Calculus 2
Midterm

Name: ____________________________

July 19, 2012

Do all problems, in any order.
Show your work!! An answer alone will not receive full credit.
No notes, texts, or calculators may be used on this exam.
You have 1 hour 35 minutes.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible Points</th>
<th>Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>80</td>
<td></td>
</tr>
</tbody>
</table>

Good Luck!!!
1. Compute the following integral:

\[
\int \frac{e^x}{\sqrt{e^{2x} + 1}} \, dx
\]

**Solution:** Let \( u = e^x \) so that \( du = e^x \, dx \) and

\[
\int \frac{e^x}{\sqrt{e^{2x} + 1}} \, dx = \int \frac{1}{\sqrt{u^2 + 1}} \, dx
\]

Now let \( u = \tan \theta \), \( 0 < \theta \leq \frac{\pi}{2} \) so \( du = \sec^2 \theta \, d\theta \) and

\[
\int \frac{1}{\sqrt{u^2 + 1}} \, dx = \int \frac{\sec^2 \theta}{\sqrt{\tan^2 \theta + 1}} \, d\theta = \int \frac{\sec^2 \theta}{\sec \theta} \, d\theta = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| = \ln |\sqrt{1 + u^2} + u| = \ln(\sqrt{1 + e^{2x}} + e^x)
\]
2. Compute the following integral:

\[ \int_0^1 x \arctan(x) \, dx \]

**Solution:** We can do this by parts with \( u = \arctan x \) and \( dv = x \) so that \( du = \frac{1}{1+x^2} \) and \( v = \frac{x^2}{2} \) so that

\[
\int_0^1 x \arctan(x) \, dx = \left[ \frac{x^2}{2} \arctan x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx \\
= \frac{1}{2} \pi - \frac{1}{2} \int_0^1 1 - \frac{1}{1+x^2} \, dx \\
= \frac{\pi}{4} - \frac{1}{2} [x]_0^1 + \frac{1}{2} [\arctan x]_0^1 \\
= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}
\]
3. Compute the following integral:

$$\int_0^\infty x e^{-x} \, dx$$

**Solution:** By parts

$$\int_0^\infty x e^{-x} \, dx = \lim_{t \to \infty} \int_0^t x e^{-x} \, dx$$

$$= \lim_{t \to \infty} \left[ -xe^{-x} \right]_0^t + \int_0^t e^{-x} \, dx$$

$$= \lim_{t \to \infty} -te^{-t} + e^{-x} \bigg|_0^t$$

$$= \lim_{t \to \infty} -te^{-t} - e^{-t} + 1$$

We know that

$$\lim_{t \to \infty} e^{-t} = 0$$

and

$$\lim_{t \to \infty} -te^{-t} = \lim_{t \to \infty} - \frac{t}{e^t} \leq \frac{1}{e^t} = 0$$

so the integral has value 1.
4. Compute the following integral:

\[ \int_{0}^{\pi} 3 \cos^2 x - \cos^3 x \, dx \]

Solution:

\[ \int_{0}^{\pi} 3 \cos^2 x - \cos^3 x \, dx = \int_{0}^{\pi} 3 \cos^2 x - \int_{0}^{\pi} \cos^3 x \, dx \]

\[ = \frac{3}{2} \int_{0}^{\pi} 1 + \cos(2x) \, dx - \int_{0}^{\pi} (1 - \sin^2 x) \cos x \, dx \]

\[ = \frac{3}{2} \left[ x + \frac{\sin(2x)}{2} \right]_{0}^{\pi} - \left[ \sin x - \frac{\sin^3 x}{3} \right]_{0}^{\pi} \]

\[ = \frac{\pi}{2} + \frac{3 \sqrt{3}}{2} - \left( \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} \right) = \frac{\pi}{2} \]
5. Consider

\[ I = \int_0^1 \cos(x^3) \, dx \]

Find a number \( n \) such that the Trapezoidal rule estimate of \( I \) will be within \( 10^{-6} \) of the true value of the definite integral. Your \( n \) need not be minimal, but you must explain why your \( n \) satisfies the stated bounds.

**Solution:** First of all let’s compute the second derivative of \( f(x) = \cos x^3 \):

\[ f'(x) = -\sin x^3 \cdot 3x^2 \]

and

\[ f''(x) = -\cos x^3 \cdot 9x^4 - \sin x^3 \cdot 6x \]

so that

\[ |f''(x)| \leq 9 + 6 = 15 \text{ for } 0 \leq x \leq 1 \]

due to the fact that \( \cos x^3 \) and \( \sin x^3 \) are bounded by \( 1 \) and \( -1 \), respectively.

hence we need to find \( n \) such that

\[ \frac{15}{12n^2} \leq 10^{-6} \]

so

\[ n^2 \geq \frac{5}{4} \times 10^6 \]

for example \( n = 2 \times 10^3 = 2000 \) will work.
6. Prove that
\[ \int_0^\infty \frac{2 \sin x}{e^{x^2} \sqrt{x^3 + 2}} \, dx \]
converges (careful!!).

**Solution:** First of all let’s divide the integral into two parts:

\[ \int_0^\infty \frac{2 \sin x}{e^{x^2} \sqrt{x^3 + 2}} \, dx = \int_0^1 \frac{2 \sin x}{e^{x^2} \sqrt{x^3 + 2}} \, dx + \int_1^\infty \frac{2 \sin x}{e^{x^2} \sqrt{x^3 + 2}} \, dx \]

the first integral is just a regular definite integral so it converges. For the second part we can use the fact that for \( x \geq 1 \) we have \( x^2 \geq x \) and so \( e^{x^2} \geq e^x \); hence

\[ -\frac{1}{e^x} \leq \frac{2 \sin x}{e^{x^2} \sqrt{x^3 + 2}} \leq \frac{1}{e^x} \]

so we just need to show that

\[ \int_1^\infty e^{-x} \, dx \]

converges. But

\[ \int_1^\infty e^{-x} \, dx = \lim_{t \to \infty} [-e^{-x}]_1^t = \frac{1}{e} \]

hence the original integral converges by the comparison theorem.
7. Let $a$ and $b$ be positive numbers. Consider the line segment connecting the origin to the point $(a, b)$. Revolve this segment around the $x$-axis and compute the area of the resulting surface. Make a drawing.

**Solution:** The line connecting $(0, 0)$ to $(a, b)$ has slope $\frac{b}{a}$ hence it has equation $y = \frac{b}{a}x$. So we have to compute

$$\int_0^a 2\pi \frac{b}{a} x \sqrt{1 + \left(\frac{b}{a}\right)^2} \, dx = 2\pi \frac{b}{a} \sqrt{1 + \left(\frac{b}{a}\right)^2} \left[\frac{x^2}{2}\right]^a_0$$

$$= \pi b \sqrt{a^2 + b^2}$$
8. Compute: \[
\int \frac{\sin x - \cos x}{\sin x + \cos x} \, dx
\]

**Solution:** Let \( u = \sin x + \cos x \) so \( du = \cos x - \sin x \) hence
\[
\int \frac{\sin x - \cos x}{\sin x + \cos x} \, dx = - \int \frac{1}{u} \, du = - \ln |u| = - \ln |\sin x + \cos x|
\]
Formula sheet
\[
\begin{align*}
\cos(2x) &= \cos^2 x - \sin^2 x \\
\sin(2x) &= 2\sin x \cos x \\
\cos^2 x &= \frac{1 + \cos(2x)}{2} \\
\sin^2 x &= \frac{1 - \cos(2x)}{2} \\
\int \sec x \, dx &= \ln |\sec x + \tan x| + C
\end{align*}
\]

The length of a curve \( y = f(x) \) between \( x = a \) and \( x = b \) is
\[
L(C) = \int_a^b \sqrt{1 + (f'(x))^2} \, dx
\]
The area of the surface obtained by revolving \( y = f(x), a \leq x \leq b \) around the \( x \)-axis is
\[
\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx
\]

The midpoint rule is \( \Delta x \left[ f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right] \) where \( \bar{x}_i \) is the midpoint of \([x_{i-1}, x_i]\).
The Trapezoidal Rule is \( \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n) \right] \).
Simpson’s Rule is \( \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right] \).

If \( E_T \) and \( E_M \) are the errors for the Trapezoidal Rule and Midpoint Rule, respectively, then
\[
|E_T| \leq \frac{K(b - a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b - a)^3}{24n^2}
\]
where \( |f''(x)| \leq K \) for \( a \leq x \leq b \).

If \( E_S \) is the error for Simpson’s Rule then \( |E_S| \leq \frac{K(b - a)^5}{180n^4} \) where \( |f^{(4)}(x)| \leq K \) for \( a \leq x \leq b \).