Calculus II HW08 (Due July 31)

31. By symmetry of the ellipse about the x- and y-axes,
\[
A = 4\int_0^\alpha y\,dx = 4\int_0^{\pi/2} b\sin\theta (-a \sin\theta)\,d\theta = 4ab\int_0^{\pi/2} \sin^2\theta\,d\theta = 4ab\int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta)\,d\theta
\]
\[
= 2ab\left[\theta - \frac{1}{3} \sin 2\theta\right]_0^{\pi/2} = 2ab\left(\frac{\pi}{2}\right) = \pi ab
\]

34. By symmetry, \(A = 4\int_0^\alpha y\,dx = 4\int_0^{\pi/2} a\sin^3\theta(-3a\cos^2\theta\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta\). Now
\[
\int \sin^4\theta \cos^2\theta\,d\theta = \int \sin^2\theta \left(\frac{1}{2}\sin^2 2\theta\right)\,d\theta = \frac{1}{8} \int (1 - \cos 2\theta) \sin^2 2\theta\,d\theta
\]
\[
= \frac{1}{8} \int \left[\frac{1}{2}(1 - \cos 4\theta) - \sin^2 2\theta \cos 2\theta\right]\,d\theta = \frac{1}{16}\theta - \frac{1}{16}\sin 4\theta - \frac{1}{48}\sin^3 2\theta + C
\]
so \(\int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{2} - \pi\). Thus, \(A = 12a^2\left(\frac{\pi}{2}\right) = \frac{3}{2} \pi a^2\).

66. \(x = e^t - t, y = 4e^{t/2}, 0 \leq t \leq 1\).
\[
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2.
\]
\[
S = \int_0^1 2\pi (e^t - t) \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi (e^t - t)(e^t + 1)\,dt
\]
\[
= 2\pi \left[\frac{1}{2}e^t + e^t - (t - 1)e^t - \frac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6)
\]

2. (a) \((1, \frac{2\pi}{4})\) \(\theta > 0: (1, \frac{2\pi}{4} - 2\pi) = (1, -\frac{2\pi}{4})\)
\(\theta < 0: (-1, \frac{2\pi}{4} - \pi) = (-1, \frac{2\pi}{4})\)

(b) \((-3, \frac{\pi}{6})\) \(\theta > 0: (-(-3), \frac{\pi}{6} + \pi) = (3, \frac{7\pi}{6})\)
\(\theta < 0: (-3, \frac{\pi}{6} + 2\pi) = (-3, \frac{13\pi}{6})\)

(c) \((1, -1)\) \(\theta = -1\) radian \(\approx -57.3^\circ\)
\(\theta > 0: (1, -1 + 2\pi)\)
\(\theta < 0: (-1, -1 + \pi)\)
4. (a) \( x = -\sqrt{2} \cos \frac{5\pi}{4} = -\sqrt{2} \left(-\frac{\sqrt{2}}{2}\right) = 1 \) and \\
\( y = -\sqrt{2} \sin \frac{5\pi}{4} = -\sqrt{2} \left(-\frac{\sqrt{2}}{2}\right) = 1 \)

gives us \((1, 1)\).

(b) \( x = 1 \cos \frac{5\pi}{2} = 1(0) = 0 \) and \\
\( y = 1 \sin \frac{5\pi}{2} = 1(1) = 1 \)

gives us \((0, 1)\).

(c) \( x = 2 \cos \left(-\frac{7\pi}{6}\right) = 2 \left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3} \) and \\
\( y = 2 \sin \left(-\frac{7\pi}{6}\right) = 2 \left(-\frac{1}{2}\right) = 1 \)

give us \((-\sqrt{3}, 1)\).

10. \(1 \leq r \leq 3, \quad \pi/6 < \theta < 5\pi/6\)

16. \( r = 4 \sec \theta \quad \Rightarrow \quad \frac{r}{\sec \theta} = 4 \quad \Rightarrow \quad r \cos \theta = 4 \quad \Rightarrow \quad x = 4\), a vertical line.

34. \( r = \ln \theta, \quad \theta \geq 1\)
37. \( r = 2 \cos 4\theta \)

54. (a) \( r = \sqrt{\theta}, \ 0 \leq \theta \leq 16\pi \). \( r \) increases as \( \theta \) increases and there are eight full revolutions. The graph must be either II or V. When \( \theta = 2\pi \), \( r = \sqrt{2\pi} \approx 2.5 \) and when \( \theta = 16\pi \), \( r = \sqrt{16\pi} \approx 7 \), so the last revolution intersects the polar axis at approximately 3 times the distance that the first revolution intersects the polar axis, which is depicted in graph V.

(b) \( r = \theta^2, \ 0 \leq \theta \leq 16\pi \). See part (a). This is graph II.

(c) \( r = \cos(\theta/3), \ 0 \leq \frac{\theta}{3} \leq 2\pi \) \( \Rightarrow \ 0 \leq \theta \leq 6\pi \), so this curve will repeat itself every \( 6\pi \) radians. 
\[
\cos\left(\frac{\theta}{3}\right) = 0 \Rightarrow \frac{\theta}{3} = \frac{\pi}{2} + \pi n \begin{align*}
\Rightarrow \theta &= \frac{3\pi}{2} + 3\pi n, \text{ so there will be two “pole” values, } \frac{3\pi}{2} \text{ and } \frac{9\pi}{2}. \end{align*}
\]
This is graph VI.

(d) \( r = 1 + 2 \cos \theta \) is a limaçon [see Exercise 53(a)] with \( c = 2 \). This is graph III.

(e) Since \(-1 \leq \sin 3\theta \leq 1, 1 \leq 2 + \sin 3\theta \leq 3, \) so \( r = 2 + \sin 3\theta \) is never 0, that is, the curve never intersects the pole. This is graph I.

(f) \( r = 1 + 2 \sin 3\theta \). Solving \( r = 0 \) will give us many “pole” values, so this is graph IV.

61. \( r = 3 \cos \theta \) \( \Rightarrow \ x = r \cos \theta = 3 \cos \theta \cos \theta, \ y = r \sin \theta = 3 \cos \theta \sin \theta \) \( \Rightarrow \)
\[
\begin{align*}
\frac{dy}{d\theta} &= -3 \sin^2 \theta + 3 \cos^2 \theta = 3 \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.
\end{align*}
\]
So the tangent is horizontal at \((\frac{3\sqrt{2}}{2}, \frac{3\pi}{4})\) and \((-\frac{3\sqrt{2}}{2}, \frac{3\pi}{4})\) \( \text{ same as } \left(\frac{3\sqrt{2}}{2}, -\frac{\pi}{4}\right) \).
\[
\frac{dx}{d\theta} = -6 \sin \theta \cos \theta = -3 \sin 2\theta = 0 \Rightarrow 2\theta = 0 \text{ or } \pi \Rightarrow \theta = 0 \text{ or } \frac{\pi}{2}.
\]
So the tangent is vertical at (3, 0) and \((0, \frac{\pi}{2})\).

8. \( r = \sin 2\theta, \ 0 \leq \theta \leq \frac{\pi}{2} \)
\[
A = \int_{0}^{\pi/2} \frac{1}{2} \sin^2 2\theta \ d\theta = \frac{1}{2} \int_{0}^{\pi/2} (1 - \cos 4\theta) \ d\theta = \frac{1}{2} \left[\theta - \frac{1}{4} \sin 4\theta\right]_{0}^{\pi/2} = \frac{1}{4} \left(\frac{\pi}{2}\right) = \frac{\pi}{8}
\]

32. \( 3 + 2 \cos \theta = 3 + 2 \sin \theta \) \( \Rightarrow \ \cos \theta = \sin \theta \) \( \Rightarrow \ \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \).
\[
A = 2 \int_{\pi/4}^{5\pi/4} \frac{1}{2} (3 + 2 \cos \theta)^2 \ d\theta = \int_{\pi/4}^{5\pi/4} (9 + 12 \cos \theta + 4 \cos^2 \theta) \ d\theta
\]
\[
= \int_{\pi/4}^{5\pi/4} \left[9 + 12 \cos \theta + 4 \cdot \frac{1}{2} (1 + \cos 2\theta)\right] \ d\theta
\]
\[
= \int_{\pi/4}^{5\pi/4} \left(11 + 12 \cos \theta + 2 \cos 2\theta\right) \ d\theta = \left[11\theta + 12 \sin \theta + \sin 2\theta\right]_{\pi/4}^{5\pi/4}
\]
\[
= \left(\frac{5\pi}{4} - 6 \sqrt{2} + 1\right) - \left(\frac{11\pi}{4} + 6 \sqrt{2} + 1\right) = 11\pi - 12 \sqrt{2}
\]
36. \( r = 0 \Rightarrow 1 + 2 \cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3} \) [for 
\( 0 \leq 3\theta \leq 2\pi \)] \( \Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9} \). The darker shaded region (from \( \theta = 0 \) to 
\( \theta = 2\pi/9 \)) represents \( \frac{1}{2} \) of the desired area plus \( \frac{1}{3} \) of the area of the inner
loop. From this area, we'll subtract \( \frac{1}{2} \) of the area of the inner loop (the lighter
shaded region from \( \theta = 2\pi/9 \) to \( \theta = \pi/3 \)), and then double that difference to
obtain the desired area.

\[
A = 2 \left[ \int_{0}^{2\pi/9} \frac{1}{2} (1 + 2 \cos 3\theta)^2 d\theta - \int_{2\pi/9}^{\pi/2} \frac{1}{2} (1 + 2 \cos 3\theta)^2 d\theta \right]
\]

Now,

\[
r^2 = (1 + 2 \cos 3\theta)^2 = 1 + 4 \cos 3\theta + 4 \cos^2 3\theta = 1 + 4 \cos 3\theta + 2 \cdot \frac{1}{2}(1 + \cos 6\theta)
\]

\[
= 1 + 4 \cos 3\theta + 2 + 2 \cos 6\theta = 3 + 4 \cos 3\theta + 2 \cos 6\theta
\]

and \( \int r^2 d\theta = 3\theta + \frac{4}{3} \sin 3\theta + \frac{2}{3} \sin 6\theta + C \), so

\[
A = \left[ \left(3\theta + \frac{4}{3} \sin 3\theta + \frac{2}{3} \sin 6\theta\right)\right]_{0}^{2\pi/9} - \left[ \left(3\theta + \frac{4}{3} \sin 3\theta + \frac{2}{3} \sin 6\theta\right)\right]_{2\pi/9}^{\pi/2}
\]

\[
= \left[ \left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{2}{3} \cdot \frac{-\sqrt{3}}{2}\right) - 0\right] - \left[ \left(\frac{\pi}{3} + 0 + 0\right) - \left(\frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{2}{3} \cdot \frac{-\sqrt{3}}{2}\right)\right]
\]

\[
= \frac{4\pi}{3} + \frac{4}{3} \sqrt{3} - \frac{1}{3} \sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3}
\]

45. \( L = \int_{a}^{b} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta = \int_{0}^{\pi} \sqrt{(2 \cos \theta)^2 + (-2 \sin \theta)^2} \, d\theta
\]

\[
= \int_{0}^{\pi} \sqrt{4(\cos^2 \theta + \sin^2 \theta)} \, d\theta = \int_{0}^{\pi} \sqrt{4} \, d\theta = [2\theta]_{0}^{\pi} = 2\pi
\]

As a check, note that the curve is a circle of radius 1, so its circumference is \( 2\pi (1) = 2\pi \).

8. \( a_n = \frac{(-1)^n n}{n! + 1} \), so \( a_1 = \frac{(-1)^1 1}{11 + 1} = -\frac{1}{2} \), and the sequence is

\[
\left\{ -1, \frac{2}{3}, -3, \frac{4}{7}, -5, \cdots \right\} = \left\{ -1, \frac{2}{3}, \frac{3}{7}, \frac{4}{25}, \frac{5}{121}, \cdots \right\}
\]

12. \( a_1 = 2, a_2 = 1, a_{n+1} = a_n - a_{n-1} \). Each term is defined in term of the two preceding terms.

\( a_2 = a_2 - a_1 = 1 - 2 = -1 \). \( a_4 = a_2 - a_2 = -1 - 1 = -2 \). \( a_5 = a_4 - a_2 = -2 - (-1) = -1 \).

\( a_6 = a_5 - a_4 = -1 - (-2) = 1 \). The sequence is \{2, 1, -1, -2, -1, \ldots\}.

24. \( a_n = \frac{n^2}{n^2 + 1} = \frac{n^2/n^2}{(n^2 + 1)/n^2} = \frac{1}{1 + 1/n^2} \), so \( a_n \to \frac{1}{1 + 0} = 1 \) as \( n \to \infty \). Converges

34. \( \lim_{n \to \infty} \frac{n}{n + \sqrt{n}} = \lim_{n \to \infty} \frac{n/n}{(n + \sqrt{n})/n} = \lim_{n \to \infty} \frac{1}{1 + 1/\sqrt{n}} = \frac{1}{1 + 0} = 1 \). Thus, \( a_n = \frac{(-1)^{n+1} n}{n + \sqrt{n}} \) has odd-numbered terms

that approach 1 and even-numbered terms that approach \(-1\) as \( n \to \infty \), and hence, the sequence \( \{a_n\} \) is divergent.
48. \( a_n = \frac{\sin 2n}{1 + \sqrt{n}} \) \( |a_n| \leq \frac{1}{1 + \sqrt{n}} \) and \( \lim_{n \to \infty} \frac{1}{1 + \sqrt{n}} = 0 \), so \( \frac{-1}{1 + \sqrt{n}} \leq a_n \leq \frac{1}{1 + \sqrt{n}} \) \( \Rightarrow \lim_{n \to \infty} a_n = 0 \) by the Squeeze Theorem. Converges.

56. \( 0 < |a_n| = \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{(n-1)} \cdot \frac{3}{n} \leq \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{n} \) [for \( n > 2 \)] = \( \frac{27}{2n} \to 0 \) as \( n \to \infty \), so by the Squeeze Theorem and Theorem 6, \( \{(−3)^n/n!\} \) converges to \( 0 \).

75. The terms of \( a_n = n(−1)^n \) alternate in sign, so the sequence is not monotonic. The first five terms are \(-1, 2, -3, 4, \text{and} -5\).

Since \( \lim_{n \to \infty} |a_n| = \lim_{n \to \infty} n = \infty \), the sequence is not bounded.

78. \( a_n = n + \frac{1}{n} \) defines an increasing sequence since the function \( g(x) = x + \frac{1}{x} \) is increasing for \( x > 1 \). \( [g'(x) = 1 - 1/x^2 > 0 \text{ for } x > 1 \] \) The sequence is unbounded since \( a_n \to \infty \) as \( n \to \infty \). (It is, however, bounded below by \( a_1 = 2 \).)

80. (a) Let \( P_n \) be the statement that \( a_{n+1} \geq a_n \) and \( a_n \leq 3 \). \( P_1 \) is obviously true. We will assume that \( P_n \) is true and then show that as a consequence \( P_{n+1} \) must also be true. \( a_{n+2} \geq a_{n+1} \ Leftrightarrow \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \ \Leftrightarrow \)

\[ 2 + a_{n+1} \geq 2 + a_n \ \Leftrightarrow \ \ a_{n+1} \geq a_n \], which is the induction hypothesis. \( a_{n+1} \leq 3 \ \Leftrightarrow \ \sqrt{2 + a_n} \leq 3 \ \Leftrightarrow \)

\[ 2 + a_n \leq 9 \ \Leftrightarrow \ \ a_n \leq 7 \], which is certainly true because we are assuming that \( a_n \leq 3 \). So \( P_n \) is true for all \( n \), and so \( a_1 \leq a_n \leq 3 \) (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, \( \lim_{n \to \infty} a_n \) exists.

(b) If \( L = \lim_{n \to \infty} a_n \), then \( \lim_{n \to \infty} a_{n+1} = L \) also, so \( L = \sqrt{2 + L} \ \Rightarrow \ L^2 = 2 + L \ \Leftrightarrow \ L^2 - L - 2 = 0 \ \Leftrightarrow \)

\( (L + 1)(L - 2) = 0 \ \Leftrightarrow \ L = 2 \) [since \( L \) can’t be negative].