Fields and Vector Spaces

1. (E) Find the multiplicative inverses of the non-zero elements in \( \mathbb{F}_7 \). (Just experimenting is probably easier than using the Euclidean algorithm.)

2. (E) Show that if \( L \subseteq K \) is a subfield then \( 1, 1 + 1, 1 + 1 + 1, \ldots \) are all elements of \( L \). (It is tempting to call these \( 1, 2, 3, \ldots \) but note that (e.g. in \( \mathbb{F}_p \)) they are not necessarily all distinct.) Deduce that \( \mathbb{F}_p \) does not have any subfields (other than itself). What do you think the smallest subfield of \( \mathbb{R} \) is?
   [Ans.: \( \mathbb{Q} \)]

3. (E) Do the following equations have solutions in the fields \( \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_3, \mathbb{F}_2 \)?
   (a) \( x^2 + 1 = 0 \)
   (b) \( x^2 - x - 1 = 0 \)

   Note: what this means in each case is this: is there an element of the given field such that if you substitute it in to this equation and do all the arithmetic in that field then you get zero? The answers for the first three fields should be easy from elementary background knowledge. The last two fields have very few elements and so you can just experiment.

4. (E) Which of the following are subspaces of the given vector space?
   (a) \( \{ x \in \mathbb{R}^3 \mid 2x_1 - x_2 + x_3 = 1 \} \subseteq \mathbb{R}^3 \)
   (b) \( \{ x \in \mathbb{R}^3 \mid x_1 = 2x_2 \} \subseteq \mathbb{R}^3 \)
   (c) \( \{ P \in P_3 \mid P(1) = 0 \} \subseteq P_3 \) where \( P_n \) denotes the vector space of real polynomials of degree \( \leq n \) in a variable \( x \).
   (d) \( \{ A \in M \mid A^T = -A \} \subseteq M \) where \( M \) is the vector space of real \( 2 \times 2 \) matrices and \( T \) denotes matrix transpose
   [Ans.: no, yes, yes, yes]

5. (E) Find all the vectors
   (a) in \( \mathbb{F}_2^3 \) that are scalar multiples of \( a = (1, 2) \)
   (b) in the subset \( U := \{ x \mid x_1 + 2x_2 = 0 \} \) of \( \mathbb{F}_2^3 \).
   (c) in \( Sp(a, b) \subseteq \mathbb{F}_2^3 \), where \( a := (1, 1, 0), b := (1, 1, 1) \).
   [Ans. (a) \((0, 0), (1, 2), (2, 1), (b) \((0, 0), (1, 1), (2, 2), (c) \((0, 0, 0), (1, 1, 0), (1, 1, 1), (0, 0, 1)\)]

6. (E) In \( \mathbb{F}_2^3 \) find all the vectors in
   (a) \( Sp(x, y) \) where \( x = (1, 1, 0), y = (0, 1, 1) \)
   (b) the subspace \( V \subseteq \mathbb{F}_2^3 \) given by \( V = \{ x \in \mathbb{F}_2^3 \mid x_1 + x_2 + x_3 = 0 \} \).

7. (E) Let \( U = \{ x \mid x_1 = 0 \} \) and \( V = \{ x \mid x_2 = 0 \} \) be subspaces of \( \mathbb{R}^3 \). What is the sum \( U + V \) of these subspaces? State the Dimension Theorem for sums of subspaces and verify it in this example. Is this an example of a direct sum?
   [Ans.: \( \mathbb{R}^3 \), no]

8. (E)
   (a) Show that in \( \mathbb{F}_3^3 \) the vectors \( (1, 1, 3), (2, 0, 2), (4, 3, 0) \) are linearly dependent.
   (b) Completethe following sentence without using any terms from linear algebra. “If \( \mathbb{R} \) were finite-dimensional as a vector space over \( \mathbb{Q} \), it would mean that there exist a finite number \( r_1, \ldots, r_n \) of . . . such that every . . . could be written as . . .”

9. (E) Give a basis of the subspace of \( \mathbb{F}_3^3 \) defined by the equation \( x_1 + x_2 + x_3 = 0 \). What is the dimension of this subspace? How many vectors are there in this subspace? Find the coordinate matrix of the vector \( v = (4, 3, 3) \) in your chosen basis.

10. (a) (E) Find all the vectors in the subspace \( V \subseteq \mathbb{F}_2^3 \) given by \( V = \{ x \in \mathbb{F}_2^3 \mid x_1 + x_2 + x_3 = 0 \} \).
17. (S) Consider a 2x2 matrix \( A \).

15. (E) Give an example of two vector spaces.

16. (E) Consider the linear map \( T \).

11. (S) How many two-dimensional subspaces does \( \mathbb{F}_2^4 \) have? You might want to think along the lines of defining such a subspace by choosing a nonzero vector, and then choosing another vector that is not a multiple of it — their span determines a subspace. Count how many ways there are of doing this and then work out how many times each subspace has been counted.

[Ans: 35]

12. Assuming that \( \mathbb{R} \) and \( \mathbb{C} \) are fields, state which of the following sets are fields:

(a) (E) all positive integers
(b) (S) all numbers \( a + b\sqrt{3} \), where \( a, b \in \mathbb{Q} \)
(c) (S) all numbers \( a + b\sqrt{5} \), \( a, b \in \mathbb{Q} \)
(d) (E) all non-integer rational numbers
(e) (S) all numbers \( a + b\sqrt{2} \), \( a, b \in \mathbb{Q} \)

[Ans: N,Y,N,N,Y]

13. (E) The field extension \( \mathbb{Q}(\sqrt{2}) \) consists of all numbers of the form \( a + b\sqrt{2} \), where \( a, b \in \mathbb{Q} \). Show that this set is closed under multiplication and addition, that it contains 0 and 1, and that it contains the multiplicative and additive inverses of \( a + b\sqrt{2} \). Write down a basis of \( \mathbb{Q}(\sqrt{2}) \) regarded as a vector space over \( \mathbb{Q} \). What is its dimension?

14. (S)

(a) Prove the following theorem: A subset \( S \) of a field \( F \) is a subfield if \( S \) contains the zero and unity of \( F \), if \( S \) is closed under addition and multiplication, and if each \( a \in S \) has its negative \( -a \) and (provided \( a \neq 0 \)) its inverse \( a^{-1} \) in \( S \).

(b) Show that the pair of conditions “0 \( \in S \) and 1 \( \in S \)” can be replaced by the single condition “\( S \) contains at least two elements”. (Hint: \( aa^{-1} = 1 \) if \( a \neq 0 \).

**Linear Transformations**

15. (E) Give an example of two vector spaces \( V, W \) over the field \( \mathbb{F}_2 \) and a linear transformation \( T : V \rightarrow W \) which is injective but not surjective. Write down the null space and range for your example, find the rank and nullity, and verify the rank theorem.

16. (E) Consider the linear map \( T : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^3 \) defined by

\[
T(x_1, x_2, x_3) = (x_1 + x_2, x_3 + x_1, x_2 + x_3).
\]

Find all vectors in the null space and all vectors in \( T(\mathbb{F}_2^3) \). Find the rank and nullity of \( T \), and verify the rank theorem. Is \( T \) injective? surjective? bijective? an isomorphism?

[Ans: \( \text{Sp}((1, 1, 1)), \text{Sp}((1, 1, 0), (1, 0, 1)) \), 2, 1, N, N, N, N]

17. (S) Consider a 2x2 matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with entries in \( \mathbb{F}_2 \).
(a) How many such matrices are there?
(b) Find a necessary and sufficient condition for $A$ to have a matrix inverse of the same form (with respect to ordinary matrix multiplication).
(c) How many of the matrices found in (a) have inverses?
(d) (Harder) Let $A$ be such an invertible matrix, and let $e_1, e_2, e_3$ be the three non-zero vectors in $\mathbb{F}_2^2$. Show that $A$ permutes these three vectors, and find the permutation explicitly for each such $A$.

[Ans (a) 16, (b) $ad - bc \neq 0$, (c) 6, (d) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\mapsto$ a particular 3-cycle, etc.]

18. (S) Let $T_1, T_2$ be two linear mappings acting from $V$ to $W$. Define their sum $T_1 + T_2$ and prove that it is linear. Prove that rank($T_1 + T_2$) $\leq$ rank($T_1$) + rank($T_2$).

[Ans $(T_1 + T_2)(v) = T_1(v) + T_2(v)$]

19. (S) Let $P_j[\mathbb{F}_3]$ denote the vector space of polynomials of degree $j$ with coefficients in $\mathbb{F}_3$.

(a) State the dimension of $P_j[\mathbb{F}_3]$.
(b) Show that $(1 + x, x + x^2, x^3)$ is a basis for $P_2[\mathbb{F}_3]$.
(c) Show that $T : P_2[\mathbb{F}_3] \rightarrow P_3[\mathbb{F}_3]$ where $T : p(x) \mapsto (x + 2)p(x)$ is a linear map.
(d) Calculate the matrix of $T$ with respect to the basis above for $P_2[\mathbb{F}_3]$ and the basis $(1, x, x^2, x^3)$ for $P_3[\mathbb{F}_3]$.

[Ans: (a) $j + 1$, (d) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}$]

20. (S) Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. It would be better to have an example which could not be done with eigenvectors. Find non-singular matrices $P, Q$ such that the matrix $PAQ$ is a diagonal matrix $D$ in which the diagonal elements are ones or zeroes, the ones coming before the zeroes,

(a) using elementary row and column operations; and
(b) by finding suitable bases of $\mathbb{R}^3$.

(Here as elsewhere, unless otherwise stated, the “diagonal” is the leading diagonal, from top left to bottom right.)

21. (S) Show that the matrix $B = \begin{pmatrix} 5 & 2 & 7 \\ -3 & 4 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ is equivalent to the matrix $A$ in Q.20, in a sense which you should state.

**Operators**

22. (E) Let $p$ be a polynomial.

(a) Suppose that the square matrix $A$ is diagonal with entries $\mu_j$. Show that $p(A)$ is diagonal with entries $p(\mu_j)$.
(b) Suppose $T, v, \lambda$ are (respectively) an operator, vector and scalar such that $Tv = \lambda v$, and let $p$ be a polynomial. Show that $p(T)(v) = p(\lambda)(v)$.

23. (E) Give an example of a vector space $V$ and a linear transformation $T$ such that $T$ cannot be represented by an upper triangular matrix with respect to any basis of $V$.

24. (S) A subspace $U$ of a vector space $V$ is said to be invariant under a transformation $T$ if $T(U) \subseteq U$.

(a) Suppose that the matrix of a linear transformation (on a two-dimensional vector space) with respect to some coordinate system is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. How many subspaces are there which are invariant under the transformation?
(b) Let $\lambda$ be an eigenvalue of a linear transformation on a vector space $V$, and $U$ the corresponding set of eigenvectors (including the zero vector). Show that $U$ is invariant under $T$. 
25. (S) Give an example of a matrix $A$ which represents a projection on $\mathbb{R}^2$. Verify that $A^2 = A$. Find the eigenvalues of your matrix $A$.

In general a linear transformation $T$ is a projection if and only if $T^2 = T$. Prove that the eigenvalues of a projection are 0 or 1.

[Ans: See Q.24a; 0,1]
26. (S) For each of the following matrices \( A \) find, when possible, a complex non-singular matrix \( P \) for which \( P^{-1}AP \) is diagonal:

(a) \[
\begin{pmatrix}
2 & 4 \\
5 & 3
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
3 & 2 \\
-2 & 3
\end{pmatrix}
\]

(c) \[
\begin{pmatrix}
1 & 2 \\
2 & -2
\end{pmatrix}
\]

(d) \[
\begin{pmatrix}
-1 & 2i \\
-2i & 2
\end{pmatrix}
\]

[Ans. e.g. (a) \( \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix} \), (b) \( \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \), (c) \( \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \)].

In the next few problems we write \( J_n(\alpha) \) for the Jordan matrix which is the \( n \times n \) matrix whose \( i, j \)-th entry is: \( \alpha \) if \( i = j \); 1 if \( j = i + 1 \) and 0 otherwise. Thus for example \( J_3(-5) = \begin{pmatrix} -5 & 1 & 0 \\
0 & -5 & 1 \\
0 & 0 & -5 \end{pmatrix} \).

27. (E) Write down \( A = J_2(1) \), where the notation is defined above. State its eigenvalue(s). Write down the matrix \( A - \lambda I \) for each eigenvalue \( \lambda \), state its rank and nullity, and hence (or otherwise) write down the geometric multiplicity of each eigenvalue. Also compute \((A - \lambda I)^2\), and hence or otherwise write down the algebraic multiplicity of each eigenvalue.

28. (S) Write down the eigenvalues of \( J_3(\alpha) \), and find their algebraic and geometric multiplicity. How does this generalize to \( J_n(\alpha) \)?

[Ans: \( \alpha \), 3, 1. \( \alpha, n, 1 \)]

29. (S) For \( k > 1 \) the \( k \)-th generalized eigenspace of \( T : V \to V \) with eigenvalue \( \lambda \) is \( E_{\lambda,k} := \{ v \in V | (T - \lambda I)^k v = 0 \} \). So, for \( k = 1 \) the generalized eigenspace is just the eigenspace in the usual sense.

(a) Show that if \( k \leq l \) then \( E_{\lambda,k} \subseteq E_{\lambda,l} \). Deduce that the algebraic multiplicity is greater than or equal to the geometric multiplicity.

(b) Let \( A = J_3(\alpha) \). Show that \( E_{\alpha,k} = V \) for \( k \geq 3 \). Describe \( E_{\alpha,2} \) and give its dimension.

(c) In general, what is \( \dim E_{\alpha,k} \) for the Jordan matrix \( J_n(\alpha) \)?

[Ans: (b) \( \text{Sp}((1,0,0),(0,1,0)) \), 2 (c) \( \min(k,n) \)]
30. (E) Give examples of each of the following:

(a) an operator on $\mathbb{C}^3$ whose minimal polynomial is $x^2$.
(b) an operator on $\mathbb{C}^4$ whose minimal polynomial is $x(x - 1)^2$.
(c) an operator on $\mathbb{C}^4$ whose characteristic and minimal polynomials are both $x(x - 1)^2(x - 3)$.
(d) an operator on $\mathbb{C}^4$ whose characteristic polynomial is $x(x - 1)^2(x - 3)$ and whose minimal polynomial is $x(x - 1)(x - 3)$.

[Ans: (a) $T(x, y, z) = (z, 0, 0)$, (b) $T(x, y, z, w) = (x + y, y, 0, 0)$, (c) $T(x, y, z, w) = (0, y + z, z, 3w)$, (d) $T(x, y, z, w) = (0, y, z, 3w)$]

31. (S) any operator $T$ on $\mathbb{C}^2$ such that $T^2 = 0$, $0$ is the only eigenvalue, and that there is a basis such that $T$ has matrix

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \text{ or } 
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]. What are the characteristic and minimal polynomials in each case?

32. (S) Suppose $T$ is an operator on a vector space $V$, and let $v \in V$. Let $p$ be the monic polynomial of smallest degree such that $p(T)v = 0$. Prove that $p$ divides the minimal polynomial of $T$.

33. (S) Suppose $T$ is the operator on $\mathbb{C}^3$ which takes $(x, y, z)$ to $(y, z, 0)$. Show that there is no operator $S$ such that $S^2 = T$.

34. (S) Prove the Cayley-Hamilton theorem for “strictly” upper triangular matrices, by direct computation. (The “strictly” here means that the diagonal values are 0.) Do the same for upper triangular matrices.

35. (S) A nilpotent operator $T$ is one for which there exists a positive integer $p$

(a) Suppose $N$ is a nilpotent operator. Prove that 0 is the only eigenvalue.
(b) Suppose 0 is the only eigenvalue of an operator $N$ on a complex vector space. Prove that $N$ is nilpotent. Given an example to show that this is not necessarily true on a real vector space.

[Ans (b) $T(x, y, z) = (0, -z, y), x, y, z \in \mathbb{R}$]

36. (S) A matrix is said to be in Jordan form if it is block-diagonal with each diagonal block being a Jordan matrix (see the description above Q.??). There may be more than one block with a given parameter value.

So for example the $7 \times 7$ matrix

\[
\begin{pmatrix}
J_3(5) & 0 & 0 \\
0 & J_1(5) & 0 \\
0 & 0 & J_3(2)
\end{pmatrix}
\]

is in Jordan form. Note also that $J_1(\alpha)$ is the $1 \times 1$ matrix (a.k.a. “number”) $\alpha$, and so a diagonal matrix is an example of Jordan form where all the blocks are of size 1.

(a) Consider a matrix $A$ in Jordan form with just two Jordan blocks $J_p(\alpha), J_q(\beta)$ where $\alpha \neq \beta$. Find the characteristic polynomial, eigenvalues, minimal polynomial, and algebraic multiplicity of each eigenvalue (i.e. the dimension of the corresponding generalized eigenspace) of $A$

(b) The same, only now assume that $\alpha = \beta$.

(c) Conjecture how this generalizes to a general matrix in Jordan form, with blocks $J_{p_i(\alpha_i)}(\alpha_j)$.

(d) (Added after the tutorial sheet was issued to the class.) In (a) - (c) above, give the geometric multiplicity of each eigenvalue.

37. (S) minimal polynomial, characteristic polynomial and dimensions of the (ordinary, not generalized) eigenspaces and such that $A \neq B$ (and neither do $A, B$ differ only by a change in the order of the blocks down the diagonal).
38. (E)

(a) Suppose \( T \) is an operator on a vector space \( V \) and \((v_1, \ldots, v_n)\) is a basis of \( V \) with respect to which the matrix of \( T \) is the Jordan matrix \( J_n(a) \). Describe the matrix of \( T \) with respect to the basis \((v_n, \ldots, v_1)\).

(b) Repeat the question if \((v_1, \ldots, v_n)\) is a basis with respect to which the matrix of \( T \) is in Jordan Form (Qu.36).

[Ans (a) The matrix \( J'_n(\alpha) = \{j'_{i,j}\} \) where \( j'_{i,j} = \alpha \) if \( i = j \), 1 if \( i = j + 1 \), 0 otherwise. (b) Block diagonal matrix with blocks \( J'_n(\alpha) \), in the opposite order.]

39. (S) Let \( V \) be an \( n \)-dimensional real vector space.

(a) Let \( V \) have basis \( \{v_1, \ldots, v_n\} \). Write down a basis of \( \mathbb{C}V \), the complexification of \( V \), verifying that your basis is indeed a linearly independent spanning set. Hence find \( \dim \mathbb{C}V \).

(b) Let \( c_1, c_2 \in \mathbb{R}, c = c_1 + ic_2 \), and \((v_1, v_2) \in \mathbb{C}V \). Show that \( c(v_1, v_2) = (c_1v_1 - c_2v_2, c_1v_2 + c_2v_1) \).

(c) Let \( T \) be an operator on \( V \), and \( \mathbb{C}T \) its complexification. Show that \( \mathbb{C}T \) is an operator on \( \mathbb{C}V \).

(d) If \( I \) is the identity on \( V \), show that \( \mathbb{C}I \) is the identity on \( \mathbb{C}V \).

[Ans. (a) \( \{(v_1, 0), \ldots, (v_n, 0)\}, n. \)]
40. (S) Let $^C T$ be the complexification of an operator $T$ on a finite dimensional real vector space $V$, and $^C V$ the complexification of $V$.

(a) If $\lambda$ is a real eigenvalue of $^C T$, show that it is also an eigenvalue of $T$.
(b) If $\lambda$ is a complex eigenvalue of $^C T$, show that its complex conjugate $\bar{\lambda}$ is also an eigenvalue of $^C T$.
(c) If $\lambda$ is a complex eigenvalue of $^C T$, it may be proved that the algebraic multiplicities of $\lambda$ and $\bar{\lambda}$ are equal. Deduce that, if $\dim V$ is odd, $^C T$ must have a real eigenvalue.

41. (P) Let $V$ be a complex $n$-dimensional space. Its decomplexification is the real vector space $^{\mathbb{R}} V = \{ v : v \in V \}$ where addition of vectors is defined as in $V$ and multiplication by real numbers is the same as the scalar product of $V$, restricted to the reals. Let $\{ v_1, \ldots, v_n \}$ be a basis of $V$. Show that $\{ v_1, \ldots, v_n, iv_1, \ldots, iv_n \}$ is a basis of $^{\mathbb{R}} V$, and write down its dimension. Is $^C (^{\mathbb{R}} V) = V$?

[Ans: No]

Bilinear and Quadratic Forms on Real Vector Spaces

42. (S)

(a) In lectures an SBF was defined by its properties of (i) symmetry, (ii) linearity in the first argument. Prove that an SBF is linear in the second entry.
(b) If $B$ is a symmetric $n \times n$ matrix, prove that $b(x, y) = x^T B y$ defines a SBF on $^{\mathbb{R}} V$, where $x, y$ are column vectors as usual.
(c) Let $b$ be a SBF on a vector space $V$. Show that $N_b := \{ x \in V | b(x, v) = 0 \text{ for all } v \in V \}$ is a subspace of $V$.

43. (E)

(a) Consider the SBF $b$ on $^{\mathbb{R}} 2$ with matrix $B = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$. Write down the corresponding quadratic form in terms of standard coordinates $x_1, x_2$ on $^{\mathbb{R}} 2$. Find a vector $v \neq 0$ in $^{\mathbb{R}} 2$ such that $b(v, v) = 0$. Give a sketch showing the regions in the plane where $b(x, x)$ is positive, negative and zero. Find the eigenvalues of the matrix, and hence or otherwise determine the type, rank and signature of $b$.
(b) Consider the SBF on $^{\mathbb{R}} 3$ with matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Write down the corresponding quadratic form in terms of standard coordinates $x_1, x_2, x_3$ on $^{\mathbb{R}} 3$. Sketch the regions in $^{\mathbb{R}} 3$ where $b(x, x)$ is positive, negative and zero. Find the eigenvalues of the matrix, and hence or otherwise determine the type, rank and signature of $b$.

[Ans: (a) $2x_1^2 + 6x_1x_2 + x_2^2, (-3 + \sqrt{7}, 2)$, $(3 + \sqrt{37})/2, (1,1), 2, 0$
(b) $x_1^2 + x_2^2 - x_3^2, \{1, 1, -1\}$, $(2,1), 3, 1$]

44. (S)

(a) Let $V$ be the vector space of $2 \times 2$ real matrices with real entries and trace zero. What is the dimension of $V$? (Read (a) right through before deciding whether to give up.) Consider the SBF (known, by the way, as the “trace form”) $b(X, Y) = \text{Trace}(XY)$ on $V$. Find the matrix of $b$ with respect to the basis $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $V$. Find the eigenvalues of the matrix, and hence or otherwise determine the type, rank and signature of $b$.
45. (S)  
(a) Suppose that the matrix of $b$ in a basis is in the standard form $$\begin{pmatrix} I_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$ Identify a $p$-dimensional subspace on which $b$ is positive definite. Identify also an $(n - p)$-dimensional subspace on which $b$ is negative semi-definite (meaning that $b(v, v) \leq 0$ for all $v$ in the subspace). By considering intersections, deduce that there is no subspace dimension larger than $p$ on which $b$ is positive definite.

(b) True or false: an SBF is non-degenerate if and only if its matrix does not have zero as an eigenvalue.

(c) What are the possible types of an SBF on $\mathbb{R}^3$ if there exists a 2-dimensional subspace on which it is negative definite?

(d) An SBF on $\mathbb{R}^n$ has type $(p, q)$. What is the largest possible dimension for a subspace $V$ such that $b(v, v) < 0$ for all nonzero $v \in V$?

(e) Same as (d), but now $b(v, v) \leq 0$ for all nonzero $v \in V$.

(f) True or false: If an SBF is positive definite on subspaces $U, U' \subseteq V$ then it is positive definite on their sum $U + U'$. Explain your answer. (Hint: take a look at Q.43.)

46. (S)  
(a) diagonal, where $S = \begin{pmatrix} 7 & -6 \\ -6 & -2 \end{pmatrix}$.

(b) Find also a non-singular matrix $P$ such that $P^T SP$ is diagonal with diagonal entries $\pm 1$ or 0.

(c) What is the type of the SBF on $\mathbb{R}^2$ given by the matrix $S$? What is its rank and what is its signature?

47. (S) Let $V$ denote the vector space of $n \times n$ real matrices.

(a) What is the dimension of $V$ and of the subspace of symmetric matrices and of the subspace of antisymmetric matrices?

(b) Let $b(X, Y) = \text{Trace}(XY)$. Show that $b(X, Y) = \sum_{j=1}^{n} \sum_{k=1}^{n} X_{jk} Y_{kj}$, and that $b$ defines an SBF on $V$.

(c) Show that $b$ is positive definite on the subspace of symmetric matrices and negative definite on the subspace of antisymmetric matrices.

(d) Find the type, rank and signature of $b$.

48. (S) Let $b$ be the SBF on $\mathbb{R}^4$ given by the matrix $B$ given in block form as $B = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. Let $A$ be a fixed $2 \times 2$ matrix. Define $U := \left\{ x \in \mathbb{R}^4 | x = \begin{pmatrix} Av \\ v \end{pmatrix}, v \in \mathbb{R}^2 \right\}$. (We are using “block form” notation above.) Show that $U$ is a subspace of $\mathbb{R}^4$ and state its dimension. Show that $b$ is identically zero on $U$ if and only if $A$ is an orthogonal matrix (i.e. iff $A^T A = I$).

49. (S) Consider the quadratic form $\beta = 2x^2 + 3y^2 + z^2 - 4xy + 2xz + 2yz$.

(a) Write down the (symmetric) matrix $B$ of this quadratic form.

(b) By evaluating determinants only, determine the type of this form.
50. (S, with H extension) Show that the origin is a critical point of
\[ f(x, y, z) = 2x^2 + y \sin y + z^2 + 2(y + z) \sin x - 2ky \sin z \]
(where \( k \) is a constant). What can you say about the nature of the critical point for different values of \( k \)?
[Ans: Min if \(-1 < k < 0\), saddle if \( k < -1 \) or if \( k > 0 \); (harder) min if \( k = 0, -1 \)]

51. (S) What is the type of the SBF with matrix \( \begin{pmatrix} -3 & 12 & -7 \\ 12 & 4 & 2 \\ -7 & 2 & 2 \end{pmatrix} \)? You do not need to evaluate a 3 × 3 determinant (or compute eigenvalues - don’t even think of it).
[Ans: (2,1)]

52. (S) What is the type of the SBF with matrix (a) \( \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \), (b) \( \begin{pmatrix} -1 & 6 & 3 \\ 6 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix} \), (c) \( \begin{pmatrix} -3 & 2 & 7 \\ 2 & -2 & 6 \\ 7 & 6 & 1 \end{pmatrix} \)?
[Ans: (a) (2,1), (b) (2,1), (c) (2,2)]

53. (S) Classify the following quadrics.
(a) \( x^2 + 2y^2 + 3z^2 + 2xy + 2xz = 1 \)
(b) \( 2xy + 2xz + 2yz = 1 \)
(c) \( x^2 + 3y^2 + 2xz + 2yz - 6z^2 = 1 \)
You may be able to do all of these using determinants if you think carefully. You can always check your answer by asking Maple for the eigenvalues.
[Ans: (a) ellipsoid, (b) hyperboloid of two sheets (c) hyperboloid of one sheet]

54. (S) Let \( \beta(x) \) be a quadratic form on \( \mathbb{R}^n \) given by a symmetric matrix \( S \). How are the maximum and minimum values of \( \beta(x) \) on the unit sphere \( x^T x = 1 \) related to the eigenvalues of \( S \)? (Hint: orthogonal change of coordinates to standard form.)
Continuing, use Lagrange multipliers to find the critical points of \( x^T S x \) subject to the constraint \( x^T x = 1 \).
[Ans: max, \( \lambda_i \), min, \( \lambda_i \); \( x_i \) (normalised eigenvector)]

55. (S) Consider the SBFs on \( \mathbb{R}^2 \) given (with respect to the standard basis) by the matrices \( B = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix} \), \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \). Show that one of these is positive definite and hence find a basis for \( \mathbb{R}^2 \) with respect to which the matrices of both the SBFs are diagonal (with the positive definite one having the identity as its matrix). Write down the change of basis matrix that diagonalises both forms.

56. (S) In each of the following cases show that \( \sim \) is an equivalence relation on the set \( X \), and describe \([x]\) for \( x \in X \).
(a) \( X = \mathbb{R}, u \sim v \iff u - v \in \mathbb{Z} \)
(b) \( X = \mathbb{Z}, a \sim b \iff a + 2b = 3k \) for some \( k \in \mathbb{Z} \)
(c) \( X = \{(a, b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \} \) (so an element of \( X \) is a pair of integers, with the second one non-zero.) \( (a, b) \sim (k, l) \iff al = bk \).
[Ans: (a) \( (x + k) | k \in \mathbb{Z} \); (b) \( (x + 3k) | k \in \mathbb{Z} \); (c) \( (k, l) | k/l = a/b \)]

57. (S)
(a) Show that if \( x \in \mathbb{R}^2 \) is non-zero then there exists an invertible \( 2 \times 2 \) matrix \( A \) such that \( Ae_1 = x \), where \( e_1 \) is the first standard basis vector in \( \mathbb{R}^2 \).

(b) Use the above to show that, given two non-zero vectors \( x, y \in \mathbb{R}^2 \), there exists an invertible \( 2 \times 2 \) matrix \( P \) such that \( y = Px \). (Hint: take \( x \) to \( e_1 \) and then \( e_1 \) to \( y \).)

(c) Let \( x \sim y \) iff there exists an invertible \( 2 \times 2 \) matrix \( A \) such that \( y = Ax \). Show that this defines an equivalence relation on \( \mathbb{R}^2 \).

(d) What are the equivalence classes for this equivalence relation? How many elements does \( \mathbb{R}^2 / \sim \) have?

[Ans: (d) \(([e_1],[0]), 2\)]

58. (S) The three assumptions for an equivalence relation are (i) reflexivity, (ii) symmetry, (iii) transitivity. The following argument appears to show that the first of these is redundant. “By (ii), \( a \sim b \Rightarrow b \sim a \), and then (iii) implies that \( a \sim a \).” Spot the flaw, and produce an example of a relation which satisfies (ii) and (iii) but not (i).

[Ans: \( X = (0,1), a \sim b \iff a + b = 0 \)]
59. (E) Let \( V \subseteq X \) be a subspace.
   (a) Check that the relation \( x \sim y \iff x - y \in V \) is symmetric and transitive.
   (b) Check that the addition defined in class for the quotient space \( X/V \) is well defined.
   (c) Check that the distributive law \( \lambda(x + y) = \lambda x + \lambda y \), for all vectors \( x, y \) and scalars \( \lambda \), holds in \( X/V \).

60. (S) Suppose \( T : X \to Y \) is a linear map and that \( V \subseteq X \) is a subspace such that \( V \subseteq \ker T \). Define a linear map \( \overline{T} : X/V \to Y \) (checking that the map you have defined is both well defined and linear) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{P} & & \downarrow{\overline{T}} \\
X/V & \xrightarrow{T} & Y
\end{array}
\]

commutes. (The vertical map is the usual one, denoted \( P \) in lectures.) Find the dimension of the kernel of \( \overline{T} \) in terms of the dimensions of \( V \) and \( \ker T \). (Hint: apply the rank theorem to \( T \) and \( \overline{T} \)).

[Ans: \( \dim \ker T - \dim V \)]

61. (S) Let \( T \) be the linear map on \( \mathbb{R}^3 \) with matrix \( \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \) relative to the standard basis \((e_1, e_2, e_3)\).
   Show that \( V = \text{Sp}(e_2) \) is invariant under \( T \) (i.e. \( T(V) \subseteq V \)). Write down a basis of \( \mathbb{R}^3/V \), and the matrix of the canonical map \( \overline{T} \) with respect to this basis.

62. (S) Consider the differentiation map \( D : P_3 \to P_3 \), where \( P_3 \) is the real vector space of polynomials of degree \( \leq 3 \) in a variable \( x \). Show that \( D \) gives rise to a linear map \( \overline{D} : P_3/V \to P_3/V \), where \( V \) is the subspace of constant polynomials. What is the matrix of \( \overline{D} \) with respect to the basis \(([x], [x^2], [x^3])\) of \( P_3/V \)?

[Ans: \( \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \)]

**Polar and Singular Value Decomposition**

63. (S) Find (by explicit calculation) all \( 2 \times 2 \) real symmetric positive definite matrices \( S \) satisfying \( S^2 = I_2 \), where \( I_2 \) is the \( 2 \times 2 \) unit matrix.
   [Ans: \( I_2 \)]

64. (S) Find the polar form \( A = S\sqrt{A^T A} \) of the matrices (a) \( \frac{1}{\sqrt{10}} \begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix} \) (b) \( \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{pmatrix} \)

[Ans: (a) \( \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \) (b) \( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \)]

65. (S) Suppose \( A \) is a \( 2 \times 2 \) symmetric matrix with unit eigenvectors \( u_1, u_2 \). If its eigenvalues are \( \lambda_1 = 3 \) and \( \lambda_2 = -2 \), what are \( U, \Sigma \) and \( V \) in the SVD \( A = U\Sigma V^T \)?
   [Ans: \( U \) has columns \( u_1, -u_2 \), \( \Sigma = \text{diag}(3, 2) \), \( V \) has columns \( u_1, u_2 \)]

66. (S) Find the singular value decomposition of \( A \) if it has orthogonal columns \( w_1, \ldots, w_n \) of lengths \( \sigma_1, \ldots, \sigma_n \) (i.e. \( w_i^T w_i = \sigma_i^2 \), and you may assume \( \sigma_i > 0 \)).
   [Ans: \( U\text{diag}(\sigma_1, \ldots, \sigma_n)I^T \), where the \( i \)th column of \( U \) is \( w_i/\sigma_i \)]

67. (S) Find the SVD of the matrix \( \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \)

68. (S) Let \( A_1, A_2 \) be real \( n \times n \) matrices. Prove that they have the same singular values iff there exist real orthogonal matrices \( S_1, S_2 \) such that \( A_1 = S_1 A_2 S_2 \).
69. (S) Find the SVD of
(a) \( \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \)
(b) \( \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \)
(c) \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \)

[Ans: (a) \( (1)(2,0,0,0) \) \( \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix} \)]

(b) \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \) \( I_3 \)

(c) \( I_2 \left( \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \right) \)

70. (S)

(a) A relation \( \sim \) on a set \( X \) is said to be circular if \( a \sim b \) and \( b \sim c \) imply \( c \sim a \), for all \( a, b, c \in X \). Prove that a relation is reflexive and circular iff it is reflexive and symmetric and transitive. \([6]\)

(b) State the properties and sizes of the matrices \( U, \Sigma, V \) in the singular value decomposition \( A = U\Sigma V^T \) of an \( m \times n \) real matrix \( A \). What is meant by singular values? \([4]\)

(c) Find the singular value decomposition of the matrix \( \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). \([7]\)