2. \( y = -t \cos t - t \implies \frac{dy}{dt} = -t(-\sin t) + \cos t(-1) - 1 = t \sin t - \cos t - 1. \)

\[
\text{LHS} = t \frac{dy}{dt} = t(t \sin t - \cos t - 1) = t^2 \sin t - t \cos t - t
\]

so \( y \) is a solution of the differential equation. Also \( y(\pi) = -\pi \cos \pi - \pi = -\pi(-1) - \pi = \pi - \pi = 0 \), so the initial condition is satisfied.

10. (a) \( y = k \implies y' = 0 \), so \( \frac{dy}{dt} = y^4 - 6y^2 + 5y^2 \implies 0 = k^4 - 6k^2 + 5k^2 \implies k^2(k^2 - 6k + 5) = 0 \implies k = 0, 1, \text{ or } 5 \)

(b) \( y \) is increasing \( \iff \frac{dy}{dt} > 0 \iff y^2(y - 1)(y - 5) > 0 \iff y \in (-\infty, 0) \cup (0, 1) \cup (5, \infty) \)

(c) \( y \) is decreasing \( \iff \frac{dy}{dt} < 0 \iff y \in (1, 5) \)

11. (a) This function is increasing and decreasing. But \( \frac{dy}{dt} = e^y(y - 1)^2 \geq 0 \) for all \( t \), implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When \( y = 1, \frac{dy}{dt} = 0 \), but the graph does not have a horizontal tangent line.

2. (a) [Image of a graph showing the direction field for the differential equation \( y' = \tan \left( \frac{\pi}{2} y \right) \).]

(b) It appears that the constant functions \( y = 0, y = 2, \) and \( y = 4 \) are equilibrium solutions. Note that these three values of \( y \) satisfy the given differential equation \( y' = \tan \left( \frac{\pi}{2} y \right) \).

3. \( y' = 2 - y \). The slopes at each point are independent of \( x \), so the slopes are the same along each line parallel to the \( x \)-axis.

Thus, III is the direction field for this equation. Note that for \( y = 2, y' = 0 \).

4. \( y' = x(2 - y) = 0 \) on the lines \( x = 0 \) and \( y = 2 \). Direction field I satisfies these conditions.

5. \( y' = x + y - 1 = 0 \) on the line \( y = -x + 1 \). Direction field IV satisfies this condition. Notice also that on the line \( y = -x \) we have \( y' = -1 \), which is true in IV.

6. \( y' = \sin x \sin y = 0 \) on the lines \( x = 0 \) and \( y = 0 \), and \( y' > 0 \) for \( 0 < x < \pi, 0 < y < \pi \). Direction field II satisfies these conditions.
\[ y' = xy - x^2 = x(y - x), \text{ so } y' = 0 \text{ for } x = 0 \text{ and } y = x. \] The slopes are positive only in the regions in quadrants I and III that are bounded by \( x = 0 \) and \( y = x \). The solution curve in the graph passes through \((0, 1)\).
19. (a) \( y' = F(x, y) = y \) and \( y(0) = 1 \) \( \Rightarrow \) \( x_0 = 0, y_0 = 1 \).

(i) \( h = 0.4 \) and \( y_1 = y_0 + hF(x_0, y_0) \) \( \Rightarrow \) \( y_1 = 1 + 0.4 \cdot 1 = 1.4 \). \( x_1 = x_0 + h = 0 + 0.4 = 0.4 \),
so \( y_1 = y(0.4) = 1.4 \).

(ii) \( h = 0.2 \) \( \Rightarrow \) \( x_1 = 0.2 \) and \( x_2 = 0.4 \), so we need to find \( y_2 \).
\( y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2 \),
\( y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44 \).

(iii) \( h = 0.1 \) \( \Rightarrow \) \( x_4 = 0.4 \), so we need to find \( y_4 \).
\( y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1 \),
\( y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21 \),
\( y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331 \),
\( y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641 \).

(b) [Graph showing \( y = e^x \) with lines for \( h = 0.1 \), \( h = 0.2 \), \( h = 0.4 \).]

We see that the estimates are underestimates since they are all below the graph of \( y = e^x \).

(c) (i) For \( h = 0.4 \): (exact value) \( - (\text{approximate value}) = e^{0.4} - 1.4 \approx 0.0918 \)

(ii) For \( h = 0.2 \): (exact value) \( - (\text{approximate value}) = e^{0.4} - 1.44 \approx 0.0518 \)

(iii) For \( h = 0.1 \): (exact value) \( - (\text{approximate value}) = e^{0.4} - 1.4641 \approx 0.0277 \)

Each time the step size is halved, the error estimate also appears to be halved (approximately).

8. \( \frac{dy}{d\theta} = \frac{e^y \sin^2 \theta}{y \sec \theta} \Rightarrow \frac{y}{e^y} \cdot dy = \frac{\sin^2 \theta}{\sec \theta} \cdot d\theta \Rightarrow \int y e^{-y} dy = \int \sin^2 \theta \cos \theta d\theta \).
Integrating the left side by parts with \( u = y, dv = e^{-y} dy \) and the right side by the substitution \( u = \sin \theta \), we obtain \( -ye^{-y} - e^{-y} = \frac{1}{2} \sin^2 \theta + C \). We cannot solve explicitly for \( y \).

14. \( y' = \frac{xy \sin x}{y + 1}, y(0) = 1 \). \( \frac{y + 1}{y} \cdot \frac{dy}{dx} = x \sin x \) \( \Rightarrow \) \( \int \left( 1 + \frac{1}{y} \right) \cdot dy = \int x \sin x \cdot dx \) \( \Rightarrow \)
\( y + \ln |y| = -x \cos x + \sin x + C \) [use parts with \( u = x, dv = \sin x \cdot dx \)]. Now \( y(0) = 1 \) \( \Rightarrow \)
\( 1 + 0 = 0 + 0 + C \) \( \Rightarrow \) \( C = 1 \), so \( y + \ln |y| = -x \cos x + \sin x + 1 \). We cannot solve explicitly for \( y \).
16. \( \frac{dP}{dt} = \sqrt{P} t \) \( \Rightarrow \) \( \frac{dP}{\sqrt{P}} = \sqrt{t} dt \) \( \Rightarrow \) \( \int P^{-1/2} dP = \int t^{1/2} dt \) \( \Rightarrow \) \( 2P^{1/2} = \frac{2}{3} t^{3/2} + C. \)

\( P(1) = 2 \) \( \Rightarrow \) \( 2\sqrt{2} = \frac{2}{3} + C \) \( \Rightarrow \) \( C = 2\sqrt{2} - \frac{2}{3} \), so \( 2P^{1/2} = \frac{2}{3} t^{3/2} + 2\sqrt{2} - \frac{2}{3} \) \( \Rightarrow \) \( \sqrt{P} = \frac{1}{2} t^{3/2} + \sqrt{2} - \frac{1}{3} \) \( \Rightarrow \)

\( P\left(\frac{1}{2} t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2 \).

29. The curves \( x^2 + 2y^2 = k^2 \) form a family of ellipses with major axis on the \( x \)-axis. Differentiating gives

\( \frac{d}{dx} (x^2 + 2y^2) = \frac{d}{dx} (k^2) \) \( \Rightarrow \) \( 2x + 4yy' = 0 \) \( \Rightarrow \) \( 4yy' = -2x \) \( \Rightarrow \) \( y' = -\frac{x}{2y} \). Thus, the slope of the tangent line at any point \((x, y)\) on one of the ellipses is \( y' = -\frac{x}{2y} \), so the orthogonal trajectories must satisfy \( y' = \frac{2y}{x} \) \( \Leftrightarrow \) \( \frac{dy}{dx} = \frac{2y}{x} \Leftrightarrow \frac{dy}{y} = 2 \frac{dx}{x} \Leftrightarrow \)

\( \int \frac{dy}{y} = 2 \int \frac{dx}{x} \Leftrightarrow \ln |y| = 2 \ln |x| + C_1 \Leftrightarrow \ln |y| = \ln |x|^2 + C_1 \Leftrightarrow \)

\( |y| = e^{\ln|x|^2+C_1} \Leftrightarrow y = \pm x^2 \cdot e^{C_1} = C x^2 \). This is a family of parabolas.

32. Differentiating \( y = \frac{x}{1 + kx} \) gives \( y' = \frac{1}{(1 + kx)^2} \cdot \frac{1}{x} \), but \( k = \frac{x - y}{xy} \), so

\( y' = \frac{1}{(1 + \frac{x-y}{y})^2} = \frac{y^2}{x^2}. \) Thus, the orthogonal trajectories must satisfy

\( y' = -\frac{x^2}{y^2} \Leftrightarrow y^2 dy = -x^2 dx \Leftrightarrow \int y^2 dy = -\int x^2 dx \Leftrightarrow \)

\( \frac{1}{2} y^3 = -\frac{1}{2} x^3 + C_1 \Leftrightarrow y^3 = C - x^3 \Leftrightarrow y = \sqrt[3]{C-x^3}. \)