Salem Numbers of Trace -2 and Traces of Totally Positive Algebraic Integers

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Abstract. Until recently, no Salem numbers were known of trace below -1. In this paper we provide several examples of trace -2, including an explicit infinite family. We establish that the minimal degree for a Salem number of trace -2 is 20, and exhibit all Salem numbers of degree 20 and trace -2. Indeed there are just two examples.
We also settle the closely-related question of the minimal degree d of a totally positive algebraic integer such that its trace is $\leq 2d-2$. This minimal degree is 10, and there are exactly three conjugate sets of degree 10 and trace 18. Their minimal polynomials enable us to prove that all except five conjugate sets of totally positive algebraic integers have absolute trace greater than $16/9$.
We end with a speculative section where we prove that, if a single polynomial with certain properties exists, then the trace problem for totally positive algebraic integers can be solved.

1 Introduction

A *Salem number* is a real algebraic integer greater than 1 whose other conjugates all lie in the closed disc $|z| \leq 1$, with at least one on the circle $|z| = 1$. Here we settle the question: what is the smallest possible degree for a Salem number of trace -2?

The problem is related to that of finding totally positive algebraic integers of degree d and trace $2d-2$. For suppose that

$$f(x) = x^d - (2d-2)x^{d-1} + \cdots$$

is the minimal polynomial of a totally positive algebraic integer. Then we apply the transformation $x = z + 1/z + 2$, and clear denominators, to produce a reciprocal polynomial

$$F(z) = z^{2d} + 2z^{2d+1} + \cdots + 2z + 1$$

which is the minimal polynomial of an algebraic integer of degree $2d$ and trace -2. The reverse transformation is a little more complicated: in $F(z)/z^d$, replace each $z^j + 1/z^j$

by $T_j(x-2)$, where T_j is the j-th Chebyshev polynomial, defined by $T_j(z+1/z) = z^j + 1/z^j$. Any roots of $f(x)$ in the interval $0 < x < 4$ are mapped to pairs of roots of $F(z)$ on the unit circle. Any roots of $f(x)$ in the interval $x > 4$ are mapped to pairs of reciprocal real positive roots of $F(z)$. We see that the problem of finding all Salem numbers of degree $2d$ and trace -2 is equivalent to that of finding all totally positive algebraic integers θ of degree d and trace $2d-2$ such that both (i) $\theta > 4$; and (ii) all other conjugates of θ are in the interval $0 < x < 4$.

The similar problem for trace -1 was settled some time ago: the smallest degree for a Salem number of trace -1 is 8, and there is just one such Salem number, having minimal polynomial

$$z^8 + z^7 - z^6 - 4z^5 - 5z^4 - 4z^3 - z^2 + z + 1 \;\; .$$

In [14] it is shown that there are infinitely many Salem numbers of trace -1, with examples of degree $2d$ for every $d \geq 4$. At that time, no examples of trace below -1 were known. We now know (see [5]) that there are infinitely many Salem numbers of every trace. In this paper we give a simpler proof that there are infinitely many Salem numbers of trace -2, using techniques from [14].

Some examples of Salem numbers of trace -2 are given in the next section, including one of degree only 26. These examples were obtained using a graphical construction described in [6] (generalising that in [4]), and using an interlacing construction described in [5] (which is greatly generalised in [7]). Bounds obtained in [11] show that to achieve trace -2 the degree must be at least 18. Further computations, announced in [14], showed that to achieve trace -2 the degree must be at least 20. This is confirmed by the computations of Sect. 3. However, we need no longer rely on these computations, as a direct proof of this is given in Sect. 4. The gap between 20 and 26 seemed tantalisingly narrow, and an improved search algorithm, detailed below, was set to work on degree 20. Luckily for us, we did not need to go up to degree 22! There are two examples at degree 20, and their minimal polynomials are given in Table 1.

Table 1. Minimal polynomials of the Salem numbers of degree 20 and trace -2

$$z^{20} + 2z^{19} + z^{18} - 3z^{17} - 9z^{16} - 15z^{15} - 18z^{14} - 18z^{13} - 16z^{12} - 14z^{11}$$
$$-13z^{10} - 14z^9 - 16z^8 - 18z^7 - 18z^6 - 15z^5 - 9z^4 - 3z^3 + z^2 + 2z + 1$$

$$z^{20} + 2z^{19} - 8z^{17} - 22z^{16} - 40z^{15} - 58z^{14} - 74z^{13} - 87z^{12} - 96z^{11}$$
$$-99z^{10} - 96z^9 - 87z^8 - 74z^7 - 58z^6 - 40z^5 - 22z^4 - 8z^3 + 2z + 1$$

The search in fact found all totally positive algebraic integers of degree 10 and trace 18. There are just three conjugate sets of these, and their minimal polynomials are displayed in Table 2. The first two of these polynomials yield Salem numbers via the transformation $x = z + 1/z + 2$; the third does not, since it has two roots greater than 4.

In [11] a lower bound is given for the absolute trace of totally positive algebraic integers. Using all three of the polynomials in Table 2, we are able to improve this bound

Table 2. Minimal polynomials of the totally positive algebraic integers of degree 10 and trace 18

$$f_1(x) = x^{10} - 18x^9 + 135x^8 - 549x^7 + 1320x^6 - 1920x^5 + 1662x^4 - 813x^3 + 206x^2 - 24x + 1$$

$$f_2(x) = x^{10} - 18x^9 + 134x^8 - 538x^7 + 1273x^6 - 1822x^5 + 1560x^4 - 766x^3 + 200x^2 - 24x + 1$$

$$f_3(x) = x^{10} - 18x^9 + 134x^8 - 537x^7 + 1265x^6 - 1798x^5 + 1526x^4 - 743x^3 + 194x^2 - 24x + 1$$

(Sect. 4). The paper then concludes with a speculative section on the trace problem for totally positive algebraic integers.

2 Examples of Salem Numbers of Trace -2

2.1 Examples from Graphs

In [4], Salem numbers were constructed using star-like trees. It is known which star-like trees have exactly one eigenvalue $\lambda > 2$, and for any such tree we define $\tau > 1$ by $\sqrt{\tau} + 1/\sqrt{\tau} = \lambda$. Then τ is a Salem number, unless λ is a rational integer. It was shown in particular that for any integer $r \geq 2$, and any integers $a_1, \ldots, a_r$, all at least 2 (and excluding certain exceptional choices), the only solutions to the equation

$$\sum_{i=1}^{r} \frac{z^{a_i - 1} - 1}{z^{a_i} - 1} = 1 + \frac{1}{z} \tag{1}$$

are a certain Salem number (or perhaps a reciprocal Pisot number), its conjugates, and possibly some roots of unity. The corresponding star-like tree has $a_1, \ldots, a_r$ vertices on its r arms.

Applying the method in [3], if one takes $r = 10$, $a_1 = 390$, $a_2 = 462$, $a_3 = 1190$, $a_4 = 1938$, $a_5 = 1995$, $a_6 = 2090$, $a_7 = 2805$, $a_8 = 4641$, $a_9 = 4862$, and $a_{10} = 5005$, one produces a Salem number of degree 23838 and trace -2. It might seem a daunting task to test a polynomial $f(z)$ of degree 23838 for irreducibility. Luckily we need only check that no roots of unity are roots of $f(z)$, and it is sufficient to test that

$$\gcd(f(z), f(-z)) = \gcd(f(z), f(z^2)) = \gcd(f(z), f(-z^2)) = 1 \ ,$$

since if ω is a root of unity, then ω is conjugate to one of $-\omega$, ω^2, $-\omega^2$.

The degree can be reduced greatly by exploiting other graphs. For example, by adding forks to the ends of some of the branches of a star-like tree, the Salem formula (1) can be generalised (again with some exclusions) to

$$\sum_{i=1}^{r} \frac{z^{a_i - 1} - 1}{z^{a_i} - 1} + \sum_{j=1}^{s} \frac{z^{b_j - 1} + 1}{z^{b_j} + 1} = 1 + \frac{1}{z} \ . \tag{2}$$

Taking $r = s = 3$, $a_1 = 66$, $a_2 = 130$, $a_3 = 238$, $b_1 = 255$, $b_2 = 273$, and $b_3 = 385$ in (2), one obtains a Salem number of trace -2 and degree 1278.

For more on producing Salem numbers from graphs, see [6]. The current record low degree for a Salem number τ of trace -2 obtained from graphs is degree 460, with τ being a root of

$$\frac{z^{69}-1}{z^{70}-1} + \frac{(z^{13}+1)(z^{182}-1)}{(z-1)(z^{195}+1)} + \frac{(z^{11}+1)(z^{220}-1)}{(z-1)(z^{231}+1)} = 1 + \frac{1}{z} \ .$$

2.2 Examples via Interlacing

Let $Q(z)$ and $P(z)$ be relatively prime polynomials with integer coefficients, and with all their roots on the unit circle. Suppose further that $P(z)$ is monic, $Q(z)$ has positive leading coefficient, and that the roots of P and Q interlace on the unit circle. This last condition means that as you progress clockwise around the unit circle, you encounter a zero of P and a zero of Q alternately. Finally we suppose that either $P(1) = 0$, or $Q(1) = 0$ and $2P(1) - Q'(1) < 0$. Then part of Proposition 4 of [5] states that $(z^2 - 1)P(z) - zQ(z)$ is the minimal polynomial of a Salem number (or perhaps a reciprocal Pisot number), possibly multiplied by a cyclotomic polynomial (i.e., a polynomial all of whose roots are roots of unity).

In the next section, we use this interlacing construction to produce an infinite family of Salem numbers of trace -2. The smallest degree of any member of this family is 38, with the Salem number being a root of

$$\frac{z^5-1}{(z^2-1)(z^3-1)} + \frac{z^{12}-1}{(z^5-1)(z^7-1)} + \frac{z^{24}-1}{(z^{11}-1)(z^{13}-1)} = z - \frac{1}{z} \ .$$

The current record via interlacing is of degree only 26. Define polynomials

$$\begin{aligned} P(z) = {} & z^{24} + 4z^{23} + 9z^{22} + 15z^{21} + 21z^{20} + 26z^{19} + 29z^{18} + 29z^{17} + 26z^{16} \\ & + 21z^{15} + 15z^{14} + 8z^{13} - 8z^{11} - 15z^{10} - 21z^9 - 26z^8 - 29z^7 - 29z^6 \\ & - 26z^5 - 21z^4 - 15z^3 - 9z^2 - 4z - 1 \ , \end{aligned}$$

$$\begin{aligned} Q(z) = {} & 2z^{24} + 7z^{23} + 14z^{22} + 21z^{21} + 27z^{20} + 31z^{19} + 33z^{18} + 33z^{17} + 32z^{16} \\ & + 31z^{15} + 31z^{14} + 31z^{13} + 31z^{12} + 31z^{11} + 31z^{10} + 31z^9 + 32z^8 + 33z^7 \\ & + 33z^6 + 31z^5 + 27z^4 + 21z^3 + 14z^2 + 7z + 2 \ . \end{aligned}$$

Then $P(z)$ and $Q(z)$ satisfy all the required conditions, and the polynomial $(z^2-1)P(z) - zQ(z)$ (which has trace -2) is in fact irreducible, and is the minimal polynomial of a Salem number of degree 26. For an explanation of the construction of this remarkable pair of polynomials, see [7].

2.3 Infinitely Many Examples

For any integer p that is coprime to $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, clearing denominators in the equation

$$\frac{z^5-1}{(z^2-1)(z^3-1)} + \frac{z^{12}-1}{(z^5-1)(z^7-1)} + \frac{z^{p+11}-1}{(z^{11}-1)(z^p-1)} = z - \frac{1}{z} \ . \tag{3}$$

gives a polynomial of trace -2. From Proposition 4 of [5], this is the minimal polynomial of a Salem number, possibly multipled a cyclotomic polynomial. We now show that in fact this polynomial is irreducible for all $p > 11$ coprime to $2\cdot 3\cdot 5\cdot 7\cdot 11$, giving infinitely many examples of Salem numbers of trace -2. All that is required is to show that no root of unity satisfies (3).

Putting $y = z^p$, (3) reads $h(z,y) = 0$, where

$$h(z,y) = \frac{z^5-1}{(z^2-1)(z^3-1)} + \frac{z^{12}-1}{(z^5-1)(z^7-1)} + \frac{yz^{11}-1}{(z^{11}-1)(y-1)} - z + \frac{1}{z} \;\; .$$

We again apply the trick that if a root of unity ω satisfies (3), then so does one of $-\omega$, ω^2, $-\omega^2$. (See [1] for more applications of this idea.) Our restriction on p implies in particular that p is odd, and hence $(-\omega)^p = -\omega^p$.

Eliminating z between $h(z,y) = 0$ and $h(-z,-y) = 0$ yields

$$(y^2+1)^2 f(y) = 0 \;\; ,$$

where $f(y)$ has no cyclotomic factors. Eliminating y instead yields

$$(z^4 - z^2 + 1)g(z) = 0 \;\; ,$$

where $g(z)$ has no cyclotomic factors. If both $z = \omega$ and $z = -\omega$ were to satisfy (3), then we would need $y = \omega^p$ to be a primitive fourth root of unity, with ω a primitive twelfth root of unity. It would follow that p is divisible by 3.

Similarly, eliminating first y and then z between $h(z,y) = 0$ and $h(z^2,y^2) = 0$ yields that for both $z = \omega$ and $z = \omega^2$ to satisfy (3) requires that ω is a primitive second, third, fifth, seventh, or eleventh root of unity, and that $\omega^p = 1$. It would follow that p is divisible by at least one of 2, 3, 5, 7, or 11.

Finally, considering $h(z,y) = 0$ and $h(-z^2,-y^2) = 0$ simultaneously shows that for both $z = \omega$ and $z = -\omega^2$ to satisfy (3) requires that p is divisible by 3.

We see that no roots of unity can satisfy (3) provided that p is prime to $2\cdot 3\cdot 5\cdot 7\cdot 11$. There are infinitely many such p, giving infinitely many Salem numbers of trace -2.

3 Totally Positive Algebraic Integers of Given Degree and Trace

3.1 The Old Search Algorithm

For ease of exposition, we consider the search for totally positive algebraic integers of degree 10 and trace 18. The algorithm clearly generalises to arbitrary degree and trace.

We seek all positive integers $a_2, \ldots, a_{10}$ such that

$$f(x) = x^{10} - 18x^9 + a_2x^8 - a_3x^7 + a_4x^6 - a_5x^5 + a_6x^4 - a_7x^3 + a_8x^2 - a_9x + a_{10}$$

has 10 distinct positive real roots. It is extremely difficult for a totally positive algebraic integer of degree 10 to have trace as small as 18, so having found all suitable a_i, nearly all of the corresponding polynomials $f(x)$ will be reducible.

For $1 \le i \le 10$, we let $f_i(x)$ be the $(10-i)$th derivative of $f(x)$. If $f(x)$ has 10 distinct positive roots, then for each i, $f_i(x)$ will have i distinct positive roots. We find all possibilities for $f(x)$ by building up from below: we list all values of a_2 such that $f_2(x)$ has 2 distinct positive roots; for each suitable a_2, we list all values of a_3 such that $f_3(x)$ has 3 distinct positive roots; and so on. Many of our candidates for the higher derivatives of $f(x)$ do not survive this lifting process: we frequently find that for some $f_i(x)$ having i positive real roots there is no choice of a_{i+1} that makes $f_{i+1}(x)$ have $i+1$ positive roots.

Given a candidate for $f_i(x)$, having i distinct positive roots, the technique of Robinson [8] (used also in [11]) to find all suitable values of a_{i+1} was to observe that the *real* values of a_{i+1} such that $f_{i+1}(x)$ has $i+1$ distinct positive roots form an interval (possibly empty), with endpoints determined by considering the values of $f_{i+1}(x)$ at its local maxima and minima. Although much more efficient than a naive brute-force search, this method requires the computation of the roots of a huge number of polynomials, using floating-point arithmetic.

3.2 The New Search Algorithm

Our new algorithm still builds up $f(x)$ from its derivatives, as in the previous section. But the endpoints of the interval for a_i are determined in a different manner, removing the need for floating-point arithmetic, and hugely speeding the search.

With notation as in the previous section, we suppose that we have a candidate for $f_i(x)$ having i distinct positive roots. We wish to identify the (possibly empty) range of values for a_{i+1} such that $f_{i+1}(x)$ has $i+1$ distinct positive roots. For ease of notation, we put $a = a_{i+1}$. We observe that $D(a)$, the discriminant of $f_{i+1}(x)$ (which is a polynomial in a, given that all higher coefficients have been selected), vanishes at the (real) endpoints of the desired interval for a. In fact *all* its roots are real: they are the numbers $f_{i+1}(\beta)$, where β is a root of f_i.

Indeed the required interval is marked by the middle two roots of $D(a)$ (with the interpretation that if D has odd degree, then we take the middle zero and the one to the left of it). For when $a = a_{i+1}$ is large and negative, f_{i+1} has either one or two real roots (depending on the parity of i), and as a is increased the number of real roots of f_{i+1} jumps by two as we pass each root of $D(a)$. When a is large and positive, the number of real roots is either one or none. The only possible interval in which the number of real roots can be as large as $i+1$ is that bounded by the middle two roots. For these roots of f_{i+1} all to be positive we require also that $a > 0$.

It might appear that the problem of root-finding has simply been transferred to a different polynomial, but note that we only need to find the roots of $D(a)$ to the nearest integer. To this end, we take some crude initial approximations to the middle two roots, then refine these using Sturm sequences to pin the roots down to the nearest integer. (The initial approximation that we used was simply to try the endpoints of the previous interval.) Then a further Sturm sequence computation for a *single* value of a_{i+1} in the interval will reveal whether or not f_{i+1} has the full $i+1$ real roots for *all* a_{i+1} in the interval. We also require $a_{i+1} > 0$, to ensure that all these roots are positive.

A further improvement is to use non-trivial lower bounds for the a_i, based on known lower bounds for the traces of totally positive algebraic integers given in [11]—see also Sect. 4. This prunes out many hopeless $f_i(x)$ with i small.

The full search took 147 hours on a 1.2GHz PC, using PARI/GP, and produced three irreducible polynomials of degree 10, trace 18, with 10 distinct positive roots, as listed in Table 2. Some 4065 reducible polynomials of degree 10 and trace 18 were found. Studying the irreducible factors of this output provides the necessary information to find all Salem numbers of trace -1 and degree ≤ 18, and also confirms that to achieve degree d and trace $\leq 2d-2$ requires $d \geq 10$ (this also follows from the result of the next section).

4 Improving the Lower Bound for the Absolute Trace of Totally Positive Algebraic Integers

In [12] it was shown that all except five conjugate sets of totally positive algebraic integers α have absolute (also called mean) trace $\mathrm{tr}(\alpha)/\deg(\alpha) > 1.7719$ ($\mathrm{tr}(\alpha)$ and $\deg(\alpha)$ being the trace and degree respectively). We can now use the three newly discovered polynomials of degree 10 to improve this bound to $1.778378 > 16/9$. The proof employs the same method as [12]: semi-infinite linear programming is used to produce the following inequality, valid for all $x > 0$

$$\begin{aligned}
x - .5455833645\log|x| - .4958676072\log|x-1| - .05892353929\log|x-2| \\
-.1846627119\log|x^2-3x+1| - .002613011520\log|x^2-4x+1| \\
-.008163503307\log|x^2-4x+2| - .09063100904\log|x^3-5x^2+6x-1| \\
-.01899914258\log|x^3-6x^2+9x-1| - .008696349375\log|x^3-6x^2+9x-3| \\
-.05794447530\log|x^4-7x^3+13x^2-7x+1| \\
-.03510719518\log|x^4-7x^3+14x^2-8x+1| \\
-.008492128216\log|x^5-9x^4+28x^3-35x^2+15x-1| \\
-.01082775244\log|x^5-9x^4+27x^3-31x^2+12x-1| \\
-.0008908117930\log|x^6-11x^5+43x^4-72x^3+51x^2-14x+1| \\
-.005949580568\log|x^7-13x^6+63x^5-143x^4+158x^3-80x^2+16x-1| \\
-.008478368652\log|f_1(x)| - .007206449910\log|f_2(x)| \\
-.01019001634\log|f_3(x)| > 1.7783786\ ,
\end{aligned} \tag{4}$$

where $f_1(x)$, $f_2(x)$, $f_3(x)$ are the three degree 10 polynomials displayed in Table 2. To prove the existence of the lower bound $\mathrm{tr}(\alpha)/\deg(\alpha) > 1.7783786$ for a totally positive nonexceptional α, we substitute for x each conjugate α_j of α, and average. Then if the minimal polynomial of α does not appear in the inequality, we get that $\mathrm{tr}(\alpha)/\deg(\alpha) > 1.7783786 + \sum c_k \log|R_k|$, where the c_k are positive, and the R_k are nonzero integer resultants. Hence $\mathrm{tr}(\alpha)/\deg(\alpha) > 1.7783786$, as claimed.

The exceptional α are those of absolute trace less than 1.7783786 whose minimal polynomial *does* appear in the above inequality, namely α having minimal polynomial $x-1$, x^2-3x+1, x^3-5x^2+6x-1, $x^4-7x^3+13x^2-7x+1$ or $x^4-7x^3+14x^2-8x+1$.

Note that, if $d = \deg(\alpha)$ and $\mathrm{tr}(\alpha) \leq 2d-2$ then, as this inequality excludes the five exceptional polynomials, we must have $16d/9 < \mathrm{tr}(\alpha) \leq 2d-2$, so that $d \geq 10$. This confirms again the computation at the end of the previous section, and checks too that there are no totally positive algebraic integers of degree 9 and trace 16.

5 A Polynomial That Would Solve the Trace Problem

5.1 Background

The trace problem for totally positive algebraic integers (called the "Schur-Siegel-Smyth trace problem" by Peter Borwein in his very nice recent book [2]), is the following.

Problem 1. Fix $\rho < 2$. Then show that all but finitely many totally positive algebraic integers β have $\mathrm{tr}(\beta)/\deg(\beta) > \rho$.

Thus here β is a zero of an irreducible monic polynomial of degree $\deg(\beta)$ with integer coefficients, whose roots are all positive, and whose sum is $\mathrm{tr}(\beta)$.

In 1918 I. Schur [9] solved the problem for $\rho < \sqrt{e} = 1.6487$. In 1943 C.L. Siegel [10] solved it for $\rho < 1.737$. In [11] (see also [12]) the problem was solved for $\rho < 1.7719$, while in the previous section we solve it for $\rho < 1.7783786$. In [13] it was shown that there was no inequality of the type (4) having a lower bound ρ for any ρ larger than $2 - 10^{-41}$. Shortly afterwards J.-P. Serre (personal communication, see "Note added in proof" in [13]), showed that there was no such inequality for any ρ larger than 1.8983021. Here we present possible further evidence against this problem being solvable for all $\rho < 2$. We prove that the existence of a single polynomial f with properties given below would imply that the problem cannot be solved for ρ sufficiently close to 2. The result is, however, highly speculative, as such a polynomial may not exist!

5.2 The Polynomial

Suppose that f is a monic polynomial of degree at least 2 with integer coefficients and all positive distinct roots such that

- $|f(0)| \geq 2$;
- between every pair of distinct roots of f there is an x with $|f(x)| \geq 2$.

Then we claim that the set of all totally positive algebraic integers contains infinitely many β whose absolute trace is no greater than $\mathrm{tr}(f)/\deg(f)$.

We now prove the claim. Let $p > 2$ be prime, ω_p a primitive p-th root of unity, and $\alpha = \omega_p + 1/\omega_p$, with conjugates α_i, and let Q be the minimal polynomial of α. Then

$$F(x) = \prod_i (f(x) - \alpha_i) = Q(f(x))$$

is a polynomial of degree $\deg(f)\deg(Q)$ and trace $\deg(Q)\mathrm{tr}(f)$, and so absolute trace $\mathrm{tr}(f)/\deg(f)$. Note that, as the α_i are in $(-2,2)$, it is clear from the graph of f that all the roots of F are real, positive and distinct. Let β be any one of them. Then $f(\beta) = \alpha_i$ for some i, so that the field $\mathbb{Q}(\beta)$ contains α_i, and hence $\deg(\beta) \geq \frac{1}{2}(p-1)$. Let $m(\beta)$ be the absolute trace of β. Then, taking the β_j to be the representatives of the conjugate sets of the roots of F, we have

$$\mathrm{tr}(F) = \sum_j \mathrm{tr}(\beta_j) = \sum_j m(\beta_j)\deg(\beta_j)$$

and so

$$\mathrm{tr}(F)/\deg(F) = \mathrm{tr}(f)/\deg(f) = \sum_j m(\beta_j) w_j$$

where the weights $w_j := \deg(\beta_j)/\deg(F)$ sum to 1. Hence at least one of the $m(\beta_j)$, $m(\beta^{(p)})$ say, is at most $\mathrm{tr}(f)/\deg(f)$.

Now let $p \to \infty$ through a sequence of primes. Then $\deg(\beta^{(p)}) \to \infty$, so that there must be infinitely many different $\beta^{(p)}$. Thus the $\beta^{(p)}$ give the required algebraic integers. This proves the claim.

Now we see that if there is an f as above with also $\mathrm{tr}(f)/\deg(f) < 2$, then the trace problem has no solution when $\rho > \mathrm{tr}(f)/\deg(f)$.

We do not know whether such a polynomial f exists. Any such f clearly must have absolute trace $\mathrm{tr}(f)/\deg(f) > 1.7783786$. Indeed, its absolute trace must be larger than any lower bound of any inequality (which might be found in the future) of the type (4). For a monic integral polynomial with all positive roots and absolute trace less than 2, the crucial quantity is

$$M_f := \min\left(|f(0)|, |f(\gamma_1)|, |f(\gamma_2)|, \ldots, |f(\gamma_{n-1})|\right) \quad ,$$

where n is the degree of f, and $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ are the roots of f'. We need $M_f \geq 2$.

There are many variants on this idea. For example, if f is a monic polynomial of degree at least 3 with integer coefficients such that between every pair of distinct roots of f there is an x with $|f(x)/x| \geq 2$, then again one can show that the set of all totally positive algebraic integers contains infinitely many β whose absolute trace is no greater than $\mathrm{tr}(f)/\deg(f)$. The argument is entirely similar, but using the polynomial $F(x) = \prod(f(x) - \alpha_i x)$.

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