Calculus 2, Spring 2010
Sample Final

May 3, 2010

1. Find the interval of convergence of the series
\[ \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} (x - 3)^n \]

2. Consider the following equality
\[ \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{-n} = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left( \frac{1}{x} \right)^n = \frac{1}{1 - x} + \frac{\frac{1}{x}}{1 - \frac{1}{x}} = 0 \]
explain why this is false for any value of \( x \).

3. Find the surface area of the surface obtained by rotating \( y = \sqrt{7 - x^2} \), \( 0 \leq x \leq 1 \) about the \( x \) axis.

4. Solve the following differential equation:
\[ \frac{dy}{dx} = 1 + y^2 \quad y(2) = 0 \]

5. Are the following series convergent or divergent? Are they absolutely convergent?
   (a) \[ \sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2 + 1} \]
   (b) \[ \sum_{n=1}^{\infty} (-1)^n \frac{n}{e^n} \]
   (c) \[ \sum_{n=1}^{\infty} (-1)^n \frac{(3n)!}{(n!)^2(2n)!} \]
\[ \sum_{n=3}^{\infty} (-1)^n \frac{1}{\sqrt{n-2}} \]

6. Suppose \( \sum_{n=0}^{\infty} c_n x^n \) converges when \( x = -4 \) and diverges when \( x = 6 \). What can you say about the convergence or divergence of

(a) \( \sum c_n \)

(b) \( \sum c_n 8^n \)

(c) \( \sum c_n (-3)^n \)

(d) \( \sum (-1)^n c_n 9^n \)

(e) \( \sum c_n 4^n \)

7. Evaluate

\[ \lim_{x \to 0} \frac{x - \arctan x}{x^3} \]

don’t use L’H.

8. Approximate

\[ \int_{0}^{1} x \cos x^3 \, dx \]

to within 10\(^{-3}\).

9. Let \( C \) be the curve \( y = x^4/4 \) with \( 0 \leq x \leq 1/2 \).

(a) Set up an integral for the length of \( C \)

(b) Using a Taylor series and term by term integration, express the integral in part (a) as a convergent infinite series. Give numerical values for the first three terms in the series and a formula for the general term of the series.

(c) Explain why the method of (b) wouldn’t work to find the length of the same curve extending from \( x = 0 \) all the way to \( x = 2 \). Give an approximate value for this length, using the trapezoidal rule with \( n = 4 \) divisions.

10. Find the Taylor series for \( \cos x \) centered at \( \pi/4 \).

11. Evaluate each of the following integrals:

(a)

\[ \int x^3 \cos x \, dx \]
(b) \[ \int \sqrt{2x - x^2} \, dx \]

(c) \[ \int (\cos^3 x)(\sin^4 x) \, dx \]

(d) \[ \int \arctan x \, dx \]

12. (a) Describe how to approximate \( e^{\frac{\pi}{10}} \) with an error of size at most 0.0001 using a partial sum of a Taylor series. Give an explicit error estimate, and write an explicit partial sum which correctly approximates \( e^{\frac{\pi}{10}} \). You may use the fact that \( e < 3 \). (Do the arithmetic needed for the error estimate; don’t do the arithmetic involved in the partial sum!)

(b) Find some interval of positive length with center at 0 so that \( \cos x \) can be replaced by \( 1 - \frac{x^2}{2} + \frac{x^4}{24} \) with an error of size at most \( \frac{1}{100} \) at any point in the interval. (Again, an error estimate is needed.) Comment: a “best possible” interval is not requested, but the interval that’s given should be supported by reasoning.

13. Write

\[ \int_0^x \sin t^2 \, dt \]

as a series. What is its interval of convergence? Approximate

\[ \int_0^{\frac{\pi}{4}} \sin t^2 \, dt \]

within 0.001 of the actual value (do not compute the corresponding partial sum)