

Applications of Frobenius n -homomorphisms

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For any linear mapping $f : A \rightarrow B$ of algebras, we define a multi-linear mapping $\Phi_k(f) : A^k \rightarrow B$ using the Frobenius recursion formula (with $\Phi_1 = f$):

$$\Phi_k(f)(a_1, \dots, a_k) = \Phi_1(a_1)\Phi_{k-1}(f)(a_2, \dots, a_k) - \sum_{j=2}^k \Phi_{k-1}(f)(a_2, \dots, a_1 a_j, \dots, a_k).$$

DEFINITION 1. [1] Let A and B be algebras with unity. The mapping f is called a Frobenius n -homomorphism if $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

The results of the present paper are based on the following theorem due to the authors [2]: *The linear mapping $f : \mathbb{C}[u_1, \dots, u_m] \rightarrow \mathbb{C}$ is a Frobenius n -homomorphism if and only if there exists a multi-set $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$ in \mathbb{C}^m such that $f(p) = p(\mathbf{x}_1) + \dots + p(\mathbf{x}_n)$.*

§1. Let $s_k : \text{Sym}^n(\mathbb{C}) \rightarrow \mathbb{C}$ be the mapping defined by the k -th Newton polynomial $s_k[b_1, b_2, \dots, b_n] = b_1^k + b_2^k + \dots + b_n^k$. For $m \in \mathbb{N}$, denote the semi-lattice $\mathbb{Z}_{\geq 0}^m$ by \mathcal{N}_m . For $\omega = (i_1, i_2, \dots, i_m) \in \mathcal{N}_m$ set $\mathbf{z}^\omega = z_1^{i_1} z_2^{i_2} \dots z_m^{i_m}$ and $|\omega| = i_1 + i_2 + \dots + i_m$. Consider a mapping $g : \mathbb{C}^m \rightarrow \text{Sym}^n(\mathbb{C})$.

PROPERTY 1. For each $k = 1, 2, \dots$ the composition $s_k \circ g : \mathbb{C}^m \rightarrow \mathbb{C}$ is a homogeneous form of degree k and is denoted $\sum_{|\omega|=k} \binom{k}{\omega} \alpha_\omega \mathbf{z}^\omega$, where $\binom{k}{\omega} = \frac{k!}{i_1! i_2! \dots i_m!}$.

If g has Property 1, define a mapping $\Phi : \mathcal{N}_m \rightarrow \mathbb{C}$ by the formula $\Phi(\omega) = \alpha_\omega$.

Set $\Phi(0) = n$, and $\Phi_1 = \Phi$, then introduce for each $\ell > 1$ the mapping $\Phi_\ell : \mathcal{N}_m^\ell \rightarrow \mathbb{C}$ using the following recursion formula:

$$\Phi_\ell(\omega_1, \omega_2, \dots, \omega_\ell) = \Phi_1(\omega_1)\Phi_{\ell-1}(\omega_2, \dots, \omega_\ell) - \sum_{j=2}^{\ell} \Phi_{\ell-1}(\omega_2, \dots, \omega_1 + \omega_j, \dots, \omega_\ell).$$

PROPERTY 2. The sequence $\{\Phi_\ell\}$ that corresponds to g truncates at the $n+1$ -th step, i.e. $\Phi_{n+1}(\omega_1, \dots, \omega_{n+1}) \equiv 0$.

§2. For a multi-set $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$ in \mathbb{C}^m , introduce a mapping $f_X : \mathbb{C}^m \rightarrow \text{Sym}^n(\mathbb{C})$ by the formula

$$f_X(\mathbf{z}) = [\langle \mathbf{z}, \mathbf{x}_1 \rangle, \dots, \langle \mathbf{z}, \mathbf{x}_n \rangle].$$

THEOREM 1. A mapping g is given by f_X for some multi-set X if and only if g has properties 1 and 2.

This result gives a version for finite measures of the “analogue of the Paley-Wiener theorem for the complex Radon transform” (in the terminology of [3]).

§3. Let

$$A(t, \mathbf{z}) = t^n - a_1(\mathbf{z})t^{n-1} + \cdots + (-1)^n a_n(\mathbf{z}),$$

where $a_k(\mathbf{z})$ is a homogeneous form of degree k . The roots $[t_1, \dots, t_n]$ of the equation $A(t, \mathbf{z}) = 0$ define a mapping $f_A : \mathbb{C}^m \rightarrow \text{Sym}^n(\mathbb{C})$.

COROLLARY 2. There is a multi-set X such that $f_A = f_X$, that is, $[t_1, \dots, t_n] = [\langle \mathbf{z}, \mathbf{x}_1 \rangle, \dots, \langle \mathbf{z}, \mathbf{x}_n \rangle]$ and therefore $A(t, \mathbf{z}) = \prod_{i=1}^n (t - \langle \mathbf{z}, \mathbf{x}_i \rangle)$ if and only if f_A has Property 2.

Thus we obtain an algorithm to test the decomposability of a homogeneous polynomial $A(t, \mathbf{z})$ into linear factors. The relation

$$\frac{d}{dt} \log A(t, \mathbf{z}) = \sum_{k \geq 0} t^{-k-1} s_k \circ f_A(\mathbf{z}) = \sum_{k \geq 0} t^{-k-1} \sum_{|\omega|=k} \binom{k}{\omega} \alpha_{\omega} \mathbf{z}^{\omega}$$

gives the values of the parameters α_{ω} , and thus defines the mapping Φ .

It is required to check the condition $\Phi_{n+1} \equiv 0$. We use the following explicit expression [1]:

$$\Phi_{\ell}(\omega_1, \dots, \omega_{\ell}) = \sum_{\sigma} \text{sign}(\sigma) \Phi_{\sigma}(\omega_1, \dots, \omega_{\ell})$$

where σ runs over the permutation group S_{ℓ} . By decomposing σ into the product of disjoint cycles $\gamma_1 \gamma_2 \cdots \gamma_q$, define $\Phi_{\sigma} = \Phi_{\gamma_1} \Phi_{\gamma_2} \cdots \Phi_{\gamma_q}$, and for $\gamma = (i_1, \dots, i_r) \in S_{\ell}$ set $\Phi_{\gamma}(\omega_1, \dots, \omega_{\ell}) = \Phi(\omega_{i_1} + \cdots + \omega_{i_r})$.

§4. Let $a_n(\mathbf{z})$ be a non-trivial homogeneous form of degree $n > 0$ as above. Choose a point $\mathbf{y} \in \mathbb{C}^m$ such that $a_n(\mathbf{y}) = 1$ and set $A_{\mathbf{y}}(t, \mathbf{z}) = a_n(\mathbf{z} + t\mathbf{y})$. Then

$$A_{\mathbf{y}}(t, \mathbf{z}) = t^n + \sum_{k=1}^n (-1)^k a_k(\mathbf{z}, \mathbf{y}) t^{n-k}.$$

By Corollary 2, the polynomial $A_{\mathbf{y}}(t, \mathbf{z})$ breaks into linear factors (and thus the form $a_n(\mathbf{z})$ is given by $\prod_{i=1}^n \langle \mathbf{z}, \mathbf{x}_i \rangle$) if and only if $f_{A_{\mathbf{y}}}$ has Property 2. So, the above criterion is applicable to the problem of decomposing forms into linear factors (see [4]).

§5. Consider the Lie algebra W_m of vector fields on \mathbb{C}^m with polynomial coefficients. The vector fields $D_{\omega, j} = \mathbf{z}^{\omega} \frac{\partial}{\partial z_j}$ form a basis for W_m .

THEOREM 3. The Lie algebra W_m acts on the functions Φ_{ℓ} by the formula

$$D_{\omega, j} \Phi_{\ell}(\omega_1, \omega_2, \dots, \omega_{\ell}) = \sum_{k=1}^{\ell} \langle \omega_k, e_j \rangle \Phi_{\ell}(\omega_1, \omega_2, \dots, \omega_k + \omega - e_j, \dots, \omega_{\ell}),$$

where e_j is the j -th unit vector.

COROLLARY 4. *To carry out a direct check of Property 2 it is sufficient to calculate the values of Φ_{n+1} on representatives of the orbits of the action of the enveloping algebra of the Lie algebra W_m .*

References

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