8. $\frac{dy}{d\theta} = e^u \sin^2 \theta \Rightarrow \frac{y}{e^u} \frac{dy}{d\theta} = \sin^2 \theta \Rightarrow \int ye^{-y} \, dy = \int \sin^2 \theta \, d\theta$. Integrating the left side by parts with $u = y$, $dv = e^{-y} \, dy$ and the right side by the substitution $u = \sin \theta$, we obtain $-ye^{-y} - e^{-y} = \frac{1}{2} \sin^2 \theta + C$. We cannot solve explicitly for $y$.

12. $\frac{dy}{dx} = \frac{y \cos x}{1 + y^2}$, $y(0) = 1$. $(1 + y^2) \frac{dy}{dx} = y \cos x \Rightarrow \frac{1 + y^2}{y} \, dy = \cos x \, dx \Rightarrow \int \left( \frac{1}{y} + 1 \right) \, dy = \int \cos x \, dx \Rightarrow \ln |y| + \frac{1}{2} y^2 = \sin x + C$. $y(0) = 1 \Rightarrow \ln 1 + \frac{1}{2} = \sin 0 + C \Rightarrow C = \frac{1}{2}$, so $\ln |y| + \frac{1}{2} y^2 = \sin x + \frac{1}{2}$.

We cannot solve explicitly for $y$.

14. $\frac{dP}{dt} = \sqrt{P} \Rightarrow dP/\sqrt{P} = \sqrt{t} \, dt \Rightarrow \int P^{-1/2} \, dP = \int \sqrt{t} \, dt \Rightarrow 2P^{1/2} = \frac{2}{3} t^{3/2} + C$.

$P(1) = 2 \Rightarrow 2\sqrt{2} = \frac{2}{3} + C \Rightarrow C = 2\sqrt{2} - \frac{2}{3}$, so $2P^{1/2} = \frac{2}{3} t^{3/2} + 2\sqrt{2} - \frac{2}{3} \Rightarrow \sqrt{P} = \frac{1}{3} t^{3/2} + \sqrt{2} - \frac{2}{3} \Rightarrow P = \left( \frac{1}{3} t^{3/2} + \sqrt{2} - \frac{2}{3} \right)^2$.

16. $xy' + y = y^2 \Rightarrow x \frac{dy}{dx} = y^2 - y \Rightarrow x \, dy = (y^2 - y) \, dx \Rightarrow \frac{dy}{y^2 - y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \Rightarrow \int \left( \frac{1}{y-1} - \frac{1}{y} \right) \, dy = \int \frac{dx}{x} \Rightarrow \ln |y-1| - \ln |y| = \ln |x| + C \Rightarrow \ln \left| \frac{y-1}{y} \right| = \ln (e^C |x|) \Rightarrow \frac{y-1}{y} = e^C |x| \Rightarrow \frac{y-1}{y} = Kx$, where $K = \pm e^C \Rightarrow 1 - \frac{1}{y} = Kx$.

$\frac{1}{y} = 1 - Kx \Rightarrow y = \frac{1}{1 - Kx}$. [The excluded cases, $y = 0$ and $y = 1$, are ruled out by the initial condition $y(1) = -1$.]

Now $y(1) = -1 \Rightarrow -1 = \frac{1}{1 - K} \Rightarrow 1 - K = -1 \Rightarrow K = 2, \text{ so } y = \frac{1}{1 - 2x}$.

29. The curves $x^2 + 2y^2 = k^2$ form a family of ellipses with major axis on the $x$-axis. Differentiating gives $\frac{d}{dx} (x^2 + 2y^2) = \frac{d}{dx} (k^2) \Rightarrow 2x + 4yy' = 0 \Rightarrow 4yy' = -2x \Rightarrow y' = \frac{-x}{2y}$. Thus, the slope of the tangent line at any point $(x, y)$ on one of the ellipses is $y' = \frac{-x}{2y}$, so the orthogonal trajectories must satisfy $y' = \frac{2y}{x} \Leftrightarrow \frac{dy}{dx} = \frac{2y}{x} \Leftrightarrow \frac{dy}{y} = 2 = \frac{dx}{x} \Leftrightarrow \int \frac{dy}{y} = 2 \int \frac{dx}{x} \Leftrightarrow \ln |y| = 2 \ln |x| + C_1 \Leftrightarrow \ln |y| = \ln |x|^2 + C_1 \Leftrightarrow |y| = e^{\ln |x|^2 + C_1} \Leftrightarrow y = \pm x^2 \cdot e^{C_1} = Cx^2$. This is a family of parabolas.
32. Differentiating $y = \frac{x}{1 + kx}$ gives $y' = \frac{1}{(1 + kx)^2}$, but $k = \frac{x - y}{xy}$, so

$$y' = \frac{1}{\left(1 + \frac{x}{y} - 1\right)^2} = \frac{y^2}{x^2}.$$ Thus, the orthogonal trajectories must satisfy

$$y' = \frac{-x^2}{y^2} \implies y^2 \, dy = -x^2 \, dx \implies \int y^2 \, dy = -\int x^2 \, dx \implies$$

$$\frac{1}{2} y^2 = -\frac{1}{2} x^3 + C_1 \iff y^2 = C - x^3 \iff y = \sqrt[3]{C - x^3}.$$  

33. If $S = \frac{dT}{dr}$, then $\frac{dS}{dr} = \frac{d^2T}{dr^2}$. The differential equation $\frac{d^2T}{dr^2} + 2 \frac{dT}{dr} = 0$ can be written as $\frac{dS}{dr} + 2 \frac{S}{r} = 0$. Thus,

$$\frac{dS}{dr} = \frac{-2S}{r} \implies \frac{dS}{S} = \frac{-2}{r} \, dr \implies \int \frac{1}{S} \, dS = \int \frac{-2}{r} \, dr \implies \ln|S| = -2 \ln|r| + C.$$  

Assuming $S = \frac{dT}{dr} > 0$ and $r > 0$, we have $S = e^{-2 \ln|r| + C} = e^{\ln r^{-2}} e^C = r^{-2} k \quad [k = e^C] \implies S = \frac{1}{r^2} k \implies \frac{dT}{dr} = \frac{1}{r^2} k \implies$

$$dT = \frac{1}{r^2} k \, dr \implies \int dT = \int \frac{1}{r^2} k \, dr \implies T(r) = -\frac{k}{r} + A.$$  

$T(1) = 15 \implies 15 = -k + A \quad (1)$ and $T(2) = 25 \implies 25 = -\frac{k}{2} + A \quad (2)$.

Now solve for $k$ and $A$: $-2(2) + (1) \implies -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

40. (a) Use 1 billion dollars as the $x$-unit and 1 day as the $t$-unit. Initially, there is $10$ billion of old currency in circulation, so all of the $50$ million returned to the banks is old. At time $t$, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\frac{10 - x(t)}{10} 0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\frac{dx}{dt} = \frac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\frac{dx}{10 - x} = 0.005 \, dt \implies \frac{-dx}{10 - x} = -0.005 \, dt \implies \ln(10 - x) = -0.005 t + c \implies 10 - x = Ce^{-0.005t},$

where $C = e^c \implies x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10(1 - e^{-0.005t})$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$$9 = 10(e^{-0.005t}) \quad \Rightarrow \quad 0.9 = 1 - e^{-0.005t} \quad \Rightarrow \quad e^{-0.005t} = 0.1 \quad \Rightarrow \quad -0.005t = -\ln 0.1 \quad \Rightarrow \quad t = 200 \ln 10 \approx 460.517 \text{ days} \approx 1.26 \text{ years}.$$
41. (a) Let \( y(t) \) be the amount of salt (in kg) after \( t \) minutes. Then \( y(0) = 15 \). The amount of liquid in the tank is 1000 L at all times, so the concentration at time \( t \) (in minutes) is \( y(t)/1000 \) kg/L and \[ \frac{dy}{dt} = -\frac{y(t)}{1000} \frac{L}{\text{min}} \left(1 \text{ L/min}\right) = -\frac{y(t)}{1000} \frac{\text{kg}}{\text{min}}. \] \[ \int \frac{dy}{y} = -\frac{1}{100} \int dt \quad \Rightarrow \quad \ln y = -\frac{t}{100} + C, \] and \( y(0) = 15 \) \( \Rightarrow \) \( \ln 15 = C \), so \( \ln y = \ln 15 - \frac{t}{100} \). It follows that \( \ln \left(\frac{y}{15}\right) = -\frac{t}{100} \) and \( \frac{y}{15} = e^{-t/100} \), so \( y = 15 e^{-t/100} \) kg.

(b) After 20 minutes, \( y = 15 e^{-20/100} = 15e^{-0.2} \approx 12.3 \) kg.

1. (a) \( \frac{dP}{dt} = 0.05 P - 0.0005 P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100) \). Comparing to Equation 1, \( \frac{dP}{dt} = kP(1 - P/K) \), we see that the carrying capacity is \( K = 100 \) and the value of \( k \) is 0.05.

(b) The slopes close to 0 occur where \( P \) is near 0 or 100. The largest slopes appear to be on the line \( P = 50 \). The solutions are increasing for \( 0 < P_0 < 100 \) and decreasing for \( P_0 > 100 \).

(c) All of the solutions approach \( P = 100 \) as \( t \) increases. As in part (b), the solutions differ since for \( 0 < P_0 < 100 \) they are increasing, and for \( P_0 > 100 \) they are decreasing. Also, some have an IP and some don’t. It appears that the solutions which have \( P_0 = 20 \) and \( P_0 = 40 \) have inflection points at \( P = 50 \).

(d) The equilibrium solutions are \( P = 0 \) (trivial solution) and \( P = 100 \). The increasing solutions move away from \( P = 0 \) and all nonzero solutions approach \( P = 100 \) as \( t \to \infty \).

8. (a) \( P(0) = P_0 = 400 \), \( P(1) = 1200 \) and \( K = 10,000 \). From the solution to the logistic differential equation \( P(t) = \frac{P_0K}{P_0 + (K - P_0)e^{-kt}} \), we get \( P = \frac{400 \cdot 10,000}{400 + (9600)e^{-kt}} = \frac{10,000}{1 + 24e^{-kt}} \). \( P(1) = 1200 \) \( \Rightarrow \) \( 1 + 24e^{-k} = \frac{100}{12} \) \( \Rightarrow \) \( e^k = \frac{298}{56} \) \( \Rightarrow \) \( k = \ln\frac{26}{11} \). So \( P = \frac{10,000}{1 + 24e^{-t \ln(26/11)}} = \frac{10,000}{1 + 24 \cdot \left(\frac{11}{36}\right)^t} \).

(b) \( 5000 = \frac{10,000}{1 + 24 \cdot \left(\frac{11}{36}\right)^t} \) \( \Rightarrow \) \( 24 \left(\frac{11}{36}\right)^t = 1 \) \( \Rightarrow \) \( t \ln \frac{11}{36} = \ln \frac{1}{24} \) \( \Rightarrow \) \( t \approx 2.68 \) years.

9. (a) \( \frac{d^2 P}{dt^2} = kP \left(1 - \frac{P}{K}\right) \Rightarrow \frac{d^2 P}{dt^2} = k \left[ P \left(- \frac{1}{K} \frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right) \frac{dP}{dt}\right] = k \frac{dP}{dt} \left(\frac{P}{K} + 1 - \frac{P}{K}\right) \)

\[ = k \left[ kP \left(1 - \frac{P}{K}\right) \right] \left(1 - \frac{2P}{K}\right) = k^2 P \left(1 - \frac{P}{K}\right) \left(1 - \frac{2P}{K}\right). \]

(b) \( P \) grows fastest when \( P'' \) has a maximum, that is, when \( P'' = 0 \). From part (a), \( P'' = 0 \) \( \iff \) \( P = 0, P = K \), or \( P = K/2 \). Since \( 0 < P < K \), we see that \( P'' = 0 \) \( \iff \) \( P = K/2 \).
5. Comparing the given equation, $y' + 2y = 2e^{2x}$, with the general form, $y' + P(x)y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives
\[ e^{2x}y' + 2e^{2x}y = 2e^{3x} \Rightarrow (e^{2x}y)' = 2e^{3x} \Rightarrow e^{2x}y = \int 2e^{3x} \, dx \Rightarrow e^{2x}y = \frac{2}{3}e^{3x} + C \Rightarrow y = \frac{2}{3}e^{x} + Ce^{-2x}. \]

15. $y' = x + y$ \Rightarrow $y' + (-1)y = x$. $I(x) = e^{\int (-1) \, dx} = e^{-x}$. Multiplying by $e^{-x}$ gives $e^{-x}y' - e^{-x}y = xe^{-x}$ \Rightarrow $(e^{-x}y)' = xe^{-x}$ \Rightarrow $e^{-x}y = \int xe^{-x} \, dx = -xe^{-x} - e^{-x} + C$ [integration by parts with $u = x, dv = e^{-x} \, dx$] \Rightarrow $y = -x - 1 + Ce^{x}$. $y(0) = 2$ \Rightarrow $-1 + C = 2$ \Rightarrow $C = 3$, so $y = -x - 1 + 3e^{x}$. 