

# The rationality of symmetric products

V M Buchstaber and E G Rees

The  $n$ th symmetric product of a variety  $V$  is denoted  $\text{Sym}^n(V)$ . A variety  $V$  is rational if it is birationally equivalent to affine space. It is well known that if  $V$  is rational then its symmetric products are also rational [N], [M], [GKZ]. However the proofs given in these references are somewhat non-constructive. The aim of this note is to write down an explicit birational equivalence. It is enough to prove that the symmetric product of affine space is rational. We work over the complex field although most of the discussion covers some other ground fields. The proof gives an explicit set of generators for the field of rational symmetric functions on  $n$  variables from  $\mathbf{C}^m$  in terms of generalised Newton polynomials. In the paper [BR2] we determined the relations satisfied by these polynomials.

We regard  $\text{Sym}^n(\mathbf{C}^m) = \overline{M(n, m)}$  as the quotient of the space  $M(n, m)$  of all  $n \times m$  complex matrices by the symmetric group  $S_n$  permuting the columns which we write as  $X\sigma$  where  $X \in M(n, m)$  and  $\sigma \in S_n$  is regarded as a permutation matrix. We will need the van der Monde matrix for which we use the notation

$$V(a_1, a_2, \dots, a_n) = \begin{pmatrix} a_1^{n-1} & a_1^{n-2} & \dots & \dots & a_1 & 1 \\ a_2^{n-1} & a_2^{n-2} & \dots & \dots & a_2 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_n^{n-1} & a_n^{n-2} & \dots & \dots & a_n & 1 \end{pmatrix}$$

For  $X = (x_{i,j}) \in M(n, m)$  and  $1 \leq r \leq m$  we define  $V_r(X) = V(x_{r,1}, x_{r,2}, \dots, x_{r,n})$  and then  $f_r(X) = X V_r(X)$ .

**Lemma 1** Each map  $f_r$  is invariant under the action of  $S_n$  and so defines a map

$$\overline{f_r} : \overline{M(n, m)} \rightarrow M(n, m).$$

Let  $\overline{M(n, m)}_r$  denote the subset consisting of those matrices where the entries in the  $r$ th row are all distinct and so the matrix  $V_r(X)$  is non-singular. It is clear that  $\overline{M(n, m)}_r$  is a Zariski open subset of  $\overline{M(n, m)}$ .

For  $A = (a_{i,j}) \in M(n, m)$  and  $1 \leq r \leq m$  we define the polynomial  $p_A(t, r)$  to be the determinant of the matrix

$$\begin{pmatrix} a_{r,n} & 1 & 0 & \dots & 0 & 0 \\ a_{r,n-1} & a_{r,n} & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{r,1} & a_{r,2} & \dots & \dots & a_{r,n} & n \\ t^n & t^{n-1} & \dots & \dots & t & 1 \end{pmatrix}$$

The roots of  $p_A(t, r)$  are denoted by  $y_{r,1}, y_{r,2}, \dots, y_{r,n}$  and the Zariski open subset  $\{A \in M(n, m) \mid y_{r,1}, y_{r,2}, \dots, y_{r,n} \text{ are distinct}\}$  is denoted  $M(n, m)_r$ .

**Lemma 2**([BR1], Proposition 8) The elements  $a_{r,k}$  are the power sums of the elements  $y_{r,1}, y_{r,2}, \dots, y_{r,n}$  so  $a_{r,n} = y_{r,1} + y_{r,2} + \dots + y_{r,n}$  and  $a_{r,1} = y_{r,1}^n + y_{r,2}^n + \dots + y_{r,n}^n$ .

**Theorem** The map  $\overline{f}_r : \overline{M(n, m)}_r \rightarrow M(n, m)_r$  is an algebraic homeomorphism.

**Proof** We first show that the image of  $\overline{f}_r$  is contained in  $M(n, m)_r$ . The  $r$ th row of the matrix  $A = XV_r(X)$  is  $a_{r,n}, a_{r,n-1}, \dots, a_{r,1}$  where  $a_{r,n-k} = x_{r,1}^k + x_{r,2}^k + \dots + x_{r,n}^k$ . Since  $x_{r,1}, x_{r,2}, \dots, x_{r,n}$  are distinct for  $X \in \overline{M(n, m)}_r$  it is clear that  $\overline{f}_r(\overline{M(n, m)}_r) \subset M(n, m)_r$ .

Define  $g_r : M(n, m)_r \rightarrow \overline{M(n, m)}_r$  by  $g_r(A) = AV(y_{r,1}, y_{r,2}, \dots, y_{r,n})^{-1}$  where  $y_{r,1}, y_{r,2}, \dots, y_{r,n}$  are the roots of the polynomial  $p_A(t, r)$  as described in Lemma 2. One easily checks that  $g_r$  and  $\overline{f}_r$  are mutual inverses.

## References

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Steklov Mathematical Institute,  
Russian Academy of Sciences,  
Gubkina Street 8,  
117966 Moscow,  
Russia.

School of Mathematics,  
James Clerk Maxwell Building,  
University of Edinburgh,  
Edinburgh EH9 3JZ,  
Scotland.

buchstab@mendeleevo.ru

E.Rees@ed.ac.uk