40. (a) \( \frac{dx}{dt} = k(\alpha - x)(b - x) \), \( \alpha \neq b \). Using partial fractions, \( \frac{1}{(\alpha - x)(b - x)} = \frac{1/(b - \alpha)}{a - x} - \frac{1/(b - \alpha)}{b - x} \), so

\[
\int \frac{dx}{(\alpha - x)(b - x)} = \int k \, dt \quad \Rightarrow \quad \frac{1}{b - \alpha} \left( -\ln |\alpha - x| + \ln |b - x| \right) = kt + C \quad \Rightarrow \quad \ln \left| \frac{b - x}{a - x} \right| = (b - \alpha)(kt + C).
\]

The concentrations \( [A] = a - x \) and \( [B] = b - x \) cannot be negative, so \( \frac{b - x}{a - x} \geq 0 \) and \( \left| \frac{b - x}{a - x} \right| = \frac{b - x}{a - x} \).

We now have \( \ln \left( \frac{b - x}{a - x} \right) = (b - \alpha)(kt + C) \). Since \( x(0) = 0 \), we get \( \ln \left( \frac{b}{a} \right) = (b - \alpha)C \). Hence,

\[
\ln \left( \frac{b - x}{a - x} \right) = (b - \alpha)kt + \ln \left( \frac{b}{a} \right) \Rightarrow \frac{b - x}{a - x} = e^{(b - \alpha)kt} \Rightarrow x = \frac{b - x}{a - x} e^{(b - \alpha)kt} - \frac{a - x}{a - x} = \frac{a b [e^{(b - \alpha)kt} - 1]}{bc [e^{(b - \alpha)kt} - 1]} \text{ moles/L}.
\]

(b) If \( b = \alpha \), then \( \frac{dx}{dt} = k(\alpha - x)^2 \), so \( \frac{dx}{(\alpha - x)^2} = \int k \, dt \text{ and } \frac{1}{\alpha - x} = kt + C \). Since \( x(0) = 0 \), we get \( C = \frac{1}{\alpha} \).

Thus, \( a - x = \frac{1}{kt + 1/\alpha} \) and \( x = a - \frac{\alpha}{akt + 1} = \frac{a^2 k t}{akt + 1} \text{ moles/L} \). Suppose \( x = [C] = a/2 \) when \( t = 20 \). Then

\[
x(20) = a/2 = \frac{a}{2} = \frac{20a^2 k}{20ak + k} \Rightarrow 40a^2 k = 20a^2 k + a \Rightarrow 20a^2 k = a \Rightarrow k = \frac{1}{20a}, \text{ so}
\]

\[
x = \frac{a^2 t/(20a)}{1 + at/(20a)} = \frac{at/20}{1 + t/20} = \frac{at}{t + 20} \text{ moles/L}.
\]

42. If \( S = \frac{dT}{dr} \), then \( \frac{dS}{dr} = \frac{d^2 T}{dr^2} \). The differential equation \( \frac{d^2 T}{dr^2} + \frac{2}{r} \frac{dT}{dr} = 0 \) can be written as \( \frac{dS}{dr} + \frac{2}{r} S = 0 \). Thus,

\[
\frac{dS}{dr} = -\frac{2S}{r} \Rightarrow \frac{dS}{S} = -\frac{2}{r} \, dr \Rightarrow \int \frac{1}{S} \, dS = \int -\frac{2}{r} \, dr \Rightarrow \ln |S| = -2 \ln |r| + C. \text{ Assuming } S = dT/dr > 0 \text{ and } r > 0, \text{ we have } S = e^{-2 \ln r + C} = e^{\ln r^{-2} e^C} = r^{-2} k \quad [k = e^C] \Rightarrow S = \frac{1}{r^2} k \Rightarrow \frac{dT}{dr} = \frac{1}{r^2} k \Rightarrow
\]

\[
dT = \frac{1}{r^2} k \, dr \Rightarrow \int dT = \int \frac{1}{r^2} k \, dr \Rightarrow T(r) = -\frac{k}{r} + A.
\]

\( T(1) = 15 \Rightarrow 15 = -k + A \) (1) and \( T(2) = 25 \Rightarrow 25 = -\frac{1}{2} k + A \) (2).

Now solve for \( k \) and \( A \):

\[
-2(2) + (1) \Rightarrow -35 = -A, \text{ so } A = 35 \text{ and } k = 20, \text{ and } T(r) = -20/r + 35.
\]

45. (a) Let \( y(t) \) be the amount of salt (in kg) after \( t \) minutes. Then \( y(0) = 15 \). The amount of liquid in the tank is \( 1000 \text{ L} \) at all times, so the concentration at time \( t \) (in minutes) is \(\frac{y(t)}{1000} \text{ kg/L} \) and \(\frac{dy}{dt} = -\frac{y(t)}{1000} \text{ kg/L} \left(\frac{10 \text{ L}}{\text{min}}\right) = -\frac{y(t)}{100} \text{ kg/min} \).

\[
\int \frac{dy}{y} = -\frac{1}{100} \int dt \Rightarrow \ln y = -\frac{t}{100} + C, \text{ and } y(0) = 15 \Rightarrow \ln 15 = C, \text{ so } \ln y = \ln 15 - \frac{t}{100}.
\]

It follows that \( \ln \left( \frac{y}{15} \right) = -\frac{t}{100} \) and \( y = y_0 e^{-t/100} \), so \( y = 15 e^{-t/100} \text{ kg/L} \).

(b) After 0 minutes, \( y = 15 e^{-0/100} = 15 e^{0.0} = 15 \text{ kg/L} \). After 20 minutes, \( y = 15 e^{-20/100} = 15 e^{-0.2} \approx 12.3 \text{ kg} \).
1. (a) \( \frac{dP}{dt} = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100). \) Comparing to Equation 4, \( \frac{dP}{dt} = kP(1 - P/M) \), we see that the carrying capacity is \( M = 100 \) and the value of \( k \) is 0.05.

(b) The slopes close to 0 occur where \( P \) is near 0 or 100. The largest slopes appear to be on the line \( P = 50 \). The solutions are increasing for \( 0 < P_0 < 100 \) and decreasing for \( P_0 > 100 \).

(d) The equilibrium solutions are \( P = 0 \) (trivial solution) and \( P = 100 \). The increasing solutions move away from \( P = 0 \) and all nonzero solutions approach \( P = 100 \) as \( t \to \infty \).

10. (a) \( P(0) = P_0 = 400, P(1) = 1200 \) and \( M = 10,000 \). From the solution to the logistic differential equation

\[
P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kt}},
\]

we get

\[
P = \frac{400 \cdot 10,000}{400 + (9600)e^{-kt}} = \frac{10,000}{\frac{1}{24}e^{-kt} + \frac{25}{9}}, \quad P(1) = 1200 \quad \Rightarrow \quad 1 + \frac{24e^{-k}}{\frac{100}{12}} \Rightarrow e^k = \frac{288}{33} \Rightarrow k = \ln \frac{288}{33}. \]

So

\[
P = \frac{10,000}{\frac{1}{12}e^{-\ln(36/11)} + \frac{25}{9}} = \frac{10,000}{1 + 24 \cdot (11/36)^t}.
\]

(b) \( 5000 = \frac{10,000}{1 + 24(11/36)^t} \Rightarrow 24(\frac{11}{36})^t = 1 \Rightarrow t \ln \frac{11}{36} = \ln \frac{1}{24} \Rightarrow t \approx 2.68 \) years.

11. (a) \( \frac{dP}{dt} = kP \left( 1 - \frac{P}{M} \right) \Rightarrow \frac{d^2P}{dt^2} = k \left[ P \left( -\frac{1}{M} \frac{dP}{dt} \right) + \left( 1 - \frac{P}{M} \right) \frac{dP}{dt} \right] = k \frac{dP}{dt} \left( \frac{P}{M} - 1 + \frac{P}{M} \right) \]

\( = k \frac{dP}{dt} \left( 1 - \frac{P}{M} \right) \left( 1 - \frac{2P}{M} \right) = k^2 P \left( 1 - \frac{P}{M} \right) \left( 1 - \frac{2P}{M} \right). \)

(b) \( P \) grows fastest when \( P' \) has a maximum, that is, when \( P'' = 0 \). From part (a), \( P'' = 0 \Leftrightarrow P = 0, P = M \), or \( P = M/2 \). Since \( 0 < P < M \), we see that \( P'' = 0 \Leftrightarrow P = M/2 \).

6. \( y' - y = e^x \Leftrightarrow y' + (-1)y = e^x \Rightarrow P(x) = -1. \) \( I(x) = e \int P(x) \, dx = e \int -1 \, dx = -e^{-x}. \) Multiplying the original differential equation by \( I(x) \) gives \( e^{-x}y' - e^{-x}y = e^0 \Rightarrow (e^{-x}y)' = 1 \Rightarrow e^{-x}y = \int 1 \, dx \Rightarrow e^{-x}y = x + C \Rightarrow y = \frac{x + C}{e^{-x}} \Rightarrow y = xe^x + Ce^x. \)
8. \( 4x^2y + x^4y' = \sin^2 x \quad \Rightarrow \quad (x^4y)' = \sin^2 x \quad \Rightarrow \quad x^4y = \int \sin^2 x \, dx \quad \Rightarrow \)

\[
x^4y = \int \sin x (1 - \cos^2 x) \, dx = \int (1 - u^2)(-du) \quad \left[ \begin{array}{c}
u = \cos x, \\
du = -\sin x \, dx \end{array} \right]
\]

\[
= \int (u^2 - 1) \, du = -\frac{1}{3}u^3 - u + C = -\frac{1}{3}u(u^2 - 3) + C = \frac{1}{3} \cos x (\cos^2 x - 3) + C 
\]

\[
y = \frac{1}{3x^2} \cos x (\cos^2 x - 3) + \frac{C}{x^4}
\]

9. Since \( P(x) = 1 \) and the coefficient is \( x \), we can write the differential equation \( xy' + y = \sqrt{x} \) in the easily integrable form \( (xy)' = \sqrt{x} \) \( \Rightarrow \quad xy = \frac{2}{3}x^{3/2} + C \) \( \Rightarrow \quad y = \frac{2}{3}\sqrt{x} + C/x \).

20. \( (x^2 + 1) \frac{dy}{dx} + 3x(y - 1) = 0 \quad \Rightarrow \quad (x^2 + 1) y' + 3xy = 3x \quad \Rightarrow \quad y' + \frac{3x}{x^2 + 1} y = \frac{3x}{x^2 + 1} \).

\[
I(x) = e^{\int \frac{3x}{x^2 + 1} \, dx} = e^{(3/2) \ln|x^2 + 1|} = \left( e^{\ln(x^2 + 1)} \right)^{3/2} = (x^2 + 1)^{3/2}. \quad \text{Multiplying by } (x^2 + 1)^{3/2} \text{ gives}
\]

\[
(x^2 + 1)^{3/2} y' + 3x(x^2 + 1)^{1/2} y = 3x(x^2 + 1)^{1/2} \quad \Rightarrow \quad \left[ (x^2 + 1)^{3/2} y \right]' = 3x(x^2 + 1)^{1/2} \quad \Rightarrow 
\]

\[
(x^2 + 1)^{3/2} y = \int 3x(x^2 + 1)^{1/2} \, dx = (x^2 + 1)^{3/2} + C \quad \Rightarrow \quad y = 1 + C(x^2 + 1)^{-3/2}. \quad \text{Since } y(0) = 2, \text{ we have}
\]

\[
2 = 1 + C(1) \quad \Rightarrow \quad C = 1 \quad \text{and hence, } y = 1 + (x^2 + 1)^{-3/2}.
\]

23. Setting \( u = y^{1-n} \), \( \frac{du}{dx} = (1-n) y^{-n} \frac{dy}{dx} \) or \( \frac{dy}{dx} = \frac{u^{n/(1-n)}}{1-n} \frac{du}{dx} \).

Then the Bernoulli differential equation becomes \( \frac{u^{n/(1-n)}}{1-n} \frac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)} \) or \( \frac{du}{dx} + (1-n)P(x)u = Q(x)(1-n) \).

9. \( x = \sqrt{t}, \quad y = 1 - t \)

(a) \[
\begin{array}{c|c|c|c|c|c}
\hline
 t & 0 & 1 & 2 & 3 & 4 \\
\hline
 x & 0 & 1 & 1.414 & 1.732 & 2 \\
 y & 1 & 0 & -1 & -2 & -3 \\
\hline
\end{array}
\]

(b) \( x = \sqrt{t} \quad \Rightarrow \quad t = x^2 \quad \Rightarrow \quad y = 1 - t = 1 - x^2. \quad \text{Since } t \geq 0, \quad x \geq 0. \)

So the curve is the right half of the parabola \( y = 1 - x^2 \).

11. (a) \( x = \sin \frac{1}{2} \theta, \quad y = \cos \frac{1}{2} \theta, \quad -\pi \leq \theta \leq \pi \). \n
\[
x^2 + y^2 = \sin^2 \frac{1}{2} \theta + \cos^2 \frac{1}{2} \theta = 1. \quad \text{For } -\pi \leq \theta \leq 0, \text{ we have}
\]

\[
-1 \leq x \leq 0 \text{ and } 0 \leq y \leq 1. \quad \text{For } 0 \leq \theta \leq \pi, \text{ we have } 0 \leq x \leq 1 \quad \text{and } 1 \geq y \geq 0. \quad \text{The graph is a semicircle.}
14. \(x = e^x - 1, y = e^{2t}\).  

\[ y = (e^x)^2 = (x+1)^2 \] and since \(x > -1\), we have the right side of the parabola \(y = (x+1)^2\).

24. (a) From the first graph, we have \(1 \leq x \leq 2\). From the second graph, we have \(-1 \leq y \leq 1\). The only choice that satisfies either of those conditions is III.  

(b) From the first graph, the values of \(x\) cycle through the values from \(-2\) to \(2\) four times. From the second graph, the values of \(y\) cycle through the values from \(-2\) to \(2\) six times. Choice I satisfies these conditions.  

(c) From the first graph, the values of \(x\) cycle through the values from \(-2\) to \(2\) three times. From the second graph, we have \(0 \leq y \leq 2\). Choice IV satisfies these conditions.  

(d) From the first graph, the values of \(x\) cycle through the values from \(-2\) to \(2\) two times. From the second graph, the values of \(y\) do the same thing. Choice II satisfies these conditions.

25. When \(t = -1\), \((x, y) = (0, -1)\). As \(t\) increases to \(0\), \(x\) decreases to \(-1\) and \(y\) increases to \(0\). As \(t\) increases from \(0\) to \(1\), \(x\) increases to \(0\) and \(y\) increases to \(1\).  

As \(t\) increases beyond \(1\), both \(x\) and \(y\) increase. For \(t < -1\), \(x\) is positive and decreasing and \(y\) is negative and increasing. We could achieve greater accuracy by estimating \(x\)- and \(y\)-values for selected values of \(t\) from the given graphs and plotting the corresponding points.

28. (a) \(x = t^3 - t + 1 = (t^2 + 1) - t > 0\) [think of the graphs of \(y = t^4 + 1\) and \(y = t\)] and \(y = t^2 \geq 0\), so these equations are matched with graph V.  

(b) \(y = \sqrt{t} \geq 0\). \(x = t^2 - 2t = t(t - 2)\) is negative for \(0 < t < 2\), so these equations are matched with graph I.  

(c) \(x = \sin 2t\) has period \(2\pi / 2 = \pi\). Note that \(y(t) = \sin(t + 2\pi + \sin 2(t + 2\pi)) = \sin(t + 2\pi + \sin 2t) = \sin(t + \sin 2t) = y(t)\), so \(y\) has period \(2\pi\).  

These equations match graph II since \(x\) cycles through the values \(-1\) to \(1\) twice as \(y\) cycles through those values once.  

(d) \(x = \cos 5t\) has period \(2\pi / 5\) and \(y = \sin 2t\) has period \(\pi\), so \(x\) will take on the values \(-1\) to \(1\), and then \(1\) to \(1\), before \(y\) takes on the values \(-1\) to \(1\). Note that when \(t = 0\), \((x, y) = (1, 0)\). These equations are matched with graph VI.  

(e) \(x = t + \sin 4t, y = t^2 + \cos 3t\). As \(t\) becomes large, \(t\) and \(t^2\) become the dominant terms in the expressions for \(x\) and \(y\), so the graph will look like the graph of \(y = x^2\), but with oscillations. These equations are matched with graph IV.  

(f) \(x = \frac{\sin 2t}{4 + t^2}, y = \frac{\cos 2t}{4 + t^2}\). As \(t \to \infty\), \(x\) and \(y\) both approach \(0\). These equations are matched with graph III.
37. (a) \( x = t^2 \Rightarrow t = x^{1/2} \), so \( y = t^2 = x^{3/2} \).

We get the entire curve \( y = x^{3/2} \) traversed in a left to right direction.

\( x = t^2, \ y = t^2 \)

(b) \( x = t^6 \Rightarrow t = x^{1/6} \), so \( y = t^4 = x^{4/6} = x^{2/3} \).

Since \( x = t^6 \geq 0 \), we only get the right half of the curve \( y = x^{2/3} \).

(c) \( x = e^{-2t} = (e^{-1})^t \) [so \( e^{-x} = x^{1/2} \)],

\( y = e^{-2t} = (e^{-1})^t = (x^{1/2})^2 = x^{1/2} \).

If \( t < 0 \), then \( x \) and \( y \) are both larger than 1. If \( t > 0 \), then \( x \) and \( y \) are between 0 and 1. Since \( x > 0 \) and \( y > 0 \), the curve never quite reaches the origin.

4. \( x = t - t^{-1}, \ y = 1 + t^2; \ t = 1 \). \( \frac{dy}{dt} = 2t, \ \frac{dx}{dt} = 1 + t^{-2} = \frac{t^2 + 1}{t^2}, \) and \( \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t \left( \frac{t^2 + 1}{t^2} \right) = \frac{2t^2}{t^2 + 1} \).

When \( t = 1 \), \( (x, y) = (0, 2) \) and \( \frac{dy}{dx} = 2 \), so an equation of the tangent to the curve at the point corresponding to \( t = 1 \) is \( y - 2 = 1(x - 0) \), or \( y = x + 2 \).

18. \( x = t^3 - 3t, \ y = t^3 - 3t^2 \). \( \frac{dy}{dt} = 3t^2 - 6t = 3t(t - 2) \), so \( \frac{dy}{dt} = 0 \iff t = 0 \) or \( t = 2 \)

\( \frac{dx}{dt} = 3t^2 - 3 = 3(t + 1)(t - 1) \),

so \( \frac{dx}{dt} = 0 \iff t = -1 \) or \( t = 1 \) \( \iff (x, y) = (2, -4) \) or \( (-2, -2) \). The curve has horizontal tangents at \( (0, 0) \) and \( (2, -4) \), and vertical tangents at \( (2, -4) \) and \( (-2, -2) \).

25. \( x = \cos t, \ y = \sin t \cos t \). \( \frac{dx}{dt} = -\sin t, \ \frac{dy}{dt} = -\sin^2 t + \cos^3 t = \cos 2t \).

\( (x, y) = (0, 0) \iff \cos t = 0 \iff t \) is an odd multiple of \( \frac{\pi}{2} \). When \( t = \frac{\pi}{2} \),

\( \frac{dx}{dt} = -1 \) and \( \frac{dy}{dt} = -1 \), so \( \frac{dy}{dx} = 1 \). When \( t = \frac{3\pi}{2} \), \( \frac{dx}{dt} = 1 \) and \( \frac{dy}{dt} = -1 \). So \( \frac{dy}{dx} = -1 \). Thus, \( y = x \) and \( y = -x \) are both tangent to the curve at \( (0, 0) \).
42. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leq t \leq 3$. $\frac{dx}{dt} = e^t - e^{-t}$ and $\frac{dy}{dt} = -2$, so

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2.$$

Thus, $L = \int_0^3 (e^t + e^{-t}) \, dt = [e^t - e^{-t}]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}$.

48. $x = 3t - t^2$, $y = 3t^2$. $\frac{dx}{dt} = 3 - 2t$ and $\frac{dy}{dt} = 6t$, so

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

$$L = \int_{\sqrt{3}}^{\sqrt{3}} (3 + 3t^2) \, dt = 2 \int_0^{\sqrt{3}} (3 + 3t^2) \, dt = 2[3t + t^3]_0^{\sqrt{3}}$$

$$= 2(3\sqrt{3} + 3\sqrt{3}) = 12\sqrt{3}.$$