 < b \le a+1$, define

$$b_k = b - \frac{b - b_0}{2^k}$$

for any $k \in \mathbf{Z}_+$. Fix a function $\eta_0 \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $0 \le \eta_0 \le 1$, $\eta_0' \ge 0$, $\eta_0(t) = 0$ for $t \le 0$, and $\eta_0(t) = 1$ for $t \ge 1$. For each integer $k > 0$, we let $\eta(t) =$

$\eta_0((t-b_k)/(b_{k+1}-b_k))$ and $\beta = 2\,((m+2)/m)^k$. Then $(b_{k+1}-b_k = (b-b_0)/2^{k+1})$ we have

$$\Big(\int_{b_{k+1}}^{a+1} dt \int_M u^{2((m+2)/m)^{k+1}}\, d\mu\Big)^{(1/2)(m/(m+2))^{k+1}}$$
$$\leq C(m)^{(1/2)(m/(m+2))^k}\Big(\frac{m+2}{m}\Big)^{k(m/(m+2))^k}\Big(\frac{2^{k+1}}{b-b_0}\Big)^{(1/2)(m/(m+2))^k}$$
$$\times\Big(\int_{b_k}^{a+1} dt \int_M u^{2((m+2)/m)^k}\, d\mu\Big)^{(1/2)(m(m+2))^k}.$$

The iteration shows that for any integer $k > 0$, we have

$$\Big(\int_{b}^{a+1} dt \int_M u^{2((m+2)/m)^{k+1}}\, d\mu\Big)^{(1/2)(m/(m+2))^{k+1}}$$
$$\leq \frac{C(m)}{(b-b_0)^{(m+2)/4}}\Big(\int_a^b dt \int_M u^2\, d\mu\Big)^{1/2}.$$

Here again $C(m)$ is a uniform constant that depends only on the Sobolev constant of the evolving metrics and the dimension m. Since the last term is uniformly bounded, this implies that as $k \to \infty$, we have

$$\sup_{b\leq t\leq a+1} u \leq \frac{C(m)}{(b-b_0)^{(m+2)/4}}\Big(\int_a^b dt \int_M u^2\, d\mu\Big)^{1/2}.$$

□

5. Uniform bounds on gauge

In order to use this uniform C^0-estimate and the flow equation (5.3) to derive the desired C^2-estimate, we still need to control the size of $\frac{\partial \rho}{\partial t}$. However, from our earlier modification above, we cannot determine $\frac{\partial \rho}{\partial t}$ at any integer point. For any noninteger points, we have a uniform bound C such that

$$\Big|\frac{\partial \rho}{\partial t}\Big|_t < C, \quad \forall\, t \neq \text{integer}.$$

Note that $\sigma(t)$ is an infinitely long broken line in $\mathrm{Aut}_r(M)$. Next, we want to further modify the curve $\sigma(t)$ by smoothing the corner at the integer points. Let us first set up some notation. Let $\jmath$ be the Lie algebra of $\mathrm{Aut}(M)$. As before, suppose that $\hbar$ is the left-invariant metric on $\mathrm{Aut}(M)$. Denote id the identity element in $\mathrm{Aut}(M)$, and denote exp the exponential map at the identity. Use B_r to denote the ball centered at the identity element with radius r.

After the modifications in the paragraph above, $\sigma(t)$ is an infinitely long broken line in $\mathrm{Aut}(M)$. We can write down this curve explicitly. For any integer

$i = 0, 1, 2, \ldots, \infty$, we have

$$\sigma(t) = \sigma(i) \cdot \exp\big((t-i)X_i\big), \quad \forall\, t \in [i, i+1]. \tag{5.1}$$

Here $\{X_i\}$ is a sequence of vector fields in $\jmath$ with a uniform upper bound C on their lengths:

$$\|X_i\|_{\hbar} \leq C, \quad \forall\, i = 0, 1, 2, \ldots, \infty. \tag{5.2}$$

Then there exists a uniform positive number $1/4 > \delta > 0$ such that for any integer $i > 0$, we have

$$\sigma(t) \in \sigma(i) \cdot B_{1/2}, \quad \forall\, t \in (i-\delta, i+\delta).$$

Note that δ depends on $\|X_i\|_{\hbar}$. Since the latter has a uniform upper bound, δ must have a uniform lower bound. We then can choose one $\delta > 0$ for all i.

At each ball $\sigma_i \cdot B_1$, we want to replace the curve segment $\sigma(t)$ ($t \in [i-\delta, i+\delta]$) by a new smooth curve $\tilde{\sigma}(t)$ such that we have the following.

(1) The two endpoints and their derivatives are not changed:*

$$\tilde{\sigma}(i \pm \delta) = \sigma(i \pm \delta)$$

and

$$\left(\Big(\frac{d}{dt}\tilde{\sigma}(t)\Big)\tilde{\sigma}(t)^{-1}\right)_{t=i\pm\delta} = \left(\Big(\frac{d}{dt}\sigma(t)\Big)\sigma(t)^{-1}\right)_{t=i\pm\delta}.$$

(2) There exists a uniform bound C' which depends only on the upper bound of $\|X_i\|_{\hbar}$ and δ such that

$$\left\|\Big(\frac{d}{dt}\tilde{\sigma}(t)\Big)\tilde{\sigma}(t)^{-1}\right\|_{\hbar} \leq C', \quad \forall\, t \in [i-\delta, i+\delta].$$

(3) For any $t \in [i-\delta, i+\delta]$, we have $\tilde{\sigma}(t) \in \sigma(i)B_1$. In other words, there exists a uniform constant C such that

$$|\tilde{\sigma}(t)\sigma(t)^{-1}|_{\hbar} < C.$$

The last step is to set $\tilde{\sigma}(t) = \sigma(t)$ for all other time t. Then the new curve $\tilde{\sigma}(t)$ has all the properties we want.

* In a Euclidean ball, we can use the fourth-order polynomial to achieve this. In any unit ball of any finite-dimensional Riemannian manifold, we can always do this uniformly, as long as the metric and other data involved are uniformly bounded.

(1) There exists a uniform constant C such that $|\tilde{\sigma}(t)\sigma(t)^{-1}| < C$ for all $t \in [0, \infty)$.
(2) There exists a uniform constant C such that

$$\left|\left(\frac{d}{dt}\tilde{\sigma}(t)\right) \cdot \tilde{\sigma}(t)^{-1}\right| < C \quad \text{for any } t \geq 0.$$

Denote $\tilde{\sigma}(t)^*\omega_1 = \omega_{\tilde{\rho}(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\tilde{\rho}(t)$. Then $\omega_{\tilde{\rho}(t)}$ is a Kähler-Einstein metric

$$\omega_{\tilde{\rho}(t)}^n = e^{-\tilde{\rho}(t)+h_\omega}\,\omega^n.$$

There exists a uniform constant C such that

$$|\rho(t) - \tilde{\rho}(t)| < C \tag{5.3}$$

and

$$\left|\frac{\partial\tilde{\rho}(t)}{\partial t}\right| < C$$

hold for all t.

Now inequality (5.3) implies that

$$\left|\log\left(\det\frac{\omega_{\rho(t)}^n}{\omega_{\tilde{\rho}(t)}^n}\right)\right| \leq C.$$

Combining this with Theorem 4.4, we arrive at the following.

THEOREM 5.1
There exist a one-parameter family of Kähler-Einstein metrics $\omega_{\tilde{\rho}(t)} = \omega + \sqrt{-1}\partial\bar{\partial}\tilde{\rho}(t)$, which is essentially parallel to the initial family of Kähler-Einstein metrics, and a uniform constant C such that the following holds:

$$|\varphi(t) - \tilde{\rho}(t)| \leq C,$$
$$-C < \log\frac{{\omega_{\varphi(t)}}^n}{{\omega_{\tilde{\rho}(t)}}^n} < C,$$

and

$$\left|\frac{\partial\tilde{\rho}(t)}{\partial t}\right| < C$$

over the entire modified Kähler-Ricci flow.

6. C^2- and higher-order derivative estimates

Consider the modified Kähler-Ricci flow

$$\frac{\partial}{\partial t}(\varphi - \tilde{\rho}) = \varphi - \tilde{\rho} + \log \frac{{\omega_{\varphi}}^n}{{\omega_{\tilde{\rho}(t)}}^n} - \frac{\partial \tilde{\rho}}{\partial t}. \tag{6.1}$$

By Theorem 5.1, we have a uniform bound on both $|(\varphi - \tilde{\rho})|$ and $\left|\frac{\partial}{\partial t}(\varphi - \tilde{\rho})\right|$. This fact plays an important role in deriving C^2-estimates on the evolved relative Kähler potential $(\varphi - \tilde{\rho})$ in this section.

THEOREM 6.1

If the C^0-norms of $|(\varphi - \tilde{\rho})|$ and $\left|\frac{\partial}{\partial t}(\varphi - \tilde{\rho})\right|$ are uniformly bounded (independent of time t), then there exists a uniform constant C such that

$$0 \le n + \tilde{\triangle}(\varphi - \tilde{\rho}) < C,$$

where $\tilde{\triangle}$ is the Laplacian operator corresponding to the evolved Kähler-Einstein metrics $\tilde{\omega}_{\rho(t)}$.

Here we follow Yau's estimate in [35], and we want to set up some notation first. Let $\triangle'$ be the Laplacian operator corresponding to the evolved Kähler metric $\omega_{\varphi(t)}$, respectively. Let $\square = \triangle' - \frac{\partial}{\partial t}$. Put $\omega_{\tilde{\rho}(t)} = \sqrt{-1} h_{\alpha\bar{\beta}}\, dz^{\alpha} \otimes z^{\bar{\beta}}$ and $\omega_{\varphi(t)} = \sqrt{-1} g'_{\alpha\bar{\beta}}\, dz^{\alpha} \otimes dz^{\bar{\beta}}$, where

$$g'_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}} + \frac{\partial^2 (\varphi(t) - \tilde{\rho}(t))}{\partial z^{\alpha} \partial z^{\bar{\beta}}}.$$

Then

$$\triangle' = \sum_{\alpha,\beta=1}^{n} g'^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^{\alpha} \partial z^{\bar{\beta}}}, \qquad \tilde{\triangle} = \sum_{\alpha,\beta=1}^{n} h^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^{\alpha} \partial z^{\bar{\beta}}},$$

and

$$\left[\frac{\partial}{\partial t}, \tilde{\triangle}\right] = - \sum_{a,b,c,d=1}^{n} h^{a\bar{b}} \frac{\partial^2 \frac{\partial \tilde{\rho}}{\partial t}}{\partial z^{c} \partial z^{\bar{b}}} h^{c\bar{d}} \frac{\partial^2}{\partial z^{a} \partial z^{\bar{d}}}.$$

Furthermore, we have

$$\tilde{\triangle} \frac{\partial \tilde{\rho}}{\partial t} = - \frac{\partial \tilde{\rho}}{\partial t}.$$

Thus the Hessian of $\frac{\partial\tilde{\rho}}{\partial t}$ with respect to the evolved Kähler-Einstein metric $\omega_{\tilde{\rho}(t)}$ is uniformly bounded from above since $\left|\frac{\partial\tilde{\rho}}{\partial t}\right|$ is uniformly bounded from above.

Proof

We want to use the maximum principle in this proof. Let us first calculate $\Box(n + \tilde{\Delta}(\varphi - \tilde{\rho}))$.

Let us choose a coordinate so that at a fixed point both $\omega_{\tilde{\rho}(t)} = \sqrt{-1}h_{\alpha\bar{\beta}}\, dz^\alpha \otimes dz^{\bar{\beta}}$ and the complex Hessian of $\varphi(t) - \tilde{\rho}(t)$ are in diagonal forms. In particular, we assume that $h_{i\bar{j}} = \delta_{i\bar{j}}$ and $(\varphi(t) - \tilde{\rho}(t))_{i\bar{j}} = \delta_{i\bar{j}}\,(\varphi(t) - \tilde{\rho}(t))_{i\bar{i}}$. Thus

$$g'^{i\bar{s}} = \frac{\delta_{i\bar{s}}}{1 + (\varphi(t) - \tilde{\rho}(t))_{i\bar{i}}}.$$

For convenience, put

$$F = \frac{\partial}{\partial t}(\varphi - \tilde{\rho}) - (\varphi - \tilde{\rho}) + \frac{\partial\tilde{\rho}}{\partial t}.$$

Note that F has a uniform bound. The modified Kähler-Ricci flow (6.1) can be reduced to

$$\log\frac{{\omega_\varphi}^n}{{\omega_{\tilde{\rho}(t)}}^n} = F$$

or, equivalently,

$$\left(\omega_{\tilde{\rho}(t)} + \partial\bar{\partial}(\varphi - \tilde{\rho})\right)^n = e^F\,\omega^n;$$

that is,

$$\log\det\left(h_{i\bar{j}} + \frac{\partial^2(\varphi - \tilde{\rho})}{\partial z_i \partial z_{\bar{j}}}\right) = F + \log\det(h_{i\bar{j}}).$$

For convenience, set

$$\psi(t) = \varphi(t) - \tilde{\rho}(t)$$

in this proof. Note that both $|\psi(t)|$ and $\left|\frac{\partial\psi(t)}{\partial t}\right|$ are uniformly bounded (see Theorem 5.1). We first follow the standard calculation of C^2-estimates in [35]. Differentiate both sides with respect to $\frac{\partial}{\partial z_k}$,

$$(g')^{i\bar{j}}\left(\frac{\partial h_{i\bar{j}}}{\partial z_k} + \frac{\partial^3\psi(t)}{\partial z_i \partial\bar{z}_j \partial z_k}\right) - h^{i\bar{j}}\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial F}{\partial z_k},$$

and differentiating again with respect to $\frac{\partial}{\partial \bar{z}_l}$ yields

$$(g')^{i\bar{j}}\Big(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l}\Big) + h^{t\bar{j}} h^{i\bar{s}} \frac{\partial h_{t\bar{s}}}{\partial \bar{z}_l} \frac{\partial h_{i\bar{j}}}{\partial z_k} - h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}$$
$$- (g')^{t\bar{j}}(g')^{i\bar{s}}\Big(\frac{\partial h_{t\bar{s}}}{\partial \bar{z}_l} + \frac{\partial^3 \psi(t)}{\partial z_t \partial \bar{z}_s \partial \bar{z}_l}\Big)\Big(\frac{\partial h_{i\bar{j}}}{\partial z_k} + \frac{\partial^3 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k}\Big) = \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l}.$$

Assume that we have normal coordinates at the given point; that is, $h_{i\bar{j}} = \delta_{ij}$, and the first-order derivatives of g vanish. Now taking the trace of both sides results in

$$\widetilde{\Delta} F = h^{k\bar{l}}(g')^{i\bar{j}}\Big(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l}\Big)$$
$$- h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}} \frac{\partial^3 \psi(t)}{\partial z_t \partial \bar{z}_s \partial \bar{z}_l} \frac{\partial^3 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k} - h^{k\bar{l}} h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}.$$

On the other hand, we also have

$$\Delta'\big(\widetilde{\Delta}\psi(t)\big) = (g')^{k\bar{l}} \frac{\partial^2}{\partial z_k \partial \bar{z}_l}\Big(h^{i\bar{j}} \frac{\partial^2 \psi(t)}{\partial z_i \partial \bar{z}_j}\Big)$$
$$= (g')^{k\bar{l}} h^{i\bar{j}} \frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} + (g')^{k\bar{l}} \frac{\partial^2 h^{i\bar{j}}}{\partial z_k \partial \bar{z}_l} \frac{\partial^2 \psi(t)}{\partial z_i \partial \bar{z}_j},$$

and we substitute $\frac{\partial^4 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l}$ in $\Delta'(\widetilde{\Delta}\psi(t))$, so that the previous equation reads

$$\Delta'\big(\widetilde{\Delta}\psi(t)\big) = -h^{k\bar{l}}(g')^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}} \frac{\partial^3 \psi(t)}{\partial z_t \partial \bar{z}_s \partial \bar{z}_l} \frac{\partial^3 \psi(t)}{\partial z_i \partial \bar{z}_j \partial z_k}$$
$$+ h^{k\bar{l}} h^{i\bar{j}} \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \widetilde{\Delta} F + (g')^{k\bar{l}} \frac{\partial^2 h^{i\bar{j}}}{\partial z_k \partial \bar{z}_l} \frac{\partial^2 \psi(t)}{\partial z_i \partial \bar{z}_j},$$

which we can rewrite after substituting $\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} = -R_{i\bar{j}k\bar{l}}$ and $\frac{\partial^2 h^{i\bar{j}}}{\partial z_k \partial \bar{z}_l} = R_{j\bar{i}k\bar{l}}$ as

$$\Delta'\big(\widetilde{\Delta}\psi(t)\big) = \widetilde{\Delta} F + h^{k\bar{l}}(g')^{t\bar{j}}(g')^{i\bar{s}} \psi(t)_{t\bar{s}l} \psi(t)_{i\bar{j}k}$$
$$+ (g')^{i\bar{j}} h^{k\bar{l}} R_{i\bar{j}k\bar{l}} - h^{i\bar{j}} h^{k\bar{l}} R_{i\bar{j}k\bar{l}} + (g')^{k\bar{l}} R_{j\bar{i}k\bar{l}} \psi(t)_{i\bar{j}}.$$

Restrict to the coordinates we chose in the beginning, so that both g and $\psi(t)$ are in diagonal form. The equation transforms to

$$\Delta'\big(\widetilde{\Delta}\psi(t)\big) = \frac{1}{1+\psi(t)_{i\bar{i}}} \frac{1}{1+\psi(t)_{j\bar{j}}} \psi(t)_{i\bar{j}k} \psi(t)_{\bar{i}j\bar{k}} + \widetilde{\Delta} F$$
$$+ R_{i\bar{i}k\bar{k}}\Big(-1 + \frac{1}{1+\psi(t)_{i\bar{i}}} + \frac{\psi(t)_{i\bar{i}}}{1+\psi(t)_{k\bar{k}}}\Big).$$

Now set $C = \inf_{i \neq k} R_{i\bar{i}k\bar{k}}$, and observe that

$$\begin{aligned}
R_{i\bar{i}k\bar{k}}\Big(-1 + \frac{1}{1+\psi(t)_{i\bar{i}}} + \frac{\psi(t)_{i\bar{i}}}{1+\psi(t)_{k\bar{k}}}\Big) &= \frac{1}{2} R_{i\bar{i}k\bar{k}} \frac{(\psi(t)_{k\bar{k}} - \psi(t)_{i\bar{i}})^2}{(1+\psi(t)_{i\bar{i}})(1+\psi(t)_{k\bar{k}})} \\
&\geq \frac{C}{2} \frac{(1+\psi(t)_{k\bar{k}} - 1 - \psi(t)_{i\bar{i}})^2}{(1+\psi(t)_{i\bar{i}})(1+\psi(t)_{k\bar{k}})} \\
&= C\Big(\frac{1+\psi(t)_{i\bar{i}}}{1+\psi(t)_{k\bar{k}}} - 1\Big),
\end{aligned}$$

which yields

$$\begin{aligned}
\Delta'(\widetilde{\Delta}\psi(t)) \geq{}& \frac{1}{(1+\psi(t)_{i\bar{i}})(1+\psi(t)_{j\bar{j}})} \psi(t)_{i\bar{j}k} \psi(t)_{\bar{i}j\bar{k}} + \widetilde{\Delta} F \\
&+ C\Big(\big(n + \widetilde{\Delta}\psi(t)\big) \sum_i \frac{1}{1+\psi(t)_{i\bar{i}}} - 1\Big).
\end{aligned}$$

We need to apply one more trick to obtain the requested estimates. Namely,

$$\begin{aligned}
&\Delta'\big(e^{-\lambda\psi(t)}(n + \widetilde{\Delta}\psi(t))\big) \\
&\quad = e^{-\lambda\psi(t)} \Delta'\big(\widetilde{\Delta}\psi(t)\big) + 2\nabla' e^{-\lambda\psi(t)} \nabla'\big(n + \widetilde{\Delta}\psi(t)\big) + \Delta'(e^{-\lambda\psi(t)})\big(n + \widetilde{\Delta}\psi(t)\big) \\
&\quad = e^{-\lambda\psi(t)} \Delta'\big(\widetilde{\Delta}\psi(t)\big) - \lambda e^{-\lambda\psi(t)} (g')^{i\bar{i}} \psi(t)_i \big(\widetilde{\Delta}\psi(t)\big)_{\bar{i}} - \lambda e^{-\lambda\psi(t)} (g')^{i\bar{i}} \psi(t)_{\bar{i}} \big(\widetilde{\Delta}\psi(t)\big)_i \\
&\qquad - \lambda e^{-\lambda\psi(t)} \Delta'\psi(t)\big(n + \widetilde{\Delta}\psi(t)\big) + \lambda^2 e^{-\lambda\psi(t)} (g')^{i\bar{i}} \psi(t)_i \psi(t)_{\bar{i}} \big(n + \widetilde{\Delta}\psi(t)\big) \\
&\quad \geq e^{-\lambda\psi(t)} \Delta'\big(\widetilde{\Delta}\psi(t)\big) - e^{-\lambda\psi(t)} (g')^{i\bar{i}} \big(n + \widetilde{\Delta}\psi(t)\big)^{-1} \big(\widetilde{\Delta}\psi(t)\big)_i \big(\widetilde{\Delta}\psi(t)\big)_{\bar{i}} \\
&\qquad - \lambda e^{-\lambda\psi(t)} \Delta'\psi(t)\big(n + \widetilde{\Delta}\psi(t)\big),
\end{aligned}$$

which follows from the Schwarz lemma applied to the middle two terms. We write out one term here (the other goes in an analogous way):

$$\begin{aligned}
&\big(\lambda e^{-(\lambda/2)\psi(t)} \psi(t)_i \big(n + \widetilde{\Delta}\psi(t)\big)^{1/2}\big)\big(e^{-(\lambda/2)\psi(t)} (\widetilde{\Delta}\psi(t))_{\bar{i}} (n + \widetilde{\Delta}\psi(t))^{-1/2}\big) \\
&\quad \leq \frac{1}{2}\big(\lambda^2 e^{-\lambda\psi(t)} \psi(t)_i \psi(t)_{\bar{i}} (n + \widetilde{\Delta}\psi(t)) \\
&\qquad\quad + e^{-\lambda\psi(t)} (\widetilde{\Delta}\psi(t))_{\bar{i}} (\widetilde{\Delta}\psi(t))_i (n + \widetilde{\Delta}\psi(t))^{-1}\big).
\end{aligned}$$

Now consider

$$\begin{aligned}
&-\big(n + \widetilde{\Delta}\psi(t)\big)^{-1} \frac{1}{1+\psi(t)_{i\bar{i}}} \big(\widetilde{\Delta}\psi(t)\big)_i \big(\widetilde{\Delta}\psi(t)\big)_{\bar{i}} + \Delta'\widetilde{\Delta}\psi(t) \\
&\quad \geq -\big(n + \widetilde{\Delta}\psi(t)\big)^{-1} \frac{1}{1+\psi(t)_{i\bar{i}}} |\psi(t)_{k\bar{k}i}|^2 + \widetilde{\Delta} F \\
&\quad + \frac{1}{1+\psi(t)_{i\bar{i}}} \frac{1}{1+\psi(t)_{k\bar{k}}} \psi(t)_{k\bar{i}\bar{j}} \psi(t)_{i\bar{k}j} + C\big(n + \widetilde{\Delta}\psi(t)\big) \frac{1}{1+\psi(t)_{i\bar{i}}}.
\end{aligned}$$

On the other hand, using the Schwarz inequality, we have

$$\begin{aligned}
&\big(n+\widetilde{\Delta}\psi(t)\big)^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}|\psi(t)_{k\bar{k}i}|^2\\
&= \big(n+\widetilde{\Delta}\psi(t)\big)^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}\Big|\frac{\psi(t)_{k\bar{k}i}}{(1+\psi(t)_{k\bar{k}})^{1/2}}\big(1+\psi(t)_{k\bar{k}}\big)^{1/2}\Big|^2\\
&\le \big(n+\widetilde{\Delta}\psi(t)\big)^{-1}\Big(\frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{k\bar{k}i}\psi(t)_{\bar{k}k\bar{i}}\Big)\big(1+\psi(t)_{l\bar{l}}\big)\\
&= \frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{k\bar{k}i}\psi(t)_{\bar{k}k\bar{i}}\\
&= \frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{i\bar{k}k}\psi(t)_{k\bar{i}\bar{k}}\\
&\le \frac{1}{1+\psi(t)_{i\bar{i}}}\frac{1}{1+\psi(t)_{k\bar{k}}}\psi(t)_{i\bar{k}j}\psi(t)_{k\bar{i}\bar{j}},
\end{aligned}$$

so that we get

$$\begin{aligned}
&-\big(n+\tilde{\Delta}\psi(t)\big)^{-1}\frac{1}{1+\psi(t)_{i\bar{i}}}\big(\widetilde{\Delta}\psi(t)\big)_i\big(\tilde{\Delta}\psi(t)\big)_{\bar{i}}+\Delta'\tilde{\Delta}\psi(t)\\
&\quad\ge \tilde{\Delta}F+C\big(n+\tilde{\Delta}\psi(t)\big)\frac{1}{1+\psi(t)_{i\bar{i}}}.
\end{aligned}$$

Putting all these together, we obtain

$$\begin{aligned}
&\triangle'\big(e^{-\lambda\psi(t)}(n+\tilde{\Delta}\psi(t))\big)\ge e^{-\lambda\psi(t)}\Big(\tilde{\Delta}F+C\big(n+\tilde{\Delta}\psi(t)\big)\sum_{i=1}^{n}\frac{1}{1+\psi(t)_{i\bar{i}}}\Big)\\
&-\lambda\, e^{-\lambda\psi(t)}\Delta'\psi(t)\big(n+\tilde{\Delta}(\varphi-\tilde{\rho})\big). \qquad (6.2)
\end{aligned}$$

Consider

$$\begin{aligned}
\tilde{\Delta}F &= \tilde{\Delta}\Big(\frac{\partial}{\partial t}\psi(t)-\psi(t)+\frac{\partial\tilde{\rho}}{\partial t}\Big)\\
&= \tilde{\Delta}\frac{\partial}{\partial t}\psi(t)-\big(n+\tilde{\Delta}\psi(t)\big)+n+\tilde{\Delta}\frac{\partial\tilde{\rho}}{\partial t}\\
&\ge \frac{\partial}{\partial t}\big(n+\tilde{\Delta}\psi(t)\big)\\
&\quad+\sum_{a,b,c,d=1}^{n}h^{a\bar{b}}\frac{\partial^2\frac{\partial\tilde{\rho}}{\partial t}}{\partial z^c\partial z^{\bar{b}}}h^{c\bar{d}}\frac{\partial^2\psi(t)}{\partial z^a\partial z^{\bar{d}}}-\big(n+\tilde{\Delta}\psi(t)\big)+n+\tilde{\Delta}\frac{\partial\tilde{\rho}}{\partial t}\\
&\ge \frac{\partial}{\partial t}\big(n+\tilde{\Delta}\psi(t)\big)-c_1\big(n+\tilde{\Delta}\psi(t)\big)-c_2
\end{aligned}$$

for some uniform constants c_1 and c_2. In the last inequality, we have used the fact that $\left|\frac{\partial\tilde\rho}{\partial t}\right|$ is uniformly bounded and

$$\left|\frac{\partial^2 \frac{\partial\tilde\rho}{\partial t}}{\partial z^c \partial z^{\bar b}}\right|_{\tilde\rho(t)} \le c_3 \cdot \left|\frac{\partial\tilde\rho}{\partial t}\right|,$$

and

$$0 < h_{c\bar d} + \frac{\partial^2 \psi(t)}{\partial z^c \partial z^{\bar d}} \le \big(n + \tilde\triangle \psi(t)\big) h_{c\bar d}$$

holds as a matrix. Here c_3 is some uniform constant. We also have

$$\begin{aligned} e^{-\lambda\psi(t)} \tilde\triangle F &\ge e^{-\lambda\psi(t)} \frac{\partial}{\partial t}\big(n + \tilde\triangle\psi(t)\big) - c_1\, e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) - c_2\, e^{-\lambda\psi(t)} \\ &\ge \frac{\partial}{\partial t}\big(e^{-\lambda\psi(t)}(n + \tilde\triangle\psi(t))\big) + \lambda \frac{\partial}{\partial t}\psi(t) e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) \\ &\quad - c_1\, e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) - c_2\, e^{-\lambda\psi(t)} \\ &\ge \frac{\partial}{\partial t}\big(e^{-\lambda\psi(t)}(n + \tilde\triangle\psi(t))\big) - (c_1 + |\lambda| c_4) e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) - c_2\, e^{-\lambda\psi(t)}. \end{aligned}$$

Here c_4 is a uniform constant. Plugging this into inequality (6.2), we obtain

$$\begin{aligned} \Box\big(e^{-\lambda\psi(t)}(n + \tilde\triangle\psi(t))\big) \ge e^{-\lambda\psi(t)}\Big(C\big(n + \tilde\triangle\psi(t)\big) \sum_{i=1}^n \frac{1}{1 + \psi(t)_{i\bar i}}\Big) \\ - \lambda\, e^{-\lambda\psi(t)} \triangle'\psi(t)\big(n + \tilde\triangle\psi(t)\big) \\ - (c_1 + |\lambda| c_4)\, e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) - c_2\, e^{-\lambda\psi(t)}. \end{aligned}$$

Now

$$\triangle'\psi(t) = n - \sum_{i=1}^n \frac{1}{1 + (\varphi - \tilde\rho)_{i\bar i}}.$$

Plugging this into the previous inequality, we obtain

$$\begin{aligned} \Box\big(e^{-\lambda\psi(t)}(n + \tilde\triangle\psi(t))\big) \ge e^{-\lambda\psi(t)}\Big((C + \lambda)\big(n + \tilde\triangle\psi(t)\big) \sum_{i=1}^n \frac{1}{1 + \psi(t)_{i\bar i}}\Big) \\ - (c_1 + |\lambda| c_4 + n)\, e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) - c_2\, e^{-\lambda\psi(t)}. \end{aligned}$$

Let $\lambda = -C + 1$; we then have

$$\begin{aligned} \Box\big(e^{-\lambda\psi(t)}(n + \tilde\triangle\psi(t))\big) \ge e^{-\lambda\psi(t)}\Big(\big(n + \tilde\triangle\psi(t)\big) \sum_{i=1}^n \frac{1}{1 + \psi(t)_{i\bar i}}\Big) \\ - c_5\, e^{-\lambda\psi(t)}\big(n + \tilde\triangle\psi(t)\big) - c_2\, e^{-\lambda(\varphi - \tilde\rho)}. \end{aligned}$$

Here c_5 is a uniform constant. Note the algebraic inequality

$$\sum_i \frac{1}{1+\psi(t)_{i\bar{i}}} \geq \Big(\frac{\sum_i (1+\psi(t)_{i\bar{i}})}{\prod_i (1+(\varphi-\tilde{\rho})_{i\bar{i}})}\Big)^{1/(n-1)}$$
$$= e^{-F/(n-1)}\big(n+\tilde{\Delta}\psi(t)\big)^{1/(n-1)}.$$

This can be verified by taking the $(n-1)$th power of both sides. So the last term in the previous relation can be estimated by

$$e^{-\lambda\psi(t)} \sum_i \frac{1}{1+\psi(t)_{i\bar{i}}}\big(n+\Delta\psi(t)\big)$$
$$\geq e^{-F/(n-1)} e^{-\lambda/(n-1)}\big(e^{-\lambda\psi(t)}(n+\tilde{\Delta}\psi(t))\big)^{n/(n-1)}.$$

Now setting $u = e^{-\lambda\psi(t)}(n+\tilde{\Delta}\psi(t))$ and recalling that $\psi(t) \leq -1$ and hence $e^{-\lambda\psi(t)} \geq 1$, we finally obtain the estimate

$$\Box u \geq -c_1 - c_2 u + c_0 u^{n/(n-1)}.$$

Assume that u achieves its maximum at x_0 and $\frac{\partial u}{\partial t}\big|_{x_0,t} \geq 0$; then at this point, $\Box u = \Delta' u - \frac{\partial u}{\partial t}\big|_{x_0,t} \leq 0$. Therefore, the maximum principle gives us an upper bound $u(x_0) \leq C$ which, in turn, gives

$$0 \leq \big(n+\tilde{\Delta}\psi(t)\big)(x) \leq e^{\lambda\psi(t)(x)} u(x_0) \leq C,$$

and hence we found a C^2-estimate of $\psi(t)$. □

PROPOSITION 6.2
Let $\tilde{\rho}(t)$ be as in Theorem 6.1. Then there exists a uniform constant C such that

$$\|\varphi(t) - \tilde{\rho}(t)\|_{C^3(\omega_{\tilde{\rho}})} \leq C.$$

Proof
Let

$$g'_{i\bar{j}} = h_{i\bar{j}} + (\varphi - \tilde{\rho})_{i\bar{j}}$$

and

$$S = \sum_{i,j,k,r,s,t=1}^{n} g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} (\varphi-\tilde{\rho})_{i\bar{j}k} (\varphi-\tilde{\rho})_{\bar{r}s\bar{t}}\,.$$

Using Calabi's computation and [35, Theorem 5.1], one can show that $S \leq C$ for some uniform constant C. Consequently, the proposition is proved. □

7. The proofs of the main results

According to Theorems 5.1 and 6.1 and Proposition 6.2, we have uniform C^3-estimates on $\varphi(t) - \tilde{\rho}(t)$ along the modified Kähler-Ricci flow. It is not difficult to prove the following.

LEMMA 7.1
For any integer $l > 0$, there exists a uniform constant C_l such that

$$\left\| D^l\big(\varphi(t) - \tilde{\rho}(t)\big)\right\|_{\omega_{\tilde{\rho}}} \leq C_l,$$

where D^l represents arbitrary lth derivatives. Consequently, there exists a uniform bound on the sectional curvature and all the derivatives of $\omega_{\varphi(t)}$. The bound may possibly depend on the order of derivatives.

Following this lemma, we can easily derive that the evolved Kähler metrics $\omega_{\varphi(t)}$ converge to a Kähler metric in the limit (by choosing a subsequence). We show that the limit is a Kähler-Einstein metric. Following Proposition 2.7 and the fact that E_0 and E_1 have a uniform lower bound, we have

$$\int_0^\infty \frac{n\sqrt{-1}}{V}\int_M \partial\frac{\partial\varphi}{\partial t}\wedge\overline{\partial}\frac{\partial\varphi}{\partial t}{\omega_\varphi}^{n-1}\,dt = E(0) - E(\infty) < C,$$
$$\int_0^\infty \frac{2}{V}\int_M \big(R(\omega_\varphi) - r\big)^2{\omega_\varphi}^n\,dt = E_1(0) - E_1(\infty) \leq C.$$

Combining this with Lemma 7.1, we prove that for almost all convergence subsequences of the evolved Kähler metrics $\omega_{\varphi(t)}$, the limit metric is of constant scalar curvature metric. From here, it is not difficult to show that any sequence of the evolved Kähler metrics has a subsequence that converges to a metric of constant scalar curvature. In the canonical class, any metric of constant scalar curvature is a Kähler-Einstein metric. We then prove the following.

THEOREM 7.2
The modified Kähler-Ricci flow converges to some Kähler-Einstein metric by taking subsequences.

To prove uniqueness of the limit by sequence, we can follow [10] to first prove the exponential decay of

$$\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2\omega_\varphi^n.$$

In other words, there exists a positive constant α and a uniform constant C such that

$$\int_M \left(\frac{\partial\varphi}{\partial t}\right)^2 \omega_\varphi^n < Ce^{-\alpha t}$$

for all evolved metrics over the Kähler-Ricci flow. Eventually, we prove the following main proposition (as in [10]).

PROPOSITION 7.3
For any integer $l > 0$, $\frac{\partial\varphi}{\partial t}$ *converges exponentially fast to zero in any* C^l*-norm. Furthermore, the Kähler-Ricci flow converges exponentially fast to a unique Kähler-Einstein metric on any Kähler-Einstein manifolds.*

8. Kähler-Einstein orbifolds

In this section we prove that any Kähler-Einstein orbifold such that there is another Kähler metric in the same Kähler class which has strictly positive bisectional curvature must be a global quotient of $\mathbb{C}P^n$ by a finite group. The simplest example of Kähler orbifolds is the global quotient of $\mathbb{C}P^n$ by a finite group. Roughly speaking, a generic Kähler orbifold is the union of a family of open sets, where each open set admits a finite covering from an open smooth Kähler manifold where a finite group acts holomorphically. (We give a precise definition later.) If it admits a Kähler-Einstein metric, then it is called a Kähler-Einstein orbifold. The goal in this section is to show that under our assumption, there exists a global branching covering with a finite group action from $\mathbb{C}P^n$ to the underlying Kähler orbifold. The organization of this section is as follows. In Section 8.1 we introduce the notion of complex orbifolds and various geometric structures associated with them. In Section 8.2 we consider the Kähler-Ricci flow on any Kähler-Einstein orbifolds. If there is another Kähler metric in the same Kähler class such that the bisectional curvature is positive, then the Kähler-Ricci flow converges and the limit metric is a Kähler-Einstein metric with positive bisectional curvature. In Section 8.3 we prove that any orbifold that admits a Kähler-Einstein metric of constant bisectional curvature must be a global quotient of $\mathbb{C}P^n$. In Section 8.4 we re-prove that any Kähler-Einstein metric with positive bisectional curvature must be of constant bisectional curvature (Berger's theorem in [3]). We also prove that if a Kähler metric is sufficiently close to a Kähler-Einstein metric on the Kähler-Ricci flow, then the positivity of bisectional curvature is preserved when taking the limit (Lemma 8.20).

8.1. Kähler orbifolds

Let us begin with the definition of uniformization system over an open connected analytic space.*

*One reference for orbifolds is Ruan [30].

Definition 8.1
Let U be a connected analytic space, let V be a connected n-dimensional smooth Kähler manifold, and let G be a finite group acting on V holomorphically. An n-dimensional uniformization system of U is a triple (V, G, π), where $\pi : V \to U$ is an analytic map inducing an identification between two analytic spaces V/G and U. Two uniformization systems (V_i, G_i, π_i), $i = 1, 2$, are isomorphic if there is a biholomorphic map $\phi : V_1 \to V_2$ and isomorphism $\lambda : G_1 \to G_2$ such that ϕ is λ-equivariant and $\pi_2 \circ \phi = \pi_1$.

In this definition, we require that the fixed point set be real codimension 2 or higher. (If the group action preserves orientation, then the fixed point must be codimension 2 or higher.) Then the nonfixed point set (the complement of the fixed point set) is locally connected, which is important for our purpose. The following proposition is immediate.

PROPOSITION 8.2
Let (V, G, π) be a uniformization system of U. For any connected open subset U' of U, (V, G, π) induces a unique isomorphism class of uniformization systems of U'.

Proof
We want to clarify what "induces" means in this proposition. For any open subset $U' \subset U$, consider the preimage $\pi^{-1}(U')$ in V. G acts by permutations on the set of connected components of $\pi^{-1}(U')$. Let V' be one of the connected components of $\pi^{-1}(U')$, let G' be the subgroup of G which fixes the component V', and let $\pi' = \pi|_{V'}$. Then (V', G', π') is an induced uniformizing system of U'. One can also show that any other induced uniformization system must be isomorphic to this one. We skip this part of the proof and refer interested readers to [30] for details. □

In light of this proposition, we can define equivalence of two uniformization systems at a single point. For any point $p \in U$, let (V_1, G_1, π_1) and (V_2, G_2, π_2) be two uniformization systems of neighborhoods U_1 and U_2 of p. We say that (V_1, G_1, π_1) and (V_2, G_2, π_2) are *equivalent* at p if they induce isomorphic uniformization systems for a smaller neighborhood $U_3 \subset U_1 \cap U_2$ of p. Next, we define a complex (Kähler) orbifold.

Definition 8.3
Let M be a connected analytic space. An n-dimensional complex orbifold structure on M is given by the following data. For any point $p \in M$, there are neighborhoods U_p and their n-dimensional uniformization systems (V_p, G_p, π_p) such that for any $q \in U_p$, (V_p, G_p, π_p) and (V_q, G_q, π_q) are equivalent at q. A point $p \in M$ is called

regular if there exists a uniformization system (V_p, G_p, π_p) over $U_p \ni p$ such that G_p is trivial; otherwise, it is called singular. The set of regular points is denoted by M_{reg}. The set of singular points is denoted by M_{sing}, and $M = M_{\mathrm{reg}} \cup M_{\mathrm{sing}}$.

Next, we define orbifold vector bundles over a complex orbifold. As before, we begin with local uniformization systems for orbifold vector bundles. Given an analytic space U which is uniformized by (V, G, π) and a complex analytic space E with a surjective holomorphic map $pr : E \to U$, a uniformization system of rank k complex vector bundle for E over U consists of the following data:

(1) a uniformization system (V, G, π) of U;

(2) a uniformization system $(V \times \mathbb{C}^k, G, \tilde{\pi})$ for E; the action of G on $V \times \mathbb{C}^k$ is an extension of the action of G on V given by $g(x, v) = (g \cdot x, \rho(x, g) \cdot v)$, where $\rho : V \times G \to GL(\mathbb{C}^k)$ is a holomorphic map satisfying

$$\rho(g \cdot x, h) \circ \rho(x, g) = \rho(x, h \circ g), \quad \forall g,\ h \in G,\ x \in V;$$

(3) the natural projection map $\tilde{pr} : V \times \mathbb{C}^k \to V$ satisfying

$$\pi \circ \tilde{pr} = pr \circ \tilde{\pi}.$$

We can similarly define isomorphisms between two uniformization systems of orbifold vector bundles for E over U. The only additional requirement is that the diffeomorphism between two copies of $V \times \mathbb{C}^k$ be linear on each fiber of $\tilde{pr} : V \times \mathbb{C}^k \to V$. Moreover, we can also define the equivalent relation between two uniformization systems of complex vector bundles at any specific point. Here is the definition of orbifold vector bundles over complex orbifolds.

Definition 8.4

Let M be a complex orbifold, and let E be a complex vector space with a surjective holomorphic map $pr : E \to M$. A rank k complex orbifold vector bundle structure on E over M consists of the following data. For each point $p \in M$, there is a unformized neighborhood U_p and a uniformization system of a rank k complex vector bundle for $pr^{-1}(U_p)$ over U_p such that for any $q \in U_p$, the rank k complex orbifold vector bundles over U_p and U_q are isomorphic in a smaller open subset $U_p \cap U_q$. Two orbifold vector bundles $pr_1 : E_1 \to M$ and $pr_2 : E_2 \to M$ are isomorphic if there is a holomorphic map $\tilde{\psi} : E_1 \to E_2$ given by $\tilde{\psi}_p : (V_{1,p} \times \mathbb{C}^k, G_{1,p}, \tilde{\pi}_{1,p}) \to (V_{2,p} \times \mathbb{C}^k, G_{2,p}, \tilde{\pi}_{2,p})$ which induces an isomorphism between $(V_{1,p}, G_{1,p}, \tilde{\pi}_{1,p})$ and $(V_{2,p}, G_{2,p}, \tilde{\pi}_{2,p})$ and is a linear isomorphism between the fibers of $\tilde{pr}_{1,p}$ and $\tilde{pr}_{2,p}$.

For a complex orbifold, one can define the tangent bundle, the cotangent bundle, and various exterior or tensor powers of these bundles. All the differential geometric quantities such as cohomology class, connections, metrics, and curvatures can be introduced on the complex orbifold.

Suppose that M is a complex orbifold as in Definition 8.3. For any $p \in M$, let $p \in U_p$ be uniformized by (V_p, G_p, π_p). When we say a metric g is defined on U_p, we really mean a metric $\overline{g}$ defined on V_p such that G_p acts on V_p by isometries. For simplicity, we say that the metric g is defined on U_p and $\pi_p^* g = \overline{g}$. This simplification makes sense especially when p is a regular point, that is, when G_p is trivial. One way to define a metric on the entire complex orbifold is first to define it on M_{reg} and then to extend it to be a metric on M with possible singularities since M_{sing} is codimension at least 2 or higher. The following gives a definition of what a smooth Kähler metric or a Kähler form on the complex orbifold is.

Definition 8.5
For any point $p \in M$, let U_p be uniformized by (V_p, G_p, π_p). A Kähler metric g (resp., a Kähler form ω) on a complex orbifold M is a smooth metric on M_{reg} such that for any $p \in M$, ${\pi_p}^* g$ (resp., Kähler form ${\pi_p}^* \omega)^*$ can extend to a smooth Kähler metric (resp., smooth Kähler form) on V_p.

Definition 8.6
A function f is called a smooth function on an orbifold M if for any $p \in M$, $f \circ \pi_p$ is a smooth function on V_p.

Similarly, one can define any tensor to be smooth on M if its preimage on each local uniformization system is smooth. Clearly, the curvature tensor and the Ricci tensor of any smooth metric on orbifolds, as well as their derivatives, are smooth tensors. A complex orbifold that admits a Kähler metric is called a Kähler orbifold.

Definition 8.7
A curve $c(t)$ on Kähler orbifold M is called *geodesic* if, near any point p on it, $c(t) \cap U_p$ can be lifted to a geodesic on V_p and at least one preimage of $c(t)$ is smooth in V_p. Here U_p is any open connected neighborhood of p over which (V_p, G_p, π_p) is a uniformization system.

Under this definition, we have the following.

*Note that ${\pi_p}^*$ is only defined away from the fixed point set of V_p. Since the fixed point set is at least codimension 2 or higher, any metric defined on the nonfixed point set of V_p has a unique smooth extension on V_p if such an extension exists. This definition essentially says that a metric is smooth in the orbifold sense if such an extension always exists in each uniformization system of the underlying Kähler orbifold structure.

PROPOSITION 8.8
Any minimizing geodesic between two regular points never passes any singular point of the Kähler orbifold.

Proof
Otherwise, we can argue that the geodesic is not minimizing. Suppose that p is a singular point, and suppose that $p \in U_p$ is a small open set that is uniformized by (V_p, G_p, π_p) with an equivariant metric g on V_p. Suppose that a portion of geodesic lying inside of U_p is $c(t) : [-\epsilon, \epsilon]$ such that $A = c(-\epsilon), B = c(\epsilon) \in U_p$, and $p = c(0)$. Assume that this geodesic is parameterized by arc length. Thus the distance between A and B is 2ϵ, while the distance between A (or B) and O is ϵ. Without loss of generality, we may assume that A and B are regular points. Suppose that $\pi^{-1}(p) = O$; $\pi^{-1}(A) = \{A_1, A_2, \dots, A_{l_1}\}$ and $\pi^{-1}(B) = \{B_1, B_2, \dots, B_{l_2}\}$. Note that $\{A_1, A_2, \dots, A_{l_1}\}$ and $\{B_1, B_2, \dots, B_{l_2}\}$ are on the sphere of radius ϵ which centers at O. If G_p is nontrivial (then the preimages of A and B are not unique; i.e., $l_1 > 1$ and $l_2 > 1$), then there is at least one pair of A_i, B_j $(1 \leq i \leq l_1, 1 \leq j \leq l_2)$ such that the distance between the two points is shorter than 2ϵ on V_p.* Suppose that this geodesic is $\tilde{C}$. Then $\pi_p(\tilde{C})$ is a geodesic (which connects A and B) whose length is shorter than 2ϵ. Thus $c(t)$ is not a minimizing geodesic between A and B since $\pi_p(A_i) = A$ and $\pi_p(B_j) = B$. □

8.2. Kähler-Ricci flow on Kähler-Einstein orbifolds
A Kähler-Einstein orbifold metric is a metric on orbifold such that the Ricci curvature is proportional to the metric. A Kähler orbifold with a Kähler-Einstein metric is called a Kähler-Einstein orbifold.

THEOREM 8.9
Let M be any Kähler-Einstein orbifold. If there is another Kähler metric in the same cohomology class which has nonnegative bisectional curvature and positive bisectional curvature at least at one point, then the Kähler-Ricci flow converges to a Kähler-Einstein metric with positive bisectional curvature.

We want to generalize our proof of Theorem 1.1 to the orbifold case. Note that the analysis for Kähler orbifolds is exactly the same as that for Kähler manifolds (see [16]). We want to show that this theorem can be proved exactly like Theorem 1.1. First, we need to set up some notation. Following Section 2.1, we use the Kähler form

*In any ball of radius 1 on any metric space, the maximum distance between any two points in the ball is 2, which is the diameter of the unit ball. Fix a point in the ball; then the minimal distance from that point to any set of points in the ball is strictly less than 2 if that set contains two or more points.

ω as a smooth Kähler form on the orbifold M. Locally on M_{reg}, it can be written as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. Denote by $\mathscr{B}$ the set of all real-valued smooth functions on M in the orbifold sense (see Definition 8.6). Then the Kähler class $[\omega]$ consists of all Kähler forms that can be expressed as

$$\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0$$

on M for some $\varphi \in \mathscr{B}$. In other words, the space of all Kähler potentials in this Kähler class is

$$\mathscr{H} = \left\{ \varphi \in \mathscr{B} \,\middle|\, \omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0 \right\}.$$

The Ricci form for ω is

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log \omega^n.$$

As in the case of smooth manifolds, $[\omega]$ is the canonical Kähler class if ω and the Ricci form $\mathrm{Ric}(\omega)$ are in the same cohomology class after proper rescaling. In the canonical Kähler class, consider the Kähler-Ricci flow

$$\frac{\partial\varphi}{\partial t} = \log \frac{{\omega_\varphi}^n}{\omega^n} + \varphi - h_\omega,$$

where h_ω is defined as in Section 2.2. Clearly, this flow preserves the structure of Kähler orbifold and, in particular, preserves the Kähler class $[\omega]$. Examining our proof of Theorem 7.2, the following three parts are crucial:

(1) the preservation of positive bisectional curvature under the Kähler-Ricci flow;
(2) the introduction of a set of new functionals E_k and new invariants $\Im_k$ $(k = 0, 1, \ldots, n)$;
(3) the uniform estimate on the diameter; consequently, the uniform control on the Sobolev constant and the Poincaré constant.

To extend these to the case of Kähler orbifolds, we really need to make sure that the following tools for geometric analysis hold in the orbifold case:

(1) maximum principle for smooth functions and tensors on Kähler orbifold (see Definition 8.7);
(2) integration by parts for smooth functions/tensors in the orbifold case;
(3) the second variation formula for any smooth geodesics (see Proposition 8.8).

By our definition of Kähler orbifolds, it is not difficult to see that the maximum principle holds on orbifolds. Thus Theorem 2.1 still holds in the orbifold case. In other words, the bisectional curvature of the evolved metric is strictly positive after the initial time if the initial metric has nonnegative bisectional curvature and positive bisectional curvature at least at one point. Moreover, the integration by parts on orbifold holds for any smooth function on M with smooth metrics in the orbifold sense. Thus our definitions of new functionals $E_0, E_1, \ldots, E_n$ can be carried over to this Kähler orbifold setting without any change. Moreover, the formula for their derivatives still holds. In particular, E_0 and E_1 are decreasing strictly under the Kähler-Ricci flow. Furthermore, the set of invariants $\Im_0, \Im_1, \ldots, \Im_n$ are well defined and vanish on any Kähler-Einstein orbifold. Since Tian's inequality holds on any Kähler-Einstein orbifold, Proposition 2.14 and Corollary 2.16 hold as well. Finally, the second variation formula for minimizing geodesic between any two regular points on Kähler orbifolds is exactly the same as the formula on smooth manifolds (see Proposition 8.8). Thus we can use the same set of ideas in Section 3 to estimate diameter;* consequently, the Sobolev constant and the Poincaré constant can be uniformly controlled as well. The rest of the arguments in our proof of Theorem 7.2 can be extended to the orbifold case directly. Thus we can prove Theorem 8.9 for Kähler-Einstein orbifolds.

8.3. Kähler-Einstein orbifolds with constant positive bisectional curvature

In this section, we want to prove the following.

THEOREM 8.10
Let M be any Kähler orbifold. If there is a Kähler-Einstein metric with constant positive bisectional curvature, then it is a global quotient of $\mathbb{C}P^n$.

Suppose that $\overline{g}$ is the standard Fubini-Study metric on $\mathbb{C}P^n$ with constant bisectional curvature. Suppose that g is a Kähler-Einstein metric on M with constant bisectional curvature. Normalize the bisectional curvature of g on M and of $\overline{g}$ on $\mathbb{C}P^n$ so that both have bisectional curvature equal to 1. Consequently, the conjugate radius of $\mathbb{C}P^n$ is π. Let p be any regular point in M. By definition, let U_p be a small neighborhood of p, and let (V_p, G_p, π_p) be the uniformization system. Since $p \in M_{\text{reg}}$, G_p is a trivial group. Consider $g' = {\pi_p}^* g$ as a Kähler metric with constant bisectional curvature on V_p. If we choose U_p sufficiently small, then (V_p, g') is an open subset of $(\mathbb{C}P^n, \overline{g})$ with the induced metric from $(\mathbb{C}P^n, \overline{g})$. In the following, we drop notation g' and use $\overline{g}$ only. Our goal is to extend π_p into a local isometric map from $\mathbb{C}P^n$ to M.

Next, we set up some notations. Denote by q the preimage of p. Consider

$$\begin{array}{ccc} & \mathbb{C}P^n & M \\ & \cup & \cup \\ \pi_p : & V_p & \rightarrowtail U_p \end{array}$$

*In the proof of Lemma 3.5, without loss of generality, we may assume that $p, q \in M_{\text{reg}}$, where the diameter $D = d(p, q)$. Furthermore, we may assume that $A_1 = B_{p,r} \subset M_{\text{reg}}$ and $A_2 = B_{q,r} \subset M_{\text{reg}}$. According to Lemma 8.8, any minimizing geodesic between A_1 and A_2 belongs to M_{reg}. Consequently, we can use Lemma 3.1 from J. Cheeger and T. H. Colding [8] to complete the diameter bounds as in the smooth case.

Now we want to lift this map π_p to a map from $\mathbb{C}P^n$ to M. First, we need to rewrite this map in a different way:

$$\begin{array}{lcl} \exp_q : T_q(\mathbb{C}P^n) & \to & \mathbb{C}P^n \\ & & \cup \\ & & V_p \\ \downarrow \mathrm{id} & & \downarrow \pi_p \\ & & U_p \\ & & \cap \\ \exp_p : T_p(M) & \to & M \end{array}$$

Set

$$\Pi = \exp_p \circ \mathrm{id} \circ \exp_q^{-1} .$$

Then at least Π is defined in V_p and

$$\Pi = \pi_p = \exp_p \circ \mathrm{id} \circ \exp_q^{-1} \quad \text{in } V_p. \tag{8.1}$$

Consider the open ball of radius π in $T_q(\mathbb{C}P^n)$ which we denote by B_π. Then $\exp_q^{-1}$ is well defined on $\exp_q(B_\pi) \subset \mathbb{C}P^n$. The image of ∂B_π under the exponential map is a projective subspace of codimension 1, which is denoted as $\mathbb{C}P^{n-1}_\infty$. Then

$$\exp_q(B_\pi) = \mathbb{C}P^n \setminus \mathbb{C}P^{n-1}(\infty).$$

We claim that we can extend the map Π in this way to $\mathbb{C}P^n \setminus \mathbb{C}P^{n-1}(\infty)$ via formula (8.1). The key step is the following lemma. (In the following arguments, we abuse notation by using letters p and q for generic points on M.)

LEMMA 8.11
*Any smooth geodesic on M can be extended uniquely and indefinitely. In particular, it can be extended uniquely (before the length π).**

Proof
Suppose that $c(t) : [0, a]$ is a geodesic defined on M with length $a > 0$. If $c(a) \in M_{\mathrm{reg}}$, then it can easily be extended as usual. If $c(a) \in U_p$ for some $p \in M_{\mathrm{sing}}$, in particular, if $c(a) \in M_{\mathrm{sing}}$, we want to extend the geodesic uniquely as well. Consider the part of geodesic $c(t) \cap U_p$, and still denote it as $c(t)$. Suppose that U_p is uniformized by (V_p, G_p, π_p). For convenience, the pullback metric $g'_p = \pi^* g$ is a smooth metric on V_p, and G_p acts isometrically on (V_p, g'_p). Consider its preimages $\tilde{c}(t)$ in V_p under

*We are interested in the unique extension up to length π since it is the conjugate radius of any Kähler metric with constant bisectional curvature 1.

π_p. (Note that π_p is a local isometric map from (V_p, g'_p) to (U_p, g), in particular, if we restrict the map to $M_{\text{reg}} \cap U_p$.) Although the preimages are not unique in V_p, each preimage $\tilde{c}(t)$ has a unique extension on V_p. More importantly, the images of these geodesic extensions on V_p under π_p are unique in U_p. Therefore, the geodesic $c(t)$ is also extendable uniquely in this setting. □

In fact, we have the following.

COROLLARY 8.12
Any geodesic in a Kähler orbifold with constant bisectional curvature can be extended long enough to become a closed geodesic. If the bisectional curvature is 1, *then the length of each closed geodesic is either* 2π *or* $2\pi / l$ *for some integers* l. *Moreover, there exists a maximum integer of all such integers* l, *denoted by* $l_{\max}$. *Then the conjugate radius of the Kähler orbifold with constant bisectional curvature* 1 *is* $\pi / l_{\max}$.

LEMMA 8.13
Π *can be extended to be a global map from* $\mathbb{C}P^n$ *to* M. *Moreover,* Π *is a local isometry from an open dense set* $(\Pi^{-1}(M_{\text{reg}}), \overline{g})$ *in* $\mathbb{C}P^n$ *to* (M_{reg}, g).

Proof
Consider the open ball $B_\pi \subset T_q(\mathbb{C}P^n)$ of radius π. Then the closure of $\exp_q(B_\pi)$ is just the whole $\mathbb{C}P^n$. By Lemma 8.11, Π can be defined in the open set $\exp_p(B_\pi)$. Next, taking the closure of this map, we define a map from $\mathbb{C}P^n$ to M. Moreover, Π is a local isometry from $(\Pi^{-1}(M_{\text{reg}}), \overline{g})$ to (M, g). □

Now we want to prove the following lemma.

LEMMA 8.14
For any point A *on* M, *there exists only a finite number of preimages on* $\mathbb{C}P^n$.

Proof
Otherwise, there exists an infinite number of preimages of the point A on $\mathbb{C}P^n$. This set of infinite number points must have a concentration point on $\mathbb{C}P^n$. In particular, for any small $\epsilon > 0$, there are at least two preimages of A such that the distance between these two points in $\mathbb{C}P^n$ is less than ϵ. Consider the image of the minimal geodesic that connects these two preimage points on $\mathbb{C}P^n$ under Π. We obtain a geodesic loop centered at point A whose length is less than ϵ. This violates the fact that the conjugate radius of (M, g) is at least $\pi / l_{\max}$ (see Corollary 8.12). Thus the lemma holds. □

LEMMA 8.15
For any $p \in M$, *let* U_p *be a small neighborhood of* p, *and let* (V_p, G_p, π_p) *be the uniformization system over* U_p. *Let* W_p *be any connected component of* $\Pi^{-1}(U_p)$. *Then there exists a finite group* G'_p *acting isometrically on* $(W_p, \overline{g})$ *such that* $(W_p, G'_p, \Pi|_{W_p})$ *is a uniformization system over* U_p, *which is equivalent to* (V_p, G_p, π_p). *(In particular, if we choose different connected components of* $\Pi^{-1}(U_p)$, *then the induced uniformization system* $(W_p, G'_p, \Pi|_{W_p})$ *is invariant up to isometries.)*

Proof
Set $f = \Pi|_W$, and set $g' = \pi^* g$. Then g' is a smooth Kähler metric with constant bisectional curvature 1 on V_p, where G_p acts isometrically on V_p with respect to this metric g'. There exists a smooth lifting of f to $\tilde{f} : W_p \to V_p$ such that $\pi_p \circ \tilde{f} = f$. It is easy to verify that $\tilde{f}$ is an isometric map from $(W_p, \overline{g})$ to (V_p, g'). Moreover, it can be proved that $\tilde{f}$ is a one-to-one map from W_p to V_p. Now consider the pullback group $G' = \tilde{f}^{-1} G_p$, and define the group action via $\tilde{f}$. Since G_p acts isometrically on (V_p, g'), G'_p acts isometrically on $(W_p, \overline{g})$. By definition, (W_p, G'_p, f) is a uniformization system over U_p which is equivalent to the original uniformization system (V_p, G_p, π_p). □

LEMMA 8.16
For any $p \in M$, *let* U_p *be a small neighborhood with* $(W_p, G_p, \Pi|_{W_p})$, *a uniformization system, where* $(W_p, \overline{g}) \subset (\mathbb{C}P^n, \overline{g})$. *Then the fixed point set of* G_p *is a totally geodesic Kähler submanifold of* $(W_p, \overline{g}) \subset (\mathbb{C}P^n, \overline{g})$.

Proof
Following properties of isometric group actions on Kähler manifolds, the lemma is proved. □

This lemma has an immediate corollary.

COROLLARY 8.17
Consider $W_{\text{sing}} = \Pi^{-1}(M_{\text{sing}})$. *Then* W_{sing} *is a union of* CP^k *for some* $k \leq n - 1$. *When* $k = 0$, *it is the preimage of isolated singular points on* M.

Proof
The only totally geodesic Kähler submanifold in $(\mathbb{C}P^n, \overline{g})$ is $\mathbb{C}P^k$ for some $k \leq n - 1$. □

Denote by $\mathrm{Aut}(\mathbb{C}P^n)$ the holomorphic transformation group of $\mathbb{C}P^n$. Then we have the following.

LEMMA 8.18
For any $p \in M$, let U_p be a small neighborhood, and let $(W_p, G_p, \Pi|_{W_p})$ be a uniformization system over U_p, where $(W_p, \overline{g}) \subset (\mathbb{C}P^n, \overline{g})$. For any $\sigma \in G_p$, it can be extended to be a group element in $\mathrm{Aut}(\mathbb{C}P^n)$, *and we still denote it as σ. Moreover, $\Pi \circ \sigma = \Pi$ on $\mathbb{C}P^n$.*

Proof
It is easy to see that σ can be extended uniquely to an element in $\mathrm{Aut}(\mathbb{C}P^n)$ which acts isometrically on $(\mathbb{C}P^n, \overline{g})$. Consider two local isometries from $(\mathbb{C}P^n, \overline{g})$ to (M, g) : Π and $\Pi \circ \sigma$. Since the two maps agree on an open set $W_p \subset \mathbb{C}P^n$, they must agree on all $\mathbb{C}P^n$. □

From now on, we may view G_p as a subgroup of $\mathrm{Aut}(\mathbb{C}P^n)$ directly. Now we are ready to give the proof of Theorem 8.10.

Proof of Theorem 8.10
For any $p \in M$, let U_p be a small neighborhood with a uniformization system $(W_p, G_p, \Pi|_{W_p})$; then Lemma 8.18 implies that G_p is a subgroup of $\mathrm{Aut}(\mathbb{C}P^n)$. If $p \in M_{\mathrm{reg}}$, then G_p is trivial. If $p_1, p_2 \in M_{\mathrm{sing}}$ and are near to each other, then $G_{p_1} = G_{p_2}$ by continuity. Consequently, for any $p_1, p_2 \in M_{\mathrm{sing}}$ such that the fixed point sets of W_{p_1} and W_{p_2} belong to the same connected component of $\Pi^{-1}(M_{\mathrm{sing}})$, we have $G_{p_1} = G_{p_2} \subset \mathrm{Aut}(\mathbb{C}P^n)$. Consider $G \subset \mathrm{Aut}(\mathbb{C}P^n)$ to be the subgroup generated by all such G_p's. Then G acts isometrically on $(\mathbb{C}P^n, \overline{g})$ and

$$\Pi \circ \sigma = \Pi, \quad \forall \sigma \in G \subset \mathrm{Aut}(\mathbb{C}P^n). \tag{8.2}$$

This induces a covering map

$$\mathbb{C}P^n / G \to M.$$

By this explicit construction, one can verify directly that this is an orbifold isomorphism. Consequently, M is a global quotient of $\mathbb{C}P^n$ by this group G. The only thing left is to show that G is of finite order, which follows directly from equation (8.2) and Lemma 8.14. □

8.4. Pinching theorem for bisectional curvature
In this section we want to prove the following lemma.

LEMMA 8.19
If g is a Kähler-Einstein metric with strictly positive bisectional curvature on a Kähler orbifold, then g has constant bisectional curvature.

This lemma was first proved by Berger [3] on Kähler manifolds. We note that his proof can be easily modified for Kähler orbifolds. For the reader's convenience, we include a proof here.

Proof
We begin with a simple observation. In any Kähler orbifold, any Kähler-Einstein metric satisfies the elliptic equation

$$\triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}}R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}}R_{q\bar{p}k\bar{j}} - R_{i\bar{j}k\bar{l}} = 0.$$

Define a new symmetric tensor $T_{i\bar{j}k\bar{l}}$ as

$$T_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}\,g_{k\bar{j}}.$$

For any fixed $\epsilon \in (0, 1/(n+1))$, we define

$$Q_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \epsilon T_{i\bar{j}k\bar{l}}.$$

Note that $T_{i\bar{j}k\bar{l}}$ is parallel in the manifold. By a direct but tedious calculation, we arrive at

$$\begin{aligned} -\triangle Q_{i\bar{j}k\bar{l}} = {} & Q_{i\bar{j}p\bar{q}}Q_{q\bar{p}k\bar{l}} + Q_{i\bar{l}p\bar{q}}Q_{q\bar{p}k\bar{j}} - Q_{i\bar{p}k\bar{q}}Q_{p\bar{j}q\bar{l}} \\ & + \epsilon(1-(n+1)\epsilon)T_{i\bar{j}k\bar{l}} - Q_{i\bar{j}k\bar{l}}. \end{aligned} \tag{8.3}$$

Suppose that the bisectional curvature of g is not constant. Note that for $\epsilon = 1/(n+1)$, we have

$$g^{i\bar{j}}Q_{i\bar{j}k\bar{l}} = g^{k\bar{l}}Q_{i\bar{j}k\bar{l}} = 0.$$

Thus, if $R_{i\bar{j}k\bar{l}} > 0$, there exists a small positive $\epsilon \in (0, 1/(n+1))$ such that $Q_{i\bar{j}k\bar{l}} \geq 0$ in the whole manifold and vanishes in some direction at some points. In other words, there exist a point $x_0 \in M$ and two vectors $v_0, w_0 \in T_{x_0}M$ such that

$$Q_{i\bar{j}k\bar{l}}(x_0)v_0^{\bar{i}}v_0^{j}w_0^{\bar{k}}w_0^{l} = 0,$$

and for any other point $x \in M$ and any other pair of vectors $v, w \in T_xM$, we have

$$Q_{i\bar{j}k\bar{l}}(x)v^{\bar{i}}v^{j}w^{\bar{k}}w^{l} \geq 0.$$

Now consider a pair of parallel vector fields v, w in a small neighborhood of x_0 such that

$$v^i_{,j} = v^i_{,\bar{j}} = w^i_{,j} = w^i_{,\bar{j}} = 0,$$

where

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial z^i} \qquad \text{and} \qquad w = \sum_{i=1}^{n} w^i \frac{\partial}{\partial z^i}.$$

Furthermore, $v(x_0) = v_0$ and $w(x_0) = w_0$. Consider the scalar function

$$Q = Q_{i\bar{j}k\bar{l}}(x) v^{\bar{i}} v^j w^{\bar{k}} w^l$$

in a neighborhood of x_0. Clearly, Q achieves the minimum in x_0. Thus the maximum principle implies that

$$\begin{aligned} -\triangle Q &= (\triangle Q_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l)_{x=x_0} \\ &= (\triangle Q_{i\bar{j}k\bar{l}}) \mid_{x=x_0} v^{\bar{i}} v^j w^{\bar{k}} w^l \leq 0. \end{aligned}$$

Plugging this into equation (8.3), we obtain

$$\begin{aligned} -\triangle Q_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l = {} & Q_{i\bar{j}p\bar{q}} Q_{q\bar{p}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l + Q_{i\bar{l}p\bar{q}} Q_{q\bar{p}k\bar{j}} v^{\bar{i}} v^j w^{\bar{k}} w^l \\ & - Q_{i\bar{p}k\bar{q}} Q_{p\bar{j}q\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l + \epsilon\big(1-(n+1)\epsilon\big) T_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l \\ & - Q_{i\bar{j}k\bar{l}} v^{\bar{i}} v^j w^{\bar{k}} w^l. \end{aligned}$$

Define the following linear operator at point x_0:

$$A_{i\bar{j}} = Q_{i\bar{j}k\bar{l}} v^{\bar{k}} v^l \qquad \text{and} \qquad C_{i\bar{j}} = Q_{k\bar{l}i\bar{j}} w^{\bar{k}} w^l$$

and

$$M_{i\bar{j}} = Q_{i\bar{p}q\bar{j}} v^p w^{\bar{q}} \qquad \text{and} \qquad M_{ij} = Q_{i\bar{p}j\bar{q}} v^p w^q.$$

Plugging these into the previous equation and evaluating at x_0, we have

$$\begin{aligned} 0 &\geq -\triangle Q_{i\bar{j}k\bar{l}}(x) v^{\bar{i}} v^j w^{\bar{k}} w^l \mid_{x=x_0} \\ &= A_{p\bar{q}} C_{q\bar{p}} + M_{p\bar{q}} M_{q\bar{p}} - M_{pq} M_{\bar{p}\bar{q}} + \epsilon\big(1-(n+1)\epsilon\big)|v|^2|w|^2. \end{aligned}$$

By a calculation of N. Mok [25], we have

$$A_{p\bar{q}} C_{q\bar{p}} \geq M_{p\bar{q}} M_{q\bar{p}} + M_{pq} M_{\bar{p}\bar{q}}.$$

Since $0 < \epsilon < 1/(n+1)$,

$$0 \geq -\triangle Q_{i\bar{j}k\bar{l}}(x_0) v^{\bar{i}} v^j w^{\bar{k}} w^l \geq \epsilon\big(1-(n+1)\epsilon\big)|v|^2|w|^2 > 0.$$

This is a contradiction! Thus $\epsilon = 1/(n+1)$. In this case,

$$g^{i\bar{j}} Q_{i\bar{j}k\bar{l}} = g^{k\bar{l}} Q_{i\bar{j}k\bar{l}} = 0$$

and

$$Q_{i\bar{j}k\bar{l}} \geq 0.$$

Thus

$$Q_{i\bar{j}k\bar{l}}(x_0) = 0.$$

Since x_0 is the minimum for $R_{i\bar{j}k\bar{l}}$, we obtain

$$R_{i\bar{j}k\bar{l}} \equiv \frac{1}{n+1} T_{i\bar{j}k\bar{l}} \equiv \frac{1}{n+1}(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}). \qquad \square$$

From the proof, we can actually prove slightly more.

LEMMA 8.20
If the Kähler-Ricci flow converges to a unique Kähler-Einstein metric exponentially fast (Proposition 8.3), and if the bisectional curvature remains positive before taking the limit, then the limit Kähler-Einstein metric has positive bisectional curvature.

Combining this lemma, Lemma 8.19, and Proposition 7.3, we can prove the following.

THEOREM 8.21
For any integer $l > 0$, $\frac{\partial \varphi}{\partial t}$ converges exponentially fast to zero in any C^l-norm. Furthermore, the Kähler-Ricci flow converges exponentially fast to a unique Kähler-Einstein metric with constant bisectional curvature on any Kähler-Einstein manifolds.

9. Concluding remarks

In this section, we want to prove Theorem 1.1, Corollary 1.3, and Theorem 1.4. Note that the proof of Theorem 1.1 and Corollary 1.3 is similar to the proof we gave in the final section of our earlier article [10] except in the final step, where we need to use Berger's theorem (Lemma 8.19) and Theorem 8.21 to show that any Kähler-Einstein metric with positive bisectional curvature must be a space form. We skip this part and just give a proof for Theorem 1.4.

Proof of Theorem 1.4
For any Kähler metric in the canonical Kähler class such that it has nonnegative bisectional curvature on M (Kähler-Einstein orbifold) but positive bisectional curvature at least at one point, we apply the Kähler-Ricci flow to this metric on M. Theorem 2.1 still holds in the orbifold case. In other words, the bisectional curvature of the evolved

metric is strictly positive over all the time. By our Theorem 8.9 and Lemma 8.20, the Kähler-Ricci flow converges exponentially to a unique Kähler-Einstein metric of positive bisectional curvature. According to Lemma 8.19, any Kähler-Einstein metric of positive bisectional curvature on a Kähler orbifold must have a constant positive bisectional curvature. Moreover, using Theorem 8.10, we arrive at the conclusion that M must be a global quotient of $\mathbb{C}P^n$ by a finite group action. □

Proof of Corollary 1.3
Theorem 1.1 or Theorem 1.4 proves that any Kähler metric with nonnegative bisectional curvature on M and positive bisectional curvature at least at one point is path connected to a Kähler-Einstein metric of constant positive bisectional curvature. Note that all the Kähler-Einstein metrics are path connected by automorphisms (see [2]). Therefore, the space of all Kähler metrics with nonnegative bisectional curvature on M and positive bisectional curvature at least at one point is path connected. Similarly, using Theorem 2.2 and our Theorem 1.4, we can show that all Kähler metrics with nonnegative curvature operator on M and positive bisectional curvature at least at one point are path connected. Note that the nonnegative curvature operator implies the nonnegative bisectional curvature. □

Remark 9.1
Combining Theorems 1.1, 2.1, and 2.2, we can easily generalize Corollary 1.3 to the case where the bisectional curvature (or curvature operator) is assumed only to be nonnegative. We can show that the flow converges exponentially fast to a unique Kähler-Einstein metric with nonnegative bisectional curvature. Then, following an earlier work of Zhong and Mok [26], the underlying manifold must be a compact symmetric homogeneous manifold. Next, we want to propose some future problems. Some of them may not be hard to solve.

QUESTION 9.2
It is clear that E_1 plays a critically important role in proving the convergence theorem of this article. Note that the Kähler-Ricci flow is a gradient-like flow for the functional E_1 (when Ricci curvature is positive under the flow). It may be interesting to study the gradient flow of E_1.

Consider the following expansion formula in t:

$$\big(\omega_\varphi + t\,\mathrm{Ric}(\omega_\varphi)\big)^n = \Big(\sum_{k=0}^{n} \sigma_k(\omega_\varphi) t^k\Big)\omega_\varphi^n.$$

Clearly, $\sigma_0(\omega_\varphi) = 1$, $\sigma_1(\omega_\varphi) = R(\omega_\varphi)$, the scalar curvature of ω_φ. The equation for the gradient flow of E_1 is

$$\frac{\partial \varphi(t)}{\partial t} = 2\Delta_\varphi R(\omega_\varphi) - (n-1)\sigma_2(\omega_\varphi) - c_1. \tag{9.1}$$

Here c_1 is some constant that depends only on the Kähler class. Clearly, this is a sixth-order parabolic equation.

QUESTION 9.3
As Remark 1.5 indicates, what we really need is the positivity of Ricci curvature along the Kähler-Ricci flow. However, it is not expected that the positivity of Ricci curvature is preserved under the Kähler-Ricci flow except on Riemann surfaces. The positivity of bisectional curvature is a technical assumption to assure the positivity of Ricci curvature. It is very interesting to extend Theorem 1.1 to metrics without the assumption on the bisectional curvature.

CONJECTURE 9.4 (Hamilton and Tian)
On a Kähler-Einstein manifold, the Kähler-Ricci flow converges to a Kähler-Einstein metric. On a general Kähler manifold with positive first Chern class, the Kähler-Ricci flow converges, at least by taking sequences, to a Kähler-Ricci soliton modulo diffeomorphism. Note that the complex structure may change in the limit and that the limit may have mild singularities.

QUESTION 9.5
Is the positivity of the sectional curvature preserved under the Kähler-Ricci flow?

QUESTION 9.6
For any holomorphic vector field and Kähler class $[\omega]$, *are the invariants* $\mathscr{I}_k(X, [\omega])$ *independent? In the noncanonical class, we expect these invariants to be different. Note that one can derive localization formulas for these invariants as was done in [34].*

QUESTION 9.7
According to our Theorem 1.4, any Kähler-Einstein orbifold with positive bisectional curvature is necessarily biholomorphic to a global quotient of $\mathbb{C}P^n$. *What happens if we drop the Kähler-Einstein condition?*

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