Frobenius n-homomorphisms and branched coverings

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Abstract We review several equivalent definitions of a Frobenius n-homomorphism and show that, under quite general conditions, the sum and composition of two Frobenius homomorphisms behave appropriately. The main result explains how a Frobenius n-homomorphism between two function spaces corresponds to a generalised n-branched covering.

§1 Frobenius $n-$homomorphisms

In [BR1],[BR2] we introduced the concept of a Frobenius n-homomorphism and studied some of its properties (these papers also contain references to related work by other authors). We recall the basic definition.

Consider a linear map $f : A \to B$ between two commutative, associative $\mathbb{C}$ algebras. The maps $\Phi_n(f) : A^{\otimes n} \to B$ are defined as follows:

Each permutation $\sigma \in \Sigma_n$ the symmetric group on n letters, can be decomposed into a product of disjoint cycles of total length n, say $\sigma = \gamma_1\gamma_2 \dots \gamma_r$. If $\gamma = (i_1 \dots i_m)$ is a cycle, let $f_\gamma(a_1, a_2, \dots, a_n) = f(a_{i_1} a_{i_2} \dots a_{i_m})$ then

$$\Phi_n(f)(a_1, a_2, \dots, a_n) = \sum_{\sigma\in\Sigma_n} \varepsilon_\sigma f_{\gamma_1}(a_1, a_2, \dots, a_n) f_{\gamma_2}(a_1, a_2, \dots, a_n) \dots f_{\gamma_r}(a_1, a_2, \dots, a_n)$$

where ε_σ is the sign of the permutation σ.

From this definition it is clear that $\Phi_n(f) : A^{\otimes n} \to B$ is n-linear and and symmetric, one can use polarisation to ease the verification of some of its properties; in other words to prove identities it is enough to consider the values of $\Phi_n(f)(a, a, \dots, a)$ for all $a \in A$ (we will sometimes abbreviate this to $\Phi_n(f)(a)$.

There is also an inductive definition (introduced by Frobenius) for the $\Phi_n(f)$ starting with $\Phi_1(f) = f$ and, for $n \geq 1$,

$$\Phi_{n+1}(f)(a_0, a_1, \dots, a_n) = f(a_0)\Phi_n(f)(a_1, a_2, \dots, a_n) - \sum_r \Phi_n(f)(a_1, a_2, \dots, a_0a_r, \dots, a_n)$$

or equivalently,

$$\Phi_{n+1}(f)(a, a, \ldots, a) = f(a)\Phi_n(f)(a, a, \ldots, a) - n\Phi_n(f)(a^2, a, \ldots, a).$$

We will also find it useful to have another equivalent definition: $\Phi_n(f)(a)$ is the determinant of the matrix

$$\begin{pmatrix} f(a) & 1 & 0 & 0 & \ldots & 0 \\ f(a^2) & f(a) & 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & & & & \vdots \\ f(a^{n-1}) & f(a^{n-2}) & f(a^{n-3}) & \ldots & f(a) & n-1 \\ f(a^n) & f(a^{n-1}) & f(a^{n-2}) & \ldots & f(a^2) & f(a) \end{pmatrix}$$

Definition 1.1 A linear map $f : A \to B$ is called a **Frobenius n-homomorphism** if it satisfies $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

Definition 1.2 An algebra A is **connected** if each equation $x(x-1)\ldots(x-k) = 0$ has only the obvious $k+1$ solutions.

This terminology is suggested by the fact that an algebra of functions on a space X has this property if and only if X is connected.

Proposition 1.3 *Let B be connected and $f : A \to B$ be such that $\Phi_{n+1}(f) \equiv 0$ then $f(1) \in \{0, 1, 2, \ldots, n\}$.*

The proof of this Proposition is identical with that of the less general result Corollary 2.5 of [BR1]. The condition $f(1) = n$ plays a crucial rôle (Proposition 2.7 of [BR1]).

Proposition 1.4 *If $\Phi_{n+1}(f) \equiv 0$ and $f(1) = k$ then $\Phi_{k+1}(f) \equiv 0$.*

We will use the following result about symmetric polynomials (it is related to some of those in [BR3], [BR4]); it helps to explain the determinant expression used above.

Proposition 1.5 *Let* $s_k = \beta_1^k + \beta_2^k + \ldots + \beta_n^k$*, then the indeterminates* β_r $(1 \leq r \leq n)$ *are the roots of the polynomial* $d(t)$ *given as the determinant of the matrix*

$$\begin{pmatrix} s_1 & 1 & 0 & 0 & \ldots & \ldots & 0 \\ s_2 & s_1 & 2 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \ldots & s_1 & n-1 & 0 \\ s_n & s_{n-1} & s_{n-2} & \ldots & s_2 & s_1 & n \\ t^n & t^{n-1} & t^{n-2} & \ldots & t^2 & t & 1 \end{pmatrix}$$

Moreover if $f : A \to B$ *is linear and* $f(a^k) = s_k$ *then the determinant* $d(t)$ *is, up to a non-zero constant multiple, equal to*

$$t^n - \Phi_1(f)t^{n-1} + \Phi_2(f)t^{n-2} - \ldots + (-1)^n\Phi_n(f).$$

Proof

As usual, let e_r denote the elementary symmetric polynomial of degree r in β_r $(1 \leq r \leq n)$. Denote the columns of the above matrix by $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n+1})$ and replace the first column by the column vector

$$\mathbf{c}_1 - e_1\mathbf{c}_2 + e_2\mathbf{c}_3 - \ldots + (-1)^n e_n\mathbf{c}_{n+1}.$$

The determinant is unchanged. Using the standard Newton formulae (see [Mac] page 20),

$$s_r - s_{r-1}e_1 + \ldots + (-1)^{r-1}s_1e_{r-1} + (-1)^r re_r = 0$$

we see that the first column of the new matrix has zero entries except for the last entry which equals

$$p(t) = t^n - e_1t^{n-1} + e_2t^{n-2} - \ldots + (-1)^n e_n.$$

So $d(t) = (-1)^n\, n!\, p(t)$ and hence the roots of $d(t)$ are the same as those of $p(t)$, namely $\{\beta_1, \beta_2, \ldots, \beta_n\}$.

Finally, to verify the result about $f : A \to B$ it is enough to consider the special case $A = \mathbb{C}[a], B = \mathbb{C}[\beta_1, \beta_2, \ldots, \beta_n]$. Then considering the appropriate sub-determinant of the above determinant that defines e_n we see that $\Phi_n(f)(a) = n!e_n$.

This leads us to yet another characterisation of Frobenius n-homomorphisms (which is closely related to some formulae in [BR3] and [BR4]).

Proposition 1.6 *A linear map $f : A \to B$ is a Frobenius n-homomorphism if and only if for each $a \in A$ there is a polynomial $p_a(t) \in B[t]$ of degree n such that*

$$\sum_{q=0}^{\infty} \frac{f(a^q)}{t^{q+1}} = \frac{\mathrm{d}}{\mathrm{d}t} \log p_a(t).$$

Proof

Given the Frobenius n-homomorphism $f : A \to B$ and $a \in A$, consider the polynomial $p_a(t) = \det(M) \in B[t]$ where M is the matrix

$$\begin{pmatrix} f(a) & 1 & 0 & 0 & \dots & \dots & 0 \\ f(a^2) & f(a) & 2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ f(a^{n-1}) & f(a^{n-2}) & f(a^{n-3}) & \dots & f(a) & n-1 & 0 \\ f(a^n) & f(a^{n-1}) & f(a^{n-2}) & \dots & f(a^2) & f(a) & n \\ t^n & t^{n-1} & t^{n-2} & \dots & t^2 & t & 1 \end{pmatrix}$$

Choose an extension $\overline{B}$ of B such that $p_a(t)$ factors completely in $\overline{B}[t]$, say $p_a(t) = (t - \beta_1)(t - \beta_2)\dots(t - \beta_n)$ with $\beta_r \in \overline{B}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \log p_a(t) = \frac{\mathrm{d}}{\mathrm{d}t} (\log(t - \beta_1) + \log(t - \beta_2) + \dots + \log(t - \beta_n))$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t} \log(t - \beta) = \frac{1}{t - \beta} = \frac{1}{t} \left(1 - \frac{1}{\frac{\beta}{t}} \right) = \frac{1}{t} + \frac{\beta}{t^2} + \frac{\beta^2}{t^3} + \dots$$

However, by the Proposition 1.5, $\beta_1^r + \beta_2^r + \dots \beta_n^r = f(a^r)$ and the result is proved.

Conversely, if

$$\sum_{q=0}^{\infty} \frac{f(a^q)}{t^{q+1}} = \frac{\mathrm{d}}{\mathrm{d}t} \log p_a(t)$$

and $p_a(t) = (t - \beta_1)(t - \beta_2)\dots(t - \beta_n)$ then $f(a^r) = \beta_1^r + \beta_2^r + \dots \beta_n^r$ and, with $r = 0$ we have $f(1) = n$; using this and Proposition 1.5 we get that the r^{th} elementary symmetric polynomial in $\beta_1, \beta_2, \dots, \beta_n$ is $\Phi_r(f)(a)$ and hence that $\Phi_{n+1}(f)(a) = 0$.

§2 Compositions

In [BR1] we showed that the sum of a Frobenius m-homomorphism and a Frobenius n-homomorphism is a Frobenius $m+n$-homomorphism and, for completeness we recall the proof. The main result of this section is to show that they also behave appropriately under composition.

For the following definition see [L].

Definition 2.1 An algebra A is **J-semi-simple** if its Jacobson radical is trivial.

This means that the intersection of all maximal ideals of A is trivial, hence, for every non-zero a in A there is an algebra map $\varphi : A \to \mathbb{C}$ with $\varphi(a) \neq 0$.

Theorem 2.2 *Let A, B, C be associative, commutative $\mathbb{C}-$algebras with C a J-semi-simple algebra. If $f : A \to B$ and $g : B \to C$ are Frobenius n and m-homomorphisms, respectively then $gf : A \to C$ is a Frobenius mn-homomorphism.*

Proof

First we prove the Theorem in the special case where $C = \mathbb{C}$. Since $f(1) = n$ and $g(1) = m$ one has $gf(1) = mn$. It is trivial to check that the theorem is true for $m = 1$. By Theorem 3.4 of [BR1], g is the sum of m ring homomorphisms, say $g = g_1 + g_2 + \ldots + g_m$ and so $gf = g_1 f + \ldots + g_m f$. By the remark above, each $g_k f$ is a Frobenius n-homomorphism and hence by Theorem 2.9 of [BR1] their sum is a Frobenius mn-homomorphism.

For any algebra map $\varphi : C \to \mathbb{C}$, it is straightforward to verify that the composition $\varphi g : B \to C \to \mathbb{C}$ is a Frobenius m-homomorphism. So $(\varphi g)f = \varphi(gf)$ is a Frobenius mn-homomorphism.

The following result will complete the proof.

Lemma 2.3 *If C is a J-semi-simple algebra, then a linear map $F : A \to C$ is a Frobenius n-ring homomorphism if and only if $\varphi F : A \to C \to \mathbb{C}$ is a Frobenius n-ring homomorphism for every algebra map $\varphi : C \to \mathbb{C}$.*

Proof

For any $\varphi : C \to \mathbb{C}$ we have:

$$\varphi \Phi_{n+1}(F)(a_1, \ldots, a_{n+1}) = \Phi_{n+1}(\varphi \circ F)(a_1, \ldots, a_{n+1}).$$

Suppose F is not a Frobenius n-homomorphism but that $\varphi \circ F$ is a Frobenius n-homomorphism for every algebra map φ, then $\Phi_{n+1}(\varphi \circ F)(a_1, \ldots, a_{n+1}) =$

0 for such a φ. But there are $(a_1, \ldots, a_{n+1})$ such that $\Phi_{n+1}(F)(a_1, \ldots, a_{n+1}) = x \neq 0$. If $\varphi : C \to \mathbb{C}$ is non-zero on x, then $\varphi \circ F$ is not a Frobenius n-homomorphism. This concludes the proof.

§3 Branched coverings

Generalised branched coverings were studied by Smith [S] and by Dold [D].

Definition 3.1 A map $h : X \to Y$ between two Hausdorff spaces is an **n-branched covering** if there is a continuous map $t : Y \to \mathrm{Sym}^n(X)$ such that

1. $x \in th(x)$ for every $x \in X$; and
2. $\mathrm{Sym}^n(h)(ty) = ny$ for every $y \in Y$.

The aim of this section is to characterise n-branched coverings in terms of Frobenius n-homomorphisms.

When X is a compact Hausdorff space, $C(X)$ will denote the algebra of continuous functions $X \to \mathbb{C}$ with the supremum norm.

Theorem 3.2 *If X, Y are compact Hausdorff spaces, then the set of all continuous Frobenius n-homomorphisms $C(X) \to C(Y)$ can be identified with the space of continuous maps $Y \to \mathrm{Sym}^n(X)$.*

Proof

Consider a map $t : Y \to \mathrm{Sym}^n(X)$, it induces a map $t^* : C(X) \to C(Y)$ where $t^*\phi(y) = \sum \phi(x_r)$ and $t(y) = [x_1, x_2, \ldots, x_n]$. By Proposition 1.5 the map t^* is a Frobenius n-homomorphism.

Conversely, suppose that $f : C(X) \to C(Y)$ be a Frobenius n-homomorphism and let $\mathcal{E}_y : C(Y) \to \mathbb{C}$ be evaluation at the point $y \in Y$ then the composition $\mathcal{E}_y f$ is also a Frobenius n-homomorphism and so, by Theorem 3.1 of [BR1] corresponds to a multi-set $[x_1, x_2, \ldots, x_n]$ in X. This defines the requires map $t : Y \to \mathrm{Sym}^n(X)$.

Example 3.3 *The linear map $C(X) \to C(X)$ given by $\phi \to n\phi$ is a Frobenius n-homomorphism and corresponds to the diagonal map $X \to \mathrm{Sym}^n(X)$.*

Example 3.4 *If $s : Y \to \mathrm{Sym}^m(X)$ and $t : Y \to \mathrm{Sym}^n(X)$ give rise to the Frobenius m, n-homomorphisms $s^*, t^* : C(X) \to C(Y)$ then the composition*

$$Y \to \mathrm{Sym}^m(X) \times \mathrm{Sym}^n(X) \to \mathrm{Sym}^{m+n}(X)$$

corresponds to $s^* + t^* : C(X) \to C(Y)$.

A continuous map $h : X \to Y$ induces a ring homomorphism $h^* : C(Y) \to C(X)$. If h is an n-sheeted (branched) covering, then as above we have a map $t^* : C(X) \to C(Y)$ which is a Frobenius n-homomorphism. We now wish to state properties of t^* which will ensure that h is such a covering.

Definition 3.5 Let A, B be commutative, associative algebras and $f : A \to B$, a ring homomorphism, then a linear map $\tau : B \to A$ is an n**-transfer** for f if

1. τ is a Frobenius n-homomorphism;
2. $\tau f : A \to A$ is multiplication by n ; and
3. $f\tau : B \to B$ is the sum of the identity and a Frobenius $(n-1)$-homomorphism.

Theorem 3.6 *If there is a continuous n-transfer τ for $h^* : C(Y) \to C(X)$, then $h : X \to Y$ is an n-branched covering.*

Proof

By the facts above, a continuous n-transfer $\tau : C(X) \to C(Y)$ corresponds to a continuous map $t : Y \to \mathrm{Sym}^n(X)$ which is such that $th : X \to \mathrm{Sym}^n(X)$ is the diagonal map and $\mathrm{Sym}^n(h)t : Y \to \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ is of the form $y \to [y_1, y_2, \ldots, y_n]$ with $y_1 = y$.

More generally, we see that the Frobenius n-homomorphism f corresponding to $t : Y \to \mathrm{Sym}^n(X)$ is the sum of Frobenius $n_1, n_2, \ldots, n_k$-homomorphisms $f_1, f_2, \ldots, f_k$ (where $n = n_1 + n_2 + \ldots + n_k$) if and only if t factors as $Y \to \mathrm{Sym}^{n_1}(X) \times \mathrm{Sym}^{n_2}(X) \times \ldots \times \mathrm{Sym}^{n_k}(X) \to \mathrm{Sym}^n(X)$ where the last map is concatenation.

Using Theorem 3.4 and Corollary 3.6 of [BR1] one can in a similar way prove

Theorem 3.7 *Let A, B be finitely generated commutative algebras, $V(A), V(B)$ the corresponding varieties and $h : V(A) \to V(B)$ be an algebraic map. Then h is an n-fold branched covering if and only if there is an n-transfer $A \to B$, for h.*

Remark 3.8 It is also straightforward, following §2, to check that the composition of continuous m, n-transfers is an mn-transfer and corresponds to the composition of two appropriate branched covering maps.

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