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KÄHLER–RICCI SOLITONS ON COMPACT COMPLEX MANIFOLDS WITH $C_1(M) > 0$

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Abstract. In this paper, we discuss the relation between the existence of Kähler–Ricci solitons and a certain functional associated to some complex Monge–Ampère equation on compact complex manifolds with positive first Chern class. In particular, we obtain a strong inequality of Moser–Trudinger type on a compact complex manifold admitting a Kähler–Ricci soliton.

0 Introduction

In this paper, we study the existence of Kähler–Ricci solitons by using properness of a certain functional. Our approach is similar to that of [T] (also see [TZ1]) for Kähler–Einstein metrics. A Kähler metric g on a compact complex manifold M with first Chern class $c_1(M) > 0$ is called a (homothetically shrinking) Kähler–Ricci soliton if there is a holomorphic vector field X on M such that the Kähler form ω_g of g satisfies

$$\mathrm{Ric}(\omega_g) - \omega_g = L_X \omega_g ,$$

where $\mathrm{Ric}(\omega_g)$ denotes the Ricci form of ω_g and L_X is the Lie derivative operator along X. In particular, if $X = 0$, such a g is a Kähler–Einstein metric. So Kähler–Ricci solitons can be regarded as a generalization of Kähler–Einstein metrics of positive scalar curvature. Ricci solitons have been studied extensively in the recent years ([K], [H], [C], [M2], [T], [TZ2,3], [WZ], etc.). One motivation is that they are very closely related to the singular behavior of limit solutions of certain PDEs which arise from geometric analysis, such as Hamilton's Ricci flow [H] and certain complex Monge–Ampère equations associated to Kähler–Einstein metrics [T]. Kähler–Ricci solitons are special Ricci solitons. It was proved recently in [TZ2] and [TZ3] that there exists at most one Kähler–Ricci soliton on any compact

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Kähler manifold with positive first Chern class, modulo holomorphic automorphisms. This extends the uniqueness theorem of Bando and Mabuchi for Kähler–Einstein metrics with positive scalar curvature [BM].

Let g be a Kähler metric on M with its Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j \, ,$$

representing $c_1(M)$. Then there is a smooth real-valued function h_g such that

$$\mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} h_g \, .$$

Suppose that X is a holomorphic vector field on M so that the integral curve K_X of the imaginary part $\mathrm{Im}(X)$ of X consists of isometries of g. By the Hodge decomposition theorem, there exists a unique smooth real-valued function $\theta_X = \theta_X(\omega_g)$ on M such that

$$\begin{cases} i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_X \, , \\ \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n = V \, . \end{cases}$$

Set

$$\mathcal{M}_X(\omega_g) = \left\{ \psi \in C^\infty(M) \;\middle|\; \omega_\psi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi > 0 \, , \ \mathrm{Im}(X)(\psi) = 0 \right\} .$$

The following functional on $\mathcal{M}_X(\omega_g)$ was introduced in [TZ3],

$$\tilde{F}_{\omega_g}(\psi) = \tilde{J}_{\omega_g}(\psi) - \frac{1}{V} \int_M \psi e^{\theta_X} \omega_g^n - \log \left(\frac{1}{V} \int_M e^{h_g - \psi} \omega_g^n \right) ,$$

where

$$\tilde{J}_{\omega_g}(\psi) = \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 \int_0^1 t(st)^k (1-st)^{n-1-k} e^{\theta_X + stX(\psi)} dt \, ds \right) \times \partial\psi \wedge \bar{\partial}\psi \wedge \omega_\psi^k \wedge \omega_g^{n-1-k}.$$

Let $K_0 (\supseteq K_X)$ be a maximal compact subgroup of the automorphisms group of M such that $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\eta \in K_0$ and any $\sigma \in K_X$. Then a Kähler–Ricci soliton with respect to X on M must be K_0-invariant by the uniqueness theorem in [TZ2]. We introduce

DEFINITION 0.1. *The functional $\tilde{F}_{\omega_g}(\,\cdot\,)$ is said to be proper with respect to X if for any sequence $\{\psi_i\}$ of K_0-invariant functions in $\mathcal{M}_X(\omega_g)$,*

$$\overline{\lim_{i \to \infty}} \tilde{F}_{\omega_g}(\psi_i) = +\infty \, ,$$

whenever $\lim_{i\to\infty} I_{\omega_g}(\psi_i) = +\infty$. Here

$$I_{\omega_g}(\psi_i) = \frac{1}{V} \int_M \psi_i (\omega_g^n - \omega_{\psi_i}^n) \, .$$

We note that the above definition is independent of the choice of ω_g since functional $\tilde{F}_{\omega_g}(\,\cdot\,)$ satisfies a co-cycle condition (cf. section 1). The following was essentially proved in [TZ2].

Theorem 0.1 [TZ2]. *Suppose that $\tilde{F}_{\omega_g}(\,\cdot\,)$ is proper with respect to a holomorphic vector field X on M. Then there is a Kähler–Ricci soliton with respect to X on M.*

This theorem says that the properness of $\tilde{F}_{\omega_g}(\,\cdot\,)$ is a sufficient condition for the existence of Kähler–Ricci solitons. Since we did not state this result as a theorem in [TZ2], we will include its proof here following [TZ2].

In [TZ3], we proved that $\tilde{F}_{\omega_g}(\,\cdot\,)$ is bounded from below if there is a Kähler–Ricci soliton on M. The main purpose of this paper is to show that $\tilde{F}_{\omega_g}(\,\cdot\,)$ is actually proper wherever M admits a Kähler–Ricci soliton. This shows that $\tilde{F}_{\omega_g}(\,\cdot\,)$ is also a necessary condition for the existence of Kähler–Ricci solitons (cf. section 5). Suppose that M admits a Kähler–Ricci soliton ω_{KS} with respect to some holomorphic vector field X. We define a weighted inner product on $C^\infty(M)$ by

$$(\phi,\psi)=\int_M \phi\psi e^{\theta_X(\omega_{KS})}\omega_{KS}^n\,,$$

and denote by

$$\Lambda_1(M,\omega_{KS})=\left\{u\in C^\infty(M)\;\middle|\;\triangle_{\omega_{KS}}u+X(u)=-u\right\}.$$

Theorem 0.2. *Let M be a compact complex manifold which admits a Kähler–Ricci soliton ω_{KS} with respect to some holomorphic vector field X and $G(\supseteq K_X)$ be a compact subgroup of K_0 with $\sigma\cdot\eta=\eta\cdot\sigma$ for any $\sigma\in K_X$ and $\eta\in G$. Suppose that any G-invariant smooth function on M is perpendicular to the space $\Lambda_1(M,\omega_{KS})$ with respect to the weighted inner product defined above. Then there are two positive numbers c and C such that for any G-invariant ψ in $\mathcal{M}_X(\omega_g)$,*

$$\tilde{F}_{\omega_{KS}}(\psi)\ge cI_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}}-C\,. \tag{0.1}$$

In particular, $\tilde{F}_{\omega_{KS}}(\,\cdot\,)$ is proper under the same assumption.

The proof of Theorem 0.2 is inspired by [T] and [TZ1], where the authors proved a fully non-linear inequality of Moser–Trudinger type for Kähler–Einstein manifolds. In fact, inequality (0.1) in Theorem 0.2 is equivalent to the following inequality,

$$\int_M e^{-\psi}\omega_{KS}^n\le C\exp\left\{\tilde{J}_{\omega_{KS}}(\psi)-c\tilde{J}_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}}-\frac{1}{V}\int_M\psi\omega_{KS}^n\right\}. \tag{0.2}$$

Inequality (0.2) generalizes the corresponding one in [T] and [TZ1]. Recall that on S^2 with the standard Riemannian metric such an inequality is just a kind of the stronger version of Moser–Trudinger inequality [A], [T].

The organization of this paper is as follows. In section 1, we give a proof of Theorem 0.1 following [TZ2]. In section 2, we give an estimate for a certain heat kernel on manifolds with modified positive Ricci curvature. In section 3, a C^0-estimate for certain complex Monge–Ampère equations is obtained. In section 4, we prove a smoothing lemma by using Hamilton's Ricci flow. Theorem 0.2 will be proved in section 5.

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1 An Analytic Criterion for Kähler–Ricci Solitons

Let (M, g) be an n-dimensional compact Kähler manifold with the first Chern class $c_1(M) > 0$. Denote by $\mathrm{Aut}^\circ(M)$ the connected component of the automorphism group of M. Let K be a maximal compact subgroup of $\mathrm{Aut}^\circ(M)$, then the Chevalley decomposition allows us to write $\mathrm{Aut}^\circ(M)$ as a semi-direct decomposition [FM],

$$\mathrm{Aut}^\circ(M) = \mathrm{Aut}_r(M) \propto R_u\,,$$

where $\mathrm{Aut}_r(M) \subset \mathrm{Aut}^\circ(M)$ is a reductive algebraic subgroup and the complexification of K, and R_u is the unipotent radical of $\mathrm{Aut}^\circ(M)$. Let $\eta_r(M)$ be the Lie subalgebra of $\mathrm{Aut}_r(M)$.

Let $X \in \eta_r(M)$ such that the one-parameter subgroup K_X generated by $\mathrm{Im}(X)$ is contained in K, where $\mathrm{Im}(X)$ denotes the imaginary part of X. Choose a K_X-invariant Kähler metric g on M with the Kähler form ω_g. Then by the Hodge decomposition theorem, there is a unique smooth real-valued function $\theta_X = \theta_X(\omega_g)$ of M such that

$$\begin{cases} i_X\omega_g = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\theta_X\,, \\ \int_M e^{\theta_X}\omega_g^n = \int_M \omega_g^n = V\,, \end{cases}$$

where $\omega_g^n = \omega_g \wedge \ldots \wedge \omega_g$. The first relation above implies

$$L_X\omega_g = di_X(\omega_g) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\theta_X\,.$$

We consider the following complex Monge–Ampère equations with parameter $t \in [0, 1]$:

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}})\exp\left\{h - \theta_X - X(\varphi) - t\varphi\right\}, \\ (g_{i\bar{j}} + \varphi_{i\bar{j}}) > 0\,, \end{cases} \tag{1.1}$$

where $h = h_g$ is a smooth real-valued function on M defined by

$$\begin{cases} \mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}h\,, \\ \int_M e^h \omega_g^n = \int_M \omega_g^n\,. \end{cases}$$

Then one can check that $\omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$ is a Kähler–Ricci soliton with respect to X, i.e. ω_ϕ satisfies

$$\mathrm{Ric}(\omega_\phi) - \omega_\phi = L_X\omega_\phi\,,$$

if and only if ϕ modulo a constant is a solution of equation $(1.1)_t$ for $t = 1$. In fact, $(1.1)_t$ is equivalent to

$$\mathrm{Ric}(\omega_{\phi_t}) - L_X(\omega_{\phi_t}) = t\omega_{\phi_t} + (1-t)\omega_g\,, \tag{1.2}$$

where ϕ_t is a solution of $(1.1)_t$.

Since

$$\frac{d}{dt}\bigg(\int_M e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n\bigg) = \int_M \big(\triangle'\dot{\phi}_t + X(\dot{\phi}_t)\big)e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n = 0\,,$$

we have

$$\int_M e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n = \int_M e^{\theta_X}\omega_g^n = V\,. \tag{1.3}$$

Integrating $(1.1)_t$ after multiplying by $e^{\theta_X + X(\phi_t)}$, it follows that

$$\int_M e^{h - t\phi_t}\omega_g^n = V\,.$$

Therefore, differentiating the above identity, we get

$$\int_M \dot{\phi}_t e^{h - t\phi_t}\omega_g^n = -\frac{1}{t}\int_M \phi_t e^{h - t\phi_t}\omega_g^n\,. \tag{1.4}$$

Set

$$\mathcal{M}(\omega_g) = \Big\{\phi \in C^\infty(M) \;\Big|\; \omega_\phi = \omega_g + \tfrac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi > 0\Big\}$$

and

$$\mathcal{M}_X(\omega_g) = \big\{\phi \in \mathcal{M}(\omega_g) \;\big|\; \mathrm{Im}(X)(\phi) = 0\big\}\,.$$

We define the following two functionals on $\mathcal{M}_X(\omega_g)$ (cf. [TZ2]):

$$\tilde{I}_{\omega_g}(\phi) = \frac{1}{V}\int_M \phi\big(e^{\theta_X}\omega_g^n - e^{\theta_X + X(\phi)}\omega_\phi^n\big)$$

and

$$\tilde{J}_{\omega_g}(\phi) = \frac{1}{V}\int_0^1\int_M \dot{\phi}_s\big(e^{\theta_X}\omega_g^n - e^{\theta_X + X(\phi_s)}\omega_{\phi_s}^n\big)\wedge ds\,, \tag{1.5}$$

where $\phi_s (0 \le s \le 1)$ is a path from 0 to ϕ in $\mathcal{M}_X(\omega_g)$. Since $\tilde{J}_{\omega_g}(\phi)$ is independent of the choice of a path, by choosing $\phi_s = s\phi$, one can show that $\tilde{J}_{\omega_g}(\phi)$ defined here coincides with one in the introduction. It is known that there are two uniform positive constants c_1 and c_2 such that

$$c_1 I_{\omega_g}(\phi) \le \tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \le c_2 I_{\omega_g}(\phi)\,, \tag{1.6}$$

where $I_{\omega_g}(\phi) = \frac{1}{V}\int_M \phi(\omega_g^n - \omega_\phi^n)$.

Let

$$\begin{aligned}\hat{F}_{\omega_g}(\phi) &= \tilde{J}_{\omega_g}(\phi) - \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n \\ &= -\frac{1}{V}\int_0^1\int_M \dot{\phi}_s e^{\theta_X + X(\phi_s)}\omega_{\phi_s}^n \wedge ds\,.\end{aligned}$$

By simple computations, one can show that for any two functions ϕ and ψ in $\mathcal{M}_X(\omega_g)$, the following co-cycle condition is satisfied,

$$\hat{F}_{\omega_g}(\psi) = \hat{F}_{\omega_g}(\phi) + \hat{F}_{\omega_\phi}(\psi - \phi)\,,$$

where $\hat{F}_{\omega_\phi}(\psi-\phi) = -\frac{1}{V}\int_0^1\int_M \dot{\phi}_s e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n \wedge ds$ and ϕ_s is a path from 0 to $(\psi-\phi)$ in $\mathcal{M}_X(\omega_g)$.

PROPOSITION 1.1. *Let ϕ_s be a solution of $(1.1)_s$ for all $s \le t$. Then*

$$\hat{F}_{\omega_g}(\phi_t) = -\frac{1}{t}\int_0^t \big(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)\big)ds\,.$$

Proof. Differentiating $(1.1)_s$ on s, we have

$$\triangle'\dot{\phi}_s + X(\dot{\phi}_s) = -(s\dot{\phi}_s + \phi_s)\,. \tag{1.7}$$

Then by using (1.7) and (1.4), we get

$$\begin{aligned}\frac{d}{ds}\big(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)\big) &= -\frac{1}{V}\int_M \phi_s \frac{d}{ds}\big(e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n\big) \\ &= -\frac{1}{V}\int_M \phi_s\big(\triangle'\dot{\phi}_s + X(\dot{\phi}_s)\big)e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n \\ &= \frac{1}{V}\int_M \phi_s(s\dot{\phi}_s+\phi_s)e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n \\ &= \frac{1}{V}\int_M \phi_s(s\dot{\phi}_s+\phi_s)e^{h-s\phi_s}\omega_g^n \\ &= \frac{1}{V}\frac{d}{ds}\bigg(\int_M(-\phi_s)e^{h-s\phi_s}\omega_g^n\bigg) + \frac{1}{V}\int_M \dot{\phi}_s e^{h-s\phi_s}\omega_g^n \\ &= \frac{1}{sV}\frac{d}{ds}\bigg(\int_M s(-\phi_s)e^{h-s\phi_s}\omega_g^n\bigg) \\ &= \frac{1}{sV}\frac{d}{ds}\bigg(\int_M s(-\phi_s)e^{\theta_X+X(\phi_s)}\omega_{\phi_s}^n\bigg)\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\tfrac{d}{ds}\big(s(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s))\big) - \big(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)\big) \\ &\qquad\qquad = \frac{1}{V}\frac{d}{ds}\bigg(\int_M s(-\phi_s)e^{\theta_X(\phi_s)}\omega_{\phi_s}^n\bigg)\,,\end{aligned} \tag{1.8}$$

and consequently,

$$\hat{F}_{\omega_g}(\phi_t) = -\frac{1}{V}\int_M \phi e^{\theta_X + X(\phi_t)}\omega_t^n - \big(\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)\big)$$
$$= -\frac{1}{t}\int_0^t \big(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)\big)ds\,. \qquad \square$$

REMARK 1.1. *It was proved in Lemma 3.2 in [TZ2] that*

$$\tfrac{d}{ds}\big(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)\big) \geq 0\,.$$

Recall that the functional $\tilde{F}_{\omega_g}(\,\cdot\,)$ is defined as

$$\tilde{F}_{\omega_g}(\psi) = \hat{F}_{\omega_g}(\psi) - \log\left(\frac{1}{V}\int_M e^{h-\psi}\omega_g^n\right)$$
$$= \tilde{J}_{\omega_g}(\psi) - \frac{1}{V}\int_M \psi e^{\theta_X}\omega_g^n - \log\left(\frac{1}{V}\int_M e^{h-\psi}\omega_g^n\right),$$

and $\tilde{F}_{\omega_g}(\,\cdot\,)$ is called proper with respect to X if for any sequence $\{\psi_i\}$ of K_0-invariant functions in $\mathcal{M}_X(\omega_g)$,

$$\overline{\lim}_{i\to\infty}\tilde{F}_{\omega_g}(\psi_i) = +\infty\,,$$

whenever $\lim_{i\to\infty} I_{\omega_g}(\psi_i) = +\infty$, where $K_0(\supseteq K_X)$ is a maximal compact subgroup of automorphisms group $\mathrm{Aut}(M)$ of M such that $\sigma\cdot\eta = \eta\cdot\sigma$ for any $\sigma \in K_X$ and $\eta \in K_0$.

Proof of Theorem 0.1. We use arguments from [TZ2]. Assuming the functional $\tilde{F}_{\omega_g}$ is proper, we shall prove the existence of a Kähler–Ricci soliton with respect to X on M. This is equivalent to proving that there is a solution of $(1.1)_t$ for $t = 1$. It suffices to prove that $I_{\omega_g}(\phi_t)$ is uniformly bounded for any solution of $(1.1)_t$ for $0 \leq t \leq 1$. This is because C^3-norm of ϕ_t can be uniformly bounded by $I_{\omega_g}(\phi_t)$ and the set of parameter t for which there exists a smooth solution of $(1.1)_t$ is non-empty and open (cf. [TZ2]). By the implicit function theorem, one can show the solution of $(1.1)_t$ varies smoothly with $t < 1$. Without the loss of generality, we may assume that the Kähler form ω_g is K_0-invariant. Then all solutions ϕ_t are all K_0-invariant.

By Proposition 1.1, we have

$$\tilde{F}_{\omega_g}(\phi_t) = -\frac{1}{t}\int_0^t \big(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)\big)ds - \log\left(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n\right)$$
$$\leq -\log\left(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n\right). \tag{1.9}$$

On the other hand, by using $(1.1)_t$ and concavity of the logarithmic function, one can deduce

$$-\log\left(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n\right) = \frac{1-t}{V}\int_M \phi_t e^{\theta_X + X(\phi_t)}\omega_{\phi_t}^n \leq \frac{1-t}{V}\int_M \phi_t e^{h-t\phi_t}\omega_g^n \leq C\,. \tag{1.10}$$

Combining (1.9) and (1.10), we get

$$\tilde{F}_{\omega_g}(\phi_t) \leq C\,.$$

Therefore, the assumption of properness on $\tilde{F}_{\omega_g}(\,\cdot\,)$ implies

$$I_{\omega_g}(\phi_t) \leq C',$$

and consequently, there is a Kähler–Ricci soliton on M with respect to X. □

2 A Heat Kernel Estimate

In this section, we give an estimate on the heat kernel of a linear elliptic operator P associated to a Kähler form ω and a holomorphic vector field X on M, where $P = P_\omega = \triangle + X(\,\cdot\,)$ is defined on the space $\mathcal{N}_X = \{u \in C^\infty(M) \mid \mathrm{Im}(X(u)) = 0\}$. As a consequence, we derive a lower bound of the Green function of P. The method here follows that of T. Mabuchi in [M1] with modifications which in turn were inspired by Li–Yau [LY]. Note that P is a self-adjoint elliptic operator on $\mathcal{N}_X$ with respect to the inner product,

$$(\phi,\psi) = \int_M \phi\psi e^{\theta_X}\omega^n.$$

LEMMA 2.1. *Let ω be a Kähler form on M and X a holomorphic vector field on M with*

$$\mathrm{Ric}(\omega) - L_X\omega \geq 0\,,$$

and

$$\triangle\theta_X \leq k\,,$$

for some positive number k, where $\theta_X = \theta_X(\omega)$ is defined as in section 1 for Kähler form ω. Let $v(x,t)$ be a positive smooth solution on $M \times (0,\infty)$ of equation $\big(P - \frac{\partial}{\partial t}\big)v = 0$. Suppose that

$$\lim_{t\to 0}\sup_{M\times\{t\}} t\big(v^{-2}\langle\overline{\partial}v, \overline{\partial}v\rangle - v^{-1}v_t\big) \leq 2n\,,$$

where $v_t = \partial v/\partial t$. Then there is a positive number C depending only on

$$m_1 = -\max_M \theta_X \text{ and } m_2 = -\min_M \theta_X$$

such that

$$v(x,t_1) \le v(y,t_2)\big(\tfrac{t_2}{t_1}\big)^{\frac{n}{C}} \exp\big\{(t_2-t_1)^{-1} r(x,y)^2/2 + C^{-1}k(t_2-t_1)\big\}\,,\ \forall t_1<t_2\,, \tag{2.1}$$

where $r(x,y)$ denotes the distance between x and y associated to metric ω.

Proof. Let $f = \ln v$ and $\hat F = t(\langle \overline\partial f, \overline\partial f\rangle - f_t)$. Then

$$\begin{aligned} P\hat F + tPf_t &\\ &= t\big(\langle \mathrm{tr}_\omega(\overline\nabla\nabla)\overline\partial f, \overline\partial f\rangle + \langle \mathrm{tr}_\omega(\overline\nabla\nabla)\partial f, \partial f\rangle \\ &\quad + X(\langle\overline\partial f, \overline\partial f\rangle) + \|\nabla\overline\nabla f\|^2 + \|\nabla\nabla f\|^2\big)\,. \end{aligned} \tag{2.2}$$

For simplicity, we choose a local holomorphic coordinate system $(z_1,\dots,z_n)$ near each point p such that $g_{i\bar j}(p) = \delta_{ij}$ and $f_{i\bar j}(p) = \delta_{ij} f_{i\bar i}(p)$. By a direct computation, one sees that

$$f_{\bar i} f_{ij\bar j} = f_{\bar i}(f_{j\bar j i} + f_l R_{i\bar l})\,,$$
$$f_i f_{\bar i j\bar j} = f_i f_{\bar j j\bar i}\,,$$

and

$$X_{\bar i}(f_j f_{\bar j})_i + f_{\bar i} f_j X_{\bar j i} = f_i (X_{\bar j} f_j)_{\bar i} + f_{\bar i}(X_{\bar j} f_j)_i\,.$$

Inserting the above identities into (2.2), we obtain

$$\begin{aligned} P\hat F &\ge t\big(\langle \partial f, \partial(Pf)\rangle + \langle \partial(Pf), \partial f\rangle\big) - tPf_t + \tfrac{t}{n}(\triangle f)^2 \\ &= 2t\big\langle \partial(Pf), \partial f\big\rangle - tPf_t + \tfrac{t}{n}(\triangle f)^2. \end{aligned} \tag{2.3}$$

Since

$$Pf - f_t = -\langle \overline\partial f, \overline\partial f\rangle\,,$$

we have

$$\hat F = -tPf\,,$$

and

$$\tfrac{\partial}{\partial t}\hat F - \tfrac{1}{t}\hat F = -tPf_t\,.$$

Hence by (2.3), we get

$$\big(P - \tfrac{\partial}{\partial t}\big)\hat F \ge -2\langle \partial \hat F, \partial f\rangle - \tfrac{1}{t}\hat F + \tfrac{t}{n}\Big(\tfrac{1}{t}\hat F + X(f)\Big)^2. \tag{2.4}$$

Let $m_1 = -\sup_{x\in M}\theta_X(x)$ and $m_2 = -\inf_{x\in M}\theta_X(x)$. As in [M1], we define a monotone-increasing function η on $[m_1, m_2]$ by

$$\eta(s) = \exp\int_m^s \frac{1}{b_0 e^{-y} - 1}\,dy\,,$$

where $b_0 = e^{m_2}(1+n)$. Then η is a solution of the ODE on $[m_1, m_2]$,

$$\tfrac{\eta''}{\eta} - \tfrac{\eta'}{\eta} = 2\Big(\tfrac{\eta'}{\eta}\Big)^2.$$

Moreover, one can check that the number C defined by

$$C = \min_{s\in[m_1,m_2]} \eta(s)^{-1}\big(1 - (1 - \eta(s)^{-1}\eta(s)'n)^2\big)$$

is positive, and

$$0 < \frac{\eta'}{\eta} \le \frac{1}{n} \,.$$

Let $F = \eta(-\theta_X)\hat{F}$. Then by (2.4), one can show that

$$\begin{aligned}(P - \frac{\partial}{\partial t})F \ge & -2\langle \partial F, \partial f\rangle - \tfrac{1}{t}F \\ & + \left(\tfrac{\eta''}{\eta} - \tfrac{\eta'}{\eta} - 2\left(\tfrac{\eta'}{\eta}\right)^2\right)\|X\|^2 F \\ & + \eta^{-1}\eta'\big(- 2X(F) - F\triangle\theta_X\big) \\ & + n^{-1}t^{-1}F^2\eta^{-1}\big(1-(1-\eta^{-1}\eta' n)^2\big) \\ & + \tfrac{t}{n}\eta\left[-\tfrac{1}{t}\eta^{-1}F(1-n\eta^{-1}\eta')^2 - X(f)\right]^2 .\end{aligned}$$

By the assumption in the lemma, we have

$$\left(P - \tfrac{\partial}{\partial t}\right) F \ge -2\langle \partial F, \partial f\rangle - 2\eta^{-1}\eta' X(F) - \left(\tfrac{1}{t} + \tfrac{k}{n}\right) F + \tfrac{C}{nt}F^2. \qquad (2.5)$$

Applying the maximal principle to the function $F(x,t)$ on $M \times (0,T]$, we get from (2.5),

$$F(x,t) \le \tilde{C}^{-1}(2n + kt)\,,$$

for any $(x,t) \in M \times (0,T]$, and consequently,

$$v(x,t)^{-2}\big\langle\overline{\partial}v(x,t), \overline{\partial}v(x,t)\big\rangle - v(x,t)^{-1}v_t(x,t) \le \tilde{C}^{-1}\left(\tfrac{2n}{t} + k\right),$$

for any $(x,t) \in M \times (0,\infty)$. Now by integrating the above estimate as in [LY], we can immediately obtain (2.1). □

Let $H = H(x,y,t)$ be a fundamental solution on $M \times M \times [0,\infty)$ of equation

$$\left(P - \tfrac{\partial}{\partial t}\right) v(x,y,t) = 0\,, \qquad (2.6)$$

i.e. H is a smooth solution of (2.6) satisfying

$$\begin{cases} H(x,y.t) = H(y,x,t)\,, \\ H(x,y,t) = \int_M H(x,z,t-s)H(z,y.s)e^{\theta_X}\omega^n, \\ \lim_{t\to 0} H(x,y,t) = \delta_x(y)\,. \end{cases}$$

By using the asymptotic behavior of H, for each fixed $x \in M$, one sees that

$$\lim_{t\to 0}\sup_{M\times\{t\}} t\big(H^{-2}\langle\overline{\partial}H(x,\cdot,t), \overline{\partial}H(x,\cdot,t) > -H^{-1}H_t\big) \le 2n\,.$$

Moreover, following an argument by Li and Yau (cf. [LY, Lemma 3.2]), we can deduce

LEMMA 2.2. *Let Z_1 and Z_2 be two measurable subset of M. Let T, δ, τ be three positive numbers with $\tau < (1+2\delta)T$. For each $x, y \in M$ and*

$0 < t \le \tau$, *denote*

$$F_{x,T}(y,t) = \int_{Z_1} H(y,\cdot,t)H(x,\cdot,T)e^{\theta_X}\omega^n .$$

Then

$$\int_{Z_2} F_{x,T}(\cdot,t)^2 e^{\theta_X}\omega^n \le \exp\left\{\frac{-r(x,Z_1)^2}{2(1+2\delta)T} + \frac{R(x,Z_2)^2}{2(1+2\delta)T-2t}\right\} F_{x,T}(x,T)\,,$$

where $r(x,Z_1) = \inf_{z\in Z_1} r(x,z)$ *and* $R(x,Z_2) = \sup_{z\in Z_2} r(x,z)$.

Proposition 2.1. *Let* $H(x,y,t)$ *be a fundamental solution on* $M\times M\times[0,\infty)$ *of equation* (2.6). *Suppose*

$$\mathrm{Ric}(\omega) - L_X\omega \ge 0\,,$$

and

$$\triangle\theta_X \le k\,,$$

for some positive number k. *Then for any* $\delta > 0$ *we have*

$$\begin{aligned}
&\widetilde{\mathrm{vol}}\big(B_x(\sqrt{t})\big)^{1/2}\widetilde{\mathrm{vol}}\big(B_y(\sqrt{t})\big)^{1/2}H(x,y,t)\\
&\le (1+\delta)^{4n/C}\exp\left\{-\frac{\{r(x,y)-\sqrt{t}\}_+^2}{4t(1+3\delta+2\delta^2)} + t\delta(2+\delta)C^{-1}k + \frac{3}{4\delta(1+\delta)} + \frac{1}{2\delta}\right\},
\end{aligned} \tag{2.7}$$

where $\widetilde{\mathrm{vol}}(B_y(\sqrt{t})) = \int_{B_y(\sqrt{t})} e^{\theta_X}\omega^n$, *and*

$$\big\{r(x,y)-\sqrt{t}\big\}_+ = \max\big\{0, r(x,y)-\sqrt{t}\big\}\,.$$

Proof. First applying Lemma 2.1 to the function $F_{x,T}(y,t)$ with $(t_1,t_2) = (T,\tau = (1+\delta)T)$, we have

$$F_{x,T}(x,T) \le F_{x,T}(y,\tau)(1+\delta)^{\frac{2n}{C}}\exp\left\{\frac{T^{-1}\delta^{-1}r(x,y)^2}{2} + C^{-1}kT\delta\right\}. \tag{2.8}$$

Let $Z_1 = B_y(\sqrt{t})$ and $Z_2 = B_x(\sqrt{t})$. Then integrating the square of the above inequality over all $y \in Z_2$ and using Lemma 2.2, it follows that

$$\begin{aligned}
&\widetilde{\mathrm{vol}}\big(B_x(\sqrt{t})\big)F_{x,T}(x,T)^2\\
&\qquad \le (1+\delta)^{4n/C}\exp\left\{\frac{-r(x,Z_1)^2}{2(1+2\delta)T} + 2C^{-1}kT\delta + \frac{3t}{2T\delta}\right\}F_{x,T}(x,T)\,,
\end{aligned}$$

and consequently,

$$\begin{aligned}
&\widetilde{\mathrm{vol}}\big(B_x(\sqrt{t})\big)\int_{Z_1} H(x,\cdot,T)^2 e^{\theta_X}\omega^n\\
&\qquad = \widetilde{\mathrm{vol}}\big(B_x(\sqrt{t})\big)F_{x,T}(x,T)\\
&\qquad \le (1+\delta)^{4n/C}\exp\left\{\frac{-r(x,Z_1)^2}{2(1+2\delta)T} + 2C^{-1}kT\delta + \frac{3t}{2T\delta}\right\}.
\end{aligned} \tag{2.9}$$

On the other hand, applying Lemma 2.1 to the function $H(x,z,t)$ in z with $(t_1,t_2)=(t,T=(1+\delta)t)$, we have for any $x,y,z\in M$,

$$H(x,y,t)^2 \le (1+\delta)^{4n/C} H(x,z,T)^2 \exp\left\{T^{-1}\delta^{-1}r(y,z)^2 + 2\tilde{C}^{-1}kT\delta\right\}.$$

Integrating this inequality over all $z\in Z_1$, and using (2.9), we can get (2.7). □

Theorem 2.1. *Let $\phi\in\mathcal{M}_X(\omega_g)$. Suppose that*

$$\mathrm{Ric}(\omega_\phi) - L_X\omega_\phi \ge \lambda\omega_\phi\,, \tag{2.10}$$

and

$$\triangle\theta_X(\omega_\phi) \le k\,,$$

for some positive numbers λ and k. Then there is a uniform constant C depending only on λ and k such that

$$\sup_M(-\phi) \le \frac{1}{V}\int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n + C\,. \tag{2.11}$$

Proof. Let $\mu_i(\mu_0=0), i=0,1,\dots,$ be the increasing sequence of eigenvalues of operator $-P=-(\triangle_{\omega_\phi}+X(\cdot))$ associated to metric ω_ϕ. Then by using the standard Bochner technique, one can obtain $\mu_1\ge\lambda$ (cf. [TZ2]). Let $G(x,y)$ be the Green function with $\int_M G(x,\cdot)e^{\theta_X(\omega_\phi)}\omega_\phi^n=0$ associated to the operator P. Then

$$G(x,y) = \int_0^\infty \left(H(x,y,t)-\tfrac{1}{V}\right)dt\,.$$

Since

$$H_0(x,y,t) = H(x,y,t) - \frac{1}{V} = \sum_{i=1}^\infty e^{-\mu_i t} f_i(x)f_i(y)\,,$$

we have

$$H_0(x,x,t+t_0) \le e^{-\mu_1 t}H_0(x,x,t_0)\,, \tag{2.12}$$

for any $t_0,t>0$, where $f_i(x)$ denote the eigenfunctions of μ_i.

In [M2], it was proved under the condition (2.10) that there is a uniform constant C_1 such that

$$\mathrm{Diam}(M,\omega_\phi) \le \tfrac{C_1}{\sqrt{\lambda}}\,.$$

Choose $t_0=\frac14\mathrm{Diam}(M,\omega_\phi)^2$. Then by Proposition 2.1 we have

$$H_0(x,x,t_0) \le C_2\,, \tag{2.13}$$

for some uniform constant C_2 depending only on λ and k.

By using (2.12) and (2.13), we get

$$\begin{aligned} G(x,y) &\ge -\int_0^{t_0}\frac{1}{V}dt - \int_{t_0}^\infty e^{-\mu_1(t-t_0)}\big(H_0(x,x,t_0)H_0(y,y,t_0)\big)^{1/2}dt \\ &\ge -C_3\,. \end{aligned}$$

Note that $\triangle_{\omega_\phi}(-\phi) \geq -n$ and

$$\sup_{\psi \in \mathcal{M}_X(\omega_g)} \|X(\psi)\|_{C^0(M)} \leq c \tag{2.14}$$

for some uniform constant $c = c(\omega_g, X)$ (cf. Lemma 5.1 in [Z]). Therefore applying the Green formula to function $-\phi$, we prove

$$\begin{aligned}
\sup_M(-\phi) &= \frac{1}{V}\int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n - \inf_M \int_M P\big(-\phi(\cdot)\big)G(x,\cdot)e^{\theta_X(\omega_\phi)}\omega_\phi^n \\
&= \frac{1}{V}\int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n - \inf_M \int_M \big(\triangle_{\omega_\phi}(-\phi) - X(\phi)\big)G(x,\cdot)e^{\theta_X(\omega_\phi)}\omega_\phi^n \\
&\leq \frac{1}{V}\int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n + C_3 V\big(n + \|X(\phi)\|_{C^0(M)}\big) \\
&\leq \frac{1}{V}\int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n + C\,. \qquad \square
\end{aligned}$$

3 A C^0-Estimate for Solutions of $(1.1)_t$

LEMMA 3.1. *There are two positive numbers c_1 and $c_2 < 1$ such that for any $\phi \in \mathcal{M}_X(\omega_g)$,*

$$c_1 \tilde{I}_{\omega_g}(\phi) \leq \tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \leq c_2 \tilde{I}_{\omega_g}(\phi)\,. \tag{3.1}$$

Proof. Let $\omega_{s\phi} = \omega + s\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$. Then one can compute

$$\begin{aligned}
&\tilde{I}_{\omega_g}(\phi) \\
&\quad = \frac{1}{V}\int_M \phi \int_0^1 \frac{d}{ds}\big(e^{\theta_X + sX(\phi)}\omega_{s\phi}^n\big) \wedge ds \\
&\quad = \frac{n\sqrt{-1}}{2\pi V}\int_0^1 ds \int_M \partial\phi \wedge \overline{\partial}\phi e^{\theta_X + sX(\phi)} \wedge \omega_{s\phi}^{n-1} \\
&\quad = \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 s^k(1-s)^{n-1-k}e^{\theta_X + sX(\phi)}ds\right) \\
&\qquad\quad \times \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \\
&\quad \leq \frac{n\sqrt{-1}}{2\pi V} C_1 \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \overline{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}.
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\tilde{J}_{\omega_g}(\phi) \\
&\quad = \frac{n\sqrt{-1}}{2\pi V}\int_0^1 dt \int_0^1 ds \int_M t\partial\phi \wedge \overline{\partial}\phi e^{\theta_X + stX(\phi)} \wedge \omega_{st\phi}^{n-1}
\end{aligned}$$

$$= \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1\int_0^1 t(st)^k(1-st)^{n-1-k}e^{\theta_X+stX(\phi)}dt\wedge ds\right) \times \partial\phi\wedge\overline{\partial}\phi\wedge\omega_\phi^k\wedge\omega_g^{n-1-k} \tag{3.3}$$

$$\geq \frac{n\sqrt{-1}}{2\pi V}C_1'\sum_{k=0}^{n-1} C_{n-1}^k\partial\phi\wedge\overline{\partial}\phi\wedge\omega_\phi^k\wedge\omega_g^{n-1-k}.$$

Combining (3.2) and (3.3), we get

$$\tilde{J}_\omega(\phi) \geq \tfrac{C_1'}{C_1}\tilde{I}_\omega(\phi),$$

and consequently, prove the second inequality of (3.1).

On the other hand, we have (cf. [TZ2]),

$$\tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) = \frac{n\sqrt{-1}}{2\pi V}\sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 s^{k+1}(1-s)^{n-1-k}e^{\theta_X+sX(\phi)}ds\right) \times \partial\phi\wedge\overline{\partial}\phi\wedge\omega_\phi^k\wedge\omega_g^{n-1-k} \tag{3.4}$$

$$\geq \frac{n\sqrt{-1}}{2\pi V}C_2\sum_{k=0}^{n-1} C_{n-1}^k\partial\phi\wedge\overline{\partial}\phi\wedge\omega_\phi^k\wedge\omega_g^{n-1-k}.$$

Hence, combining (3.2) and (3.4), we also prove the first inequality of (3.1). □

PROPOSITION 3.1. *Let $\phi = \phi_t (t \geq t_0 > 0)$ be a solution of equation $(1.1)_t$. Then there are two uniform constants C_1 and C_2 depending only on X and t_0 such that*

$$\operatorname{osc}_M \phi \leq C_1 \int_M \phi(\omega_g^n - \omega_\phi^n) + C_2 . \tag{3.5}$$

Proof. Let $\theta_X' = \theta_X(\omega_\phi)$. First we note

$$\triangle\theta_X' = -\theta_X' - X(h_{\omega_\phi}) + c,$$

for some constant c. Clearly $c \leq \operatorname{osc}_M |\theta_X'|$, since θ_X' changes the sign. By using the fact (cf. the relation (1.2)),

$$h_{\omega_\phi} = \theta_X' - (1-t)\phi + \text{const.},$$

we have

$$\begin{aligned}\triangle\theta_X' &= -\theta_X' - \|X\|_{\omega_\phi} + (1-t)X(\phi) + c\\ &\leq 2\operatorname{osc}_M|\theta_X| + 3|X(\phi)| \leq C_1',\end{aligned}$$

for some uniform constant C_1'. Applying Theorem 2.1, we see that there is some uniform constant C_2' depending only on X and t_0 such that

$$\sup_M(-\phi) \leq \frac{1}{V}\int_M (-\phi)e^{\theta_X'}\omega_\phi^n + C_2'. \tag{3.6}$$

On the other hand, by using the Green formula, we have

$$\sup_M \phi \leq \frac{1}{V}\int_M \phi e^{\theta_X}\omega_g^n + C_3' \tag{3.7}$$

for some uniform constant C_3' (cf. Lemma 5.3 in [TZ3]). Hence, combing (3.6) and (3.7), we get

$$\operatorname{osc}_M \phi \leq \frac{1}{V}\int_M \phi(e^{\theta_X}\omega_g^n - e^{\theta_X'}\omega_\phi^n) + C_2' + C_3' .$$

By using (1.6) and (3.1), we prove (3.5). □

4 A Smoothing Lemma

In this section, following an approach in [T], we will prove a smoothing lemma by using Hamilton's Ricci flow. This lemma will be used in the proof of Theorem 0.2. Let ω be any Kähler form in $c_1(M) > 0$ such that

$$\begin{cases} \operatorname{Ric}(\omega) - L_X\omega \geq (1-\epsilon)\omega , \\ \big|X(h_\omega - \theta_X(\omega))\big| \leq \epsilon c_1 , \end{cases} \tag{4.1}$$

for some constant c_1 and $0 < \epsilon < 1$. We consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \log\Big(\frac{(\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial u)^n}{\omega^n}\Big) + u - h_\omega + \theta_X(\omega) , \\ u|_{t=0} = 0 . \end{cases} \tag{4.2}$$

Note that eq. (4.2) is the scalar version of the modified Kähler–Ricci flow

$$\tfrac{\partial}{\partial t}\omega_t = -\operatorname{Ric}(\omega_t) + \omega_t + L_X\omega_t .$$

Here $\omega_t = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_t$, and $u_t = u(x,\cdot)$. Denote by $h_t = h_{\omega_t}$ and $\theta_t = \theta_X(\omega_t)$, then it follows from the above equation and maximum principle that

$$h_t - \theta_t = -\tfrac{\partial u}{\partial t} + \tilde c_t$$

where $\tilde c_t$ depends on t only. Also, $u_0 = 0$ and hence $\tilde c_0 = 0$.

We list a few basic estimates for the solution $u(x,t)$. Differentiating (4.2), we get

$$\tfrac{\partial}{\partial t}\big(\tfrac{\partial u}{\partial t}\big) = (\Delta + X)\big(\tfrac{\partial u}{\partial t}\big) + \left(\frac{\partial u}{\partial t}\right) . \tag{4.3}$$

Applying the maximum principle, we have

LEMMA 4.1. *Let u_t be a solution of* (4.2). *Then*

$$\|u_t\|_{C^0} \leq e^t\big\|h_\omega - \theta_X(\omega)\big\|_{C^0} ,$$

and

$$\big\|\tfrac{\partial u}{\partial t}\big\|_{C^0} \leq e^t\big\|h_\omega - \theta_X(\omega)\big\|_{C^0} .$$

LEMMA 4.2.

$$(\Delta + X)(h_t - \theta_t) \geq -(c_1 + n)\epsilon e^t.$$

Proof. From eq. (4.3), we have

$$\begin{aligned}\frac{\partial}{\partial t}\left((\Delta + X)\left(\frac{\partial u}{\partial t}\right)\right) &= (\Delta + X)^2\left(\frac{\partial u}{\partial t}\right) + (\Delta + X)\left(\frac{\partial u}{\partial t}\right) - \left|\nabla\bar{\nabla}\left(\frac{\partial u}{\partial t}\right)\right|^2 \\ &\leq (\Delta + X)^2\left(\frac{\partial u}{\partial t}\right) + (\Delta + X)\left(\frac{\partial u}{\partial t}\right).\end{aligned}$$

It follows from the maximum principle that

$$(\Delta + X)(h_t - \theta_t) \geq e^t \inf_M (\Delta + X)\big(h_\omega - \theta_X(\omega)\big).$$

On the other hand, (4.1) implies that, at $t = 0$,

$$(\Delta + X)(h_\omega - \theta_X(\omega)) \geq -(c_1 + n)\epsilon.$$

Then the lemma follows directly. □

LEMMA 4.3.

$$\left\|\frac{\partial}{\partial t}u\right\|_{C^0}^2 + t\left\|\nabla\left(\frac{\partial}{\partial t}u\right)\right\|_{C^0}^2 \leq e^{2t}\left\|h_\omega - \theta_X(\omega)\right\|_{C^0}^2.$$

Proof. By direct computations, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 = (\Delta + X)\left(\frac{\partial u}{\partial t}\right)^2 - \left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2 + 2\left(\frac{\partial u}{\partial t}\right)^2,$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2\right) \\ &\quad = (\Delta + X)\left(\left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2\right) - \left|\nabla\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2 - \left|\nabla\bar{\nabla}\left(\frac{\partial u}{\partial t}\right)\right|^2 + \left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + t\left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2\right) \\ &\quad \leq (\Delta + X)\left(\left(\frac{\partial u}{\partial t}\right)^2 + t\left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2\right) + 2\left(\left(\frac{\partial u}{\partial t}\right)^2 + t\left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2\right).\end{aligned}$$

Lemma 4.3 follows from the maximum principle again. □

Set

$$v = h_1 - \theta_1 - \frac{1}{V}\int_M (h_1 - \theta_1)e^{\theta_1}{\omega_1}^n.$$

LEMMA 4.4.

$$\|v\|_{L^2}^2 \leq \frac{2(c_1 + n)e^2 V}{\lambda_1}\epsilon\left\|h_\omega - \theta_X(\omega)\right\|_{C^0}.$$

Proof. By Lemma 4.2, we have

$$(\Delta + X)v + (c_1 + n)e\epsilon \geq 0.$$

It follows that

$$\begin{aligned}\int_M \left|(\Delta + X)v + (c_1 + n)e\epsilon\right|e^{\theta_1}{\omega_1}^n &= \int_M \big((\Delta + X)v + (c_1 + n)e\epsilon\big)e^{\theta_1}{\omega_1}^n \\ &= (c_1 + n)e\epsilon V.\end{aligned}$$

Hence, by applying the Poincaré inequality and Lemma 4.1, we have

$$\begin{aligned}\frac{\lambda_1}{V}\int_M |v|^2 e^{\theta_1}{\omega_1}^n &\le \frac{1}{V}\int_M |\nabla v|^2 e^{\theta_1}\omega_1^n\\ &= \frac{1}{V}\int_M (-v)(\Delta+X)ve^{\theta_1}{\omega_1}^n\\ &= \frac{1}{V}\int_M (-v)\big[(\Delta+X)v+c_1e\epsilon\big]e^{\theta_1}{\omega_1}^n\\ &\le \frac{1}{V}\|v\|_{C^0}\int_M \big((\Delta+X)v+(c_1+n)e\epsilon\big)e^{\theta_1}{\omega_1}^n\\ &\le 2e^2(c_1+n)\epsilon\big\|h_\omega-\theta_X(\omega)\big\|_{C^0}\,.\end{aligned}$$

This shows the lemma is true. □

Lemma 4.5. *We have*

$$\|v\|_{C^0}\le C(n,c_1a,\lambda_1)\big(1+\|h_\omega-\theta_X(\omega)\|_{C^0}\big)\epsilon^{1/2(n+1)}, \tag{4.4}$$

provided that the following condition holds: there exists a constant $a>0$ such that for any $x_0\in M$ and $0<r<1$,

$$\big(B_r(x_0)\big)\ge ar^{2n} \tag{4.5}$$

with respect to the metric ω_1.

Proof. Pick $r=\epsilon^{1/2(n+1)}$ and cover M by geodesic balls of radius r. For any $x\in M$, we have $x\in B_r(x_0)$ for some $x_0\in M$. Now

$$\begin{aligned}\inf_{B_r(x_0)}|v|^2\epsilon^{\frac{n}{n+1}} &\le \frac{1}{a}\int_{B_r(x_0)}|v|^2e^{\theta_1}{\omega_1}^n\\ &\le \frac{2(c_1+n)e^2V^2}{a\lambda_1}\epsilon\big\|h_\omega-\theta_X(\omega)\big\|_{C^0}\,.\end{aligned}$$

Hence

$$\inf_{B_r(x_0)}|v|\le C(n,c_1,a,\lambda_1)\epsilon^{1/2(n+1)}\big\|h_\omega-\theta_X(\omega)\big\|_{C^0}^{1/2}.$$

Assuming $\inf_{B_r(x_0)}|v|=v(x_0')$, then

$$\begin{aligned}|v(x)| &\le \big|v(x)-v(x_0')\big|+v(x_0')\le r\sup_{B_r(x_0)}|\nabla v|+v(x_0')\\ &\le e\big\|h_\omega-\theta_X(\omega)\big\|_{C^0}\epsilon^{1/2(n+1)}+C\epsilon^{1/2(n+1)}\big\|h_\omega-\theta_X(\omega)\big\|_{C^0}^{1/2}\\ &\le C'\big(1+\|h_\omega-\theta_X(\omega)\|_{C^0}\big)\epsilon^{1/2(n+1)}.\end{aligned}$$

This finishes the proof of Lemma 4.5. □

Proposition 4.1 (Smoothing lemma). *Let $\omega\in c_1(M)>0$ be any Kähler metric satisfying* (4.1). *Then there is another Käher form $\omega'=\omega+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u$ such that*

(1) $\|u\|_{C^0} \le e\|h_\omega - \theta_X(\omega)\|$

(2) $\|h' - \theta'\|_{C^{1/2}} \le C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{4(n+1)}}$, *where* $C(n, c_1, a, \lambda_1)$ *is a constant depending only on the dimension* n, *the Poincare constant* λ_1, *constants* c_1 *and* a *appeared in* (4.5).

Proof. We shall prove that ω_1 satisfies the above two conditions of the proposition under the assumption (4.5). By Lemma 4.1, it suffices to check the second condition only.

Since

$$\frac{1}{V}\int_M e^{h'-\theta'} e^{\theta_1} {\omega_1}^n = 1\,,$$

by (4.4) in Lemma 4.5, we have

$$\|h' - \theta'\|_{C^0} \le C(n, c_1, a, \lambda_1)\big(1 + \|h_\omega - \theta_X(\omega)\|_{C^0}\big)\epsilon^{1/2(n+1)}. \tag{4.6}$$

For any two points x, y in M, if the distance $d(x, y) \le \epsilon^{1/2(n+1)}$, then Lemma 4.3 implies that $|\nabla(h' - \theta')| \le e||h_\omega - \theta_X(\omega)||_{C^0}$ and hence

$$\frac{|(h' - \theta')(x) - (h' - \theta')(y)|}{\sqrt{d(x, y)}} \le e\big\|h_\omega - \theta_X(\omega)\big\|_{C^0}\epsilon^{1/4(n+1)}.$$

On the other hand, if $d(x, y) \ge \epsilon^{1/2(n+1)}$ then (4.6) implies that

$$\frac{|(h' - \theta')(x) - (h' - \theta')(y)|}{\sqrt{d(x, y)}} \le C(n, c_1, a, \lambda_1)\big(1 + \|h_\omega - \theta_X(\omega)\|_{C^0}\big)\epsilon^{1/4(n+1)}.$$

This completes the proof of Proposition 4.1. □

5 Properness of $\tilde{F}_{\omega_{KS}}(\psi)$

We are now ready to prove Theorem 0.2 stated in the introduction. The method here is analogous to one in [T] for Kähler–Einstein manifolds with positive scalar curvature.

Proof of Theorem 0.2. Let ω_{KS} be the Kähler form of Kähler–Ricci soliton g_{KS} and $\omega_g = \omega_\psi = \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi$. We consider the complex Monge–Ampère equations with parameter $t \in [0, 1]$:

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\left\{h_g - \theta_X - X(\varphi) - t\varphi\right\}, \\ (g_{i\bar{j}} + \varphi_{i\bar{j}}) > 0\,. \end{cases} \tag{5.1}$$

Clearly, $-\psi$ modulo a constant is a solution of $(5.1)_t$ for $t = 1$. Since ψ is G-invariant, by the implicit function theorem, there are G-invariant solutions of $(5.1)_t$ for t sufficiently close to 1. In fact, in [TZ2], it was proved that there are G-invariant solutions of $(5.1)_t$ for any $t \in [0, 1]$. This is because

$\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)$ is nondecreasing in t (cf. Remark 1.1 in section 1), and consequently the C^3-norm of φ_t can be uniformly bounded (cf. [TZ2], [Y]).

Put $\omega_t = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_t$. Then $\omega_1 = \omega_{KS}$. Moreover, by (1.2), we have

$$\begin{cases} h_{\omega_t} - \theta_X(\omega_t) = -(1-t)\varphi_t + c_t\,, \\ \mathrm{Ric}(\omega_t) - L_X(\omega_t) = t\omega_t + (1-t)\omega \geq t\omega_t\,, \end{cases}$$

where c_t is determined by

$$\int_M e^{-(1-t)\varphi_t + c_t} e^{\theta_X(\omega_t)}\omega_t^n = V\,.$$

In particular,

$$|c_t| \leq (1-t)\|\varphi_t\|_{C^0(M)}\,, \quad \big\|h_{\omega_t} - \theta_X(\omega_t)\big\|_{C^0(M)} \leq 2(1-t)\|\varphi_t\|_{C^0(M)}\,,$$

and

$$\begin{aligned} \big|X(h_{\omega_t} - \theta_X(\omega_t))\big| &= \big|(1-t)X(\varphi_t)\big| \\ &\leq (1-t)\big(|X(\varphi_t - \varphi_1)| + |X(\psi)|\big) \leq (1-t)c_1(\omega_{KS}, X)\,. \end{aligned}$$

Hence by applying Proposition 4.1 to each ω_t, we obtain a modified Kähler form $\omega_t' = \omega_t + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u_t$ satisfying

$$\|u_t\|_{C^0(M)} \leq 2e(1-t)\|\varphi_t\|_{C^0(M)}\,,$$

$$\|h_{\omega_t'}\|_{C^{1/2}} \leq C(n, c_1, a, \lambda_1)\big(1 + \|(1-t)\varphi_t\|_{C^0(M)}\big)(1-t)^{1/4(n+1)}.$$

As before, there are $\tilde{\psi}_t$ such that $\omega_{KS} = \omega_t' + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\tilde{\psi}_t$ and

$$\omega_{KS}^n = (\omega_t')^n e^{h_{\omega_t'} - \theta_X(\omega_t') - X(\tilde{\psi}_t) - \tilde{\psi}_t}\,.$$

It follows from the maximum principle that

$$\varphi_t = \varphi_1 - \tilde{\psi}_t + \mu_t\,, \tag{5.2}$$

where μ_t are constants with

$$|\mu_t| \leq 2(e+1)(1-t)\|\varphi_t\|_{C^0(M)} + c_1(\omega_{KS}, X)\,. \tag{5.3}$$

Hence, φ_t is uniformly equivalent to φ_1 as long as $\tilde{\psi}_t$ is uniformly bounded. Consider the operator $\Phi_t : C^{2,1/2}(M) \to C^{0,1/2}$ by

$$\Phi_t(\tilde{\psi}) = \log\left(\frac{\big(\omega_{KS} - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\tilde{\psi}\big)^n}{\omega_{KS}^n}\right) + h_{\omega_t'} - \theta_X(\omega_t') - X(\tilde{\psi}) - \tilde{\psi}\,.$$

Its linearization at $\tilde{\psi} = 0$ is $(-\triangle - 1 - X(\,\cdot\,))$, so it is invertible in the space of G-invariant functions by the assumption of Theorem 0.2. Then by the implicit function theorem, there is a $\delta > 0$, such that if the Hölder norm $\|h_{\omega_t'}\|_{C^{1/2}(\omega_{KS})}$ with respect to ω_{KS} is less than δ, then there is a unique $\tilde{\psi}_t$ such that $\Phi_t(\tilde{\psi}_t) = 0$ and $\|\tilde{\psi}_t\|_{C^{2,1/2}} \leq C(\delta)$.

We observe that

$$\lambda_{1,\omega'} \geq 2^{-n-1}\lambda_{1,\omega_{KS}}\,, \quad a \geq \tfrac{1}{2^{2n}}a_0\,,$$

whenever $\frac{1}{2}\omega_{KS} \leq \omega' \leq 2\omega_{KS}$, where a is a constant appeared in (4.5) and a_0 is a constant such that

$$a_0 r^{2n} \leq_{\omega_{KS}} \big(B_r(x)\big)\,, \quad \forall\, x \in M\,.$$

Now we choose t_0 such that $(1-t_0) \leq (\delta/4C_0)^{4(n+1)}$ and

$$(1-t_0)\|\varphi_{t_0}\|_{C^0(M)}(1-t_0)^{1/4(n+1)} = \tfrac{\delta}{4C_0}\,, \tag{5.4}$$

where $C_0 = C_0(n, c_1, a_0, \lambda_{1,\omega_{KS}})$. Then by Proposition 4.1 and the above argument, one can prove that for any $t \geq t_0$, we have

$$\|u_t\|_{C^0(M)} \leq 2e(1-t)\|\varphi_t\|_{C^0}\,,\ \|\tilde{\psi}_t\|_{C^0(M)} \leq \tfrac{1}{4}\,.$$

Therefore, by using (5.2) and (5.3), we get

$$\|\varphi_1 - \varphi_t\|_{C^0(M)} \leq 6e(1-t)\|\varphi_t\|_{C^0(M)} + c_2\,, \tag{5.5}$$

and

$$\tfrac{1}{2}\|\varphi_1\|_{C^0} - c_2 \leq \|\varphi_t\|_{C^0(M)} \leq 2\|\varphi_1\|_{C^0(M)} + c_3\,, \tag{5.6}$$

for some uniform constants c_2 and c_3, as long as $1-t \leq \min\{1/12e, 1-t_0\}$.

Since $\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)$ is nondecreasing, by (3.1), we have

$$\begin{aligned}
\tilde{F}_{\omega_{KS}}(\psi) &= -\tilde{F}_{\omega_g}(\varphi_1)\\
&= \int_0^1 \big(\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)\big)dt\\
&\geq (1-t)\big(\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)\big)\\
&\geq \varepsilon(1-t)\tilde{J}_{\omega_g}(\varphi_t)\,,
\end{aligned}$$

where $\varepsilon > 0$ is a uniform constant. On the other hand, by using the co-cycle condition of $\hat{F}_{\omega_g}(\,\cdot\,)$, we have

$$\begin{aligned}
\tilde{J}_{\omega_g}(\varphi_t) &= \tilde{J}_{\omega_g}(\varphi_1) - \frac{1}{V}\int_M (\varphi_1 - \varphi_t)e^{\theta_X}\omega_g^n + \hat{F}_{\varphi_1}(\varphi_t - \varphi_1)\\
&\geq \tilde{J}_{\omega_g}(\varphi_1) - \frac{1}{V}\int_M (\varphi_1 - \varphi_t)(e^{\theta_X}\omega_g^n - e^{\theta_X + X(\varphi_1)}\omega_{\varphi_1}^n)\\
&\geq \tilde{J}_{\omega_g}(\varphi_1) - \mathrm{osc}_M(\varphi_1 - \varphi_t)\,.
\end{aligned}$$

It follows by (1.6) and (3.1),

$$\tilde{J}_{\omega_g}(\varphi_t) \geq \varepsilon' I_{\omega_g}(\varphi_1) - \mathrm{osc}_M(\varphi_1 - \varphi_t)\,.$$

Thus by (5.5) and (5.6), we get

$$\begin{aligned}
\tilde{F}_{\omega_{KS}}(\psi) &\geq \varepsilon\varepsilon'(1-t)I_{\omega_g}(\varphi_1) - (1-t)\,\mathrm{osc}_M(\varphi_t - \varphi_1)\\
&\geq \varepsilon\varepsilon'(1-t)I_{\omega_g}(\varphi_1) - 12e(1-t)^2\|\varphi_1\|_{C^0(M)} - C_1
\end{aligned} \tag{5.7}$$

$$= \varepsilon\varepsilon'(1-t)I_{\omega_{KS}}(\psi) - 12e(1-t)^2 \operatorname{osc}_M \psi - C_1 , \quad \text{for any } t \geq t_0 .$$

Therefore, in case

$$\operatorname{osc}_M \psi \leq \tilde{C}\big(1 + I_{\omega_{KE}}(\psi)\big)$$

for some uniform constant $\tilde{C}$, then by choosing $t{=}t_0$ in (5.7) and using (5.4), we see that there are two positive numbers C and C' such that

$$\tilde{F}_{\omega_{KS}}(\psi) \geq C I_{\omega_{KS}}(\psi)^{1/4n+5} - C', \tag{5.8}$$

and consequently this would prove the theorem.

In the general case, we shall use a trick in [TZ1]. First by Proposition 3.1, we have for any $t \geq 1/2$,

$$\operatorname{osc}_M(\varphi_t - \varphi_1) \leq C_2\big(1 + I_{\omega_{KS}}(\varphi_t - \varphi_1)\big) .$$

Set $\psi' = \varphi_t - \varphi_1$. Then applying inequality (5.8) to function ψ', we get

$$\begin{aligned} \tilde{F}_{\omega_g}(\varphi_t) - \tilde{F}_{\omega_g}(\varphi_1) &= \tilde{F}_{\omega_{KS}}(\psi') \\ &\geq C_3 I_{\omega_{KS}}(\psi')^{1/4n+5} - C_4 . \end{aligned} \tag{5.9}$$

On the other hand, by integrating (1.8) from t to 1, we have

$$\begin{aligned} &\hat{F}_{\omega_g}(\varphi_1) - \hat{F}_{\omega_g}(\varphi_t) \\ &\geq \tilde{J}_{\omega_g}(\varphi_1) - \frac{1}{V}\int_M \varphi_1 e^{\theta_X}\omega_g^n - t\left(\tilde{J}_{\omega_g}(\varphi_t) - \frac{1}{V}\int_M \varphi_t e^{\theta_X}\omega_g^n\right) \\ &\geq -(1-t)\big(\tilde{I}_{\omega_g}(\varphi_1) - \tilde{J}_{\omega_g}(\varphi_1)\big) \\ &\geq -C_5(1-t)I_{\omega_g}(\varphi_1) = -C_5(1-t)I_{\omega_{KS}}(\psi) . \end{aligned} \tag{5.10}$$

By using the concavity of the logarithmic function and (3.6), we also have

$$\begin{aligned} -\log\left(\frac{1}{V}\int_M e^{h-\varphi_t}\omega_g^n\right) &\leq \frac{1-t}{V}\int_M \varphi_t e^{\theta_X(\omega_t)}\omega_{\varphi_t}^n \\ &\leq -\frac{1-t}{V}\sup_M(-\varphi_t) + C_6 \leq C_6 . \end{aligned} \tag{5.11}$$

Hence combining (5.10) and (5.11), we get

$$\tilde{F}_{\omega_g}(\varphi_t) - \tilde{F}_{\omega_g}(\varphi_1) \leq C_5(1-t)I_{\omega_{KS}}(\psi) + C_6 . \tag{5.12}$$

From (5.9) and (5.12), we deduce

$$(1-t)I_{\omega_{KS}}(\psi) \geq c_3 \operatorname{osc}_M(\varphi_t - \varphi_1)^{1/4n+5} - c_4 .$$

Then as in (5.7), we prove (cf. [TZ1]),

$$\begin{aligned} &\tilde{F}_{\omega_{KS}}(\psi) \\ &\geq \varepsilon\varepsilon'(1-t)I_{\omega_{KS}}(\psi) - (1-t)\operatorname{osc}_M(\varphi_t - \varphi_1) \\ &\geq \varepsilon\varepsilon'(1{-}t)I_{\omega_{KS}}(\psi) - (1{-}t)(c_3^{-1})^{4n+5}\big((1{-}t)I_{\omega_{KS}}(\psi){+}c_4\big)^{4n+5} \\ &\geq cI_{\omega_{KS}}(\psi)^{1/4n+5} - C , \end{aligned} \tag{5.13}$$

for some small positive number c and large number C. Thus Theorem 0.2 is proved. □

REMARK 5.1. *By* (1.6) *and* (3.1), *we see that the inequality* (5.13) *is equivalent to the following non-linear inequality of Moser–Trudinger type,*

$$\int_M e^{-\psi}\omega_{KS}^n \le C\exp\left\{\tilde J_{\omega_{KS}}(\psi) - c\tilde J_{\omega_{KS}}(\psi)^{1/4n+5} - \frac{1}{V}\int_M \psi\omega_{KS}^n\right\}$$

for some positive numbers c *and* C.

REMARK 5.2. *In view of results in [T] and [TZ1], one should be able to generalize Theorem 0.2 by proving the following:*

$$\tilde F_{\omega_{KS}}(\psi) \ge cI_{\omega_{KS}}(\psi)^{1/4n+5} - C$$

holds for any $\psi \in \Lambda_1(M,\omega_{KS})^\perp$. *In the proof of Theorem 0.2, we used a technical assumption on subgroup* $G(\subseteq K_0)$ *in order to apply the implicit function theorem. We also notice from Theorem 0.2 that* (5.13) *holds for any almost plurisubharmonic function on a Kähler–Einstein manifold without any nontrivial holomorphic vector field.*

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