EXPLICIT FORMULAS FOR THE MAHLER MEASURE OF FAMILIES OF MULTIVARIABLE POLYNOMIALS

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ABSTRACT. In this note we first present an explicit result for the Mahler measure of the real polynomial $xP(y)+P(z)$, where $|P(e^{i\theta})|$ is monotonic for θ in $[0,\pi]$. We then give an explicit formula for the Mahler measure of the n-variable Laurent polynomial $(x_1+x_1^{-1})\cdots(x_{n-2}+x_{n-2}^{-1})+2^{n-3}(x_{n-1}+x_n)$, for $n\geqslant 3$.

1. INTRODUCTION

The *Mahler measure* $m(P)$ of a Laurent polynomial $P(x_1,\dots,x_n)$ is defined as

$$m(P)=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\log(|P(e^{i\theta_1},\dots,e^{i\theta_n})|)\,d\theta_1\cdots d\theta_n.$$

For polynomials in more than one variable, it is in general a challenging problem to evaluate their Mahler measures in closed form. For such a polynomial picked at random, its Mahler measure is unlikely, given our current state of knowledge, to be able to be found explicitly. However, substantial progress had been made recently in finding polynomials whose Mahler measure can be evaluated. See Bertin [Be1], Boyd ([Bo2], [Bo3], [Bo4], [Bo5]), Boyd and Rodriguez-Villegas [BR-V], Lalín [L], Vandervelde [V], and [S]. In this note I present two families of such polynomials, one family in three variables (Theorem 1), and one in $n\geqslant 3$ variables (Theorem 3).

Theorem 1. *Suppose that $P(z)\in\mathbb{R}[z]$ is such that $|P(e^{i\theta})|$ is monotonic for θ in $[0,\pi]$. Then*

$$m(xP(y)+P(z))=m(P)+\frac{2\varepsilon}{\pi^2}\left(\sum_{|\alpha|\leqslant 1}(\mathrm{Li}_3(\alpha)-\mathrm{Li}_3(-\alpha))+\sum_{|\alpha|>1}\left(\mathrm{Li}_3\left(\frac{1}{\alpha}\right)-\mathrm{Li}_3\left(-\frac{1}{\alpha}\right)\right)\right),$$

where the sums are taken over the specified zeros α of P, with multiplicities. Here $\varepsilon=1$ if $|P(e^{i\theta})|$ is monotonic increasing, and -1 if it is monotonic decreasing.

There are many examples of polynomials having this monotonic property. For instance, it is a routine matter to verify that a monic real quadratic has the monotonic increasing property iff its zeros $x+iy$ both lie in the region

$$\mathcal{R}=\{x\geqslant 2\}\cup\{0<x<2,|y|\leqslant|x-1|\sqrt{x/(2-x)}\}.$$

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Hence a sufficient condition for a polynomial to have the monotonic property is that its zeros all lie in $\mathcal{R}$, or all lie in $-\mathcal{R}$.

The following trivial variant of an earlier result, due independently to Vandervelde [V] (in full generality) and [S] (for b, c real), is a special case of the theorem that we will need.

Corollary 2. *For $b, c \in \mathbb{C}$ not both 0, with $|b| \geqslant |c|$, we have*

$$m(b(x+x^{-1})+cy(z+z^{-1})) = \log(|b|) + \frac{2}{\pi^2}\left(\mathrm{Li}_3\left(\left|\tfrac{c}{b}\right|\right) - \mathrm{Li}_3\left(-\left|\tfrac{c}{b}\right|\right)\right).$$

Our next result seems to be the first known example of a nontrivial evaluation of the Mahler measure of an n-variable polynomial with integer coefficients for every $n \geqslant 3$. Here 'nontrivial' means that the Mahler measure seems not to be the Mahler measure of a polynomial with integer coefficients in fewer than n variables. (For instance, the evaluation $m(N + x_1 + \cdots + x_n) = \log N = m(N + x)$ for $N \geqslant n$ would count as a trivial evaluation, as would $m(P(x_1 \times \cdots \times x_n)) = m(P(x))$ for a 1-variable polynomial P.)

Theorem 3. *Let $n \geqslant 3$. Then*

$$\begin{aligned} &m((x_1+x_1^{-1})\cdots(x_{n-2}+x_{n-2}^{-1}) + 2^{n-3}(x_{n-1}+x_n)) \\ &\qquad = (n-3)\cdot\log 2 + \left(\frac{2}{\pi}\right)^{n-1} \cdot {}_{n+1}F_n\left(\{\tfrac12,\tfrac12,\tfrac12,1,\ldots,1\},\{\tfrac32,\ldots,\tfrac32\},1\right)\end{aligned}$$

(where the generalised hypergeometric function has $n-2$ parameters equal to 1, and n parameters equal to $\frac32$).

The 'classical' result $m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$ (see [Bo1]) is a consequence of both theorems: in Corollary 2 take $b = c = 1$ (see [V] or [S]), while in Theorem 3 take $n = 3$, giving

$$\frac{4}{\pi^2}\cdot {}_4F_3\left(\{\frac12,\frac12,\frac12,1\},\{\frac32,\frac32,\frac32\},1\right) = \frac{4}{\pi^2}\sum_{j \text{ odd}} \frac{1}{j^3} = \frac{7}{2\pi^2}\zeta(3).$$

2. PROOFS

We first need an integral evaluation.

Lemma 4. *We have for $\rho \in \mathbb{C}$*

$$\int_0^\pi \theta \log|e^{i\theta} - \rho|\, d\theta = \begin{cases} \Re\left(\mathrm{Li}_3(\rho) - \mathrm{Li}_3(-\rho)\right), & \text{if } |\rho| \leqslant 1 \\ \frac{\pi^2}{2}\log|\rho| + \Re\left(\mathrm{Li}_3\left(\frac{1}{\rho}\right) - \mathrm{Li}_3\left(-\frac{1}{\rho}\right)\right), & \text{if } |\rho| > 1. \end{cases}$$

Proof. If $|\rho| \leqslant 1$ the integral equals

$$\begin{aligned}\int_0^\pi \theta \log|1 - e^{i\theta}\rho|\, d\theta &= -\Re\sum_{j=1}^\infty \frac{\rho^j}{j}\int_0^\pi \theta e^{ij\theta} d\theta \\ &= 2\Re\sum_{j \text{ odd}} \frac{\rho^j}{j^3} = \Re\left(\mathrm{Li}_3(\rho) - \mathrm{Li}_3(-\rho)\right).\end{aligned}$$

We can also write the integral as $\int_0^\pi \theta(\log|\rho| + \log|1 - \frac{e^{i\theta}}{\rho}|)\,d\theta$, so that if $|\rho| > 1$ the result follows from the first case. □

Proof of Theorem 1. Assume first that $\varepsilon = 1$. Then, since (an easy consequence of Jensen's Theorem) we know that $m(ax+b) = \log(\max(|a|, |b|)$, it follows that

$$m(xP(y) + P(z)) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log(\max(|P(e^{i\theta})|, |P(e^{i\varphi})|))\,d\theta\,d\varphi$$

and, using the fact that $\max(|P(e^{i\theta})|, |P(e^{i\varphi})|) = \max(|P(e^{i\max(\theta,\varphi)})|$, this

$$\begin{aligned} &= \frac{2}{\pi^2}\int_0^\pi \theta\log|P(e^{i\theta})|\,d\theta \\ &= \frac{2}{\pi^2}\sum_j\int_0^\pi \theta\log|e^{i\theta} - \alpha_j|\,d\theta, \end{aligned}$$

where $P(z) = \prod_j(z - \alpha_j)$. Then the result for $\varepsilon = 1$ follows from Lemma 4. (We don't need to take real parts in the end, because of the nonreal zeros of P occurring in complex conjugate pairs.)

If, instead, $\varepsilon = -1$, then $|P(-e^{i\theta})|$ is monotonic increasing on $[0, \pi]$. Because $m(xP(y) + P(z)) = m(xP(-y) + P(-z))$ (see for example [S, equation (3.13)]), the result holds with the zeros α_j of $P(z)$ replaced by the zeros $-\alpha_j$ of $P(-z)$, giving the result in this case. □

Proof of Corollary 2. We first note, as in [V], that because

$$b(x + x^{-1}) + cy(z + z^{-1}) = \omega(|b|(x + x^{-1}) + |c|(\omega' y)(z + z^{-1}))$$

for some ω, ω' of modulus 1, and for any polynomial R we then have $m(\omega R(x, \omega' y, z)) = m(R(x, y, z))$, it follows that $m(b(x+x^{-1}) + cy(z+z^{-1})) = m(|b|(x+x^{-1}) + |c|y(z+z^{-1}))$, so that we have reduced the problem to the case $b, c > 0$. Then

$$\begin{aligned} m(b(x + x^{-1}) + cy(z + z^{-1})) &= m(b(x^2 + 1) + c(x^2(x^{-1}yz) + xyz^{-1})) \\ &= m(x(cy + b) + cz + b). \end{aligned}$$

using [S, Lemma 7] to replace $(x^2, x^{-1}yz, xyz^{-1})$ by (x, y, z). Finally, since $|ce^{i\theta} + b|$ is monotonic decreasing on $[0, \pi]$, and $\alpha = -b/c$, the result follows. □

Proof of Theorem 3. One idea of the proof is to remove the distinction (for example in Corollary 2) between the variables of the polynomial, and the parameters determining its coefficients. Thus, we make the parameters functions of new variables, and then integrate over these variables to obtain the Mahler measure of a polynomial in a larger number of variables. This idea was recently used to good effect by Lalín [L] for computing some new explicit Mahler measures of polynomials in up to 5 variables.

For convenience we work with the polynomial

$$Q(x, y, z, z_1, \dots, z_{n-3}) = 2^{n-3}(x + x^{-1}) + y(z + z^{-1})(z_1 + z_1^{-1})\cdots(z_{n-3} + z_{n-3}^{-1}).$$

It is easy to verify, using the measure-invariant transformations used in the proof of Lemma 4, that its measure is the same as that of the polynomial in the statement of the theorem.

We apply Corollary 2 with $b = 2^{n-3}$ and $c = (z_1 + z_1^{-1}) \cdots (z_{n-3} + z_{n-3}^{-1})$. Putting $z_1 = e^{i\theta_1}, \ldots, z_{n-3} = e^{i\theta_{n-3}}$ we have $c/2^{n-3} = \cos\theta_1 \cdots \cos\theta_{n-3}$. Hence

$$\begin{aligned} m(Q) &= (n-3) \cdot \log 2 + \frac{2}{\pi^2}\left(\frac{2}{\pi}\right)^{n-3} \int_0^{\frac{\pi}{2}} d\theta_1 \cdots \int_0^{\frac{\pi}{2}} d\theta_{n-3} \cdot 2 \sum_{j \text{ odd}} \frac{(\cos\theta_1 \cdots \cos\theta_{n-3})^j}{j^3} \\ &= (n-3) \cdot \log 2 + \left(\frac{2}{\pi}\right)^{n-1} \sum_{j \text{ odd}} \frac{\left(\int_0^{\pi/2} \cos^j \theta \, d\theta\right)^{n-3}}{j^3} \\ &= (n-3) \cdot \log 2 + \left(\frac{2}{\pi}\right)^{n-1} \sum_{k=0}^{\infty} \left(\frac{4^k}{\binom{2k}{k}}\right)^{n-3} \frac{1}{(2k+1)^n}, \end{aligned}$$

using the fact that $\int_0^{\pi/2} \cos^{2k+1}\theta \, d\theta = \frac{4^k}{(2k+1)\binom{2k}{k}}$. Now

$$\left(\frac{4^k}{\binom{2k}{k}}\right)^{n-3} \frac{1}{(2k+1)^n} = \frac{\left(\frac{1}{2}\right)_k^3 (1)_k^{n-2}}{\left(\frac{3}{2}\right)_k^n k!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$. Hence from the definition of the generalised hypergeometric function

$${}_rF_m(\{a_1, \ldots, a_r\}, \{b_1, \ldots, b_m\}, z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_m)_k} \frac{z^k}{k!}$$

we obtain the result claimed.

□

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