1. This graph represents a direction field for the differential equation

\[ y' = y \left(1 - \frac{1}{4}y^2\right) \]

(a) Find the equilibrium solutions. For each of them, state if it is a stable or unstable solution.

To find the equilibrium solutions we set \( y' = 0 \). \( y = 2 \) and \( y = -2 \) are stable solutions, whereas \( y = 0 \) is an unstable solution.

(b) Sketch on the graph above the solution curve \( f(x) \) passing through the point \((0, 1)\).

Find the following limits:
• \( \lim_{x \rightarrow \infty} f(x) = 2 \)
• \( \lim_{x \rightarrow -\infty} f(x) = 0 \)

(c) Sketch on the graph above the solution curve \( g(x) \) passing through the point \((0, -1)\). Find the following limits:
• \( \lim_{x \rightarrow \infty} g(x) = -2 \)
• \( \lim_{x \rightarrow -\infty} g(x) = 0 \)

Most people sketched the curves correctly, so I won’t do it. You just had to follow the tangent segments and be careful not to cross equilibrium solutions.

2. A bacterial culture contains 5 grams of germs at midnight. It contains 12 grams of germs at 3:00 am. How many grams of germs does it contain at 2:00 am? You can assume that the rate of growth is proportional to the amount of germs present at any time. (i.e. start by setting up a differential equation - even if you remember the general formula, you have to solve the differential equation from the beginning)

We start with \( \frac{dP}{dt} = kP \) and this is a separable equation so we get that either \( P = 0 \) or

\[
\int \frac{1}{P} dP = \int kt \Rightarrow \ln |P| = kt + C \Rightarrow P = \pm e^C e^{kt}
\]

and adding back the solution \( P = 0 \) we get that the most general solution is

\[ P = Ae^{kt} \]

where \( A \) is a real number.

By using \( P(0) = 5 \) we get \( A = 5 \). By using \( P(3) = 12 \) we get \( 12 = 5e^{3k} \) hence \( k = \frac{1}{3} \ln \left( \frac{12}{5} \right) \). Hence

\[ P(2) = 5e^{\frac{2}{3} \ln \left( \frac{12}{5} \right)} = 5e^{\ln \left( \frac{12}{5}^{\frac{2}{3}} \right)} = 5 \left( \frac{12}{5} \right)^{\frac{2}{3}} \]

3. Find the solution to the differential equation

\[ \frac{dy}{dx} = \frac{1 + x}{xy} \]

with initial condition \( y(1) = -2 \).

This is a separable equation hence you get

\[ \int y \, dy = \int \frac{1 + x}{x} \, dx \]

hence

\[ \frac{y^2}{2} = \int \frac{1}{x} + 1 \, dx = \ln |x| + x + C \]
The initial condition gives us

\[ 2 = 1 + C \Rightarrow C = 1 \]

hence the solution is

\[ y = \pm \sqrt{2(\ln |x| + x + 1)} \]

but since the initial condition is negative we have to take the negative square root hence

\[ y = -\sqrt{2(\ln |x| + x + 1)} \]

4. Are the following integrals convergent or divergent? Justify your answer. You can use results about the convergence of integrals of functions of the form \(1/x^p\). Anything else must be justified.

(a) \[ \int_0^\infty e^{-x^4} \, dx \]

First of all we can write

\[ \int_0^\infty e^{-x^4} \, dx = \int_0^1 e^{-x^4} \, dx + \int_1^\infty e^{-x^4} \, dx \]

the first integral is the integral of a continuous function on an interval, hence the result is just a number and we just need to worry about the integral from 1 to \(\infty\). But for \(x \geq 1\) we have \(-x^4 \leq -x\) hence \(e^{-x^4} \leq e^{-x}\); now

\[ \int_1^\infty e^{-x} \, dx = \lim_{t \to \infty} [-e^{-x}]_1^t = \lim_{t \to \infty} \left( \frac{1}{e} - e^{-t} \right) = \frac{1}{e} \]

hence \(\int_1^\infty e^{-x} \, dx\) is convergent and by the comparison theorem so is \(\int_1^\infty e^{-x^4} \, dx\). Hence our integral is convergent.

(b) \[ \int_2^\infty \frac{x + 2}{x^8 + x^6 + 2} \, dx \]

We have

\[ \frac{x + 2}{x^8 + x^6 + 2} \leq \frac{x + 2}{x^8} \leq \frac{2x}{x^8} \]

and

\[ \int_2^\infty \frac{2x}{x^8} = 2 \int_2^\infty \frac{1}{x^7} \]

which is convergent, hence by comparison theorem our integral is convergent.
5. A parametric curve is defined by the equation

\[ x(t) = 3t^2 + 1, \quad y(t) = t^3 - 3t \]

Sketch this curve by first plotting \( x(t) \) and \( y(t) \).
Find the length of the loop.

(I am omitting the graphs of \( x(t) \) and \( y(t) \) which you had to draw as well) The curve passes by the point \((10, 0)\) at \( t = \pm \sqrt{3} \) hence to find the length of the curve we have to compute

\[
\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{36t^2 + (3t^2 - 3)^2}
\]

\[
= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{36t^2 + 9t^4 - 18t^2 + 9}
\]

\[
= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{9t^4 + 18t^2 + 9}
\]

\[
= \int_{-\sqrt{3}}^{\sqrt{3}} 3(t^2 + 1) = [t^3 + 3t]_{-\sqrt{3}}^{\sqrt{3}} = 12\sqrt{3}
\]

6. The error bound for the trapezoidal estimate is

\[ |E_T| \leq \frac{K(b - a)^3}{12n^2} \]

State what \( K \) is in this formula. Consider

\[
I = \int_{0.4}^{1} \frac{1}{9} \sin(x^2) \, dx
\]
Find some number of subdivisions, \( n \), so that the Trapezoidal estimate of \( I \) will be within \( 10^{-6} \) of the true value of the definite integral. Do not compute the trapezoidal estimate. Your \( n \) doesn’t need to be optimal, but you need to carefully justify why such an \( n \) is appropriate.

You can use the following formula: 
\[
|A(x) - B(x)| \leq |A(x)| + |B(x)|
\]

The \( K \) is the formula is a number such that 
\[
|f''(x)| \leq K \text{ for all } a \leq x \leq b.
\]

We have
\[
f'(x) = \frac{1}{9} \cos(x^2)2x
\]
\[
f''(x) = \frac{1}{9} \left(-\sin(x^2)4x^2 + 2\cos(x^2)\right)
\]

hence
\[
|f''(x)| = \frac{1}{9} \left|2\cos(x^2) - \sin(x^2)4x^2\right| \leq \frac{1}{9} \left|4x^2\sin(x^2)\right| + |2\cos(x^2)| \leq \frac{1}{9}(4 + 2) = \frac{2}{3}
\]

and we can take \( K = \frac{2}{3} \).

So we need
\[
\frac{K(b - a)^3}{12n^2} \leq 10^{-6} \Rightarrow \frac{2}{3 \cdot 12n^2} \leq 10^{-6} \Rightarrow n^2 \geq \frac{10^6}{18}
\]

7. Salt water enters a 20 gallon tank at a rate of 5 gal/min and leaves at the same rate from the bottom. A rotor keeps the solution well mixed. Write \( x(t) \) for the number of pounds of salt in the tank at time \( t \). Suppose at \( t = 0 \) the tank is full of fresh water. The concentration of the salt water being added is \( q(t) \) lb/gal.

(a) Write the differential equation governing \( x(t) \)

We have: salt in = 5q(t) and salt out = \( \frac{5}{20} x(t) \) hence our differential equation is
\[
\frac{dx}{dt} = 5q(t) - \frac{1}{4} x(t)
\]

with initial condition \( x(0) = 0 \).

(b) Suppose that the concentration \( q(t) \) of the salt being added varies with time according to the expression \( q(t) = 2 + t \). How much salt will be in the tank at time \( t \)?

Our differential equation is now
\[
\frac{dx}{dt} = 5(2 + t) - \frac{1}{4} x(t)
\]

this is a linear equation so let’s put it in the standard form
\[
\frac{dx}{dt} + \frac{1}{4} x(t) = 5(2 + t)
\]

then \( \int \frac{1}{4} dt = \frac{t}{4} \) and we can multiply both sides by \( I(x) = e^{\frac{t}{4}} \) to get
\[
\frac{d}{dt} \left(e^{\frac{t}{4}} x(t)\right) = 5(2 + t)e^{\frac{t}{4}}
\]
hence
\[ e^{-t}x(t) = \int 5(2 + t)e^t \, dt = 40e^t + 20te^t - 80e^t + C = 20te^t - 40e^t + C \]
hence
\[ x(t) = 20t - 40 + Ce^{-t} \]
and the initial condition \( x(0) = 0 \) gives \( C = 40 \) so
\[ x(t) = 20t - 40 + 40e^{-t} \].