Sample exercises for the Final

July 1, 2009

1. Compute the following indefinite integrals:

   (a) \[
   \int x \sin(3x^2 + 2) \, dx
   \]
   
   Let \( u = 3x^2 + 2 \) then \( du = 6x \, dx \) hence \( \frac{du}{6} = 6 \, dx \). Therefore
   \[
   \int x \sin(3x^2 + 2) \, dx = \int \sin u \frac{du}{6} = \frac{1}{6} \cos u + C = \frac{1}{6} \cos(3x^2 + 2) + C
   \]

   (b) \[
   \int \frac{x + 3}{x^2} \, dx
   \]
   \[
   \int \frac{x + 3}{x^2} \, dx = \int \frac{x}{x^2} \, dx + \int \frac{3}{x^2} \, dx = \ln|x| - 3 \frac{1}{x} + C
   \]

   (c) \[
   \int e^{\sqrt{x}} \frac{1}{\sqrt{x}} \, dx
   \]
Let \( u = e^{\sqrt{x}} \), then
\[
du = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} \, dx
\]
therefore
\[
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = \int 2 \, du = 2u + C = 2e^{\sqrt{x}} + C
\]

(d)
\[
\int \frac{1 + 2x}{\sqrt{1 - x^2}} \, dx
\]

First of all let’s divide the integral in two parts:
\[
\int \frac{1 + 2x}{\sqrt{1 - x^2}} = \int \frac{1}{\sqrt{1 - x^2}} + \int \frac{2x}{\sqrt{1 - x^2}}
\]
we know how to deal with the first part, the antiderivative is \( \arcsin \). For the second part, let’s make the substitution \( u = 1 - x^2 \). Then \( du = -2x \, dx \) therefore \( -du = 2x \, dx \). Hence (for the second part):
\[
\int \frac{2x}{\sqrt{1 - x^2}} = \int \frac{-1}{\sqrt{u}} \, du
\]
\[
= \int -u^{-\frac{1}{2}} \, du
\]
\[
= -2u^{\frac{1}{2}} + C
\]
\[
= -2\sqrt{1 - x^2} + C
\]

Hence the integral we began is
\[
\int \frac{1 + 2x}{\sqrt{1 - x^2}} = \arcsin x - 2\sqrt{1 - x^2} + C
\]

2. Compute the following integrals:

(a)
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x \cos x}{1 + x^2} \, dx
\]
This function is odd and the integral is between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \). Hence the answer is 0.
(b) 

\[ \int_{0}^{\pi} \frac{\sin x}{\sqrt{\cos x}} \, dx \]

Let’s make the substitution \( u = \cos x \). Then \( du = -\sin x \, dx \) and \( u(0) = \cos 0 = 1, u(\pi/2) = \cos(\pi/2) = 0 \)

\[ \int_{0}^{\pi} \frac{\sin x}{\sqrt{\cos x}} \, dx = - \int_{1}^{0} \frac{1}{\sqrt{u}} \, du \]

\[ = - \int_{1}^{0} u^{-\frac{1}{2}} \, du \]

\[ = -\left[ 2\sqrt{u} \right]_{1}^{0} = 2 \]

(c) 

\[ \int_{0}^{3} |x^2 - 4| \, dx \]

Let’s start by noticing that \( x^2 - 4 = (x - 2)(x + 2) \). We are interested in the interval \([0, 3]\). In this interval \( x + 2 \) is always > 0 but \( x - 2 > 0 \) for \( x \in (2, 3] \), and \( x - 2 < 0 \) for \( x \in [0, 2) \). Hence we have to split the integral into two parts!

\[ \int_{0}^{3} |x^2 - 4| \, dx = \int_{0}^{2} |x^2 - 4| \, dx + \int_{2}^{3} |x^2 - 4| \, dx \]

\[ = \int_{0}^{2} -(x^2 - 4) \, dx + \int_{2}^{3} (x^2 - 4) \, dx \]

\[ = \left[ -\left( \frac{x^3}{3} - 4x \right) \right]_{0}^{2} + \left[ \left( \frac{x^3}{3} - 4x \right) \right]_{2}^{3} \]

\[ = -\frac{8}{3} - 8 + \left( 9 - 12 - \left( \frac{8}{3} - 8 \right) \right) \]

\[ = 13 - \frac{16}{3} \]

(d) 

\[ \int_{-1}^{1} \frac{x^2}{\sqrt{1-x}} \, dx \]
Let \( u = 1 - x \). Then \( du = -dx \). We still have to deal with the \( x^2 \) on the numerator: since \( u = 1 - x \) get \( x = 1 - u \) and \( x^2 = 1 - 2u + u^2 \).

We still have to compute the endpoints!! So \( u(-1) = 1 - (-1) = 2 \) and \( u(1) = 1 - 1 = 0 \). Don’t get worried by the fact that we are computing an integral where the upper limit of integration is smaller than the lower one (that’s just minus the integral the other way around, so I’ll just flip it). By substituting all of the information we have we get

\[
\int_{-1}^{1} \frac{x^2}{\sqrt{1-x}} \, dx = \int_{2}^{0} \frac{1 - 2u + u^2}{\sqrt{u}} \, du
\]

\[
= \int_{0}^{2} (1 - 2u + u^2)u^{-\frac{1}{2}} \, du
\]

\[
= \int_{0}^{2} (u^{-\frac{1}{2}} - 2u^\frac{1}{2} + u^\frac{3}{2})
\]

\[
= \left[ 2u^{\frac{1}{2}} - \frac{2}{3}u^\frac{3}{2} + \frac{5}{2}u^\frac{5}{2} \right]_0
\]

\[
= 2\sqrt{2} - \frac{4}{3}\sqrt{8} + \frac{2}{5}\sqrt{32}
\]

\[
= 2\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{2}{5}4\sqrt{2}
\]

3. Estimate the number \( \ln 1.04 \).

Estimate the number \( \ln 3 \) (remember: \( e \sim 2.7 \)).

**Solution:** We will use linear approximation. Remember the formula for linear approximation:

\[
f(x) \sim f(a) + f'(a)(x - a)
\]

Also, remember that \( \frac{d}{dx} \ln x = \frac{1}{x} \). To estimate \( \ln 1.04 \) we can use \( a = 1 \). Then

\[
\ln 1.04 \sim \ln 1 + \frac{1}{1}0.04 = 0.04
\]

To estimate \( \ln 3 \) we can use \( a = e \). Then we get

\[
\ln 3 \sim \ln e + \frac{1}{e}0.3 \sim 1 + 0.4 \cdot 0.3 = 1.12
\]

4. State the fundamental theorem of calculus.

Use it to compute

\[
\frac{d}{dx} \int_{x}^{3x-1} \tan(2t - 1)\sqrt{t} \, dt
\]
Is this computation correct:

\[
\int_{-1}^{2} \frac{1}{x^2} \, dx = \left[ \frac{-1}{x} \right]_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}
\]

**Solution:** Let \( f \) be a continuous function on \([a, b]\). Let

\[ g(x) = \int_{a}^{x} f(t) \, dt \quad a \leq x \leq b \]

Then \( g \) is continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) = f(x) \).

Moreover, if \( F \) is any antiderivative of \( f \) then

\[ \int_{a}^{b} f(t) \, dt = F(b) - F(a) \]

To compute the derivative we first have to put the integral in better shape.

\[
\int_{x}^{3x-1} \tan(2t - 1) \sqrt{t} \, dt = \int_{x}^{0} \tan(2t - 1) \sqrt{t} \, dt + \int_{0}^{3x-1} \tan(2t - 1) \sqrt{t} \, dt
\]

\[ = -\int_{0}^{x} \tan(2t - 1) \sqrt{t} \, dt + \int_{0}^{3x-1} \tan(2t - 1) \sqrt{t} \, dt \]

Now we have to take the derivative. Remember to apply the chain rule for the second summand! (if you set \( g(x) = \int_{0}^{x} \tan(2t - 1) \sqrt{t} \, dt \) then the second summand is \( g(3x - 1)! \))

\[
\frac{d}{dx} \int_{x}^{3x-1} \tan(2t - 1) \sqrt{t} \, dt = -\tan(2x - 1) \sqrt{x} + 3 \tan(2(3x - 1) - 1) \sqrt{3x - 1}
\]

\[ = -\tan(2x - 1) \sqrt{x} + 3 \tan(6x - 3) \sqrt{3x - 1} \]

As for what is wrong with the computation: we cannot apply the fundamental theorem of Calculus in this case since \( \frac{1}{x^2} \) is not continuous on \([-1, 2]\).

5. Sketch the graph of the function

\[ f(x) = \frac{x + 1}{\sqrt{x^2 + 1}} \]

**Solution:** Don’t worry, if I ask you to sketch a function on the final I will remind you of the various steps. Let’s start!

(a) the domain is all real numbers since \( x^2 + 1 > 0 \) for all real numbers.
(b) \( f(0) = 1 \) is the \( y \)-intercept. \( f(x) = 0 \iff x + 1 = 0 \iff x = -1 \) so \( x = -1 \) is the only \( x \)-intercept.

(c) \( f \) is neither even nor odd nor periodic

(d) \( f \) does not have any horizontal or vertical asymptotes. To compute the limits for \( x \) going to infinity first notice that

\[
 f(x) = \frac{x (1 + \frac{1}{x})}{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}}
\]

Now remember: \( \sqrt{x^2} \) is equal to \( x \) if \( x > 0 \) but to \( -x \) when \( x < 0 \)!! Hence

\[
 \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{x (1 + \frac{1}{x})}{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}} =
\]

\[
 = \lim_{x \to +\infty} \frac{x (1 + \frac{1}{x})}{x \sqrt{1 + \frac{1}{x^2}}} = \lim_{x \to +\infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}}} = 1
\]

whereas

\[
 \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x (1 + \frac{1}{x})}{\sqrt{x^2} \sqrt{1 + \frac{1}{x^2}}} =
\]

\[
 = \lim_{x \to -\infty} \frac{x (1 + \frac{1}{x})}{-x \sqrt{1 + \frac{1}{x^2}}} = -\lim_{x \to -\infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}}} = -1
\]

(e) The first derivative of \( f \) is

\[
 f'(x) = \frac{\sqrt{x^2 + 1} - (x + 1) \frac{x}{x^2 + 1}}{x^2 + 1}
\]

\[
 = \frac{x^2 + 1 - x^2 - x}{(x^2 + 1)^{\frac{3}{2}}}
\]

\[
 = \frac{1 - x}{(x^2 + 1)^{\frac{3}{2}}}
\]

the denominator is always positive. The numerator is negative for \( x > 1 \) and positive otherwise. Hence:

\( f \) is increasing for \( x \in (-\infty, 1) \)

\( f \) is decreasing for \( x \in (1, +\infty) \)
(f) By the first derivative test, \( f \) has a local max at 1, and \( f(1) = 1 \).

(g) Let’s compute the second derivative:

\[
f''(x) = \frac{-(x^2 + 1)^{3/2} - (1 - x)^{3/2}(x^2 + 1)^{1/2}}{(x^2 + 1)^3}
\]

\[
= \frac{-(x^2 + 1) - (1 - x)^{3/2}}{(x^2 + 1)^{3/2}(x^2 + 1)^{1/2}}
\]

\[
= \frac{-x^2 - 1 - (1 - x)3x}{(x^2 + 1)^{7/2}}
\]

\[
= \frac{2x^2 - 3x - 1}{2(x^2 + 1)^{7/2}}
\]

the two roots of the numerator are \( \frac{3 + \sqrt{17}}{4} \) and \( \frac{3 - \sqrt{17}}{4} \). These two are the points of inflection. Don’t get scared! (although I promise I will be nicer on the final!). \( \sqrt{17} \) is about 4 but a little bigger, and that’s all we care about. Hence \( \frac{3 - \sqrt{17}}{4} \) is about \(-\frac{1}{4}\) and \( \frac{3 + \sqrt{17}}{4} \) is a little smaller than 2. Hence we found:

\( f \) is concave down on \( \left( \frac{3 - \sqrt{17}}{4}, \frac{3 + \sqrt{17}}{4} \right) \)

\( f \) is concave up everywhere else. And here is the plot (thanks Maple!)

6. Sketch the graph of a function that has 3 local extrema and 5 critical points. How many inflection points does you graph have? I’ll discuss this one in class.
7. Sketch the graph of the function

\[ f(x) = \ln(1 + x^2) \]

(a) the domain in \( \mathbb{R} \)

(b) the \( y \)-intercept is \( f(0) = \ln 1 = 0 \)

\[ \ln(1 + x^2) = 0 \Rightarrow 1 + x^2 = 1 \Rightarrow x^2 = 0 \Rightarrow x = 0 \] so \((0, 0)\) is the only \( x \)-intercept

(c) \( f(-x) = \ln(1 + (-x)^2) = \ln(1 + x^2) = f(x) \) so \( f \) is even

(d)

\[ \lim_{x \to +\infty} \ln(1 + x^2) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} \ln(1 + x^2) = -\infty \]

There are no vertical asymptotes

(e) The first derivative is

\[ f'(x) = \frac{2x}{1 + x^2} \]

hence \( f'(x) > 0 \) for \( x > 0 \) and \( f'(x) < 0 \) for \( x < 0 \).

\( f \) is decreasing on \((-\infty, 0)\) and increasing on \((0, +\infty)\).

(f) By the first derivative test, \( f \) has a min at \( x = 0 \) and we already know \( f(0) = 0 \)

(g) The second derivative is

\[ f''(x) = \frac{2(1 + x^2) - 2x^2}{(1 + x^2)^2} = \frac{2 + 2x^2 - 4x^2}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2} = \frac{2(1 + x)(1 - x)}{(1 + x^2)^2} \]

Hence \( f''(x) > 0 \) for \( x \in (-1, 1) \) and \( f''(x) < 0 \) for \( x \in (-\infty, -1) \) and \((1, +\infty)\).

Therefore \( f \) is concave up for \( x \in (-1, 1) \) and concave down for \( x \in (-\infty, -1) \) and \((1, +\infty)\).
8. If $f$ is continuous and $\int_1^{22} f(x)\,dx = 3$, compute

$$\int_0^7 f(3x+1)\,dx$$

**Solution:** This is a nice way to see if you can do substitution integrals. Let’s compute $\int_0^7 f(3x+1)\,dx$ by making the substitution $u = 3x + 1$ then $du = 3\,dx$ hence $\frac{du}{3} = dx$. Moreover, $u(0) = 1$ and $u(7) = 22$. Hence

$$\int_0^7 f(3x+1)\,dx = \int_1^{22} f(u)\frac{du}{3} = \frac{1}{3} \int_1^{22} f(u)\,du = \frac{1}{3} \cdot 3 = 1$$

Here is the graph:

9. State Rolle’s theorem.

   Give an example of a function $f$ such that:
   
   - $f(a) = f(b)$ for some $a, b \in \mathbb{R}$;
   - $f$ is continuous;
   - $f$ is not differentiable on $[a, b]$;
   - The conclusion of Rolle’s theorem does not hold.
Solution: Let $f$ be a function continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose $f(a) = f(b)$. Then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

For the example, let’s use $f(x) = |x|$ on the interval $[-1, 1]$. Then $f$ is continuous on $[-1, 1]$, but $f$ is not differentiable at $x = 0$. Also, $| -1 | = |1 | = 1$. The conclusion of Rolle’s theorem does not hold because there is no $c \in [-1, 1]$ such that $f'(c) = 0$, in fact we know that

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

and $f'(x)$ does not exist for $x = 0$. What does this show? This shows us that the hypothesis “$f$ is differentiable everywhere in $(a, b)$ is really important, otherwise the theorem is false!

10. Compute the following limits:

(a) \[
\lim_{x \to +\infty} xe^{-x} = \lim_{x \to +\infty} xe^{-x} = \lim_{x \to +\infty} \frac{x}{e^x}
\]

so I can apply l’Hospital’s rule and get (differentiating top and bottom)

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

(b) \[
\lim_{x \to 0} \frac{\arctan x}{x}
\]

Numerator and denominator tend to zero for $x$ approaching 0. Hence this is an indeterminate form of type $\frac{0}{0}$ and applying l’Hospital’s rule we get

$$\lim_{x \to 0} \frac{\arctan x}{x} = \lim_{x \to 0} \frac{1}{1 + x^2} = 1$$

(c) \[
\lim_{x \to +\infty} (\ln(x) - \ln(x + 1))
\]

Let’s apply the rules of logarithms:

$$\lim_{x \to +\infty} (\ln(x) - \ln(x + 1)) = \lim_{x \to +\infty} \ln \left( \frac{x}{x + 1} \right)$$

as $x \to +\infty$, $\frac{x}{x + 1}$ tends to 1, hence $\ln \left( \frac{x}{x + 1} \right)$ tends to $\ln 1 = 0$ because $\ln$ is continuous (remember the definition of continuity?)
11. An ant is crawling along a path that is exactly the graph of the function \( y = 3x^2 \). She starts at the origin \((0,0)\). Her \(x\)-coordinate is changing at the rate of 10 cm per minute. How fast is her distance from the origin changing when her \(y\)-coordinate is 27?

**Solution:** First of all let’s make a picture:

![Graph of the function](image)

What do we know:

\[
\frac{dx}{dt} = 10 \text{cm/s}
\]

Let \(S\)=distance of the ant from the origin. Then

\[
S = \sqrt{x^2 + y^2} = \sqrt{x^2 + (3x^2)^2} = \sqrt{x^2 + 9x^4}
\]

I expressed \(S\) as a function of a unique variable, \(x\), and now I can use implicit differentiation by considering both \(D\) and \(x\) to be functions of \(t\):

\[
\frac{dS}{dt} = \frac{d}{dt} \sqrt{x^2 + 9x^4} = \frac{1}{2\sqrt{x^2 + 9x^4}} \left( 2x \frac{dx}{dt} + 36x^3 \frac{dx}{dt} \right)
\]

Now we just have to realize that if \(y = 27\) then \(3x^2 = 27\), so \(x = 3\). By substituting this and \(\frac{dx}{dt} = 10\text{cm/s}\) into out expression for \(S\) we get

\[
\frac{dS}{dt} = \frac{1630}{\sqrt{82}}
\]

12. Suppose that \(f\) is continuous and differentiable on all of \(\mathbb{R}\). Suppose \(f'(x) > 0\) for all \(x \in [0,1]\) and \(f(0) = 0\). Is \(f(1)\) positive, negative, or zero? Explain.

**Solution** Since \(f'(x) > 0\) for all \(x \in [0,1]\), this means \(f\) is strictly increasing on the interval \([0,1]\). Hence \(f(1)\) must be positive.
13. Find the volume of the solid obtained by considering the region bounded by 
\( y = x^3 \) and \( x = 1 \) and \( y = 0 \) and and rotating it along the line \( y = -2 \).

**Solution:** We will integrate with respect to \( y \). The cross section at height \( y \) is an annulus of inner radius \( 2 + \sqrt[3]{y} \) and the outer radius is 3. Hence get

\[
V(S) = \int_{0}^{1} \pi (3^2 - (2 + \sqrt[3]{y})^2) \\
= \int_{0}^{1} \pi (9 - y^{\frac{2}{3}}) \, dy \\
= \pi \left[ 9y - \frac{3}{5}y^{\frac{5}{3}} \right]_{0}^{1} = \pi \left( 9 - \frac{3}{5} \right)
\]

14. Consider the following trapezoid:

![Trapezoid Diagram]

(\( b \) and \( l \) are fixed numbers, \( B \) and \( \theta \) are not). Find the angle \( \theta \) that maximizes the area.

Let \( h \) be the height of the trapezoid. Let \( A \) be the area of the trapezoid. Then we know

\[
A = \frac{(b + B)h}{2}
\]

We know that \( b \) is a fixed number so we just have to express \( B \) and \( h \) is terms of the constants \( b \) and \( l \) and the one variable \( \theta \).

First of all

\[
h = l \cdot \cos(\theta - \frac{\pi}{2})
\]

just to make things look better I can notice that \( \cos(\theta - \frac{\pi}{2}) = \cos(\theta) \cos(-\frac{\pi}{2}) + \sin(\theta) \sin(-\frac{\pi}{2}) = \sin(\theta) \) hence \( h = l \cdot \sin \theta \) but I don’t even need to do this! (it just gets slightly more complicated otherwise). Also

\[
B = b + 2l \cdot \sin(\theta - \frac{\pi}{2}) = b - 2l \cos \theta
\]
Hence we find

\[ A(\theta) = \frac{(2b - 2l \cdot \cos \theta)l \cdot \sin \theta}{2} = (b - l \cos \theta)l \sin \theta = bl \sin \theta - l^2 \cos \theta \sin \theta \]

The domain for \( \theta \) is \( \theta \in [0, \pi] \). This is a close interval and we know how to find the max: we just check critical points and the endpoints of the interval.

For the endpoints: \( A(0) = 0 \) and \( A(\pi) = 0 \).

Let’s differentiate \( A(\theta) \):

\[ A'(\theta) = bl \cos \theta - l^2 (-\sin^2 \theta + \cos^2 \theta) = bl \cos \theta + l^2 \sin^2 \theta - l^2 \cos^2 \theta \]

Wow, this is hard!! Let’s substitute \( \sin^2 \theta = 1 - \cos^2 \theta \) so that we get a quadratic equation in \( \cos \theta \):

\[ A'(\theta) = bl \cos \theta + l^2 - 2l^2 \cos^2 \theta = l(b \cos \theta + l - 2l \cos^2 \theta) \]

hence \( A'(\theta) = 0 \) when

\[ \cos \theta = \frac{b \pm \sqrt{b^2 + 8l^2}}{4l} \]

OK, so we know there has to be at least one critical point (Rolle’s theorem!). But are there two of them?

Well, notice that, since \( b, l > 0 \), \( b^2 + 8l^2 = b^2 + 4l^2 + 4l^2 > 4l^2 \) hence \( \sqrt{b^2 + 8l^2} > \sqrt{4l^2} = 2l \). Hence

\[ \frac{b + \sqrt{b^2 + 4l^2}}{2l} > \frac{2l}{2l} = 1 \]

so there can’t be a number \( \theta \) such that \( \cos \theta = \frac{b + \sqrt{b^2 + 4l^2}}{2l} \)!!!

Hence the critical point must be \( \theta = \arccos \frac{b - \sqrt{b^2 + 4l^2}}{2l} \). And this is definitely the max since the area of the trapezoid is bigger than zero at least for some \( \theta \) so there must be a local max. (or just think of it like that: the area of this theta must really be bigger than zero-proof by picture!).

OK, this was way too hard. But it was fun, wasn’t it? In the final I would at least give you specific numbers to deal with. Actually, this looks like a nice and hard extra credit problem! so don’t worry if you couldn’t do this by yourself.
15. Find the area enclosed between the two curves \( x = 2y^2 \) and \( x = 4 + y^2 \).

**Solution:** We have to integrate with respect to \( y \). But first let’s find the points in which the two parabolas meet: \( 2y^2 = 4 + y^2 \Rightarrow y^2 = 4 \) hence \( y = 2 \) and \( y = -2 \). Therefore

\[
A = \int_{-2}^{2} (4 + y^2 - 2y^2) \, dy = \int_{-2}^{2} (4 - y^2) \, dy = \left[ 4y - \frac{y^3}{3} \right]_{-2}^{2} = 16 - \frac{16}{3}
\]