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COMPUTING THE N−BODY PROBLEM

Some Computational N−Body Techniques in Astronomy

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OUTLINE

1. Introduction

2. Algorithmic issues
   - Force evaluation
   - Particle pushing
   - Data analysis

3. Hardware issues

4. Numerical analysis issues

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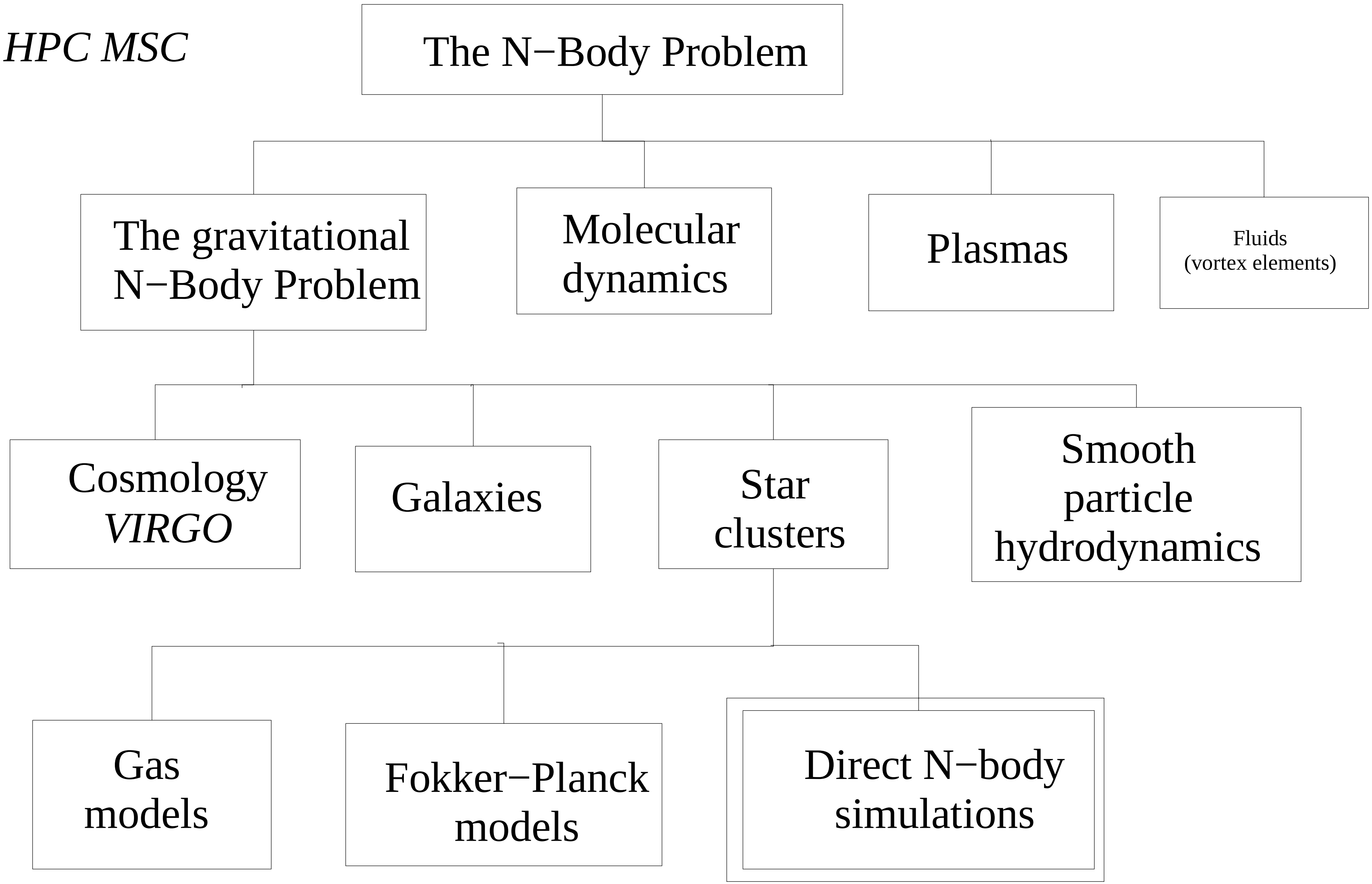
The N−Body Problem
The gravitational N−Body Problem
Molecular dynamics
Plasmas
Fluids (vortex elements)
Cosmology *VIRGO*
Galaxies
Star clusters
Smooth particle hydrodynamics
Gas models
Fokker−Planck models
Direct N−body simulations

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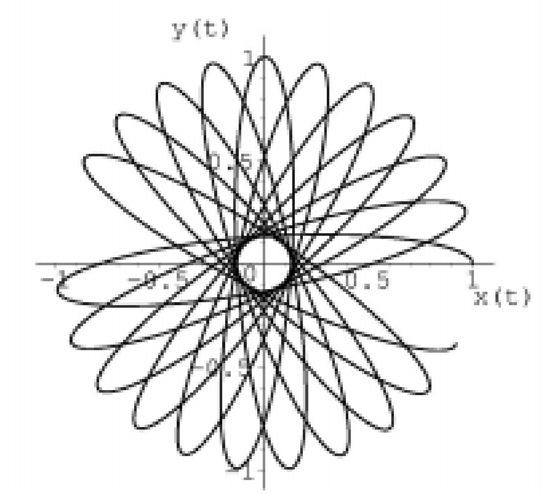
Credit: Michael Rich,
Kenneth Mighell, and
James D. Neill (Columbia University), and
Wendy Freedman (Carnegie Observatories) and
NASA

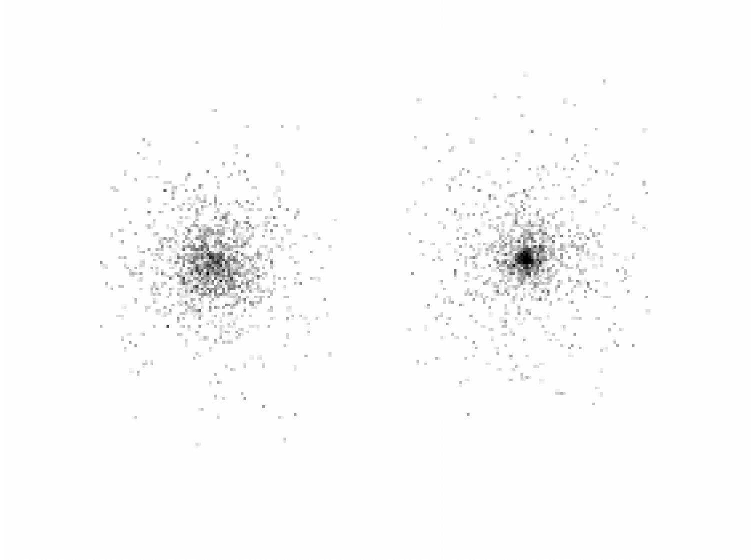
Basic Star Cluster Dynamics

- Quasi−equilibrium



- Orbital motions 10^6 years

- Long−term evolution 10^9 years



- Ratio of timescales $\propto$ N

- Binaries: periods down to <1s

THE N−BODY EQUATIONS

$$\ddot{\mathbf{r}}_i = -G \sum_{j \neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Complexity : $O(N^3)$ because

- N terms
- N equations
- ratio of time scales $\propto$ N

The progress of star cluster simulations

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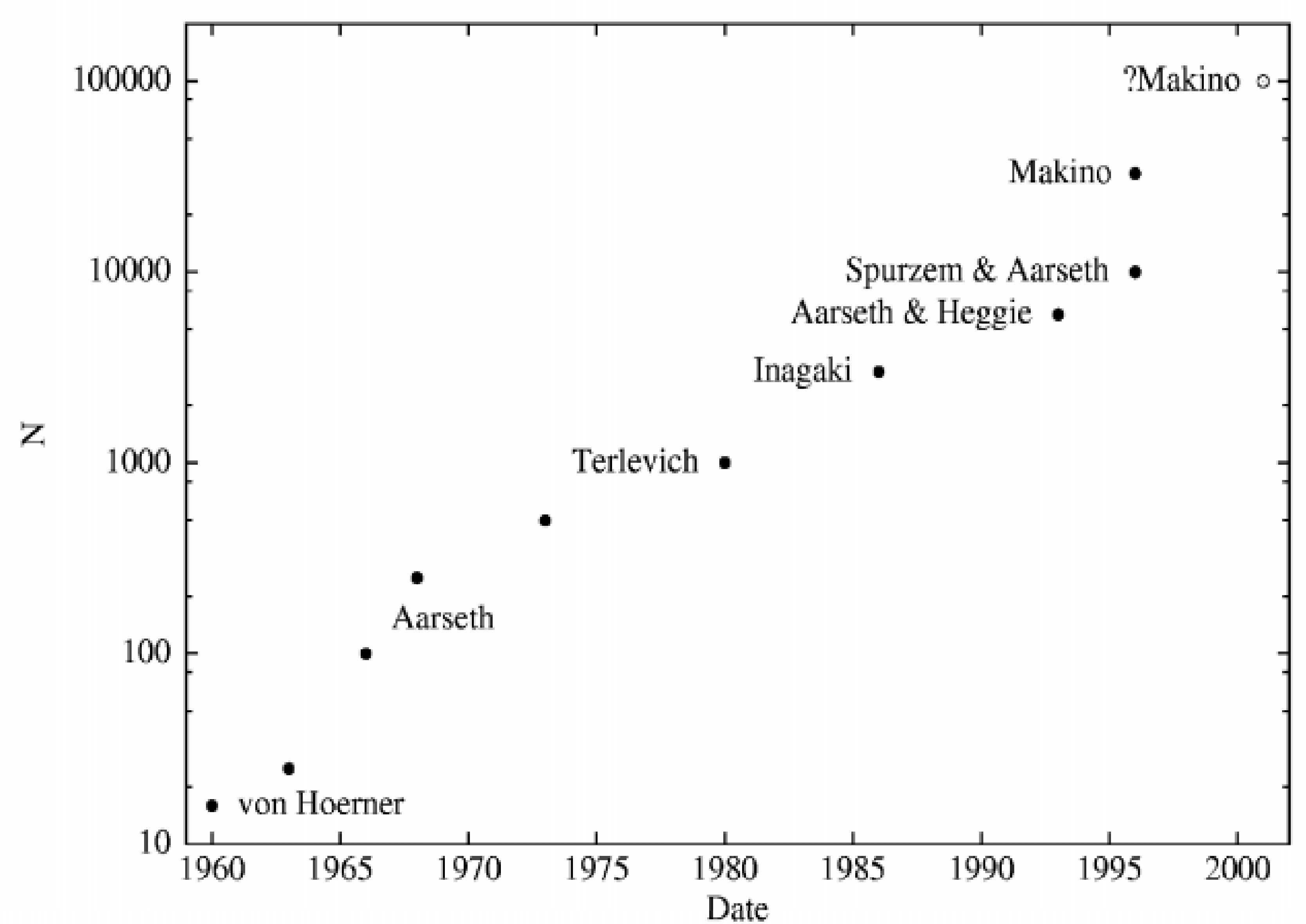

SOFTWARE ISSUES

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(a) time stepping in the N-body problem

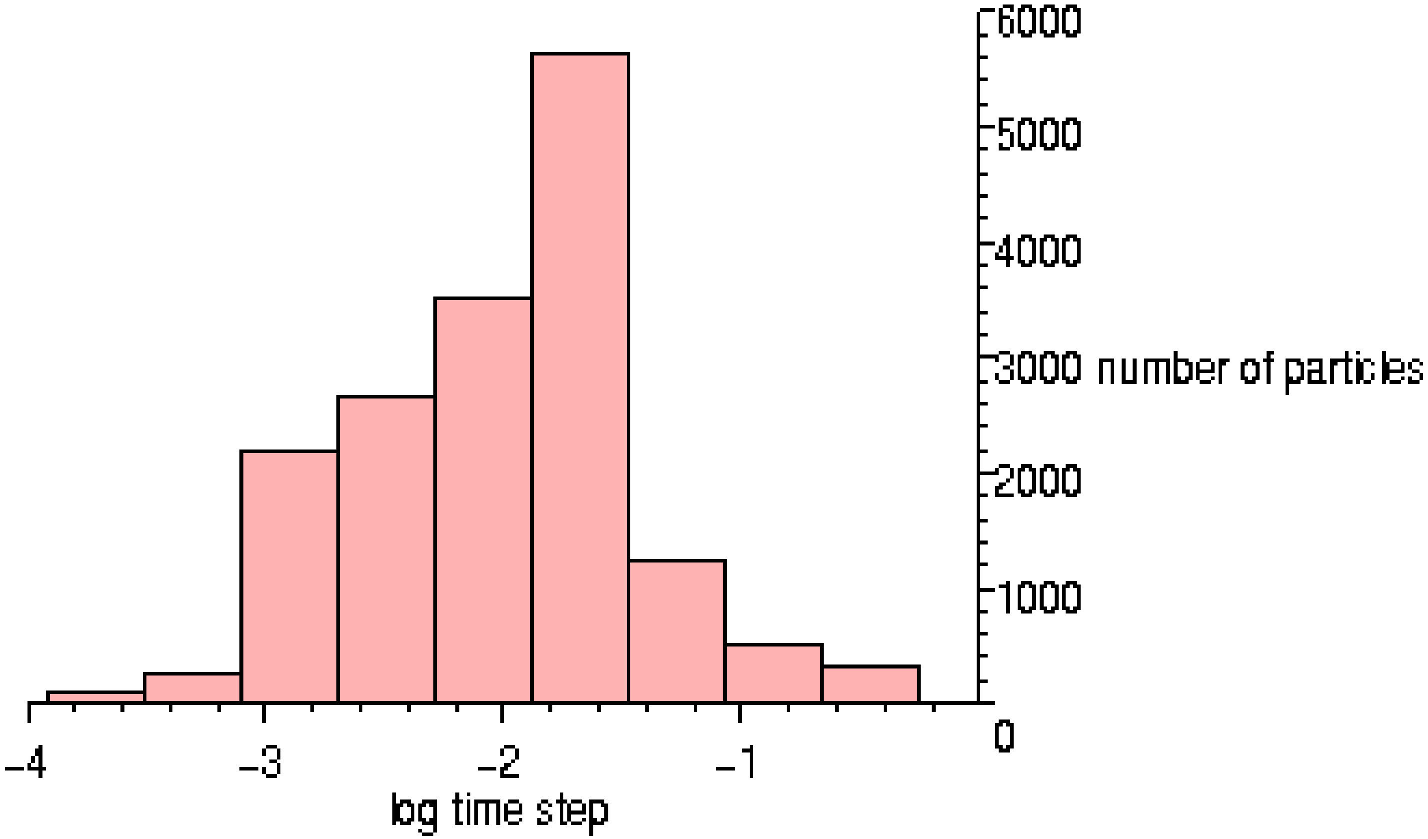
Solve $\ddot{\mathbf{r}} = \mathbf{f}$ with Hermite integrator:

1. Given $\mathbf{r}_n$, $\mathbf{v}_n$, $\mathbf{f}_n$, $\dot{\mathbf{f}}_n$ at t_n
2. predict at t_{n+1}
3. compute $\mathbf{f}_{n+1}$, $\dot{\mathbf{f}}_{n+1}$ at t_{n+1}
4. correct $\mathbf{r}_{n+1}$, $\mathbf{v}_{n+1}$ at t_{n+1}

Time step

$$h = \sqrt{\eta \frac{\frac{f}{\dot{f}} + \frac{\dot{f}}{\ddot{f}}}{\frac{f^{(3)}}{\ddot{f}} + \frac{\ddot{f}}{\dot{f}}}}$$

Individual time steps



SOFTWARE ISSUES

(b) force evaluation

Direct summation

$$\mathrm{f}_i = -G \sum_{particles} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Complexity $O(N^2)$

See hardware issues, later

Hierarchical methods

Sort particles by proximity into clumps (using tree) and replace by

$$\mathrm{f}_i \simeq -G \sum_{clumps} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

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Jernigan & Porter

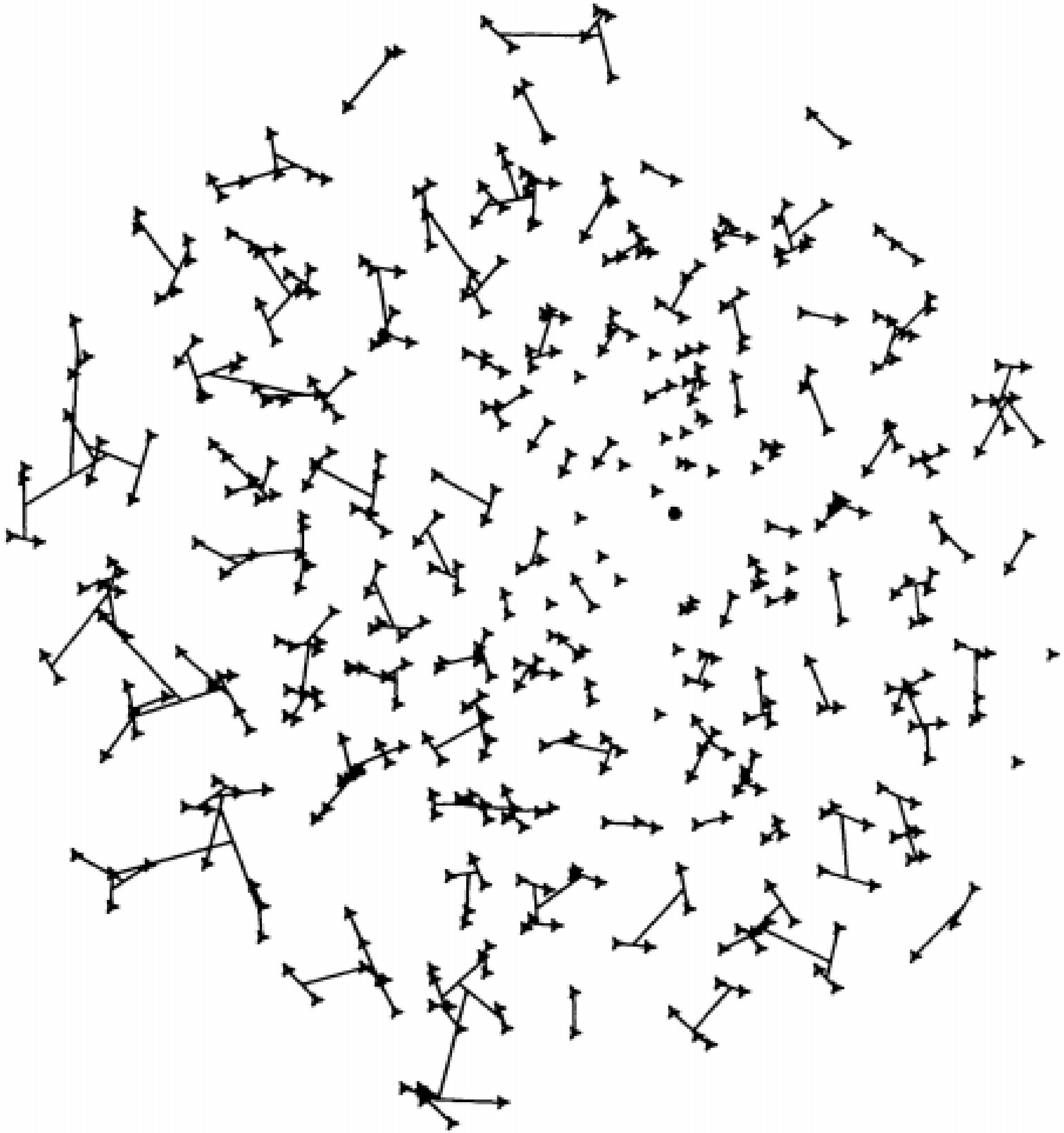

FIG. 6.—Sub-trees corresponding to the force terms generated for the circled particle in a force scan with $\Delta A_B = 0.01$. The system consists of 512 particles uniformly distributed in a two-dimensional disk.

Hierarchical force evaluation: comment

Efficient if

- required accuracy is low; or
- N is huge

But

- Accuracy requirements dictated by need to simulate relaxation.
- N dictated by available computing time.

Tree and N−body codes compared

McMillan & Aarseth

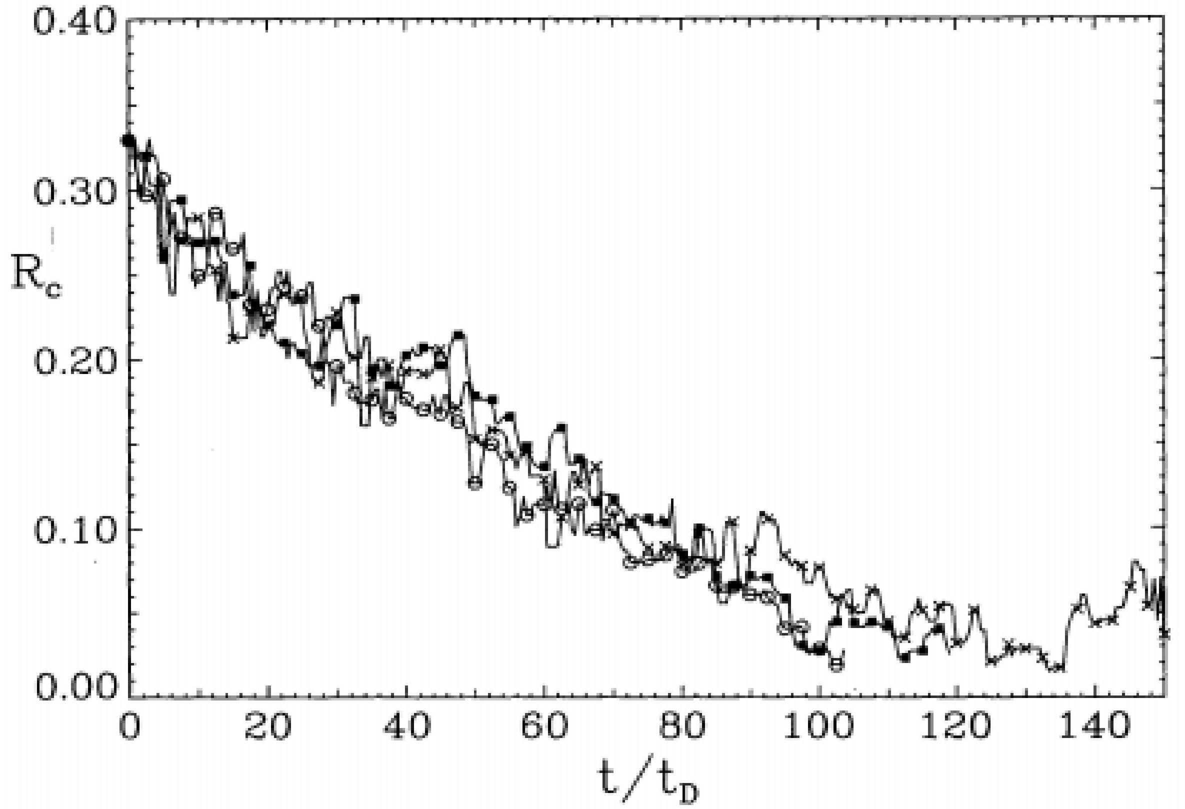

FIG. 5.—Core collapse in a 1024 body system. The three curves show, respectively, the time variation of the core radius found by (1) NBODY5 (*crosses*), (2) the tree code, with $\theta = 0.5$ and $\eta_T = 0.1$ (*open circles*), and (3) the tree code, with $\theta = 0.7$ and $\eta_T = 1.0$ (*filled squares*). Times are measured in units of the standard crossing time t_D. No significant difference exists between the tree-based and direct-summation results, even with the larger tolerance.

Dealing with singularities

$$\ddot{\mathbf{r}}_i = -G \sum_{j \neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

EXAMPLE: THE TWO-BODY PROBLEM

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3}$$

Define $\frac{d\tau}{dt} = \frac{1}{r}$.

Map $\mathbf{r} = (x, y, z) \to q = 0 + ix + jy + kz$
$\in$ quaternions.

Then

$$\frac{d^2q}{d\tau^2} = \frac{1}{2}hq,$$

where h is constant.

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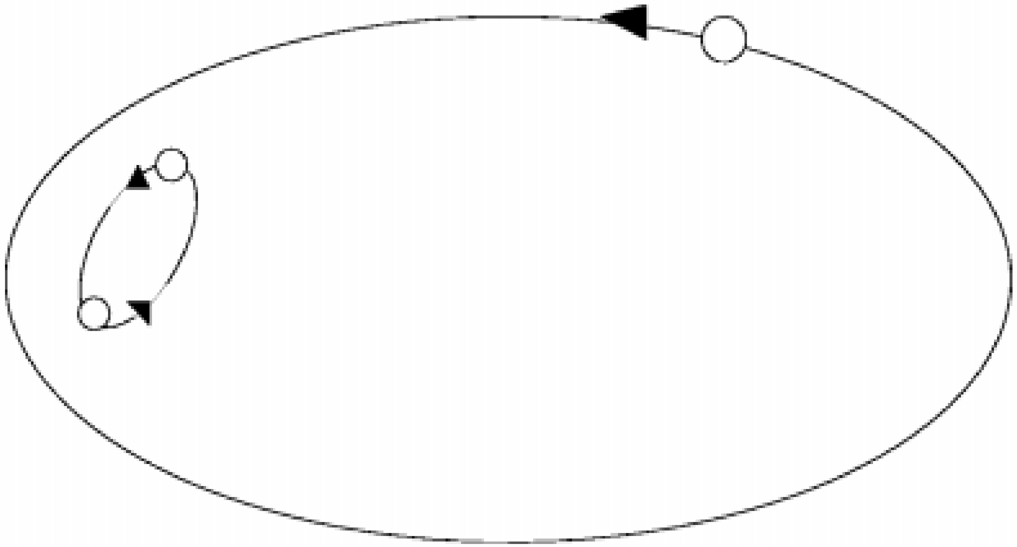
THREE-BODY SINGULARITY

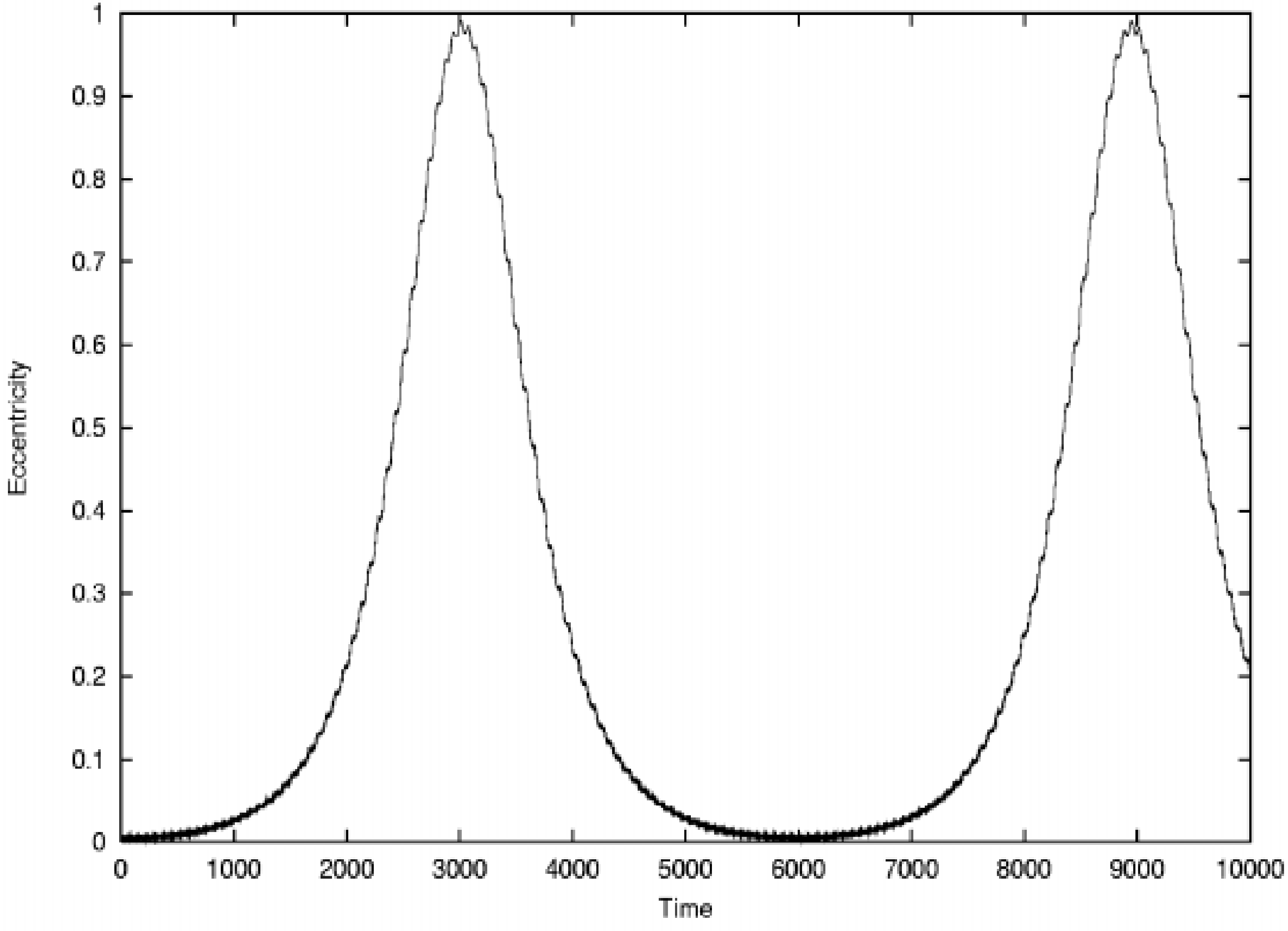
Hierarchical Triples



Three times scales

1. Inner period P_{in}
2. Outer period P_{out}
3. Secular period $\sim P_{out}^2/P_{in}$

Long−term behaviour of a triple



Remedy:

"Slow-down" treatment (Mikkola & Aarseth)

Replace true Hamiltonian for motion of binary

$$H = H_b + H_p$$

where

H_b is Hamiltonian of unperturbed binary

H_p is perturbation of third body

by

$$H = k^{-1} H_b + H_p$$

where k is slow-down factor.

HARDWARE OPTIONS

1. Workstations

•100 Mflop

•£1000

•available 24 hours

•N ~ 2000

•Many runs

HARDWARE OPTIONS

2. General−purpose HPC

- 100 Gflop
- Communications bound
- £1000000’s
- available 10% of time
- N ~ 10000
- O(1) runs

HARDWARE OPTIONS

3. Special−purpose hardware (GRAPE*)

•1 Tflop

•£50000

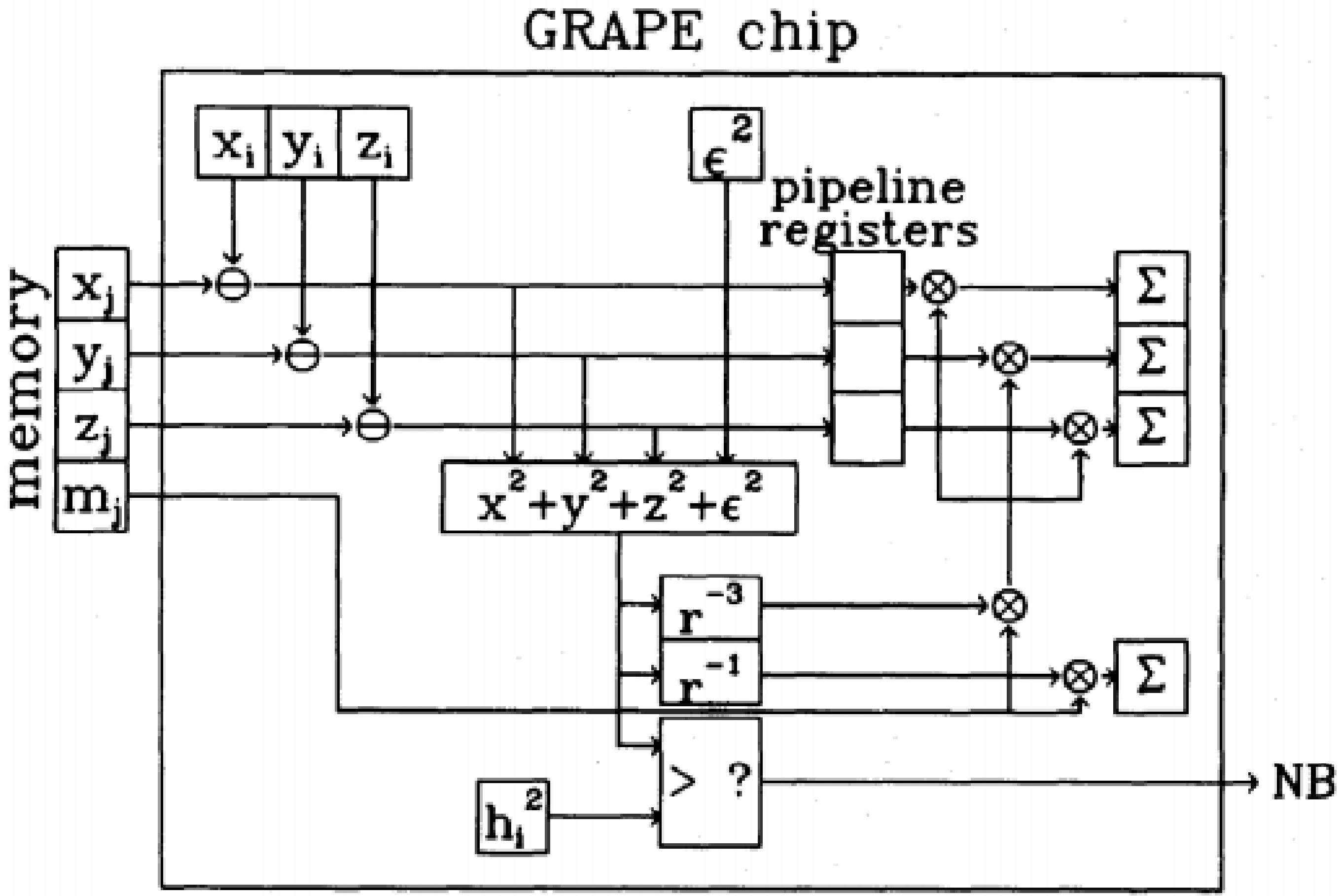
•available 24 hours

•N ~ 50000

•Many runs

*GRAvity PipE

Schematic of a simple GRAPE chip (GRAPE−3)



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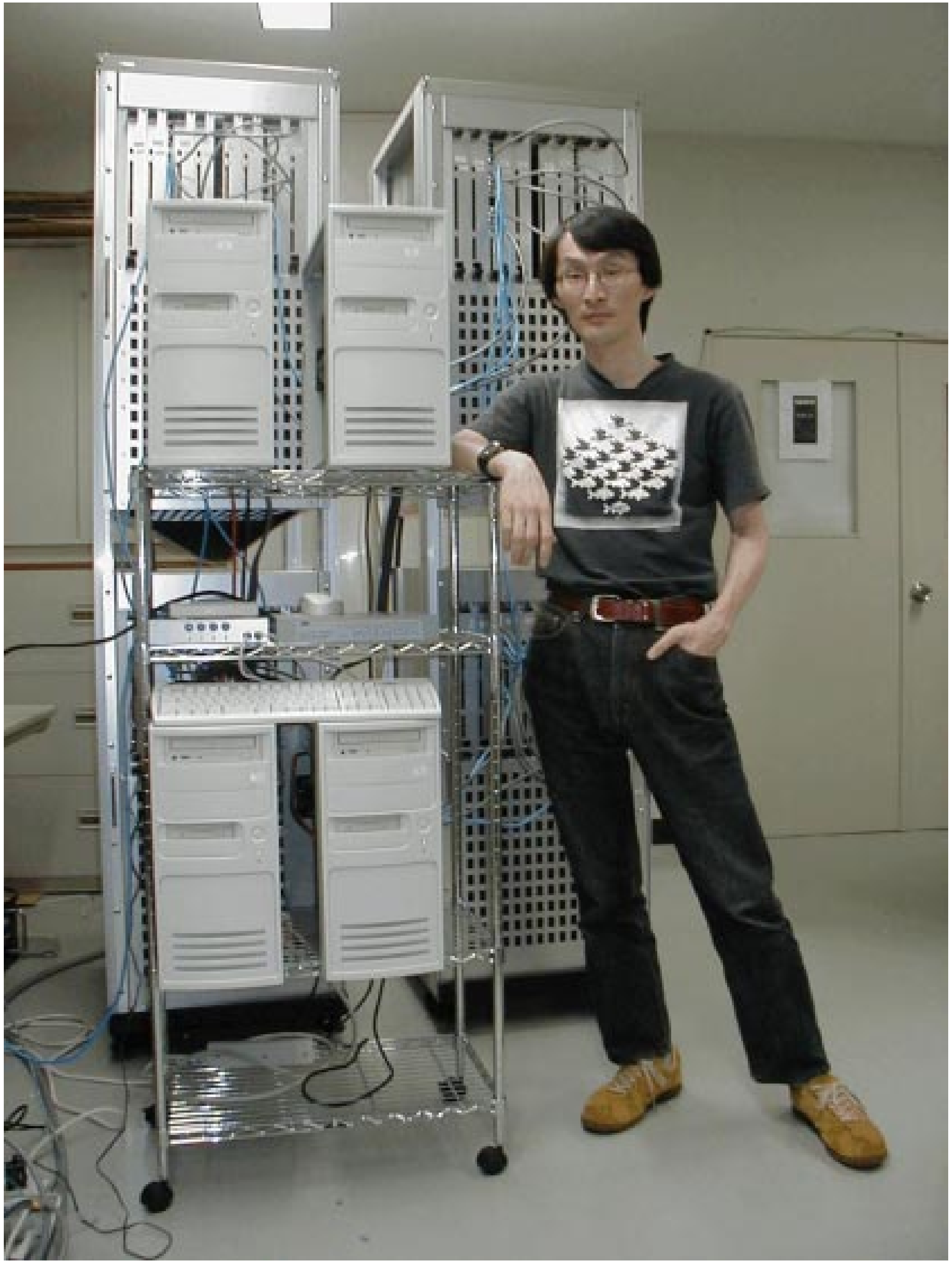
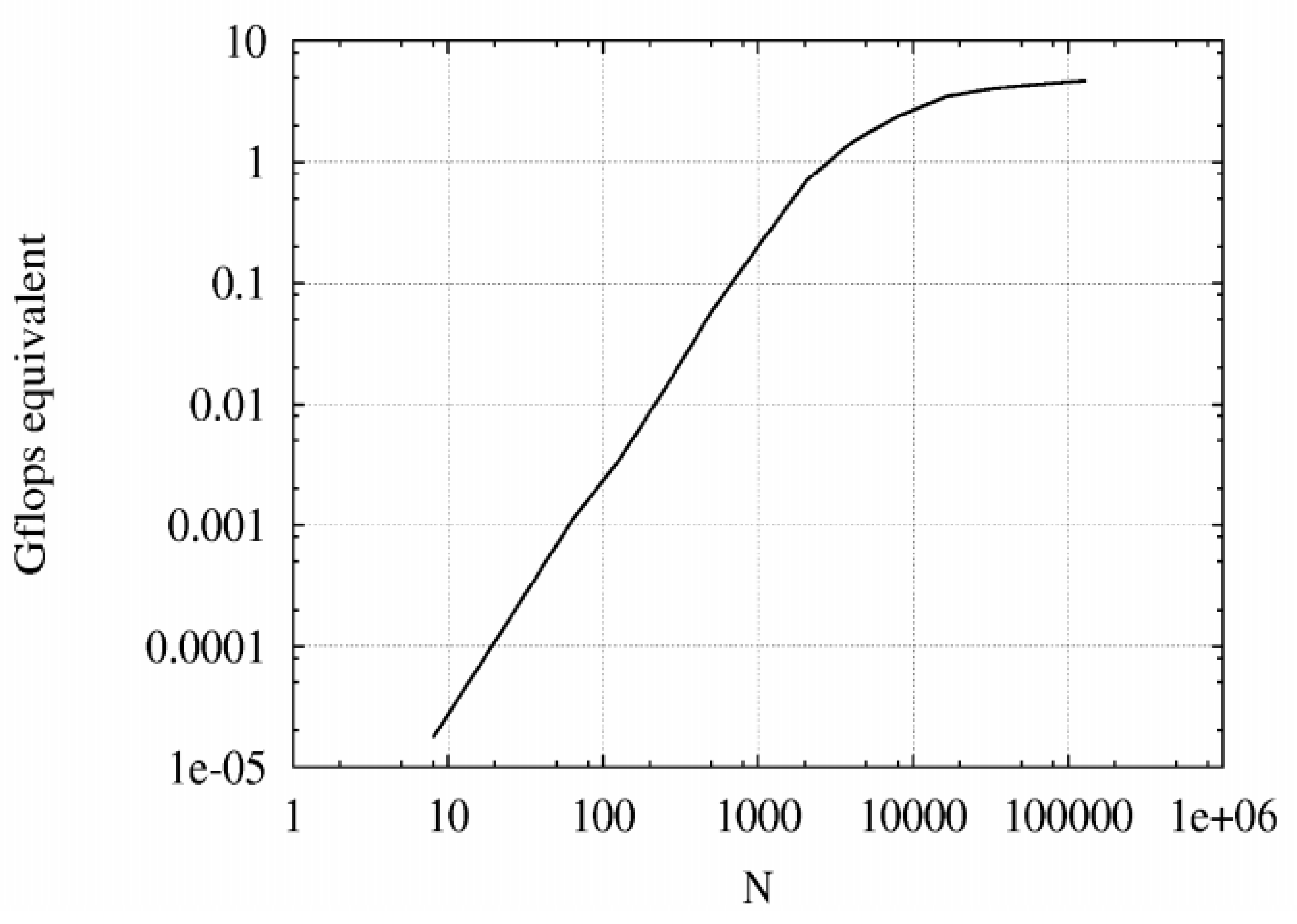
GRAPE6 and Jun Makino

2001

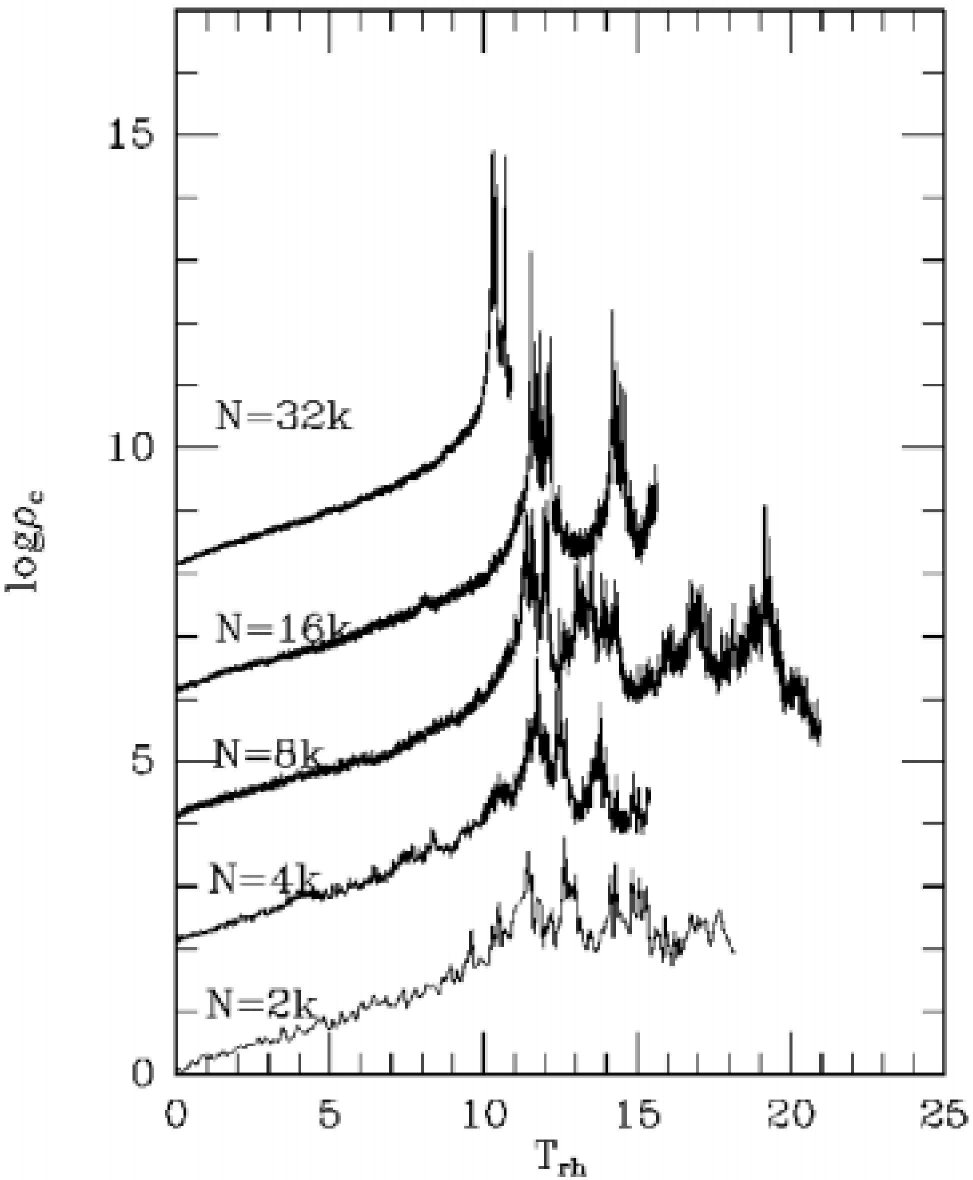
100Tflop



GRAPE 3 Efficiency



Visualisation I



Visualisation II

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Inside the Hayden dome....

Exponential growth of errors

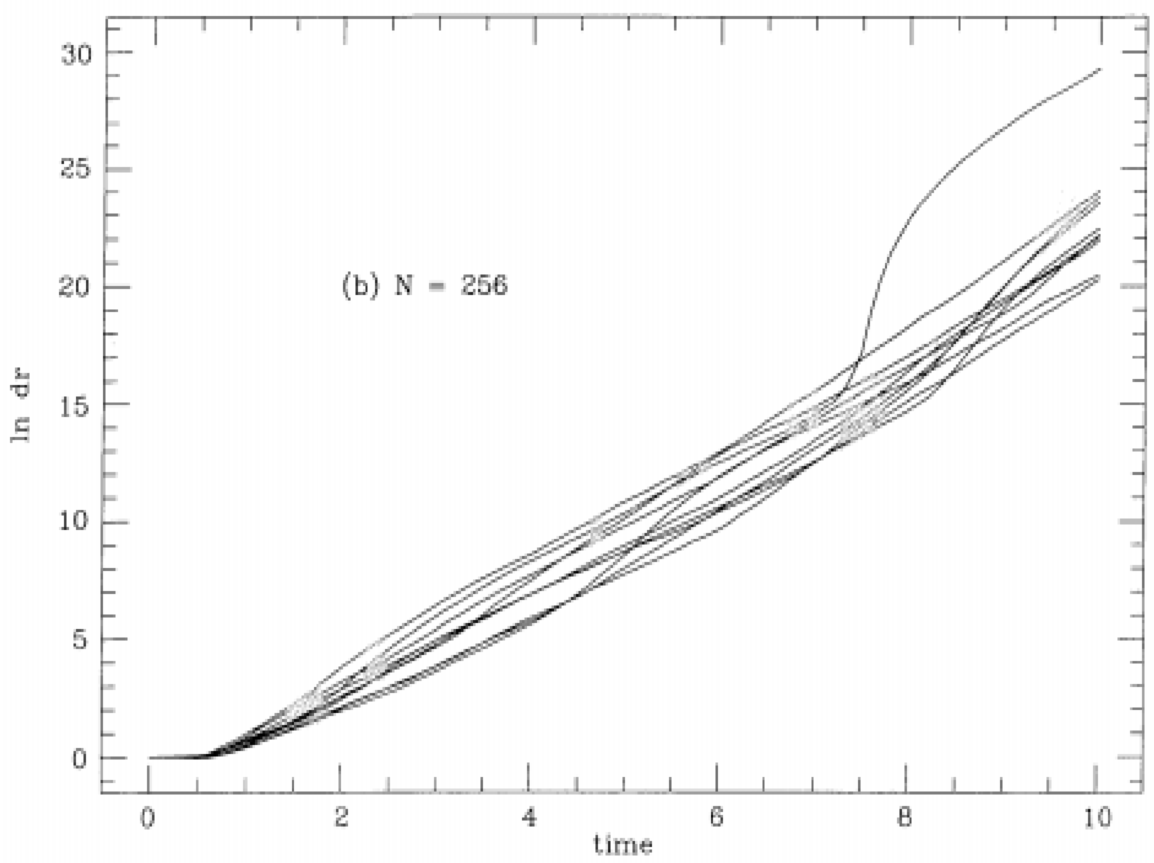

FIG. 7*b*

Simulations last thousands of e−folding times.

Do the results mean anything?

Summary

Mature subject

Driven by needs of astrophysics

Incorporates interesting mathematics

An elaborate software solution

Culture change − theorists become instrument builders

A challenging problem for numerical analysis