Sample exercises for Midterm 1

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1. Find the domain and range of the following functions:
   a. \( f(x) = \arcsin(x) \)
   b. \( f(x) = \sqrt{4 - x} \)
   c. \( f(x) = |\cos(2x + 1)| \)

Solution:
   a. We did this one in class. The arcsin is the inverse function of \( \sin \) so it has domain \([-1, 1]\). The range is \([-\frac{\pi}{2}, \frac{\pi}{2}]\) because we had to restrict the domain of the sine before we could invert it.
   b. To find the domain I have to ask for whatever is inside the square root to be positive or zero. Therefore I need \( 4 - x \geq 0 \) i.e. \( x \leq 4 \). To find the range, think graphically: to find the graph of \( f \) we start from the graph of \( \sqrt{x} \), shift to the left by 4 and then take the symmetric with respect to the y-axis. We didn’t affect the range in any way! so the range is equal to the range of \( \sqrt{x} \) which is \([0, +\infty)\). (all of this could have been showed by a graph!)
   c. The domain here is not a problem since the cosine is defined for all \( x \in \mathbb{R} \). What about the range? Again, taking \( \cos(2x + 1) \) means we are only stretching and shifting along the x-axis, so the range does not change and it’s \([-1, 1]\). But then we are taking the absolute value so the range of \( f(x) = |\cos(2x + 1)| \) is \([0, 1]\).

2. a. Define the inverse of a function (make sure to say when this is defined)
   b. Let

\[
  f(x) = \exp\left(\frac{4x - 1}{2x + 3}\right)
\]

Find the inverse of \( f \).

Solution:
a. Suppose $f$ is an injective function. Then the inverse function of $f$ is the function whose domain=range of $f$, and such that

$$f^{-1}(x) = y \iff x = f(y)$$

b. Set $y = f(x)$. Then we get (by definition, the logarithm is the inverse function of the exponential)

$$y = \exp\left(\frac{4x - 1}{2x + 3}\right) \Rightarrow \ln y = \frac{4x - 1}{2x + 3}$$

And now we can multiply both sides by the denominator and simplify:

$$(2x + 3) \ln y = 4x - 1$$

$$2x \ln y + 3 \ln y = 4x - 1$$

$$x(2 \ln y - 4) = -3 \ln y - 1$$

$$x = \frac{-3 \ln y - 1}{2 \ln y - 4}$$

if you want you can put it in a nicer form and get

$$x = f^{-1}(y) = \frac{3 \ln y + 1}{2(2 - \ln y)}$$

3. Compute the following limits or explain why the limit does not exist:

a. 

$$\lim_{x \to +\infty} \frac{2x^2 + 3x + 8}{5x^2 + 10x - 3}$$

b. 

$$\lim_{x \to +\infty} \frac{2x^2 + 3x + 8}{5x^3 + 7x^2 + 10x - 3}$$

c. 

$$\lim_{t \to 7} \frac{\sqrt{t + 2} - 3}{t - 7}$$

d. 

$$\lim_{x \to -1} 2|2 - 2x| + 1$$
e. \[ \lim_{x \to +\infty} \cos(x) + \frac{1}{x} \]

f. \[ \lim_{x \to 5} \frac{x^2 - 25}{x - 5} \]

g. \[ \lim_{x \to 10} f(x) \]

where

\[
 f(x) = \begin{cases} 
 x & x < 10 \\
 -1 & x = 10 \\
 2x - 10 & x > 10 
\end{cases}
\]

h. \[ \lim_{x \to 2} \frac{|3x - 6|}{x - 2} \]

**Solution:**

a. This is similar to the one we had in the Quiz. So I factor out \( x^2 \) and get

\[
 \lim_{x \to +\infty} \frac{2x^2 + 3x + 8}{5x^2 + 10x - 3} = \lim_{x \to +\infty} \frac{x^2(2 + \frac{3}{x} + \frac{8}{x^2})}{x^2(5 + \frac{10}{x} - \frac{3}{x^2})} \\
 = \lim_{x \to +\infty} \frac{2 + \frac{3}{x} + \frac{8}{x^2}}{5 + \frac{10}{x} - \frac{3}{x^2}} \\
 = \frac{2}{5}
\]

where I used the quotient rule and the fact that \( \lim_{x \to +\infty} \frac{\text{constant}}{x^n} = 0 \) for every positive \( n \).

b. This is similar to the previous one, except that now the two polynomial at the numerator and denominator have different degrees. The point is that we don’t like the denominator to go to infinity, nor do we like it when it goes to zero. So **the strategy is to factor out the biggest exponent that appears in the denominator**. Then we’ll be left with a denominator that approaches a finite constant and so we’ll only have to worry about the numerator. This will allow you to be able to solve all
limits of this type!!

In this case I get

\[
\lim_{x \to +\infty} \frac{2x^2 + 3x + 8}{5x^3 + 7x^2 + 10x - 3} = \lim_{x \to +\infty} \frac{x^3 \left( \frac{2}{x} + \frac{3}{x^2} + \frac{8}{x^3} \right)}{x^3 \left( 5 + \frac{7}{x} + \frac{10}{x^2} - \frac{3}{x^3} \right)}
\]

again we can divide numerator and denominator by \(x^3\) and get

\[
\lim_{x \to +\infty} \frac{\frac{2}{x} + \frac{3}{x^2} + \frac{8}{x^3}}{5 + \frac{7}{x} + \frac{10}{x^2} - \frac{3}{x^3}}
\]

the denominator is approaching a constant (i.e. 5), that’s what we wanted. The numerator is going to zero as \(x \to +\infty\). So by using the quotient rule we can conclude that the limit is zero.

c. Can I apply the quotient rule? No, because the limit of the denominator is zero. So let’s use our “rationalize” strategy:

\[
\frac{\sqrt{t + 2} - 3}{t - 7} = \frac{\sqrt{t + 2} - 3}{t - 7} \cdot \frac{\sqrt{t + 2} + 3}{\sqrt{t + 2} + 3}
\]

\[
= \frac{t + 2 - 9}{(t - 7)(\sqrt{t + 2} + 3)}
\]

\[
= \frac{t - 7}{(t - 7)(\sqrt{t + 2} + 3)}
\]

\[
= \frac{1}{\sqrt{t + 2} + 3}
\]

Again I can cancel the \((t - 7)\) safely since I am taking the limit for \(t \to 7\) hence I can assume \(t \neq 7\).

Now that looks much better because the denominator will approach a finite quantity at \(t \to 7\): in fact \(\lim_{x \to 7} \sqrt{t + 2} + 3 = \sqrt{9} + 3 = 6\) (those are continuous functions, so I can just plug in). Hence I can use the quotient law and get

\[
\lim_{t \to 7} \frac{\sqrt{t + 2} - 3}{t - 7} = \lim_{t \to 7} \frac{1}{\sqrt{t + 2} + 3} = \frac{1}{6}
\]

d. The absolute value is continuous, remember? So we can just plug in! \(|2 - 2x| = 0\) so the limit is 0 + 1 = 1.

e. This is tricky. \(\cos(x)\) looks like it’s oscillating indefinitely between -1 and 1, so it looks like the \(\lim_{x \to +\infty} \cos(x)\) doesn’t exist. Will adding \(\frac{1}{x}\) change something? No, because \(\lim_{x \to +\infty} \frac{1}{x}\) is zero, so at infinity the behaviour of \(\cos x\) won’t be modified by adding \(\frac{1}{x}\). So the limit does not exist.
f. This one is not hard. We just have to notice that the numerator factors!

\[
\lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x - 5)(x + 5)}{x - 5} = \lim_{x \to 5} x + 5 = 10
\]

g. First of all, we don’t care about what \( f(10) \) is since we’re taking the limit for \( x \to 10 \). Like every time we have a piecewise function, we have to compute the left and right limit separately. For the left hand limit: notice that, for \( x < 10 \), \( f(x) = x \) so

\[
\lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} x = 10
\]

For the right hand limit: for \( x > 10 \), \( f(x) = 2x - 10 \) hence

\[
\lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} 2x - 10 = 20 - 10 = 10
\]

The left and right hand limits coincide. Hence the limit \( \lim_{x \to 10} f(x) \) exists and is equal to 10.

h. This is similar to the limit I gave in quiz 1 (except that there I only asked for the right hand limit).

**General rule for dealing with the absolute value of \(|x - a|\):**

- If I am taking the limit for \( x \to b \) and \( b > a \), it means that I only have to consider values of \( x \) that are “next to” \( b \) and since \( b - a > 0 \) I can assume that \( x - a \) is also \( > 0 \). So in this case \(|x - a| = x - a\).
- If I am taking the limit for \( x \to b \) and \( b < a \), it means that I only have to consider values of \( x \) that are “next to” \( b \) and since \( b - a < 0 \) I can assume that \( x - a \) is also \( < 0 \). So in this case \(|x - a| = -(x - a)\).
- If I am taking the limit for \( x \to a \), then I have to compute the left and right limits separately!! When I compute the left limit, I know that \( x < a \) hence \(|x - a| = -(x - a)\). When I compute the right limit, I know that \( x > a \) hence \(|x - a| = x - a\). If the left and right hand limits coincide, then I found the limit. Otherwise the limit does not exist.

In my case, \(|3x - 6| = 3|x - 2|\) so this is an example of the third case. So I have to take left and right hand limits.

\[
\lim_{x \to 2^-} \frac{|3x - 6|}{x - 2} = \lim_{x \to 2^-} \frac{3|x - 2|}{x - 2} = \lim_{x \to 2^-} \frac{-3(x - 2)}{x - 2} = -3
\]

\[
\lim_{x \to 2^+} \frac{|3x - 6|}{x - 2} = \lim_{x \to 2^+} \frac{3|x - 2|}{x - 2} = \lim_{x \to 2^+} \frac{3(x - 2)}{x - 2} = 3
\]

the two limits are different, so the limit does not exist.
4.  
   a. State the squeeze theorem.
   
   b. Use it to find
   \[
   \lim_{x \to +\infty} \frac{\sin x + x + 1}{2x + 1}
   \]

   Solution:

   a. If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( a \) (except possibly at \( a \) and

   \[
   \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L
   \]

   then

   \[
   \lim_{x \to a} g(x) = L
   \]

   b. This is not a rational function because there’s a \( \sin x \) in there, but let’s see if factoring out an \( x \) both on numerator and denominator can help.

   \[
   \lim_{x \to +\infty} \frac{\sin x + x + 1}{2x + 1} = \lim_{x \to +\infty} \frac{x(\frac{\sin x}{x} + 1 + \frac{1}{x})}{x(2 + \frac{1}{x})}
   \]

   we are taking the limit for \( x \to +\infty \) so we can safely assume \( x \neq 0 \). So we can divide by \( x \) and get

   \[
   \lim_{x \to +\infty} \frac{\frac{\sin x}{x} + 1 + \frac{1}{x}}{2 + \frac{1}{x}}
   \]

   Well, we know how to deal with everything except for \( \frac{\sin(x)}{x} \). The exercise tells us to use the intermediate value theorem: (I stated it for limit at a point, but it still holds for limits at infinity!!!).

   \[
   -1 \leq \sin x \leq 1 \Rightarrow -\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}
   \]

   (it’s ok to multiply by \( \frac{1}{x} \) since \( x \to \infty \) so we can assume \( x > 0 \).

   But now \( \lim_{x \to +\infty} -\frac{1}{x} = \lim_{x \to +\infty} \frac{1}{x} = 0 \) hence \( \lim_{x \to +\infty} \frac{\sin(x)}{x} = 0 \). Hence

   \[
   \lim_{x \to +\infty} \frac{\frac{\sin x}{x} + 1 + \frac{1}{x}}{2 + \frac{1}{x}} = \frac{0 + 1 + 0}{2 + 0} = \frac{1}{2}
   \]

5.  
   a. State the Intermediate Value theorem.
   
   b. Use it to show that \( p(x) = x^3 + 4x^2 - 2x - 4 \) has a root.

   Solution:
a. Suppose that \( f \) is continuous on \([a, b]\) with \( f(a) \neq f(b) \).
Let \( N \) be a number such that \( f(a) < N < f(b) \) (if \( f(a) < f(b) \)) or \( f(b) < N < f(a) \) (if \( f(b) < f(a) \)).
Then there exists a \( c \) in \((a, b)\) such that \( f(c) = N \).
(Notice that \( c \) might not be unique!)

b. I don’t remember the formula to find roots of polynomials of degree 3 so I don’t want to do that. Let’s use the intermediate value theorem instead!
All I need is to find two numbers \( a \) and \( b \) such that \( p(\text{one of them}) < 0 \) and \( p(\text{one of them}) > 0 \). Then there’ll be a root in between!
Let’s see what happens if we try to plug in easy numbers (e.g. 0,1,-1,2,-2,.... we can stop as soon as we find two numbers as above).
\( p(0) = -4 < 0 \), and \( p(1) = 1 + 4 - 2 - 4 = -1 < 0 \) (so this is not good), \( p(-1) = -1 + 4 + 2 - 4 = 1 > 0 \). Therefore there will be a root between -1 and 0!

6. State the \( \epsilon - \delta \) definition of limit and use it to show that
\[
\lim_{x \to 1} 3x + 1 = 4
\]

Solution:
\[
\lim_{x \to a} f(x) = L
\]
if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that, if \( |x-a| < \delta \), then \( |f(x) - L| < \epsilon \)
Let’s prove that \( \lim_{x \to 1} 3x + 1 = 4 \). Fix \( \epsilon > 0 \). We need to find \( \delta \) such that if \( |x-1| < \delta \), then \( |f(x) - 4| < \epsilon \).
To guess the solution: we need \( |f(x) - 4| < \epsilon \) hence \( |3x + 1 - 4| < \epsilon \) hence \( 3|x-1| < \epsilon \).
This suggests we should take \( \delta = \frac{\epsilon}{3} \). Now let’s prove that the limit is 4: again, choose any \( \epsilon > 0 \). Choose \( \delta = \frac{\epsilon}{3} \) so let
\[
|x-1| < \frac{\epsilon}{3}
\]
then
\[
|f(x) - 4| = |3x + 1 - 4| = |3x - 3| = 3|x-1| < \epsilon
\]
this concludes the proof.

7. Let
\[
f(x) = \begin{cases} 
  x + 1 & \text{if } x < -1 \\
  \frac{1}{x^2 + 4x + 3} + \frac{1}{x^2 - 1} + \frac{2}{3} & -1 < x < 0 \\
  x \sin\left(\frac{1}{x^2}\right) & x > 0
\end{cases}
\]
Is it possible to set $f(-1) = a$ for some $a$ to make $f$ continuous at $x = -1$?

Is it possible to set $f(0) = b$ for some $b$ to make $f$ continuous at $x = 0$?

**Solution:** We have to compute the left and right hand limits for $x = -1$ and $x = 0$ and see if they are equal.

For $x = -1$:

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{1}{(x + 1)(x + 3)} + \frac{1}{(x - 1)(x + 1)} + \frac{2}{3}$$

$$= \lim_{x \to -1^+} \frac{x - 1 + x + 3}{(x + 1)(x + 3)(x - 1)} + \frac{2}{3}$$

$$= \lim_{x \to -1^+} \frac{2x + 2}{(x + 1)(x + 3)(x - 1)} + \frac{2}{3}$$

$$= \lim_{x \to -1^+} \frac{2(x + 1)}{(x + 1)(x + 3)(x - 1)} + \frac{2}{3}$$

$$= \lim_{x \to -1^+} \frac{2}{(x - 1)(x + 3)} + \frac{2}{3} = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}$$

and

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x + 1 = 0$$

So the left and right limits do not coincide, hence the limit of $f$ for $x \to -1$ does not exist, therefore $f$ cannot be continuous at $-1$. For $x = 0$:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x^2 + 4x + 3} + \frac{1}{x^2 - 1} + \frac{2}{3} = \frac{1}{3} - 1 + \frac{2}{3} = 0$$

(here I just plugged in $x = 0$ because 0 is in the domain of all rational functions!)

For the limit from the right we have to use the squeeze theorem. You know that

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$$

since I am approaching from the right $x$ is positive and then I can multiply everywhere by $x$ and get

$$-x \leq x \sin\left(\frac{1}{x^2}\right) \leq x$$

Now you just have to use the fact that $\lim_{x \to 0^-} (-x) = \lim_{x \to 0^+} x = 0$ and by the squeeze theorem you get

$$\lim_{x \to 0^+} x \sin\left(\frac{1}{x^2}\right) = 0$$

So the two limits coincide. Hence if I set $f(0) = 0$ my function $f$ will be continuous at 0.
8. State what it means for \( f \) to be continuous at a point.

Give an example (graphically or algebraically) of a function with domain \( \mathbb{R} \) that is continuous everywhere except at 1 and -1. Prove, using the definition of continuity that \( \sqrt{x^2 + 1} \) is continuous at \( x = 1 \). You can use the limit laws but you cannot use any theorem about continuous functions.

**Solution:** A function is continuous at a point \( a \) if \( \lim_{x \to a} f(x) = f(a) \).

Example: \( f(x) = \frac{1}{(x-1)(x+1)} \) is continuous on its domain since it’s a rational function, but it is not at \( x = 1, -1 \): here you get infinite limits and actually the right hand limit and the left hand limit will not agree. (but any time the limit is infinite the function is doomed to be discontinuous!). Any graph with a break at -1 and 1 would have been good.

To show that \( f(x) = \sqrt{x^2 + 1} \) is continuous at \( x = 1 \): we want to prove that \( \lim_{x \to 1} \sqrt{x^2 + 1} = \sqrt{1^2 + 1} = \sqrt{2} \) (this is the definition of continuity) and we can do this using the limit laws:

\[
\lim_{x \to 1} \sqrt{x^2 + 1} = \sqrt{\lim_{x \to 1} x^2 + 1} = \sqrt{\lim_{x \to 1} x^2 + \lim_{x \to 1} 1} = \sqrt{1^2 + 1} = \sqrt{2} = f(1)
\]

(where I used square root rule, sum rule, power rule, constant rule).

So the function is continuous at \( x = 1 \). In fact, you can do this for every \( a \in \mathbb{R} \) in exactly the same way thus showing that \( f \) is continuous on all of \( \mathbb{R} \).