Sample exercises for the Final

June 30, 2009

1. Compute the following indefinite integrals:

(a) \[ \int x \sin(3x^2 + 2) \, dx \]

(b) \[ \int \frac{x + 3}{x^2} \, dx \]

(c) \[ \int e^{\sqrt{x}} \, dx \]

(d) \[ \int \frac{1 + 2x}{\sqrt{1 - x^2}} \, dx \]

2. Compute the following integrals:

(a) \[ \int_{-\pi/2}^{\pi/2} \frac{x \cos x}{1 + x^4} \, dx \]

(b) \[ \int_{0}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} \, dx \]
3. Estimate the number \( \ln 1.04 \).
Estimate the number \( \ln 3 \) (remember: \( e \sim 2.7 \)).

4. State the fundamental theorem of calculus.
Use it to compute
\[
\frac{d}{dx} \int_{x}^{3x-1} \tan(2t - 1) \sqrt{t} dt
\]
Is this computation correct:
\[
\int_{-1}^{2} \frac{1}{x^2} dx = \left[ \frac{-1}{x} \right]_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}
\]

5. Sketch the graph of the function
\[
f(x) = \frac{x + 1}{\sqrt{x^2 + 1}}
\]

6. Sketch the graph of a function that has 3 local extrema and 5 critical points.
How many inflection points does your graph have?

7. Sketch the graph of the function
\[
f(x) = \ln(1 + x^2)
\]

8. If \( f \) is continuous and \( \int_{1}^{22} f(x) dx = 3 \), compute
\[
\int_{0}^{7} f(3x + 1) dx
\]

9. State Rolle’s theorem.
Give an example of a function \( f \) such that:
• \( f(a) = f(b) \) for some \( a, b \in \mathbb{R} \);
• \( f \) is continuous;
• \( f \) is not differentiable on \([a, b]\);
• The conclusion of Rolle’s theorem does not hold.

10. Compute the following limits:

(a) \[
\lim_{x \to +\infty} xe^{-x}
\]

(b) \[
\lim_{x \to 0} \frac{\arctan{x}}{x}
\]

(c) \[
\lim_{x \to +\infty} (\ln(x) - \ln(x + 1))
\]

11. An ant is crawling along a path that is exactly the graph of the function \( y = 3x^2 \). She starts at the origin \((0, 0)\). Her \( x \)-coordinate is changing at the rate of 10 cm per minute. How fast is her distance from the origin changing when her \( y \)-coordinate is 27?

12. Suppose that \( f \) is continuous and differentiable on all of \( \mathbb{R} \). Suppose \( f'(x) > 0 \) for all \( x \in [0, 1] \) and \( f(0) = 0 \). Is \( f(1) \) positive, negative, or zero? Explain.

13. Find the volume of the solid obtained by considering the region bounded by \( y = x^3 \) and \( x = 1 \) and \( y = 0 \) and and rotating it along the line \( y = -2 \).
14. Consider the following trapezoid:

\[ \begin{array}{c}
\text{l} \\
\theta \\
\text{b} \\
\theta \\
\text{l}
\end{array} \]

(b and \( l \) are fixed numbers, \( B \) and \( \theta \) are not). Find the angle \( \theta \) that minimizes the area.

15. Find the area enclosed between the two curves \( x = 2y^2 \) and \( x = 4 + y^2 \).