Utility functions & Convex conjugates

Def. (Utility function) A utility function will be a continuous, strictly increasing, strictly concave and differentiable $U : (0,\infty) \to \mathbb{R}$ s.t.

$$U'(\infty) := \lim_{x\to\infty} U'(x) = 0 \quad , \quad U'(0+) := \lim_{x \searrow 0} U'(x) = \infty$$

or $U : \mathbb{R} \to \mathbb{R}$ s.t.

$$U'(\infty) = 0 \quad , \quad U'(-\infty) := \lim_{x\to -\infty} U'(x) = \infty .$$

Classical Examples:

1) Logarithmic $\quad U(x) = \log(x) \quad , \quad x \in \mathbb{R}^+$

2) Power $\quad U(x) = \frac{1}{p} x^p \quad , p<1, \ p\neq 0 \ , \ x \in \mathbb{R}^+$

3) Exponential $\quad U(x) = -\exp(-\alpha x) \ , \ \alpha>0, \ x\in\mathbb{R}.$

We shall call U' the "marginal utility" and we shall write I for the inverse of U' (which is cont. and strict increasing):

$$U'(I(y)) = I(U'(y)) = y \qquad \forall y > 0 .$$

Def. (Convex conjugate of utility fn.) . The convex dual (or conjugate) $V : \mathbb{R}^+ \to \mathbb{R}$ of U is defined by

$$V(y) = \sup_{x \in \mathrm{Dom}(U)} [U(x) - xy] \quad , \quad y > 0 .$$

Exercise: Show that 1) $V(y) = U(I(y)) - y\, I(y)$.

~~Show that~~ 2) V is convex, decreasing, cont. diff'ble

3) s.t. $V(y) \geq U(x) - xy$ with equality iff $x = I(y)$.

4) $V'(y) = -I(y)$,

5) $U(x) = \inf_{y>0} [V(y) + xy] = V(U'(x)) + xU'(x)$
$x \in \mathrm{Dom}(x)$.

Exercise: Calculate I and V for logarithmic, exponentia
and power utilities.

Defn (time dependent utility) A continuous $U : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$
s.t. $U(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}$ is a utility function $\forall t$
will be called TIME DEPENDENT UTILITY FUNCTION.

Example If U is a utility fn then

$$(t,x) \mapsto e^{-\delta t} U(x)$$

is a time dependent utility.

Model of a market

Risk free asset $dB_t = r_t B_t \, dt$, $B_0 = 1$

(we may also write $S^0_t = B_t$)

and Risky assets $S^1, S^2, \dots, S^m$:

$$dS^i_t = S^i_t \Big\{ \mu^i_t \, dt + \sum_{j=1}^{n} S^i_t \sigma^{ij} \, dW^j_t \, .$$

Here $W = (W^1, \dots, W^n)$ is a Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ generating $\mathcal{F}_t := \sigma\{s \le t : W_s\}$.

We assume $r, \mu^i \in \mathcal{A}$ and $\sigma^{ij} \in \mathcal{G}$.

The discount process is

$$D_t := \frac{1}{B_t} = \exp\left(-\int_0^t r_s \, ds\right).$$

Recall that if there is a measure $\mathbb{Q}$ s.t. all the traded assets are (local) martingales then for any contingent claim $X \in L^2(\mathcal{F}_T)$ we have its arbitrage free price given by

$$p_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} X \,\Big|\, \mathcal{F}_t\right]$$

and if $d\mathbb{Q} = \Lambda_T d\mathbb{P}$ then (Bayes rule)

$$p_t = \frac{\mathbb{E}^{\mathbb{P}}\left[\frac{B_t}{B_T} X \Lambda_T \,\Big|\, \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\Lambda_T \,|\, \mathcal{F}_t\right]} \, .$$

We know that the ~~EMM~~ RN-measure is given via the "market-price-of-risk" process φ via

$$\Lambda_t := \exp\left(-\int_0^t \varphi_s \, dW_s - \frac{1}{2}\int_0^t \varphi_s^2 \, ds\right)$$

with Λ being a true martingale. So $\mathbb{E}[\Lambda_T \mid \mathcal{F}_t] = \Lambda_t$.

Hence

$$p_t = \mathbb{E}^{\mathbb{P}}\left[\frac{B_t}{B_T} \frac{\Lambda_T}{\Lambda_t} X \mid \mathcal{F}_t\right]$$

$$= \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r_s \, ds} e^{-\int_t^T \varphi_s \, dW_s - \frac{1}{2}\int_t^T \varphi_s^2 \, ds} X \mid \mathcal{F}_t\right].$$

For any φ fixed (~~as~~ $\mathbb{Q}$ is, in general, not unique, so market price of risk is not unique) we define the DEFLATOR process Y as

$$Y_t := D_t \Lambda_t = e^{-\int_0^T r_s \, ds} e^{-\int_0^T \varphi_s \, dW_s - \frac{1}{2}\int_0^T \varphi_s^2 \, ds}.$$

Then

$$p_t = \mathbb{E}^{\mathbb{P}}\left[\frac{Y_T}{Y_t} X \mid \mathcal{F}_t\right].$$

We note that the deflator process sats.

$$dY_t = -Y_t r_t \, dt ~~+~~ - Y_t \varphi_t \, dW_t.$$

Now let us consider an investment portfolio, value X_t and π^i_t is the amount of cash is risky asset i. Take $X_0 = x > 0$.

$$dX_t = \sum_{i=1}^{m} \frac{\pi^i_t}{S^i_t} dS^i_t + \frac{X_t - \sum_{i=1}^{m} \pi^i_t}{B_t} dB_t - C_t\, dt,$$

where C is the consumption. We see that

$$dX_t = \left(\sum_i \pi^i_t \mu^i_t - C_t + r_t X_t - \sum_{i=1}^{m} r_t \pi^i_t \right) dt$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \pi^i_t \sigma^{ij}_t dW^j_t$$

$$= \left(\pi^T_t (\mu - r) + r_t X_t - C_t \right) dt + \pi^T_t \sigma_t dW_t .$$

Now the discounted portfolio value is

$$d(D_t X_t) = D_t dX_t - r_t D_t X_t dt$$

$$= D_t \left(\pi^T_t (\mu_t - r_t) - C_t \right) dt + D_t \pi^T_t \sigma_t dW_t .$$

We know that for any ~~mar~~ risk-neutral measure we must have the market-price-of-risk satisfy

$$\mu_t - r_t = \sigma_t \varphi_t$$

and so

$$d(D_t X_t) = D_t \left(\pi^T_t \sigma_t \varphi_t - C_t \right) dt + D_t \pi^T_t \sigma_t dW_t$$

and so

$$d(\cancel{X_t}\, Y_t X_t) = Y_t\, dX_t + X_t\, dY_t + dX_t\, dY_t$$

$$\cancel{= Y_t D_t (\pi_t^T \sigma_t \varphi_t - C_t)\, dt + Y_t D_t \pi_t^T \sigma_t\, dW_t - X_t Y_t}$$

$$= Y_t \left(\pi_t^T \sigma_t \varphi_t + r X_t - C_t \right) dt + Y_t\, \pi_t^T \sigma_t\, dW_t$$
$$- X_t Y_t r_t\, dt - X_t Y_t \varphi_t\, dW_t - Y_t \varphi_t \pi_t \sigma_t\, dt$$

$$= Y_t \left(\pi_t^T \sigma_t - X_t \varphi_t \right) dW_t - C_t Y_t\, dt .$$

In integral form:

$$X_t Y_t + \int_0^t C_s Y_s\, ds = x + \int_0^t Y_s \left(\pi_s^T \sigma_s - X_s \varphi_s \right) dW_s .$$

So the process

$$(\omega, t) \mapsto (X_t Y_t)(\omega) + \int_0^t (C_s Y_s)(\omega)\, ds$$

is a local Martingale (under $\mathbb{P}$). thus it will be a supermartingale and so

$$\mathbb{E}\left[X_t Y_t + \int_0^t C_s Y_s\, ds \right] \leq x .$$

(~~we can always write X_t~~ recall we always assume that π_t is s.t. $X_t > 0$ if $x > 0$. Since $Y > 0$ and $C > 0$ the ~~supp~~ loc. mart. is bdd from below by 0)

UTILITY FROM ~~A COMPLETE MARKET~~ TERMINAL WEALTH AND NO CONSUMPTION IN A COMPLETE MARKET

Consider the case $m=n$ & μ, r, σ are constant.

We have $\varphi = \sigma^{-1}(\mu - r)$ and the market is complete (every contingent claim is replicable).

We take $\nu^i_t = \frac{\pi^i_t}{X_t}$ to be the fraction of wealth ~~number of units~~ in the risky asset i, then (consumption being $c \equiv 0$)

$$X_t Y_t = x + \int_0^t X_s Y_s \left(\nu_s^T \sigma_s - \varphi_s \right) dW_s$$

and $\mathbb{E}[X_t Y_t] \le x$. (Budget constraint)

Let C be some contingent claim s.t. $\mathbb{E}[C Y_T] = x$. Then (market is complete) there is a trading strategy (control) ν s.t.

1) $X^\nu_T = C$

2) $X^\nu_t = X_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_T} C \,\middle|\, \mathcal{F}_t\right]$

$$= \mathbb{E}^{\mathbb{P}}\left[\frac{Y_T}{Y_t} C \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{Y_T}{Y_t} X_T \,\middle|\, \mathcal{F}_t\right].$$

The trading strategy ν is given by martingale rep. thm & is unique.

For this trading strategy ν we have equality in the budget constraint.

$$\mathbb{E}\left[X_T^{x,\nu} Y_T\right] = x.$$

Now consider, $y > 0$ (and $x > 0$ always)

$$\mathbb{E}\left[U(X_T)\right] = \mathbb{E}\left[U(X_T)\right] + y\Big(\underbrace{\mathbb{E}[X_T Y_T]}_{\substack{\leq x \\ \forall \nu \in \mathcal{U}}} - \mathbb{E}[X_T Y_T]\Big)$$

$$\leq \mathbb{E}\left[U(X_T)\right] + y\left(x - \mathbb{E}[X_T Y_T]\right)$$

equality holds $\Leftrightarrow \mathbb{E}[X_T Y_T] = x$.

$$= \mathbb{E}\left[U(X_T) - y Y_T X_T\right] + xy$$

$$\leq \mathbb{E}\left[V(y Y_T)\right] + xy$$

used 3) on top of p. 2; we will have equality $\Leftrightarrow X_T = I(y Y_T)$ a.s.

$\boxed{\text{Now let } v(y) := \mathbb{E}\left[V(y Y_T)\right].}$ We get

$$\mathbb{E}\left[U(X_T)\right] \leq v(y) + xy$$

and

$$\mathbb{E}\left[U(\hat{X}_T)\right] = v(y) + xy$$

iff $\mathbb{E}[\hat{X}_T Y_T] = x$ and $\hat{X}_T = I(y Y_T)$.

So there is $\hat{v} \in \mathcal{U}$ s.t.

$$J(x,\hat{v}) = \mathbb{E}\left[U(\hat{X}_T) \mid X_0 = x\right]$$
$$= \mathbb{E}\left[V(yY_T)\right] + xy,$$

provided that $\mathbb{E}[\hat{X}_T Y_T] = x$ and $\hat{X}_T = I(yY_T)$.

For any other $v \in \mathcal{U}$ we have

$$J(x,v) = \mathbb{E}\left[U(X_T^v) \mid X_0 = x\right] \le \underbrace{\mathbb{E}\left[V(yY_T)\right] + xy}_{= J(x,\hat{v})}.$$

So we see that $J(x,v) \le J(x,\hat{v})$ $\forall v \in \mathcal{U}$ (and so $\hat{v}$ is optimal) as long as $\mathbb{E}[\hat{X}_T Y_T] = x$ and $\boxed{\hat{X}_T = I(yY_T)}$ [2].

Thus with the help of the convex conjugate V of the utility function U we have the DUAL PROBLEM with a DUAL VALUE FUNCTION

$$v(y) := \mathbb{E}\left[V(yY_T)\right], \quad y > 0.$$

Now, we know that

$$\mathbb{E}\left[V(yY_T)\right] = \mathbb{E}\left[U(I(yY_T)) - yY_T I(yY_T)\right],$$

↑ bottom of p. 1, Exercise, part 1)

so by taking

$$\boxed{\begin{aligned} F(y) &:= \mathbb{E}\left[U(I(yY_T))\right]; \\ \mathcal{X}(y) &:= \mathbb{E}\left[Y_T I(yY_T)\right] \end{aligned}}$$ [3]

we get the following representation for the dual value function:

$$\boxed{v(y) = F(y) - y\,\mathcal{X}(y).} \tag{4}$$

Recalling the two optimality criteria:

$$1)\ \ E[\hat{X}_T Y_T] = x \ ; \quad 2)\ \ \hat{X}_T = I(y Y_T),$$

inserting 2) into 1) we get

$$E\big[I(y Y_T)\,Y_T\big] = x \iff \mathcal{X}(y) = x.$$

We now note that I is cont. & strictly increasing. Since $Y_T > 0$ we get that $y \mapsto \mathcal{X}(y) = E\big[Y_T I(y Y_T)\big]$ must also be cont. and strict. increasing. Hence $\mathcal{X}^{-1}$ exists and we shall write

$$\boxed{\mathcal{Y} = \mathcal{X}^{-1}.} \tag{5}$$

Then

$$\boxed{\mathcal{X}(y) = x \iff y = \mathcal{Y}(x)} \tag{6}$$

With 2) above we have that

$$\hat{X}_T = \hat{X}_T^* = I(\mathcal{Y}(x) Y_T).$$

Recall from p. 7. that

$$\hat{X}_t = E\left[\frac{Y_T}{Y_t}\hat{X}_T \,\Big|\, \mathcal{F}_t\right].$$

Now from MRT $(Y_0 = 1)$

$$\boxed{Y_t \hat{X}_t = x + \int_0^t \psi_s\, dW_s} \tag{7}$$

Recall from p.6 that (since $c \equiv 0$), $\hat{\pi}_t = \hat{X}_t \hat{v}_t$

~~$d(X_t Y_t) = Y_t \{ \hat{X}_t (v_t^T \sigma_t$~~

$$d(\hat{X}_t Y_t) = Y_t \hat{X}_t (\hat{v}_t^T \sigma_t - \varphi_t) dW_t .$$

now we also have $d(\hat{X}_t Y_t) = \psi_t \, dW_t$ i.e.

$$\psi_t \equiv Y_t \hat{X}_t (\hat{v}_t^T \sigma_t - \varphi_t) \iff \hat{v}_t = \sigma_t^{-1} \frac{(\psi_t + \varphi_t Y_t \hat{X}_t)}{Y_t \hat{X}_t}$$

So

$$\hat{X}_t Y_t = x + \int_0^t \hat{X}_s Y_s (\hat{v}_s^T \sigma_s - \varphi_s) dW_s .$$

$$\boxed{\text{Thus we have optimal control } \hat{v}_t = (\sigma_t^T)^{-1} \left(\frac{\psi_t}{Y_t \hat{X}_t} + \varphi_t \right)} \quad 8$$

and the value function (see middle of p.9, p.10;2), p.9 bottom

$$u(x) = \sup_{v \in \mathcal{U}} J(x,v) = J(x,\hat{v}) = \mathbb{E}\left[U(\hat{X}_T^x) \right]$$

$$= \mathbb{E}\left[U(I(y(x) Y_T)) \right] = F(y(x))$$

$$\boxed{u(x) = F(y(x))} \quad 9$$

$$= v(y(x)) + y(x) \underbrace{\mathcal{X}(y(x))}_{=x}$$

$$= v(y(x)) + x\, y(x) .$$

Moreover, from p.8.

$$u(x) = \mathbb{E}\left[U(\hat{X}_T) \right] = \mathbb{E}\left[U(\hat{X}_T) - y Y_T X_T \right] + xy$$

$$\le \mathbb{E}\left[V(y Y_T) \right] + xy = v(y) + xy . \quad \forall x>0, \; y>0$$

p.2; 3) with $x \leftrightarrow \hat{X}_T$; $y \leftrightarrow y Y_T$

$$\implies \sup_{x>0} \left[u(x) - xy \right] \le v(y) \qquad (*)$$

OTOH

$$\cancel{u(\mathcal{X}(y)) = F(\underbrace{\mathcal{Y}(\mathcal{X}(y))}_{=y}) = v}$$

$$u(\mathcal{X}(y)) = v(\underbrace{\mathcal{Y}(\mathcal{X}(y))}_{=y}) + y\,\mathcal{X}(y) = v(y) + y\,\mathcal{X}(y).$$

$$\iff \quad v(y) = u(\mathcal{X}(y)) - y\,\mathcal{X}(y) \le \sup_{x>0}\,[u(x) - yx] \qquad (**)$$

So from $(*)$ & $(**)$ we get that

$$\boxed{v(y) = \sup_{x>0}\,[u(x) - xy]\,; \quad y > 0.} \qquad \boxed{10}$$

This means that u and v are conjugate.

Recall that $v(y) = \mathbb{E}[V(yY_T)]$

$$\Rightarrow v'(y) = \mathbb{E}[V'(yY_T)Y_T]$$

$$= -\mathbb{E}[I(yY_T)Y_T]$$

(p. 2, 4)

$$= -\mathcal{X}(y).$$

(p. 9, bottom)

$$\boxed{v' = -\mathcal{X}} \qquad \boxed{11}$$

Recall that $u(x) = v(\mathcal{Y}(x)) + x\,\mathcal{Y}(x)$

$$\Rightarrow u'(x) = v'(\mathcal{Y}(x))\mathcal{Y}'(x) + x\,\mathcal{Y}'(x) + \mathcal{Y}(x)$$

$$= -x\,\mathcal{Y}'(x) + x\,\mathcal{Y}'(x) + \mathcal{Y}(x)$$

$$= \mathcal{Y}(x)$$

$$\boxed{u' = \mathcal{Y}} \qquad \boxed{12}$$

We thus have the following theorem.

THEOREM (Complete mkt., no consumtion, utility from terminal wealth)

Let the payoff function (primal value fn.) be

$$u(x) := \sup_{v \in \mathcal{U}} J(x,v) = \sup_{v \in \mathcal{U}} \mathbb{E}[U(X_T)]. \qquad (*)$$

Define the dual value fn. by ($\boxed{1}$, $\boxed{3}$, $\boxed{4}$)

$$v(y) := \mathbb{E}[V(yY_T)] = F(y) - y\,\mathcal{X}(y), \quad y>0$$

where $\mathcal{X}(y) := \mathbb{E}[Y_T I(yY_T)]$, $F(y) := \mathbb{E}[U(I(yY_T))]$

Define $\hat{X}_T$ by ($\boxed{2}$)

$$\hat{X}_T := I(yY_T),$$

with y determined by ($\boxed{6}$) $\mathcal{X}(y) = x \iff y = \mathcal{Y}(x)$,

where $\mathcal{Y} := \mathcal{X}^{-1}$.

Then: 1) The value functions are conjugate ($\boxed{10}$)

$$v(y) = \sup_{x>0} [u(x) - xy],$$

$$u(x) = \inf_{y>0} [v(y) + xy].$$

2) The optimal terminal wealth is $\hat{X}_T$ and

$$u(x) = J(x,\hat{v}) = \mathbb{E}[U(\hat{X}^x_T)]$$

for a control $\hat{v}$. Here

$$\hat{X}_t = \frac{1}{Y_t}\mathbb{E}[\hat{X}_T Y_T \mid \mathcal{F}_t] = \frac{1}{Y_t}\mathbb{E}[I(\mathcal{Y}(x)Y_T)Y_T \mid \mathcal{F}_t]$$

and ($\boxed{8}$)

$$\hat{v} = (\sigma_t^T)^{-1}\left(\frac{\psi_t}{Y_t\hat{X}_t} + \varphi_t\right), \quad \text{where } \varphi_t \text{ is MPR}$$

and ψ comes from MRT for

$$\hat{X}_t Y_t = x + \int_0^t \psi_s \, dW_s.$$

3) We have ($\boxed{9}$) that $u(x) = F(\mathcal{Y}(x))$.

4) We have ($\boxed{11}$, $\boxed{12}$) that $v' = -\mathcal{X}$; $u' = \mathcal{Y}$

and so $X = -v'(y)$; $y = u'(x)$.

Example: Take $U(x) = \ln x$. Then $U'(x) = \frac{1}{x}$,

so $I(y) = 1/y$. The optimal portfolio value

is $\hat{X}_T = I(y\, Y_T) = \dfrac{1}{y\, Y_T}$.

We know that y is determined by $\mathcal{X}(y) = x$

i.e. $x = \mathbb{E}[\hat{X}_T Y_T] = \mathbb{E}\left[\frac{1}{y Y_T} Y_T\right] = \frac{1}{y}$.

So $\hat{X}_T = \dfrac{x}{Y_T}$; $\hat{X}_t = \dfrac{1}{Y_t} \mathbb{E}[Y_T \hat{X}_T \mid \mathcal{F}_t]$

$$= \frac{1}{Y_t} x \Rightarrow \hat{X}_t Y_t = x + \int_0^t 0 \, dW_s$$

The optimal strategy is

$$\hat{\pi}_t = (\sigma_t^T)^{-1} \varphi_t .$$

Now $u(x) = \mathbb{E}[\log \hat{X}_T] = \mathbb{E}\left[\log \frac{x}{Y_T}\right] = \log x - \mathbb{E}[\log Y_T]$

Recall that $Y = DZ$;

$$\log Y_T = -\int_0^T \left(r_t + \frac{1}{2}|\varphi_t|^2\right) dt - \int_0^T \varphi_t \, dW_t .$$

For reasonable MPR (i.e. $\varphi \in \mathcal{H}$) we get

$$u(x) = \log x + \mathbb{E}\left[\int_0^T \left(r_t + \frac{1}{2}|\varphi_t|^2\right) dt\right] .$$