

QUANTUM CHARACTER VARIETIES AND MORITA THEORY FOR BRAIDED TENSOR CATEGORIES

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ABSTRACT. In these lectures we review the Morita (i.e. representation) theory of tensor and braided tensor categories, and discuss applications to topological field theory.

[These are the drafts as I go, they have not been proof-read or finalized. Most notably, the bibliography will be filled in later. Some of the more technical/standard exposition is lifted directly from various of my joint papers.]

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1. LECTURE I: OVERVIEW

The goal of these lecture notes is to give an informal exposition of a number of recent applications of “higher Morita theory” – i.e. the representation theory of presentable linear, tensor, and braided tensor categories – to low-dimensional topology. The bridges to topology are the twin frameworks of fully local (a.k.a) fully extended topological field theory (in dimension 2, 3, and 4), and of factorization homology (in these lectures specialized to dimension 2). There is much more than can be said on this topic in these lectures, so we will address the topology setup and its applications during the first lecture, in as colloquial a manner as possible, and spend the second and third lectures on the development of both foundational and computational tools for working with higher Morita theory.

In the direction of topology, we will cover: factorization homology, and the construction of classical and quantized character varieties, the notion of fully extended topological field theories, higher dualizability and the cobordism hypothesis.

In the direction of higher Morita theory, we will cover: the notion of locally presentable categories, tensor and braided tensor categories; modules and bimodule categories, central

tensor categories, relative tensor products, monadic reconstruction results; the Morita 3-category of tensor categories

1.1. Classical character varieties and character stacks. Fix a compact, oriented topological surface Σ , with or without boundary. Hence, Σ is homeomorphic to $\Sigma_{g,r}$, the standard surface of genus g and with r disks removed – however we do not fix such a homeomorphism. Fix also a reductive group G , with a choice of Killing form (i.e, a non-degenerate, G -invariant bilinear pairing) $\kappa_{\mathfrak{g}} \otimes \mathfrak{g} \rightarrow \mathbb{C}$. The theory is already sufficiently rich in the case of $G = SL_2$.

Associated to the data of Σ and G we have the *character variety*:

$$\begin{aligned} \mathrm{Ch}_G(\Sigma) &:= \{\rho : \pi_1(\Sigma) \rightarrow G\}/G, \\ &\cong \{G\text{-local systems on } \Sigma\}/\mathrm{iso}. \end{aligned}$$

where, we recall that a G -local system is a principle G bundle E with a flat connection $\nabla \in \Omega(\Sigma, ad(E))$. The equivalence between the two definitions is by taking monodromies of the connection (at a fixed base-point). We will need to more carefully distinguish between the *moduli stack*, and the quotient variety; along the way we will recall an explicit presentation of both moduli spaces.

Suppose that Σ has at least one boundary component, and choose the basepoint $p \in \Sigma$ lying on that component. Then the *framed* (a.k.a. gauge-fixed) moduli space is:

$$\begin{aligned} \mathrm{Ch}_G^{fr}(\Sigma) &= \{\rho : \pi_1(\Sigma) \rightarrow G\}, \\ &\cong \{G\text{-local systems on } \Sigma, \text{ with a trivialization } E_p \cong G.\}/\mathrm{iso}. \end{aligned}$$

Recall that $\pi_1(\Sigma)$ is a free group of rank $2g + r - 1$, hence we have $\mathrm{Ch}_G^{fr}(\Sigma) \cong G^{2g+r-1} = \mathrm{Spec}(\mathcal{O}_{g,r})$, the affine algebraic variety with coordinate algebra,

$$\mathcal{O}_{g,r} = \underbrace{\mathcal{O}(G) \otimes \cdots \otimes \mathcal{O}(G)}_{2g+r-1}.$$

We note that the natural G -action on Ch_G^{fr} by conjugation induces the structure of a G -module on $\mathcal{O}_{g,r}$. By definition, the character variety is $\mathrm{Spec}(\mathcal{O}_{g,r}^G)$, the affine algebraic variety with coordinate algebra the subalgebra of G -invariant functions in $\mathcal{O}_{g,r}$. As we will see throughout these lectures, this variety loses too much information, and should be replaced by the *character stack*¹. This simply means that instead of studying modules for the invariant algebra, we study “equivariant modules” M for $\mathcal{O}_{g,r}$, i.e. M is a G -representation, and an $\mathcal{O}_{g,r}$ -module, and the multiplication map $\mathcal{O}_{g,r} \otimes M \rightarrow M$ is a homomorphism of G -representations. In the sheaf-theoretic language of algebraic geometry, we have:

$$\begin{array}{ccc} & & \mathcal{QC}(\mathrm{Ch}_G^{fr}(\Sigma)) \\ & \nearrow \text{Forget} & =_{\mathcal{O}(G)_{g,r}\text{-mod}} \\ \mathcal{QC}(\mathrm{Ch}_G(\Sigma)) & & \\ =_{\mathcal{O}_{g,r}\text{-mod}_{\mathrm{Rep} G}} & & \\ & \searrow \text{Invar.} & \mathcal{QC}(\overline{\mathrm{Ch}}_G(\Sigma)) \\ & & =_{\mathcal{O}_{g,r}^G\text{-mod}} \end{array},$$

¹We do not assume any familiarity with formalism of stacks, and will develop the very few ideas we need as we go.

Hence, we use $\overline{\text{Ch}}_G(\Sigma)$ to denote the quotient variety, and $\text{Ch}_G(\Sigma)$ to denote the stack.

1.1.1. Why stacks? Character varieties, while the most elementary to define, are deficient in a few ways. The moduli problem defined by studying G -local systems is, in some precise sense smooth – one way to say this is that the universal classifying space BG is a smooth (2-shifted) symplectic stack. This means that when treated carefully (suitably stacky and suitably derived), character stacks may be considered also to be smooth, and they enjoy the abstract properties of smooth varieties. By contrast, character varieties are singular in even very simple cases, and these singularities can be traced directly to careless treatment of self-intersections, and of stabilizers.

Exercise 1.1. Let $G = SL_2$. Show that

$$\text{Ch}_G^{fr}(S^1 \times I) = G, \quad \overline{\text{Ch}}_G(S^1 \times I) = \mathbb{C}^\times / (\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}$$

$$\text{Ch}_G^{fr}(T^2) = \{(A, B) \in G \times G \mid AB = BA\}, \quad \overline{\text{Ch}}_G(T^2) = (\mathbb{C}^\times \times \mathbb{C}^\times) / (\mathbb{Z}/2),$$

and show that the latter variety is singular at the points $(\pm 1, \pm 1)$. Hints: use the presentation of π_1 in each case, and for $\overline{\text{Ch}}_G$ use (generic) diagonalizability of matrices, and (generic) simultaneous diagonalizability of commuting matrices.

More pragmatically, we will see in these lectures that character *stacks* (and their quantizations):

- (1) contain strictly more information than character *varieties* (and their quantizations), yet
- (2) are easier to compute with algebraically, and
- (3) define a fully local topological field theory, and most importantly
- (4) involve beautiful instances of higher algebra.

1.2. Atiyah-Bott/Goldman Poisson bracket. All three versions of the moduli space carry a fundamentally important *Poisson bracket*, constructed independently by Atiyah-Bott (in differential/ G -local systems terms), and by Goldman (in more algebraic terms), and subsequently reformulated by Fock-Rosly (in representation-theoretic terms). Recall that a Poisson bracket on an affine variety X is a Lie bracket $\{, \}$ on $\mathcal{O}(X)$, such that $f \mapsto \{f, -\}$ defines a Lie algebra homomorphism to vector fields on X .

The basic ingredients of each construction is the same:

- (1) The Poincare pairing on first cohomology of Σ , and
- (2) The Killing form on $\mathfrak{g} = \text{Lie}(G)$.

To give a brief idea: in case Σ is closed, the Atiyah-Bott construction begins by identifying $T_E \text{Ch}_G(\Sigma) = \Omega^1(\Sigma, \text{ad}_E)$. The Poisson bracket is given instead in terms of a symplectic pairing via the composition,

$$\Omega^1(\Sigma, \text{ad}_E) \otimes \Omega^1(\Sigma, \text{ad}_E) \xrightarrow{\wedge} \Omega^2(\Sigma, \text{ad}_E) \xrightarrow{\kappa} \Omega^2(\Sigma, \mathbb{C}) \xrightarrow{\int_\Sigma} \mathbb{C}.$$

That is, we use wedge product on differential forms then the Killing form, and then integration over the surface, to define the symplectic pairing, hence upon dualizing, the Poisson bivector.

By contrast, Goldman’s construction relies on group cohomology of $\pi_1(\Sigma)$ for punctured surface, and was completed by Alekseev-Malkin-Meinrenken for closed surfaces using “group-valued Hamiltonian reduction”. Meanwhile Fock-Rosly present the Poisson bracket on explicitly using classical r -matrices, given a “ribbon graph” decomposition of Σ (more on this

later). At the level of character varieties, the Poisson bracket can be described using graphical calculus of G -colored ribbons, as was shown by Turaev ($G = SL_2$, and Roche-Szenes (general G).

The Poisson brackets so constructed are *natural* in a very strong sense: they pushforward under embeddings of surfaces. When the surface is closed, the Poisson bracket is non-degenerate; given a 3-manifold M with $\partial M = \Sigma$, $\text{Ch}_G(M)$ defines a Lagrangian inside of $\text{Ch}_G(\Sigma)$.

1.3. A tale of three quantizations. Suppose that A is an algebra (typically non-commutative) defined over formal power series $\mathbb{C}[[\hbar]]$, that A is in fact free as a $\mathbb{C}[[\hbar]]$ -module, and that upon setting $\hbar = 0$ the algebra $A_0 = A \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}$ becomes commutative. Then we may define, for $a, b \in A_0$ the Poisson bracket,

$$\{a, b\} = \frac{\tilde{a}\tilde{b} - \tilde{b}\tilde{a}}{\hbar} \pmod{\hbar},$$

where \tilde{x} denotes an arbitrary lift of x to A (it is an exercise to see the Poisson bracket is independent of this choice).

We say in this case that A is a deformation quantization of A_0 with its Poisson bracket, and we say that A_0 is a degeneration of A . We note that degenerations are unique, but deformation quantizations are not at all so. An important problem, the subject of these lectures, is to give a deformation quantization which enjoys all the naturality of $\text{Ch}_G(\Sigma)$. Such quantizations feature in the quantum geometric Langlands, hence by Kapustin-Witten's construction, to quantization of $N = 4, d = 4$ super Yang-Mills theory, and more directly to the Chern-Simons theory of knots and 3-manifolds, and as we shall see to the Crane-Yetter-Kauffman invariants of 3- and 4-manifolds.

Let us review here three distinct mechanisms for constructing such deformation quantizations which have appeared in the literature in the past 25 years, and compare and contrast their pro's and con's.

1.3.1. Skein modules, skein algebras, and skein categories. Recall that many knot invariants such as the Alexander and Jones polynomials, as well as the Kauffmann and HOMFLY polynomials which refine them, are defined using "skein relations". For example, the Kauffmann polynomial satisfies the relations

$$\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle, \quad \langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rangle = A \langle \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \rangle + A^{-1} \langle \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} \rangle,$$

which hold for any link $K \subset S^3$, and the equations indicate that the involved links are identical outside the depicted ball, and differ as indicated within it. We can rephrase this by defining the skein module, $\text{SkMod}(S^3)$ to be the vector space formally spanned by all links in S^3 , modulo isotopy and the linear relations (and their variants for the other polynomials). In these terms, we have that $\text{SkMod}(S^3) = \mathbb{C}[A, A^{-1}]$, which implies at once that we can reduce an arbitrary knot to a multiple of the unknot (because the skein module is at most one-dimensional), and that this is a well-defined invariant of the knot (because the space is at least one-dimensional). Given this perspective, it is natural then to define skein modules for arbitrary oriented 3-manifolds.

In [], Turaev proved:

Theorem 1.2. *Let $G = SL_2$, and let $\text{SkAlg}(\Sigma)$ denote the skein module of $\Sigma \times I$, equipped with the algebra structure coming from stacking in the I -direction. Then $\text{SkAlg}(\Sigma)$ is a deformation quantization of the character variety of Σ , with the Atiyah-Bott-Goldman Poisson bracket.*

The proof relies on an explicit basis for the classical character variety. Using the Cayley-Hamilton identity for 2×2 matrices, one can identify a basis of $\mathcal{O}(\overline{\text{Ch}}_G(\Sigma))$ with the set of crossingless multi-loop diagrams drawn on Σ . Multiplication is given by first superimposing loops, and then recursively reducing to crossingless multi-loops using Cayley-Hamilton, which in graphical form becomes²:

$$\langle L \cup \bigcirc \rangle = 2\langle L \rangle, \quad \langle \text{crossing} \rangle = \langle \text{no crossing} \rangle + \langle \text{other crossing} \rangle,$$

Turaev’s proof therefore consists of showing that the deformation defined by $\text{SkMod}(\Sigma)$ admits the same basis as a free module over $\mathbb{C}[A, A^{-1}]$ (while this is a natural and beautiful statement, its proof should not be considered easy!!). Hence, at least in the SL_2 case, it is known that to quantize the character variety, one may look to skein algebras.

Pros (+) and cons (-) of skein approach:

- (+) Clear topological meaning
- (+) No unnatural choices on Σ .
- (+) Obvious extension to 3-manifolds.
- (-) The 3-manifold extension is not flat in q .
- (-) Only captures $\overline{\text{Ch}}_G(\Sigma)$.
- (-) Does not define a TFT.

1.3.2. Quantum cluster algebras. Fock and Goncharov’s proposal to quantize character varieties treats the presence of stabilizers by introducing something between the framed variety and the quotient variety, which we call the *decorated* character variety³. The algebraic input is now the group G , its Borel subgroup B and its unipotent radical N , and we will also use the Weyl group W . The topological data now is a decorated surface $\tilde{\Sigma} = (\Sigma, \Sigma_T)$, consisting of the surface Σ together with a subset $\Sigma_T \subset \Sigma$, which must be a union of disks on the interior and half-disks on the boundary. We denote $\Sigma_G = \Sigma \setminus \overline{\Sigma_T}$, and $\Sigma_B = \Sigma \setminus (\Sigma_G \cup \Sigma_T)$.⁴

A decorated local system on $\tilde{\Sigma} = (\Sigma, \Sigma_T)$ is the data of a local system on Σ , together with a reduction to B on Σ_B and to T on Σ_T . Unwinding a bit, this means that, in addition to the local system data, we fix the data of an ‘affine flag’ $F \in G/N$ for each region S_T , and require further that the monodromies around interior points preserve the underlying flag $\pi(F) \in G/B$.

We denote by

$$\text{Ch}_G^{dec}(\tilde{\Sigma}) = \{\text{Decorated } G\text{-local systems on } \tilde{\Sigma}\}/iso,$$

the corresponding moduli stack, and as before by $\overline{\text{Ch}}_G^{dec}$ the associated quotient variety.

The key to Fock and Goncharov’s construction is that Ch_G^{dec} has open charts where the G action is actually free, and hence where Ch_G^{dec} and $\overline{\text{Ch}}_G^{dec}$ coincide, and moreover take

²hence in these conventions we degenerate at $A = -1$, rather than $A = 1$, for reasons we won’t get into here.

³Fock and Goncharov called their construction rather the *framed* character variety, but we will use the word *decorated* instead to avoid confusion

⁴Fock and Goncharov considered rather “marked points” on the boundary, and “punctures” on the interior; this is up to isotopy the same data, the reason for the shift will become apparent later.

an extremely simple form. Specifically, for each triangulation $\underline{\Delta}$ of Σ , with vertices in the region S_T , and on each triangle a spelling of the longest word $\omega_0 \in W$ in terms of simple reflections – for SL_2 this is unique. This extra data determines (in a way I won't spell out here) an open subset isomorphic to $(\mathbb{C}^\times)^d$, where $d = \dots$ is the dimension of $\overline{\text{Ch}}_G^{dec}$. Moreover, on this chart, the Atiyah-Bott/Goldman bracket is *log-canonical*, meaning $\{z_i, z_j\} = a_{ij}z_i z_j$ for some skew-symmetric integer matrix $A = (a_{ij})$; they like to encode the matrix A as a quiver. The transition functions between different charts moreover take an especially nice form, which can be expressed combinatorially as a “mutation”; the data of these charts and mutations is what is called a cluster variety. They define their quantization by simply declaring $z_i z_j = q^{\frac{a_{ij}}{2}} z_j z_i$ (which is clearly a deformation quantization of each chart), and by hand define q -deformations of the mutations.

Pros (+) and cons (-) of cluster approach:

- (+) Explicit and combinatorial local structure.
- (+) Flatness is built in.
- (-) Requires choices of S_T and triangulations – topological meaning is unclear.
- (-) The global structure is mysterious – need to glue together all quantum charts.
- (+) Cluster positivity, Unitarity (see Shapiro's talk next week).

1.3.3. Alekseev-Grosse-Schomerus algebras. Following the “RTT” presentations known to exist for quantum coordinate algebras, Alekseev introduced (and subsequently studied, with Grosse and Schomerus) certain explicit deformation quantizations of the algebras $\mathcal{O}_{g,r}$ which we will denote simply $A_{g,r}$. These are defined as a braiding-twisted tensor product of $2g + r - 1$ copies of an algebra $\mathcal{O}_q(G)$ – the ad-equivariant quantized coordinate ring of G ⁵ They are constructed via an explicit presentation as follows:

- (1) The Fock-Rosly Poisson bracket has a matrix presentation, where each the Poisson bracket $\{X_i, X_j\}$ between pairs of coordinate matrices

$$X_i, X_j \in A_1, B_1, \dots, A_g, B_g, P_1, \dots, P_{r-1}$$

is expressed explicitly using the “classical r -matrices” (the same which define quantum groups).

- (2) Replacing sums with products and classical r -matrices on G with quantum R -matrices for $U_q(\mathfrak{g})$ defines a deformation quantization of the Fock-Rosly Poisson bracket.

Pros (+) and cons (-) of AGS approach:

- (+) Tools for representation theory of $U_q(\mathfrak{g})$.
- (+) Relation to \mathcal{D} -modules and geometric representation theory.
- (+) Flatness in q is built into construction.
- (-) Only for framed $\text{Ch}_G^{fr}(\Sigma)$, where Σ is a surface with boundary. Unclear how to remove framing, or to seal punctures.
- (-) Required a choice of ribbon graph decomposition to define it.

1.4. Factorization homology. To summarize the preceding section: each of these three traditional approaches exposes a beautiful facet of the rich theory of quantizations of character varieties, and each is *a priori* a distinct perspective. Before the advent of factorization homology, it remained a long-standing question how to unify the three constructions: by

⁵Hence, $\mathcal{O}_q(G)$ quantizes the ad-equivariant Semenov-Tian-Shansky, Poisson bracket, and is not to be confused with the FRT quantization of G – often also denoted by $\mathcal{O}_q(G)$, which quantizes the Sklyanin bracket.

virtue of each one being very explicit, and the cluster and AGS constructions depending crucially on extra choices, it is very thorny to make direct comparisons. The non-canonicity of the cluster and AGS constructions further prevent them from defining a 3D TFT, while the skein construction suggests a TFT construction, but it works only when enhanced to *skein categories* (discussed later on).

These lectures will focus on a new construction of quantizations of character varieties, which unifies the three perspectives, allowing us to keep all the (+)'s and lose all the (-)'s. The factorization homology construction:

- (1) takes as basic algebraic input a braided tensor category \mathcal{A} (e.g. $\mathcal{A} = \text{Rep}_q(G)$),
- (2) uses modern homotopical/ higher categorical methods, thereby
- (3) describes the character stacks and their quantizations as primary objects, the varieties as secondary objects, and thereby
- (4) outputs categorical invariants of surfaces, which generalize Hochschild homology of algebras, and
- (5) recover each of the three traditions described above as special cases, and along the way
- (6) requires us to learn Morita theory for tensor and braided tensor categories.

The basic topological input is:

Definition 1.3. The $(2, 1)$ -category Mfld_{fr}^2 (resp. Mfld_{or}^2) has:

- As its objects, oriented (resp. framed) surfaces,
- As the 1-morphisms from S to T , all framed (resp. oriented) embeddings $S \hookrightarrow T$,
- As the 2-morphisms, the isotopies of embeddings, themselves considered modulo isotopies of isotopies.

The disjoint union of surfaces equips Mfld^2 with the structure of a symmetric monoidal bicategory.

Definition 1.4. The bicategory $\text{Disk}_{fr/or}^2$ is the full subcategory of $\text{Mfld}_{fr/or}^2$ whose objects are finite disjoint unions of framed/oriented disks.

Remark 1.5. Let us remark in passing that the notion of framed/oriented embeddings is not the most obvious one: for example a framed embedding is an embedding, together with the data of an isotopy between the push-forward framing and the framing on the image.

Remark 1.6. We will make free use the notion of bicategory – also known as a 2-category in other sources – in these notes, but we will not recall complete definitions. Hence, we will discuss objects, morphism, and 2-morphisms in a bicategory. By a $(2, 1)$ -category, we will mean a bicategory in which all 2-morphisms are invertible – this makes life much easier, and is all that is needed in most of these notes. See NCatLab for a quick review. Later in the notes we will discuss (∞, n) -categories [**HigherAlgebra**], for $n = 1, 2, 3, 4, \dots$, which we will treat as a black-box, because all arguments to be presented in any detail can be reduced to arguments in bicategories.

We will delay until tomorrow a precise definition. For today, it is enough to work with the following informal (and incomplete) definition:

Definition 1.7 (Ayala-Francis, Lurie). Fix a braided tensor category \mathcal{A} . The factorization homology of surfaces with coefficients \mathcal{A} is:

- (1) A functorial assignment,

$$Z : \text{Mfld}_{fr/or}^2 \rightarrow \mathcal{W},$$

where \mathcal{W} is some “world” (=symmetric monoidal higher category) in which to do algebra. For example $\mathcal{W} = (\text{Vect}, \otimes_{\mathbf{k}}), (\text{Cat}, \times), (\mathbf{Pr}, \boxtimes), \dots$

- (2) It is monoidal for disjoint unions, and functorial for *embeddings*, and their *isotopies*:

$$(i : M \hookrightarrow N) \mapsto (Z(i) : Z(M) \rightarrow Z(N)),$$

$$(\gamma : i \rightarrow j) \mapsto (Z(\gamma) : Z(i) \xrightarrow{\sim} Z(j)).$$

- (3) We have an identification $Z(\mathbb{D}) = \mathcal{A}$, with its braided tensor structure coming from applying (2) to disk inclusions.

- (4) The assignment Z satisfies *excision*:

$$Z(\Sigma_1 \sqcup_{P \times I} \Sigma_2) = Z(\Sigma_1) \boxtimes_{Z(P \times I)} Z(\Sigma_2),$$

- (5) The functor Z is *canonically determined* by properties (1) - (4).

Definition 1.8. (Ben-Zvi–Brochier-J) The quantum character variety $Z_q(\Sigma)$ is the presentable (i.e. $\mathcal{W} = \mathbf{Pr}$) factorization homology of Σ with coefficients in the braided tensor category $\text{Rep}_q(G)$ of locally finite-dimensional $U_q(\mathfrak{g})$ -modules.

Example 1.9. An important object of $Z_q(\Sigma)$ is given simply by the empty disk embedding $\emptyset \hookrightarrow \Sigma$, which determines a functor $\text{Vect} \rightarrow \Sigma$, hence an object we denote $\text{Dist}_\Sigma \in Z_q(\Sigma)$, which we will see quantizes the structure sheaf in $\mathcal{QC}(\text{Ch}_G(\Sigma))$.

The excision property is on the one hand eminently computable, and on the other hand, uniquely determines the theory. This allows us to relate it to the traditional approaches:

- (1) (BZBJ) For a surface Σ with at least one boundary component, we have (many) equivalences of categories,

$$Z_q(\Sigma) \simeq A_\Sigma\text{-mod}_{\text{Rep}_q(G)},$$

for a canonical algebra object in $A_\Sigma \in \text{Rep}_q(G)$. A ribbon graph presentation of Σ determines an isomorphism $A_\Sigma \cong A_{g,r}$ with the AGS algebras.

- (2) (BZBJ) For a closed surface Σ obtained by gluing a disk to a once-punctured Σ° , we have an equivalence of categories between $Z_q(\Sigma)$ and the quantum Hamiltonian reduction of A_{Σ° with respect to a canonical “quantum moment map.”
- (3) (J-Le-Schrader-Shapiro) Fix in addition to $\mathcal{A} = \text{Rep}_q(G)$ the braided tensor category $\text{Rep}_q(T)$, and the tensor category $\text{Rep}_q(B)$. Then the factorization homology of *decorated* surfaces with coefficients in the coefficients $(\text{Rep}_q(G), \text{Rep}_q(B), \text{Rep}_q(T))$ recovers (generalizes, and amends) the Fock-Goncharov cluster quantization.
- (4) (Cooke) For any surface Σ we have a canonical equivalence of categories,

$$Z_q^{cp}(\Sigma) \simeq \text{SkCat}(\Sigma),$$

between the subcategory of compact-projective objects (=quantum vector bundles) of $Z_q(\Sigma)$, and the so-called “skein category” of Σ . Equivalently, taking Yoneda free cocompletions, we have:

$$Z_q(\Sigma) \simeq \widehat{\text{SkCat}}(\Sigma).$$

This uses a roadmap proposed by Johnson-Freyd, building on ‘blob homology’ ideas of Morrison and Walker.

(5) (Cooke) In particular, we have an isomorphism,

$$\text{SkAlg}(\Sigma) \cong \text{End}(\text{Dist}_\Sigma),$$

for a canonical “distinguished object of Σ ,

Beyond recovering and thereby unifying the three traditional approaches to quantization of character varieties, the mechanism of factorization homology offers an important extension to higher (and lower) dimensions.

- (1) (Scheimbauer) The assignment Z extends *down* to define a fully local (a.k.a) fully extended 2D TFT.
- (2) (Brochier-J-Snyder) The assignment Z extends *up* to define fully local 3D (rigid) and 4D (fusion) topological field theories. This relies on works of (Johnson-Freyd–Scheimbauer and Haugseng in topology and Brandenburg–Chivrasitu–Johnson-Freyd in category theory.
- (3) (Costello-Gwilliam, Elliot-Safronov, Williams, ...) “Topological twists” of SUSY quantum field theories give rise to factorization homology theories, hence fully local topological field theories. Alternatively, holomorphic twists give rise to holomorphic factorization algebras, hence conformal field theories.
- (4) (Ayala-Francis, AF-Tanaka, AF-Rozenblum) The role of *algebras* and their Morita theoreis (e.g. of braided tensor categories in the coming lectures) can be replaced by completely general higher categories, giving a *constructive* proof of the cobordism hypothesis.

Prediction: In the coming two decades, the works (3) and (4) above and their ramifications will reduce the entire era of topological field theory to a historical footnote in the saga of factorization homology.

2. LECTURE II: RECONSTRUCTION THEOREMS FOR TENSOR AND BRAIDED TENSOR CATEGORIES

2.1. Finite categorical linear algebra. The kinds of categories we will consider in these lectures are *large*, akin to the categories of representations of algebraic groups, in contrast to *small* categories, like the categories of finite-dimensional representations of finite groups, where many of these results are easier and more well-known. The large world (namely the notion of locally presentable categories) takes some set-up, but then things work essentially the same. To motivate the coming definitions, let’s recall some easy facts about finite semi-simple abelian \mathbf{k} -linear categories.

Suppose for the moment that \mathcal{C} is a finite and semi-simple abelian and \mathbf{k} -linear category, in other words that we can write \mathcal{C} as a finite sum,

$$\mathcal{C} = \text{Vect}^{fd} \oplus \dots \oplus \text{Vect}^{fd},$$

of copies of the category Vect . Suppose \mathcal{D} is another such category, and that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a linear functor. Let X_1, \dots, X_n and Y_1, \dots, Y_m denote the simple objects of \mathcal{C} and \mathcal{D} , respectively. In this generality F *always* has a left and right adjoint⁶, determined uniquely by the isomorphisms,

$$\text{Hom}_{\mathcal{C}}(X, F^R(Y)) = \text{Hom}_{\mathcal{D}}(F(X), Y), \quad \text{Hom}_{\mathcal{C}}(F^L(X), Y) = \text{Hom}_{\mathcal{D}}(X, F(Y)).$$

⁶If you’re familiar with theorems about existence of adjoints, this might seem surprising, but the point is that exactness is implied by semi-simplicity and \mathbf{k} -linearity.

In fact we can write a formulas for F^R and F^L ; we have:

$$F^R(Y) = \bigoplus_{X_i} \text{Hom}_{\mathcal{D}}(F(X_i), Y) \otimes X_i, \quad F^L(Y) = \bigoplus_{X_i} \text{Hom}_{\mathcal{D}}(Y, F(X_i))^* \otimes X_i$$

Remark 2.1. A useful analogy is that \mathcal{C} is a categorical analog of a finite-dimensional Hilbert space, with the inner product $\langle -, - \rangle$ given by the Hom pairing. Then F^L and F^R are the categorical analog of the transpose matrix. Indeed, writing

$$F(X_i) = \bigoplus_j N_i^j Y_j \implies F^R(Y_j) = \bigoplus_i N_i^j X_i,$$

so it just transposes the non-negative integer matrix N_i^j of multiplicities.

Let's phrase monadic reconstruction in this easy setting. Recall that one formulation of the definition of adjoint functors is that we have a unit $\eta : \text{id}_{\mathcal{C}} \rightarrow F^R F$, and we have a counit $\epsilon : F F^R \rightarrow \text{id}_{\mathcal{D}}$, satisfying some natural axioms. We can use these maps to regard $F^R F$ as a ‘‘monad’’, i.e. a *unital algebra* in $\text{End}(\mathcal{C})$: the unit is η , and the multiplication is:

$$(F^R F)(F^R F) \xrightarrow{\epsilon} F^R F$$

Exercise 2.2. Show that the product defined above is associative.

Definition 2.3. A $F^R F$ -module in \mathcal{C} is an object X of \mathcal{C} , together with a morphism in \mathcal{C} , $F^R F(X) \rightarrow X$, which is associative in the obvious sense. Denote by $F^R F\text{-mod}_{\mathcal{C}}$ the category of such modules.

Note that the functor F^R itself defines a functor $\widetilde{F}^R : \mathcal{D} \rightarrow F^R F\text{-mod}_{\mathcal{C}}$, since for $Y \in \mathcal{D}$, we have $(F^R F)(F^R(Y)) \xrightarrow{\epsilon} F^R(Y)$.

Theorem 2.4 (Barr-Beck monadicity). *Suppose that F^R is conservative (i.e. that $F^R(Y) = 0 \iff Y = 0$). Then the functor,*

$$\widetilde{F}^R : \mathcal{D} \rightarrow F^R F\text{-mod}_{\mathcal{C}},$$

is an equivalence of categories.

If the conditions of the theorem are satisfied we say that F (alternatively F^R , or the pair (F, F^R)) is *monadic*.

Exercise 2.5. Dually, one can instead reconstruct \mathcal{C} in terms of the *co-algebra* $F F^R$ in $\text{End}(\mathcal{D})$ (a ‘co-monad’). Work out the statement of this theorem.

Remark 2.6. Note that by the transpose formula, F^R is conservative if and only if every simple object appears as a summand of $F(X_i)$ for some simple $X_i \in \mathcal{C}$. The monadicity theorem can be understood as a categorification of the obvious statement that, if V and W are finite-dimensional Hilbert spaces, then a linear map $f : V \rightarrow W$ is surjective if and only if its adjoint is injective, and in this case $f^T f$ defines a projection operator onto a subspace of V , which is isomorphic to W via the inclusion f^T

The monadicity theorem and its generalizations, despite being very easy to prove, are of utmost utility: In a typical situation, we will presume to understand \mathcal{C} very well, and \mathcal{D} not at all, so the theorem allows us to describe the entire category \mathcal{D} *internally* in terms of \mathcal{C} . It is the most useful in the setting of *tensor categories* and their *module categories*

Definition 2.7. A *finite semisimple tensor category* is a finite semisimple category \mathcal{A} equipped with a linear functor

$$\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A},$$

a distinguished object $\mathbf{1}_{\mathcal{A}}$ (the unit) and natural isomorphism

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

and

$$l_x : \mathbf{1}_{\mathcal{A}} \otimes x \rightarrow x \qquad r_x : x \otimes \mathbf{1}_{\mathcal{A}} \rightarrow x$$

such that the following diagrams commute:

$$\begin{array}{ccc} & (x \otimes (y \otimes z)) \otimes w & \\ \alpha_{x,y,z} \otimes \text{id}_w \nearrow & & \searrow \alpha_{x,y \otimes z,w} \\ ((x \otimes y) \otimes z) \otimes w & & x \otimes ((y \otimes z) \otimes w) \\ \alpha_{x \otimes y,z,w} \downarrow & & \downarrow \text{id}_x \otimes \alpha_{y,z,w} \\ (x \otimes y) \otimes (z \otimes w) & \xrightarrow{\alpha_{x,y,z \otimes w}} & x \otimes (y \otimes (z \otimes w)) \end{array}$$

$$\begin{array}{ccc} (x \otimes \mathbf{1}_{\mathcal{A}}) \otimes y & \xrightarrow{\alpha_{x,\mathbf{1}_{\mathcal{A}},y}} & x \otimes (\mathbf{1}_{\mathcal{A}} \otimes y) \\ & \searrow r_x & \swarrow l_x \\ & x \otimes y & \end{array}$$

Remark 2.8. As customary, we suppress α , l and r when those are clear from the context, as they can be uniquely filled in.

Definition 2.9. We say that the tensor category is *rigid* if, for every $X \in \mathcal{A}$ the left and right adjoints to the functor $Y \mapsto Y \otimes X$ are *representable*, i.e. we want there to exist objects X^* and *X (the left and right duals), with evaluation $\epsilon : X^* \otimes X \rightarrow \mathbf{1}$, and coevaluation $\eta : \mathbf{1} \rightarrow X \otimes X^*$ satisfying the usual adjunction axioms.

Remark 2.10. A rigid finite semi-simple tensor category is commonly called a *multi-fusion* category; if the unit is a simple object, it is called a *fusion* category.

Definition 2.11. Let \mathcal{A} be a finite semisimple tensor category. A left (resp, right) \mathcal{A} -module is a finite semisimple category \mathcal{M} equipped with a linear functor, $\otimes : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$ (resp, $\otimes : \mathcal{M} \times \mathcal{A} \rightarrow \mathcal{M}$), together with an associativity constraint and a natural isomorphism $\mathbf{1}_{\mathcal{A}} \otimes m \rightarrow m$ for $m \in \mathcal{M}$ making the analogous pentagon diagram commute.

Definition 2.12. Let \mathcal{M}, \mathcal{N} be two module categories over \mathcal{A} . Then a *left module functor*, or \mathcal{A} -linear functor, is a pair of a functor $F : \mathcal{M} \rightarrow \mathcal{N}$ and a natural isomorphism $f : F(a \otimes m) \rightarrow a \otimes F(m)$ for $a \in \mathcal{A}, m \in \mathcal{M}$ making the obvious diagrams commute. Right module and bimodule functors are defined similarly.

Exercise 2.13. Let \mathcal{A} be a rigid tensor category, and let \mathcal{M} and \mathcal{N} be (left, say) module categories, with a module functor $F : \mathcal{M} \rightarrow \mathcal{N}$. Then the right adjoint $F^R : \mathcal{N} \rightarrow \mathcal{M}$ carries a canonical module structure.

Definition 2.14. Let \mathcal{A} be a rigid tensor category, and let \mathcal{M} be its right module category. Let $m \in \mathcal{M}$ be an object, and define $\text{act}_m : \mathcal{A} \rightarrow \mathcal{M}$ by $\text{act}_m(X) = X \otimes m$. We define the *internal hom functor*

$$\underline{\text{Hom}}(m, n) = \text{act}_m^R(n), \quad \underline{\text{End}}(m) = \underline{\text{Hom}}(m, m)$$

Exercise 2.15. By playing with adjunction formulas, construct associative “composition maps”,

$$\underline{\text{Hom}}(n, o) \otimes \underline{\text{Hom}}(m, n) \rightarrow \underline{\text{Hom}}(m, o),$$

hence construct in particular an algebra structure on $\underline{\text{End}}(m)$, and identify the action of tensoring by this algebra with the action of the abstract monad $(\text{act}_m^R \text{act}_R)$.

Theorem 2.16. *Suppose that m generates \mathcal{M} as an \mathcal{A} -module, i.e. that act_m^R is conservative. Then we have an equivalence of module categories,*

$$\mathcal{M} \simeq \underline{\text{End}}(m)\text{-mod}_{\mathcal{A}}$$

Exercise 2.17. Take $\mathcal{A} = \text{Vect}^{fd}$, and suppose that \mathcal{C} is a finite semi-simple category, regarded as an \mathcal{A} -module category. Show that $\bigoplus_i X_i$ is a generator of \mathcal{C} , and hence define an equivalence,

$$\mathcal{C} \simeq \left(\bigoplus_i \mathcal{C} \right)\text{-mod}.$$

Exercise 2.18. Take $\mathcal{A} = \text{Rep}(G)$ for a finite group G . Consider \mathcal{A} as an $\mathcal{A} \otimes \mathcal{A}^{mop}$ -module category via $(X, Y) \cdot Z = X \otimes Z \otimes Y$. Show that the monad for this module gives the function algebra $\mathbb{C}[G]$, (with pointwise multiplication and not convolution!) as a $G \times G$ -module, and that upon applying the multiplication functor again, we get $\mathbf{k}[G]$ with its adjoint action.

Exercise 2.19. Work out a variant of the previous exercise but for the comonadic ‘forgetful’ functor $G\text{-mod} \rightarrow \text{Vect}$ to obtain the group algebra (with convolution structure).

Exercise 2.20. Repeat the previous two exercises for an arbitrary finite-dimensional Hopf algebra

2.2. Presentable categorical linear algebra. Fix a field \mathbf{k} (for many, but not all definitions and construction, \mathbf{k} could be a ring instead). In these lectures we will contend with \mathbf{k} -linear categories which are neither finite nor semi-simple, and therefore existence of adjoints is not guaranteed. However, we will develop tools to work with adjoint functors and use these to prove reconstruction theorems for them.

Definition 2.21. A category is said to be \mathbf{k} -linear if it is enriched and tensored over the category of \mathbf{k} -vector spaces.

Definition 2.22. Given a diagram $D = (d_i, f_j)$ in a category \mathcal{C} , its *colimit*, is the *initial* object $\text{colim}(D) \in \mathcal{C}$, together with morphisms $d_i \rightarrow \text{colim}(D)$, commuting with all f_j ’s. This means given any other candidate object $\text{colim}(D)'$, with its morphisms $d_i \rightarrow \text{colim}(D)'$, we have a unique morphism $\text{colim}(D) \rightarrow \text{colim}(D)'$ making the obvious diagram commute.

Definition 2.23. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *cocontinuous* if, for any diagram D , the comparison morphism $\text{colim } F(D_i) \rightarrow F(\text{colim } D_i)$ is an isomorphism.

Exercise 2.24. Produce the necessary diagram D realizing each of the following as examples of colimits: the cokernel of a morphism $f : X \rightarrow Y$, the direct sums $V \oplus W$, and tensor products $V \otimes W$ of vector spaces, the relative tensor product $M \otimes_R N$ of modules over a ring.

Definition 2.25. Let \mathcal{C} be a \mathbf{k} -linear category. An object $c \in \mathcal{C}$ is called

- *compact-projective*, if $\text{Hom}(c, -)$ is cocontinuous.
- *a generator* if $\text{Hom}(c, -)$ is conservative (injective on objects) and faithful (injective on morphisms)

Exercise 2.26. Let A be a (possibly infinite-dimensional) \mathbf{k} -algebra. Show that the compact objects of $A\text{-mod}$ are the finitely presented A -modules (modules which have finitely many generators, and finitely many relations, equivalently cokernels of morphisms between finite-rank free modules), while compact-projectives are direct summand of finite-rank free modules.

Definition 2.27. We say that a \mathbf{k} -linear category

- is *locally finitely presentable* if it admits arbitrary small colimits, and every object is a filtered colimits of compact objects, the collection of which form an essentially small category.
- *has enough compact projectives* if it admits arbitrary small colimits, and every object is a filtered colimits of compact objects, the collection of which form an essentially small category.

Remark 2.28. A typical filtered colimit to keep in mind is an infinite direct sum, or more generally a directed system. A typical finite colimit is a cokernel of a morphism. An arbitrary colimit can be written as a filtered system of finite colimits, hence compact-projective could be rephrased as “commutes with finite colimits” and “commutes with filtered colimits”.

Remark 2.29. In some loose sense a cokernel of a map $f : X \rightarrow Y$ can be thought of a the “difference” between X and Y , and the direct sum is their sum. Hence a locally presentable category is akin to an infinite-dimensional Hilbert space, where infinite sums are allowed, but we must contend with convergence issues.

To get a feel for the definitions, the best thing is to solve some exercises:

Exercise 2.30. Show that a category which has enough compact projectives in the above sense is in particular locally finitely presentable.

Exercise 2.31. Show that the category $A\text{-mod}$ has enough compact projectives, hence is locally presentable, for an arbitrary algebra A (hint: show that every A -module is a colimit of copies of A).

Exercise 2.32. Suppose that X is a compact projective generator. Then show that $\text{Hom}(X, -) : \mathcal{C} \rightarrow \text{End}(X)^{op}$ defines an equivalence of categories.

Definition 2.33. We denote by:

- **Pr** the (2,1)-category of locally presentable categories, cocontinuous functors and natural isomorphisms.
- **Pr^o** the full subcategory consisting of categories having enough compact projectives.

The 2-category \mathbf{Cat} ($= \mathbf{Cat}_{\mathbf{k}}$) of \mathbf{k} -linear categories is symmetric monoidal: Given \mathbf{k} -linear categories \mathcal{C} and \mathcal{D} , their tensor product $\mathcal{C} \otimes \mathcal{D}$ has as its objects pairs of objects of \mathcal{C} and \mathcal{D} , and morphisms defined by

$$\mathrm{Hom}_{\mathcal{C} \otimes \mathcal{D}}((c_1, d_1), (c_2, d_2)) := \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \otimes_{\mathbf{k}} \mathrm{Hom}_{\mathcal{D}}(d_1, d_2).$$

However, the \mathbf{k} -linear tensor product of two categories in \mathbf{Pr} is not again in \mathbf{Pr} . Each of \mathbf{Pr} and \mathbf{Pr}° nevertheless admit a natural symmetric monoidal structure, extending the \mathbf{k} -linear tensor product, and defined as follows:

Definition 2.34. The Deligne–Kelly [**Kelly1982**; **Deligne2007**; **Franco2013**] tensor product of categories $\mathcal{C}, \mathcal{D} \in \mathbf{Pr}$ is another category $\mathcal{C} \boxtimes \mathcal{D} \in \mathbf{Pr}$, equipped with a linear functor $\mathcal{C} \otimes \mathcal{D} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$, cocontinuous in each variable, which is moreover universal for this property, in the sense that we have an equivalence of groupoids:

$$\mathrm{Hom}_{\mathbf{Pr}}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{E})^\times \simeq \mathrm{Lin}^{c,c}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$$

where on the left hand side we throw away non-invertible natural transformations and $\mathrm{Lin}^{c,c}$ denotes the groupoid whose objects are \mathbf{k} -linear functors which are co-continuous in each variable, and whose morphisms are natural isomorphisms.

Proposition 2.35 ([**Franco2013**] Lemma 8, [**Kelly1982**] Chapter 6, [**Caviglia**]). *The Deligne–Kelly tensor product $\mathcal{C} \boxtimes \mathcal{D}$ exists and is again locally presentable.*

Corollary 2.36. *The Deligne–Kelly tensor product of two categories with enough compact projectives again has enough compact projectives, and is generated by the pure tensor products of the compact projective generators in each component.*

Remark 2.37. The Deligne–Kelly tensor product contains the “pure tensor products” of pairs of objects in the two categories, plus arbitrary colimits freely built from those. Hence it is analogous to the completed tensor product of Hilbert spaces.

Remark 2.38. In a bicategory such as \mathbf{Pr} there is the notion of a bicolimit, over a diagram D which now has objects, 1-morphisms and (necessarily, invertible) 2-morphisms. In fact \mathbf{Pr} and \mathbf{Pr}° are “2-cocomplete”, meaning that given any such diagram the desired colimit exists. This is a deep fact of enriched category theory.

Exercise 2.39. Construct the diagram D realizing $\mathcal{C} \boxtimes \mathcal{D}$ as a bicolimit.

The following elementary facts about locally presentable categories will be of the utmost importance and utility. They are a special case of the Adjoint Functor Theorem, but are much easier to prove.

Theorem 2.40. *Every cocontinuous functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between locally presentable categories (i.e. every morphism in \mathbf{Pr}) has a right adjoint F^R , and this F^R is itself cocontinuous if F preserves compact-projective objects.*

Remark 2.41. Once again, intuition about Hilbert spaces applies, but it’s a bit more subtle:

- In presentable categories, Hom spaces are allowed to be infinite-dimensional.
- We lose symmetry of the pairing, $\mathrm{Hom}(X, Y) \not\cong \mathrm{Hom}(Y, X)$.
- Related to this, there is a very different behavior between left adjoints and right adjoints.
- A functor which preserves compact objects can be thought of as analogous to a bounded linear map. However, not every cocontinuous functor preserves compacts.

- While the adjoint of a compact-preserving functor is co-continuous, it is not again compact preserving.

2.3. Presentable tensor categories and their module categories. Tensor categories in presentable categories are defined just as in the finite semi-simple case, except that the multiplication functor \otimes is required to be co-continuous in addition to \mathbf{k} -linear.

Definition 2.42. A tensor category is cp-rigid if the underlying category has enough compact projectives, and all compact-projective objects have left and right duals in the sense of Definition ??.

Definition 2.43. Given a tensor category \mathcal{C} , we define the *multiplication opposite* \mathcal{C}^{mop} as the tensor category with reversed multiplication and inverse associativity constraint.

The machinery of locally presentable tensor categories allows one to prove the most optimistic generalizations of the monadicity theorems:

Theorem 2.44. *Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a cocontinuous functor, and that its right adjoint F^R is itself cocontinuous and conservative. Then we have an equivalence,*

$$\mathcal{D} \simeq F^R F\text{-mod}_{\mathcal{C}}.$$

Corollary 2.45. *Suppose \mathcal{A} is a cp-rigid tensor category, that \mathcal{M} is its right module category, and that $m \in \mathcal{M}$ is an \mathcal{A} -projective \mathcal{A} -generator, meaning that act_m^R is conservative and cocontinuous. Then we have an equivalence of \mathcal{A} -module categories,*

$$\mathcal{M} \simeq \text{End}(m)\text{-mod}_{\mathcal{A}}.$$

Definition 2.46. Let \mathcal{A}, \mathcal{B} be tensor categories. An $\mathcal{A} - \mathcal{B}$ -bimodule is an \mathbf{Pr} category \mathcal{M} which is simultaneously a left \mathcal{A} -module and a right \mathcal{B} -module, together with an isomorphism

$$\Gamma : (a \otimes m) \otimes b \longrightarrow a \otimes (m \otimes b)$$

making the obvious diagram commute.

Definition 2.47. Let \mathcal{A} be a tensor category and \mathcal{M}, \mathcal{N} be a right and a left \mathcal{A} -module category respectively. An \mathcal{A} -balanced functor is a pair of a functor F from $\mathcal{M} \boxtimes \mathcal{N}$ to some \mathbf{Pr} category \mathcal{E} and a natural transformation

$$f : F((m \otimes a) \boxtimes n) \cong F(m \boxtimes (a \otimes n))$$

for $a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}$ making the obvious diagrams commute.

Definition 2.48. Let \mathcal{A} be a tensor category and let \mathcal{M} and \mathcal{N} be a right and left \mathcal{A} -module, respectively. The balanced (or relative) Deligne–Kelly tensor product is another category $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N} \in \mathbf{Pr}$ together with an \mathcal{A} -balanced functor $\boxtimes_{\mathcal{A}} : \mathcal{M} \boxtimes \mathcal{N} \rightarrow \mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ which induces a natural equivalence of categories between balanced functor out of $\mathcal{M} \boxtimes \mathcal{N}$ to some category \mathcal{E} , and morphisms in \mathbf{Pr} from $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ to \mathcal{E} . The existence of the balanced Deligne–Kelly tensor product follows from the cocompleteness of \mathbf{Pr} , see [Ben-Zvi2015]. Constructions of the balanced tensor product in special cases appear in [Etingof2010; Davydov2013; Douglas2014].

2.4. Presentable braided tensor categories and their factorization homology.

Definition 2.49. A braided tensor category is a tensor category $(\mathcal{A}, \otimes, \alpha)$ together with a natural automorphism β of $- \otimes -$ making the following diagrams commute:

$$\begin{array}{ccccc}
 & & x \otimes (y \otimes z) & \xrightarrow{\beta_{x,y \otimes z}} & (y \otimes z) \otimes x \\
 & \nearrow \alpha_{x,y,z} & & & \searrow \alpha_{y,z,x} \\
 (x \otimes y) \otimes z & & & & & y \otimes (z \otimes x) \\
 & \searrow \beta_{x,y} \otimes \text{id}_z & & & \nearrow \text{id}_y \otimes \beta_{x,z} \\
 & & (y \otimes x) \otimes z & \xrightarrow{\alpha_{y,x,z}} & y \otimes (x \otimes z) \\
 & & & & & \\
 & & (x \otimes y) \otimes z & \xrightarrow{\beta_{x \otimes y,z}} & z \otimes (x \otimes y) \\
 & \nearrow \alpha_{x,y,z}^{-1} & & & \searrow \alpha_{z,x,y}^{-1} \\
 x \otimes (y \otimes z) & & & & & (z \otimes x) \otimes y \\
 & \searrow \text{id}_x \otimes \beta_{y,z} & & & \nearrow \beta_{x,z} \otimes \text{id}_y \\
 & & x \otimes (z \otimes y) & \xrightarrow{\alpha_{x,z,y}^{-1}} & (x \otimes z) \otimes y
 \end{array}$$

Proposition 2.50. *The data of a braided (resp. ribbon) tensor category \mathcal{A} determines a functor (also denoted \mathcal{A}) $\mathcal{A} : \text{Disk}_{fr}^2 \rightarrow \mathbf{Pr}$ (resp. $\mathcal{A} : \text{Disk}_{or}^2 \rightarrow \mathbf{Pr}$).*

Convention 2.51. We fix the following data in applying Proposition 2.50: We denote by \mathbb{D} the standard unit disk with the right-handed orientation. The tensor product, braiding, and ribbon element are given, respectively, by: the left-to-right embedding of a pair of disks along the x -axis, isotopy interchanging those disks by rotating 180 degrees anti-clockwise, and by the oriented isotopy on \mathbb{D} rotating by 360 degrees. These are each depicted in Figure ??.

Definition 2.52. The factorization homology is the left Kan extension,

$$\begin{array}{ccc}
 \text{Disk}^2 & \xrightarrow{\mathcal{A}} & \text{Cat} \\
 & \searrow & \nearrow f\mathcal{A} \\
 & & \text{Mfld}^2
 \end{array}$$

An equivalent reformulation of the left Kan extension is to say that the factorization homology of Σ with coefficients in \mathcal{A} is:

- (1) a category, $\int_{\Sigma} \mathcal{A}$, together with
- (2) functors, $\int_i \mathcal{A} : \mathcal{A}^k \rightarrow \int_{\Sigma} \mathcal{A}$, for every disk inclusion $\mathbb{D}^{\sqcup k} \rightarrow \Sigma$, and
- (3) natural isomorphisms, $\int_{\gamma} \mathcal{A} : \int_i \mathcal{A} \rightarrow \int_j \mathcal{A}$, for every isotopy $\gamma : i \rightarrow j$, and
- (4) further coherence natural isomorphisms, making all commuting diagrams of disk embeddings commute in \mathcal{W} , and finally,

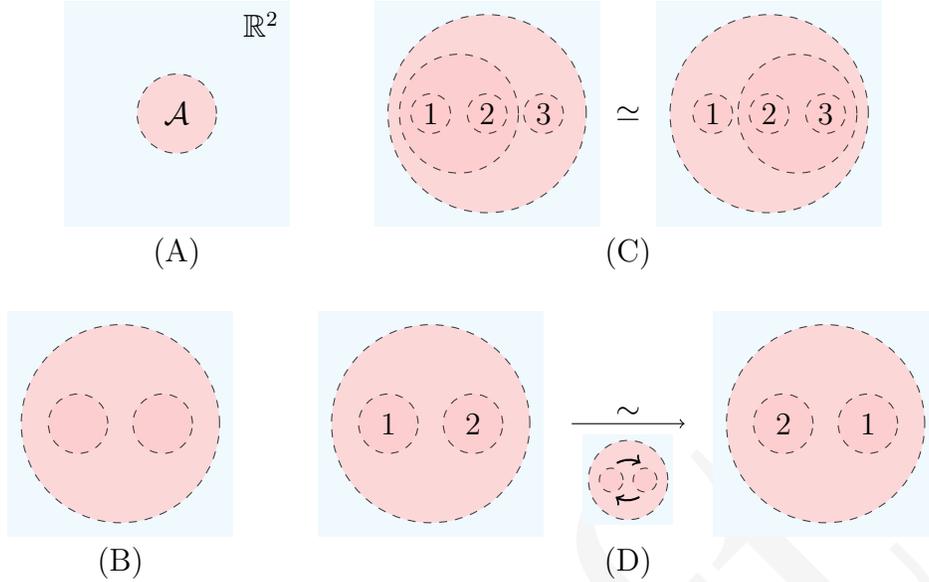


FIGURE 1. **Braided tensor categories as functors** $\text{Disk}_{fr}^2 \rightarrow \text{Cat}$. (A) depicts a basic open set; its inclusion onto \mathbb{R}^2 is a retract, so it is assigned the category \mathcal{A} canonically. (B) depicts an embedding $\mathbb{D}^2 \sqcup \mathbb{D}^2 \hookrightarrow \mathbb{D}^2$; this induces the product functor $T : \mathcal{A} \boxtimes \mathcal{A} \rightarrow \mathcal{A}$. (C) depicts an isotopy (with this choice of representatives, it an identity) between two composite disk embeddings $\mathbb{D} \sqcup \mathbb{D} \sqcup \mathbb{D} \hookrightarrow \mathbb{D}$; this induces the associator natural isomorphism α on \mathcal{A} . (D) depicts an isotopy between two disk inclusions; this induces the braiding isomorphism σ on \mathcal{A} .

(5) a canonical functor $\int_{\Sigma} \mathcal{A} \rightarrow \mathcal{C}$, for any \mathcal{C} equipped with data (1)-(3).

Thus, the factorization homology of Σ with coefficients in \mathcal{A} is the initial category receiving functors indexed by disk embeddings, compatibly with the braided tensor structure on \mathcal{A} and with isotopies. This is a very nice and natural definition, but it isn't one we can work with. We will see in the next two lectures how to compute with it.

3. LECTURE III: THE MORITA 4-CATEGORY OF BRAIDED TENSOR CATEGORIES AND THE COBORDISM HYPOTHESIS

Definition 3.1. Let \mathcal{A} be a braided tensor category. We define *braiding reverse* of \mathcal{A} to be the braided tensor category \mathcal{A}^{bop} to have the same underlying tensor category, but with braiding isomorphism $\sigma_{V,W}$ replaced by $\sigma_{W,V}^{-1}$.

Remark 3.2. The opposite braiding comes from reflecting the discs about the x -axis in the E_2 operad and can be thought of as the opposite in the second multiplication direction. The opposite in the first multiplication direction is \mathcal{A}^{mop} with the reversed tensor product and the braiding given by $\sigma_{V,W}^{-1} : V \otimes^{op} W = W \otimes V \rightarrow V \otimes W = W \otimes^{op} V$. Again \mathcal{A}^{mop} corresponds to a reflection, this time about the y -axis. It is not difficult to see that \mathcal{A}^{mop} and \mathcal{A}^{bop} are isomorphic using the braiding. Note that the double opposite $\mathcal{A}^{mopbop} = \mathcal{A}^{bopmop}$ has underlying tensor category \mathcal{A}^{mop} but with the braiding given by $\sigma_{W,V}$. Since \mathcal{A}^{mopbop}

corresponds to rotation by 180-degrees it is orientation preserving and should not be thought of as an opposite (it is isomorphic to the original \mathcal{A} and not to either of the opposites).

Finally, let us recall the following well-known construction of a braided tensor category out of a tensor category.

Definition 3.3. Let $(\mathcal{C}, \otimes, \alpha)$ be a tensor category. Then its *Drinfeld center*, or simply *center*, is a braided tensor category $Z(\mathcal{C})$ defined as follows

- objects are pairs (y, β) where β is a natural isomorphism

$$\beta_x : x \otimes y \longrightarrow y \otimes x$$

making the obvious analog of the second diagram in Definition 2.49 commutes.

- a morphism $(y, \beta) \rightarrow (y', \beta')$ is a morphism $f : y \rightarrow y'$ such that

$$\forall x \in \mathcal{C}, (f \otimes \text{id}_x)\beta_x = \beta'_x(\text{id}_x \otimes f)$$

- the tensor product of (y, β) and (y', β') is the pair $(y \otimes y', \tilde{\beta})$ where $\tilde{\beta}$ is defined by the first diagram of Definition 2.49 with y' instead of z
- the braiding of $(y, \beta) \otimes (y', \beta')$ is simply given by β'_y .

Remark 3.4. Once again this turns out to be a particular case of the general formalism of E_n -algebras: to any E_n -algebra in a sufficiently nice symmetric monoidal 1-category (typically, non-discrete) one associates its Hochschild cohomology (also called its center), which has a natural structure of an E_{n+1} -algebra.

The following properties of the center are straightforward and well-known:

Proposition 3.5. *Let \mathcal{C} be a tensor category. The assignment $(y, \beta) \mapsto (y, \beta^{-1})$ induces a braided tensor equivalence*

$$Z(\mathcal{C}^{\text{mop}}) \longrightarrow Z(\mathcal{C})^{\text{bop}}.$$

Proposition 3.6. *Let \mathcal{A} be a braided tensor category with braiding β . Then there are braided tensor functors*

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & Z(\mathcal{A}) \\ x & \longmapsto & (x, \beta_{-,x}) \end{array} \qquad \begin{array}{ccc} \mathcal{A}^{\text{bop}} & \longrightarrow & Z(\mathcal{A}) \\ x & \longmapsto & (x, \beta_{x,-}^{-1}) \end{array}$$

which assemble into a single braided tensor functor $\mathcal{A} \boxtimes \mathcal{A}^{\text{bop}} \rightarrow Z(\mathcal{A})$.

It is a general fact that the category of modules over an E_2 -algebra is an E_1 , i.e. monoidal, category. Specializing in the case at hand this recovers the following well-known

Proposition 3.7. *Let \mathcal{A} be a braided tensor category with braiding β .*

- *Every left \mathcal{A} -module category \mathcal{M} (which we assume to be strict) has a canonical structure of a right \mathcal{A} -module, with the same action, and associativity constraint given, for $a, b \in \mathcal{A}, m \in \mathcal{M}$*

$$(a \otimes b) \otimes m \xrightarrow{\beta_{a,b}} b \otimes (a \otimes m).$$

- *Given two left \mathcal{A} -modules \mathcal{M}, \mathcal{N} , the balanced tensor product $\mathcal{M} \boxtimes_{\mathcal{A}} \mathcal{N}$ where \mathcal{M} is given the above right module structure, turns the category of left \mathcal{A} -modules into a monoidal 2-category.*