School of Mathematics

Unblocking Heuristics for Warmstarting Interior Point Methods

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Warmstarting Interior Point Methods

- Simplex Method is easy and efficient to warmstart
- IPM Warmstart is seen as difficult to impossible

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Some research (among others)

- Mitchell, Todd '92
- Hippolito '93
- Lustig, Marsten, Shanno '94
- Gondzio '98
- Gondzio, Vial '99
- Yıldırım, Wright '02
- Gondzio, Grothey '03

Renewed interest recently

- John,Yıldırım '06
- Benson,Shanno '06

⇒ Two aims

- Evaluate different warmstarting approaches
- Warmstarting IPMs is possible

Interior Point Methods

$$\min c^\top x \text{ s.t. } \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \qquad \text{(LP)}$$

Optimality conditions:

$$\begin{aligned} c - A^\top y - z &= 0 \\ Ax &= b \\ XZe &= 0 \quad (\mu e) \\ x, z &\geq 0 \end{aligned}$$

$\Rightarrow$ Newton Step:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \xi_c \\ \xi_b \\ r_{xz} \end{bmatrix} = \begin{bmatrix} c - A^\top y - z \\ b - Ax \\ \mu e - XZe \end{bmatrix} \qquad \text{(NS-LP)}$$

$\Rightarrow$ Reduced to

$$\begin{bmatrix} -\Theta & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_c - X^{-1} r_{xz} \\ \xi_b \end{bmatrix}$$

where $\Theta = X^{-1}Z, \quad X = \text{diag}(x), \quad Z = \text{diag}(z)$

Interior Point Methods II

- choose x_0, y_0, z_0, μ_0
- solve (NS-LP) for $\Delta x, \Delta z, \Delta y$
- compute stepsizes

$$\begin{aligned} \alpha_P &= \max_{\alpha>0}\{\alpha : x + \alpha\Delta x \geq 0\} \\ \alpha_D &= \max_{\alpha>0}\{\alpha : z + \alpha\Delta z \geq 0\} \end{aligned}$$

- take step

$$\begin{aligned} x_+ &= x + 0.995\alpha_P\Delta x \\ y_+ &= y + 0.995\alpha_D\Delta y \\ z_+ &= z + 0.995\alpha_D\Delta z \end{aligned}$$

- update μ:

$$\mu_+ = \gamma\frac{x_+^\top z_+}{n}, \quad 0 < \gamma < 1$$

Corrector steps: Mehrotra corrector

- Given: *Predictor* Step $(\Delta x_p, \Delta y_p, \Delta z_p)$
- Solve:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x_c \\ \Delta y_c \\ \Delta z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (\mu_{new} - \mu_{old})e + \Delta x_p^\top \Delta z_p \end{bmatrix} \quad \left(= \begin{bmatrix} \xi_c \\ \xi_b \\ \mu e - XZe \end{bmatrix} \right)$$

Note that $\Delta x_p^\top \Delta z_p = (x + \Delta x_p)^\top (z + \Delta z_p) - \mu e$
(= residual in $\mu e - XZe = 0$ after predictor step is taken)

- Use search direction $\Delta x = \Delta x_p + \Delta x_c$
  $\Delta z = \Delta z_p + \Delta z_c$

Corrector steps: Higher-Order corrector I

- Given: *Aggregated* Predictor Step: $(\Delta x_p, \Delta y_p, \Delta z_p)$
- Solve:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x_c \\ \Delta y_c \\ \Delta z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \Delta x_p^\top \Delta z_p \end{bmatrix}$$

- Use direction $\Delta x'_p = \Delta x_p + \Delta x_c$

$$\Delta z'_p = \Delta z_p + \Delta z_c$$

- Repeat (as long as improvement)

Corrector steps: Higher-Order corrector II

- Given: *Aggregated* Predictor Step: $(\Delta x_p, \Delta y_p, \Delta z_p)$
- Solve:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x_c \\ \Delta y_c \\ \Delta z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m(\Delta x_p^\top \Delta z_p) \end{bmatrix}$$

where m is a (componentwise) modificator function

$$m(\Delta x_{p,i} \Delta z_{p,i}) = \begin{cases} \Delta x_{p,i} \Delta z_{p,i} & (x_i + \Delta x_{p,i})(z_i + \Delta z_{p,i}) < \mu/10 \\ -5 * \mu & (x_i + \Delta x_{p,i})(z_i + \Delta z_{p,i}) > 10\mu \\ 0 & \text{else} \end{cases}$$

- Use direction $\Delta x'_p = \Delta x_p + \Delta x_c$
  $\Delta z'_p = \Delta z_p + \Delta z_c$
- Repeat (as long as improvement)

$\Rightarrow$ Higher order corrector as unblocking step.

Warmstarting Interior Point Methods

Aim: Use information from solution process of

$$\min c^\top x \text{ s.t. } \begin{array}{rcl} Ax & = & b \\ x & \geq & 0 \end{array} \qquad \text{(original-LP)}$$

to construct a starting point for (nearby problem)

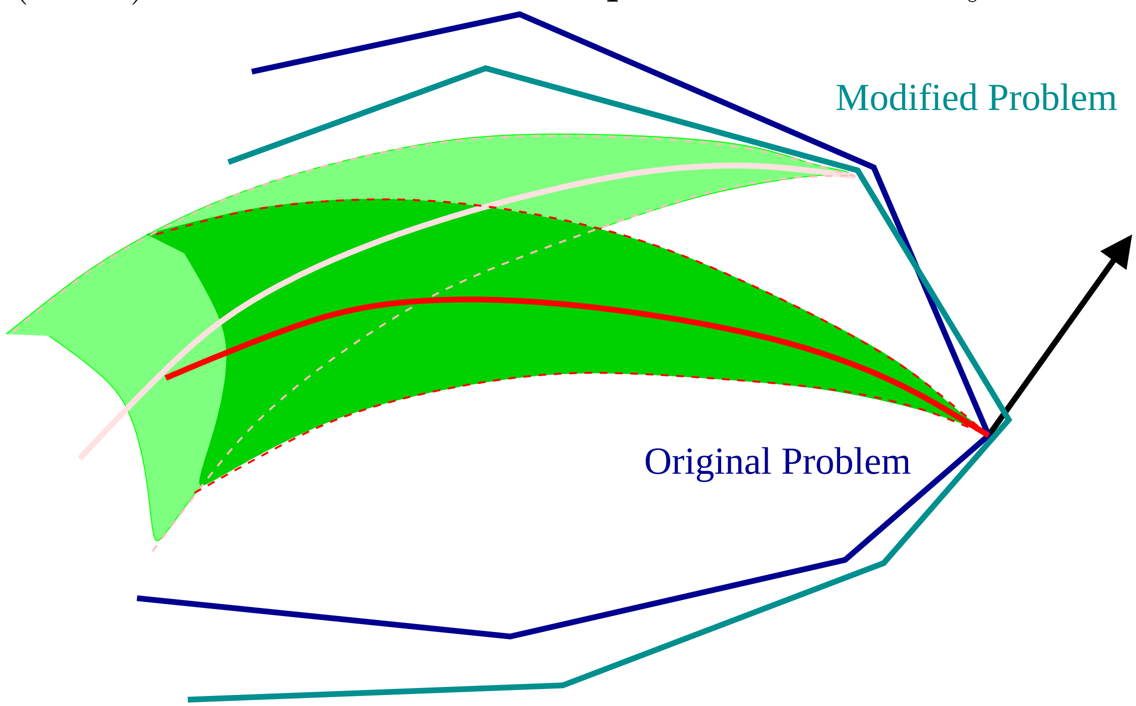
$$\min \tilde{c}^\top x \text{ s.t. } \begin{array}{rcl} \tilde{A}x & = & \tilde{b} \\ x & \geq & 0 \end{array} \qquad \text{(perturbed-LP)}$$

where $\tilde{A} \approx A, \tilde{b} \approx b, \tilde{c} \approx c$

- It is **not** a good idea to use the solution of (original-LP) to start (perturbed-LP).
- *Unlike for the Simplex Method!*

Why?

Hippolito (1993): Search direction is parallel to nearby constraints

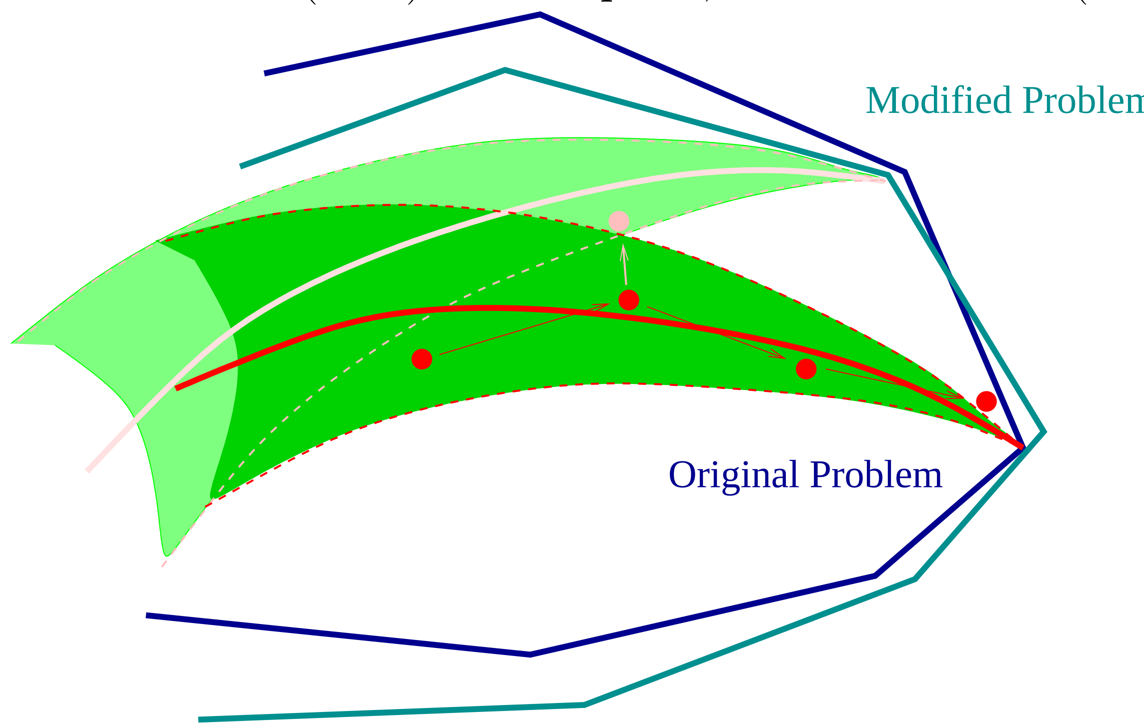


⇒ only small step in search direction can be taken

⇒ *blocking* of step

Warmstarting Heuristics

Idea: Start close to the (new) central path, not close to the (old) solution



⇒ Start from a previous iterate and do additional *modification* step.

⇒ *Blocking* might still be a problem.

Warmstarting Strategies

Three components:

- Choose a $\hat{\mu}$-**iterate** from previous solve
- Do a **modification** step
- Have **unblocking** strategy

Questions:

- How to choose $\hat{\mu}$?
- What is an appropriate modificator step?
- What to do about blocking?

Theoretical Guidance:

Step is non-blocking if (i.e. full step $\alpha_P = \alpha_D = 1$ is feasible) if

- Current point (x, z) is well centered (r_{xz} small).
- Primal/dual residuals $(\xi_b, , \xi_c)$ are small.
- Larger μ allows for larger residuals.

$\Rightarrow$ **Heuristics:**

- Retrace iterates of original problem until appropriate μ-center is found (large μ)
- Only correct for part of the residual at every step (small ξ_b, ξ_c)
- Re-center iterates (small r_{xz})

Modification Steps

- **Additional Centering Step**:
  Before returning advanced center **or** at start of modified problem.

- **Adjustment of z**:
  New residual $\tilde{\xi}_c = \tilde{c} - \tilde{A}^\top y - z$ can be reduced by changing z.
  Subject to safeguards: $z_i \geq \sqrt{\mu}$

- **Least Squares Step** (Yıldırım, Wright '02).

Unblocking Heuristics

- **Split Directions**
  Compute separate search directions to correct for ξ_b, ξ_c, r_{xz}.

- **Higher Order Correctors**

- Unblocking by **Sensitivity Analysis**

Separate Directions

Split the computation of the Newton Step

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \xi_c \\ \xi_b \\ r_{xz} \end{bmatrix}$$

into three separate parts:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \left[\begin{array}{c|c|c} \Delta x_p & \Delta x_d & \Delta x_c \\ \Delta y_p & \Delta y_d & \Delta y_c \\ \Delta z_p & \Delta z_d & \Delta z_c \end{array}\right] = \left[\begin{array}{c|c|c} \xi_c & 0 & 0 \\ 0 & \xi_b & 0 \\ 0 & 0 & r_{xz} \end{array}\right]$$

to compute three separate search directions: Δ_p, Δ_d, Δ_c.

$\Rightarrow$ Correct for primal feasibility, dual feasibility, centrality separately:

- Only correct for as much of ξ_b, ξ_c as can be done without blocking.
- Make full centrality step

Separate Directions Algorithm

- Calculate directions $(\Delta_p, \Delta_d, \Delta_c)$.
- Find maximal steps in every direction:

$$\begin{aligned} \alpha^p &= \min\left\{\max_{\alpha>0}\{\alpha : x + \alpha\Delta x_p \geq 0\}, \max_{\alpha>0}\{\alpha : z + \alpha\Delta z_p \geq 0\}\right\} \\ \alpha^d &= \min\left\{\max_{\alpha>0}\{\alpha : x + \alpha\Delta x_d \geq 0\}, \max_{\alpha>0}\{\alpha : z + \alpha\Delta z_d \geq 0\}\right\} \end{aligned}$$

- Combine into trial search direction:

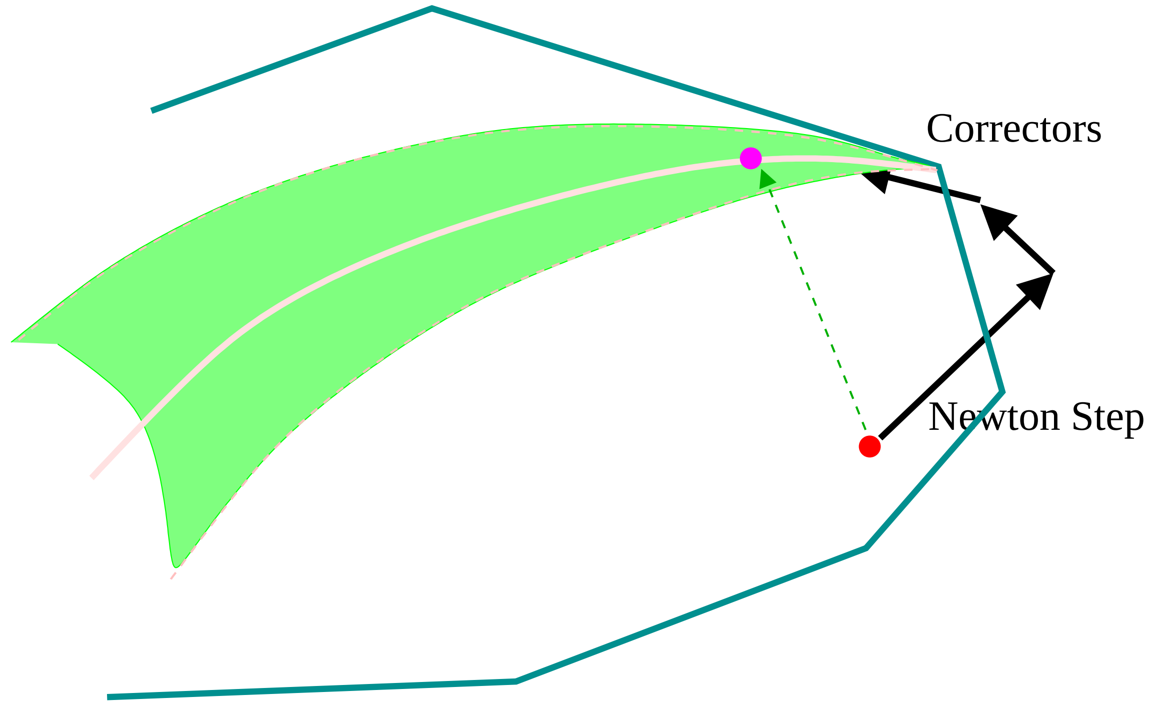
$$\begin{aligned} \Delta x &= \Delta x_c + \min\{5\alpha^p, 1\}\Delta x_p + \min\{5\alpha^d, 1\}\Delta x_d \\ \Delta z &= \Delta z_c + \min\{5\alpha^p, 1\}\Delta z_p + \min\{5\alpha^d, 1\}\Delta z_d \end{aligned}$$

- And use this for predictor/corrector steps

Higher Order Correctors to Unblock Directions

Idea:

- Newton direction might block (close to constraints)
- (Higher Order) Correctors correct for this



Higher Order Correctors to Unblock Directions

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x_c \\ \Delta y_c \\ \Delta z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m(\Delta x_p^\top \Delta z_p) \end{bmatrix}$$

$$m(\Delta x_{p,i} \Delta z_{p,i}) = \begin{cases} \Delta x_{p,i} \Delta z_{p,i} & (x_i + \Delta x_{p,i})(z_i + \Delta z_{p,i}) < \mu/10 \\ -5 * \mu & (x_i + \Delta x_{p,i})(z_i + \Delta z_{p,i}) > 10\mu \\ 0 & \text{else} \end{cases}$$

Conflicting goals:

- Attempt to reach central path
  - can take larger step at next iteration
  - central path is away from boundary (reduce blocking)
- But using large r_{xz} will lead to blocking step

Higher Order Correctors:

- Aim towards central path
- But do not force if adjustment to large

Sensitivity Analysis

The step equation

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \xi_c \\ \xi_b \\ r_{xz} \end{bmatrix} = \begin{bmatrix} c - A^\top y - z \\ b - Ax \\ \mu e - XZe \end{bmatrix} \qquad \text{(NS-LP)}$$

Implies a functional relationship

$$(\Delta x, \Delta y, \Delta z) = F(x, y, z, \mu)$$

$\Rightarrow$ Can get derivatives (sensitivity information) $\frac{\partial \Delta x_i}{\partial x_j}, \frac{\partial \Delta z_i}{\partial x_j}, \frac{\partial \Delta x_i}{\partial z_j}, \frac{\partial \Delta z_i}{\partial z_j}$

Use this to unblock the step

- How to get derivatives ?
- How to use them to unblock the direction ?

Sensitivity Analysis: Derivatives

Differentiate the step equation (w.r.t. x_i) to obtain

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \frac{\partial \Delta x}{\partial x_i} \\ \frac{\partial \Delta y}{\partial x_i} \\ \frac{\partial \Delta z}{\partial x_i} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_i}\xi_c \\ \frac{\partial}{\partial x_i}\xi_b \\ -\Delta Z e_i + \frac{\partial}{\partial x_i} r_{xz} \end{bmatrix}$$

And similarly for $\frac{\partial}{\partial z_i}$

$\Rightarrow$

- System matrix is the same as for step equation
- one additional backsolve yields sensitivity for one x_i, z_i component.

Sensitivity Analysis: Unblocking

Computation of complete sensitivity is computationally expensive

$\Rightarrow$ **Heuristic:**

For every blocking component i of $\Delta x, \Delta z$:

- Get sensitivity for change of x_i, z_i
- Find *most efficient* unblocking change to current point (x_i, z_i) subject to *safeguards*:
  - No change outside $[x_i/10, 10x_i]$
  - No change larger than $\mathcal{O}(\max\{\|\xi_c\|, \|\xi_b\|\})$.
  - No introduction of new blocking components

Numerical Experiments: NETLIB

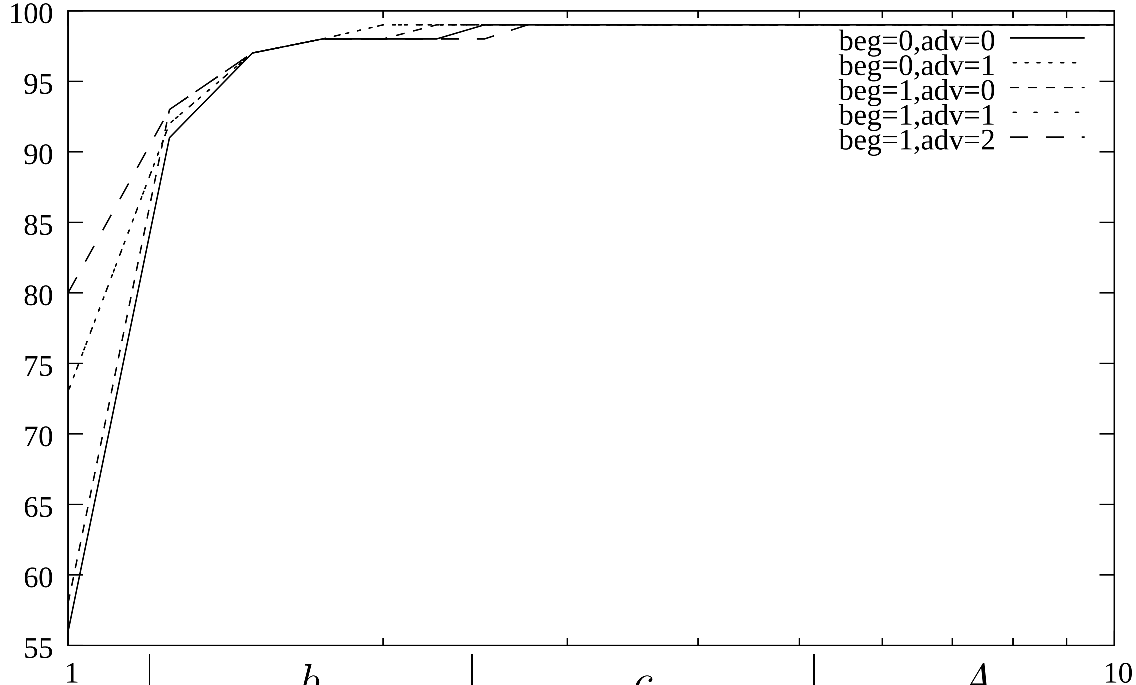
Testset proposed by Benson, Shanno '06:

- Take small to mid-scale instances of NETLIB LP library
- Randomly perturb problem data in b, c or A
- Perturb 10% (at least 20) of components on average.
- Perturb by $0.001, 0.01, 0.1$.

Our results:

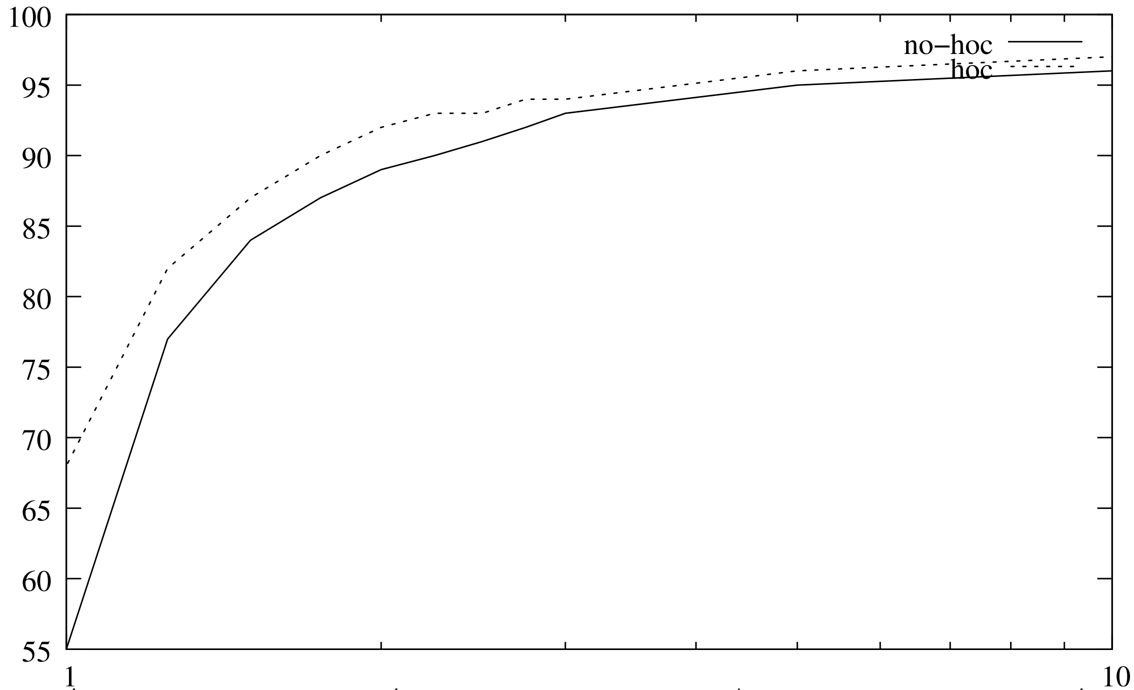
- Choose 10 random instances (and take average)
- All problems warmstarted with $\hat{\mu} = 10^{-2}$.

Results: Additional Centering Steps



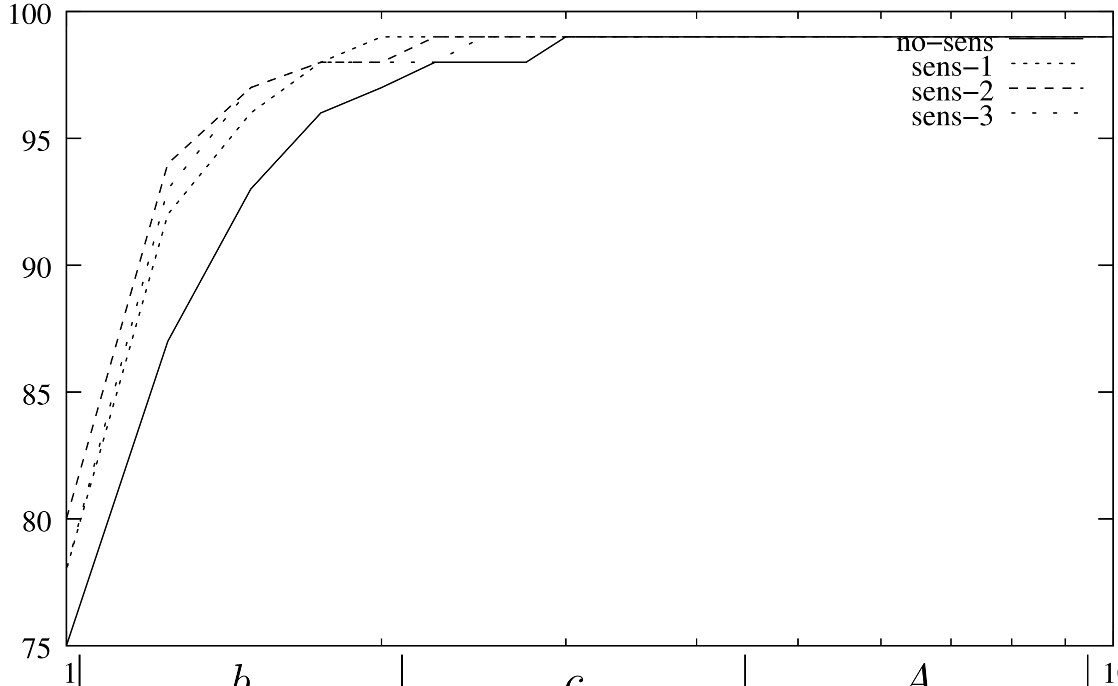
	b			c			A			
	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001	total
beg=0,end=0	6.1	5.4	5.3	10.2	7.4	6.4	8.0	6.3	7.1	7.0
beg=1,end=0	6.0	5.3	5.2	10.0	7.1	6.2	7.9	6.8	7.0	6.9
beg=0,end=1	6.1	4.9	5.0	10.4	6.9	5.9	8.5	6.4	6.8	6.8
beg=1,end=1	5.8	5.0	5.0	10.2	7.0	6.0	8.4	6.2	6.5	6.8
beg=1,end=2	5.7	4.8	5.1	10.5	6.8	5.7	8.8	6.3	6.4	6.8

Results: Higher Order Correctors



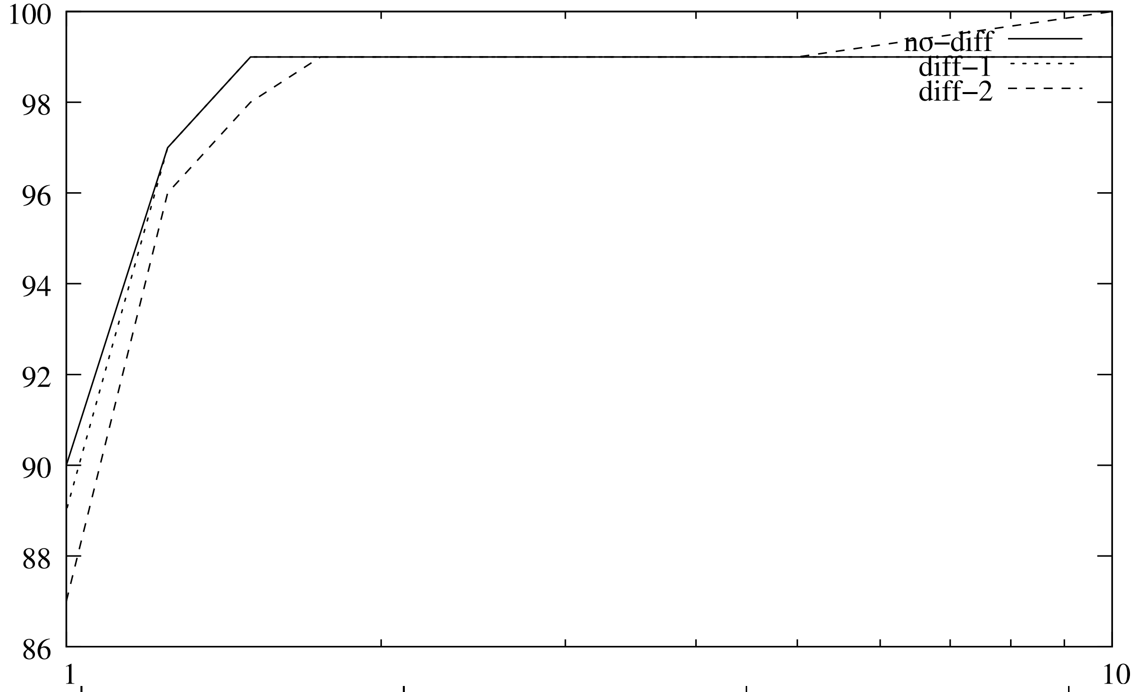
	b			c			A			
	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001	total
no-hoc	6.1	5.0	5.0	14.9	8.7	6.2	11.5	7.0	6.6	8.0
hoc	5.7	4.7	5.2	11.6	7.1	5.8	9.4	6.4	6.5	7.0

Results: Sensitivity Steps



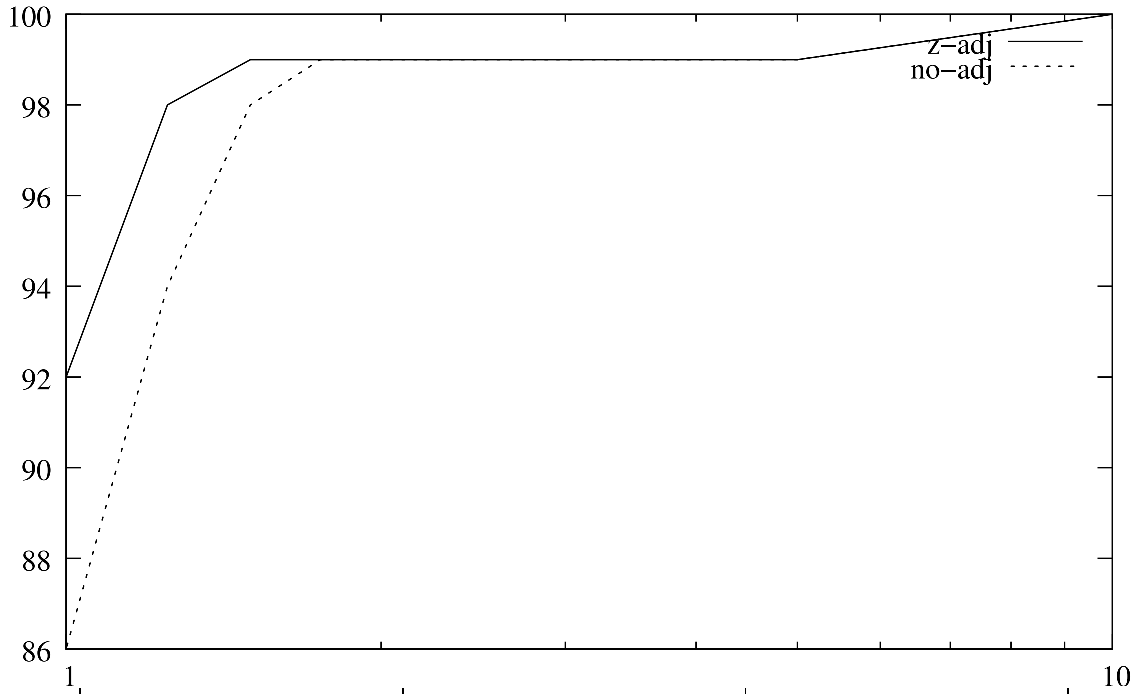
	b			c			A			
	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001	total
no-sens	6.1	5.0	5.0	14.9	8.7	6.2	11.5	7.0	6.6	8.0
sens-1	5.9	4.9	5.0	13.2	7.9	6.0	10.4	6.8	6.7	7.6
sens-2	5.8	5.0	5.0	11.9	8.0	6.1	9.7	6.7	6.9	7.4
sens-3	5.8	5.2	5.1	11.5	7.7	5.8	9.7	6.8	6.8	7.3

Results: Spliting Direction



	b			c			A			
	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001	total
no-diff	5.7	4.7	5.2	11.6	7.1	5.8	9.4	6.4	6.5	7.0
diff-1	5.7	4.8	5.2	12.0	6.9	5.8	9.4	6.5	6.5	7.1
diff-2	5.8	4.8	5.2	11.9	7.2	5.7	9.8	6.5	6.5	7.2

Results: z-Adjustment



	b			c			A			
	0.1	0.01	0.001	0.1	0.01	0.001	0.1	0.01	0.001	total
no-adj	5.7	4.7	5.2	11.6	7.1	5.8	9.4	6.4	6.5	7.0
z-adj	5.7	4.8	5.1	10.5	6.8	5.7	8.8	6.3	6.4	6.8

Results (Best Warmstart) - perturbations in b

Problem	0.1			0.01			0.001		
	cold	warm	per	cold	warm	perc	cold	warm	per
ADLITTLE	10.0			10.0			11.4		
AFIRO	10.1			10.1			10.1		
AGG2	16.1			16.2			16.1		
AGG3	15.7			15.5			16.0		
BANDM	13.8			14.0			13.5		
BEACONFD	-			-			-		
BLEND	9.0			9.0			9.0		
BOEING1	19.3			21.5			19.1		
BORE3D	-			-			-		
BRANDY	-			-			-		
DEGEN2	-			-			-		
E226	16.0			15.8			15.0		
GROW15	13.0			13.0			13.0		
GROW7	12.0			12.0			12.0		
ISRAEL	21.0			20.5			19.9		
KB2	17.7			17.4			17.2		
LOTFI	19.3			20.0			20.0		
RECIPELP	14.0			14.0			14.5		
SC105	12.0			12.0			12.0		
SC205	12.0			12.0			12.0		
SC50A	11.0			11.0			11.0		
SC50B	10.0			10.0			12.1		
SCAGR25	12.0			11.9			12.7		
SCAGR7	10.1			9.9			9.8		
SCFXM1	14.6			15.2			14.1		
SCSD1	9.9			10.3			10.2		
SCTAP1	14.7			14.9			15.6		
SHARE1B	21.5			20.8			21.3		
SHARE2B	9.3			9.2			9.1		
STOCFOR1	13.5			13.0			15.4		
Average	13.8			13.8			13.9		

Results (Best Warmstart) - perturbations in b

Problem	0.1			0.01			0.001		
	cold	warm	per	cold	warm	perc	cold	warm	per
ADLITTLE	10.0	6.0	40.0	10.0	5.0	50.0	11.4	6.0	47.3
AFIRO	10.1	4.2	58.4	10.1	4.3	57.4	10.1	4.3	57.4
AGG2	16.1	4.6	71.4	16.2	4.0	75.3	16.1	4.0	75.1
AGG3	15.7	5.6	64.3	15.5	5.0	67.7	16.0	5.0	68.7
BANDM	13.8	8.2	40.5	14.0	4.1	70.7	13.5	4.0	70.3
BEACONFD	-	-	-	-	-	-	-	-	-
BLEND	9.0	4.0	55.5	9.0	4.3	52.2	9.0	4.2	53.3
BOEING1	19.3	7.2	62.6	21.5	8.3	61.3	19.1	5.1	73.2
BORE3D	-	-	-	-	-	-	-	-	-
BRANDY	-	-	-	-	-	-	-	-	-
DEGEN2	-	-	-	-	-	-	-	-	-
E226	16.0	12.8	20.0	15.8	5.0	68.3	15.0	4.8	68.0
GROW15	13.0	4.0	69.2	13.0	4.0	69.2	13.0	4.0	69.2
GROW7	12.0	4.0	66.6	12.0	4.0	66.6	12.0	4.0	66.6
ISRAEL	21.0	6.9	67.1	20.5	4.0	80.4	19.9	4.0	79.8
KB2	17.7	5.0	71.7	17.4	5.0	71.2	17.2	5.0	70.9
LOTFI	19.3	6.8	64.7	20.0	5.7	71.5	20.0	5.8	71.0
RECIPELP	14.0	7.0	50.0	14.0	7.0	50.0	14.5	10.8	25.5
SC105	12.0	5.0	58.3	12.0	5.1	57.5	12.0	5.0	58.3
SC205	12.0	5.2	56.6	12.0	5.0	58.3	12.0	5.0	58.3
SC50A	11.0	4.0	63.6	11.0	4.0	63.6	11.0	4.0	63.6
SC50B	10.0	4.2	58.0	10.0	4.0	60.0	12.1	14.2	-17.3
SCAGR25	12.0	4.8	60.0	11.9	4.1	65.5	12.7	4.0	68.5
SCAGR7	10.1	4.1	59.4	9.9	4.0	59.5	9.8	4.0	59.1
SCFXM1	14.6	5.0	65.7	15.2	5.8	61.8	14.1	4.1	70.9
SCSD1	9.9	9.5	4.0	10.3	5.9	42.7	10.2	5.1	50.0
SCTAP1	14.7	6.0	59.1	14.9	5.0	66.4	15.6	5.3	66.0
SHARE1B	21.5	5.8	73.0	20.8	5.4	74.0	21.3	5.0	76.5
SHARE2B	9.3	5.2	44.0	9.2	5.1	44.5	9.1	5.1	43.9
STOCFOR1	13.5	5.4	60.0	13.0	5.1	60.7	15.4	5.3	65.5
Average	13.8	5.8	56.3	13.8	4.9	62.6	13.9	5.3	60.0

Results (Best Warmstart) - perturbations in c

Problem	0.1			0.01			0.001		
	cold	warm	per	cold	warm	perc	cold	warm	per
ADLITTLE	10.3	7.3	29.1	10.1	5.2	48.5	10.4	5.0	51.9
AFIRO	10.3	5.3	48.5	10.3	4.8	53.3	10.7	4.8	55.1
AGG2	16.7	6.6	60.4	16.4	4.8	70.7	16.0	4.1	74.3
AGG3	16.0	6.9	56.8	16.0	5.3	66.8	15.9	4.9	69.1
BANDM	13.7	14.2	-3.6	13.9	5.2	62.5	13.6	4.0	70.5
BEACONFD	10.1	4.7	53.4	10.0	4.0	60.0	11.0	4.8	56.3
BLEND	9.4	7.3	22.3	9.0	4.6	48.8	9.0	4.3	52.2
BOEING1	19.6	24.2	-23.4	19.6	8.6	56.1	19.1	5.8	69.6
BORE3D	12.9	6.1	52.7	13.2	4.4	66.6	13.2	4.2	68.1
BRANDY	15.2	8.7	42.7	15.5	4.3	72.2	15.3	4.0	73.8
DEGEN2	9.8	4.5	54.0	10.0	4.8	52.0	10.0	5.0	50.0
E226	15.6	15.0	3.8	15.2	9.0	40.7	15.1	4.5	70.1
GROW15	22.9	13.7	40.1	22.9	9.2	59.8	17.7	11.0	37.8
GROW7	18.9	14.3	24.3	19.9	12.4	37.6	23.6	17.5	25.8
ISRAEL	20.4	7.7	62.2	21.0	4.2	80.0	21.1	4.3	79.6
KB2	17.8	6.8	61.7	17.9	5.0	72.0	18.0	5.0	72.2
LOTFI	19.0	30.7	-61.5	23.0	20.9	9.1	22.4	12.7	43.3
RECIPELP	-	-	-	-	-	-	-	-	-
SC105	11.4	15.4	-35.0	11.8	5.9	50.0	11.5	5.0	56.5
SC205	12.7	20.9	-64.5	13.1	18.2	-38.9	12.1	6.7	44.6
SC50A	11.2	6.8	39.2	11.0	4.1	62.7	11.0	4.0	63.6
SC50B	10.3	7.2	30.0	10.0	4.4	56.0	10.0	4.0	60.0
SCAGR25	12.0	4.7	60.8	12.4	4.4	64.5	13.0	4.0	69.2
SCAGR7	10.1	4.8	52.4	9.9	4.1	58.5	10.0	4.0	60.0
SCFXM1	14.4	7.4	48.6	14.0	4.0	71.4	14.0	4.0	71.4
SCSD1	9.5	5.2	45.2	9.2	5.0	45.6	9.0	5.0	44.4
SCTAP1	16.2	6.6	59.2	16.1	5.8	63.9	15.8	6.0	62.0
SHARE1B	22.6	8.9	60.6	21.9	6.0	72.6	20.9	5.5	73.6
SHARE2B	9.2	7.2	21.7	9.0	5.0	44.4	9.1	5.0	45.0
STOCFOR1	12.8	5.0	60.9	13.0	5.0	61.5	14.4	5.0	65.2
Average	14.2	9.8	31.1	14.3	6.5	54.1	14.2	5.7	59.8

Results (Best Warmstart) - perturbations in A

Problem	0.1			0.01			0.001		
	cold	warm	per	cold	warm	perc	cold	warm	per
ADLITTLE	10.8	9.4	12.9	10.5	5.0	52.3	10.4	5.0	51.9
AFIRO	10.1	5.0	50.4	10.0	4.1	59.0	10.0	4.0	60.0
AGG2	15.9	5.3	66.6	16.0	4.2	73.7	16.2	4.0	75.3
AGG3	15.2	6.3	58.5	15.7	5.2	66.8	16.1	5.0	68.9
BANDM	13.8	7.9	42.7	13.8	4.4	68.1	13.4	4.1	69.4
BEACONFD	10.1	4.8	52.4	10.0	4.0	60.0	10.0	4.0	60.0
BLEND	9.0	9.5	-5.5	9.2	5.3	42.3	9.0	4.4	51.1
BOEING1	19.3	5.2	73.0	19.6	5.0	74.4	19.8	5.0	74.7
BORE3D	15.0	4.0	73.3	13.9	4.0	71.2	13.6	4.0	70.5
BRANDY	14.2	14.1	0.7	17.8	15.4	13.4	28.1	18.8	33.0
DEGEN2	11.1	13.4	-20.7	29.2	30.5	-4.4	93.0	86.0	7.5
E226	15.5	10.2	34.1	15.1	4.9	67.5	15.0	4.1	72.6
GROW15	20.2	12.9	36.1	15.3	11.3	26.1	13.4	5.0	62.6
GROW7	24.0	16.1	32.9	17.1	8.8	48.5	13.5	6.4	52.5
ISRAEL	19.8	5.4	72.7	20.0	4.0	80.0	19.9	4.0	79.8
KB2	18.2	15.3	15.9	18.2	5.1	71.9	17.8	5.0	71.9
LOTFI	20.0	7.1	64.5	25.8	12.3	52.3	50.1	36.2	27.7
RECIPELP	13.9	7.1	48.9	13.9	6.6	52.5	14.0	6.0	57.1
SC105	11.8	7.1	39.8	11.5	5.0	56.5	12.0	5.0	58.3
SC205	12.6	7.7	38.8	12.0	5.0	58.3	12.0	5.0	58.3
SC50A	11.1	7.1	36.0	11.0	4.0	63.6	11.0	4.0	63.6
SC50B	10.0	5.1	49.0	10.0	4.0	60.0	10.0	4.0	60.0
SCAGR25	11.7	9.4	19.6	11.8	4.3	63.5	12.5	4.3	65.6
SCAGR7	10.1	6.5	35.6	10.0	4.0	60.0	9.7	4.0	58.7
SCFXM1	15.2	8.0	47.3	14.9	4.6	69.1	14.4	5.0	65.2
SCSD1	9.1	6.3	30.7	9.3	5.2	44.0	9.2	4.8	47.8
SCTAP1	14.2	9.5	33.0	15.6	6.2	60.2	15.1	5.2	65.5
SHARE1B	21.0	9.4	55.2	21.2	7.0	66.9	22.1	5.6	74.6
SHARE2B	9.6	9.9	-3.1	9.2	5.7	38.0	9.0	5.0	44.4
STOCFOR1	11.5	5.8	49.5	12.3	5.2	57.7	12.1	5.1	57.8
Average	14.1	8.4	38.0	14.7	6.7	55.8	17.7	8.9	58.9

Results (Best Warmstart) - all perturbations

Problem	b			c			A		
	cold	warm	per	cold	warm	perc	cold	warm	per
ADLITTLE	10.4	5.6	46.1	10.2	5.8	43.1	10.5	6.4	39.0
AFIRO	10.1	4.2	58.4	10.4	4.9	52.8	10.0	4.3	57.0
AGG2	16.1	4.2	73.9	16.3	5.1	68.7	16.0	4.5	71.8
AGG3	15.7	5.2	66.8	15.9	5.7	64.1	15.6	5.5	64.7
BANDM	13.7	5.4	60.5	13.7	7.8	43.0	13.6	5.4	60.2
BEACONFD	-	-	-	10.3	4.5	56.3	10.0	4.2	58.0
BLEND	9.0	4.1	54.4	9.1	5.4	40.6	9.0	6.4	28.8
BOEING1	19.9	6.8	65.8	19.4	12.8	34.0	19.5	5.0	74.3
BORE3D	-	-	-	13.1	4.9	62.5	14.1	4.0	71.6
BRANDY	-	-	-	15.3	5.6	63.3	20.0	16.1	19.5
DEGEN2	-	-	-	9.9	4.7	52.5	44.4	43.3	2.4
E226	15.6	7.5	51.9	15.3	9.5	37.9	15.2	6.4	57.8
GROW15	13.0	4.0	69.2	21.1	11.3	46.4	16.3	9.7	40.4
GROW7	12.0	4.0	66.6	20.8	14.7	29.3	18.2	10.4	42.8
ISRAEL	20.4	4.9	75.9	20.8	5.4	74.0	19.9	4.4	77.8
KB2	17.4	5.0	71.2	17.9	5.6	68.7	18.0	8.4	53.3
LOTFI	19.7	6.1	69.0	21.4	21.4	0.0	31.9	18.5	42.0
RECIPELP	14.1	8.2	41.8	-	-	-	13.9	6.5	53.2
SC105	12.0	5.0	58.3	11.5	8.7	24.3	11.7	5.7	51.2
SC205	12.0	5.0	58.3	12.6	15.2	-20.6	12.2	5.9	51.6
SC50A	11.0	4.0	63.6	11.0	4.9	55.4	11.0	5.0	54.5
SC50B	10.7	7.4	30.8	10.1	5.2	48.5	10.0	4.3	57.0
SCAGR25	12.2	4.3	64.7	12.4	4.3	65.3	12.0	6.0	50.0
SCAGR7	9.9	4.0	59.5	10.0	4.3	57.0	9.9	4.8	51.5
SCFXM1	14.6	4.9	66.4	14.1	5.1	63.8	14.8	5.8	60.8
SCSD1	10.1	6.8	32.6	9.2	5.0	45.6	9.2	5.4	41.3
SCTAP1	15.0	5.4	64.0	16.0	6.1	61.8	14.9	6.9	53.6
SHARE1B	21.2	5.4	74.5	21.8	6.8	68.8	21.4	7.3	65.8
SHARE2B	9.2	5.1	44.5	9.1	5.7	37.3	9.2	6.8	26.0
STOCFOR1	13.9	5.2	62.5	13.4	5.0	62.6	11.9	5.3	55.4
Average	13.8	5.3	59.6	14.2	7.3	48.4	15.5	8.0	50.9

Warmstarting Large Scale Problems

SQP: Capacitated MCNF:

$$\begin{array}{ll} \min & \sum_{(i,j)\in\mathcal{E}} \frac{x_{ij}}{K_{ij}-x_{ij}} \\ \text{s. t.} & \sum_{k\in\mathcal{D}} x_{ij}^{(k)} \leq K_{ij}, \ \forall (i,j) \in \mathcal{E}, \\ & N x^{(k)} = d^{(k)}, \quad \forall k \in \mathcal{D}, \\ & x^{(k)} \geq 0, \qquad \forall k \in \mathcal{D}. \end{array}$$

- Nonlinear objective, linear constraints
- Solve by SQP, subproblems solved by IPM
- Use warmstarts for each QP problem in sequence

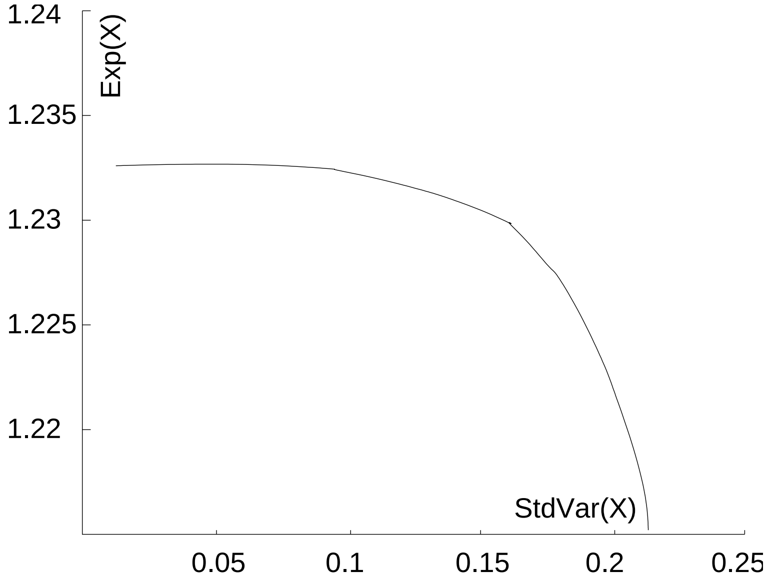
Application: Efficient Frontier (Portfolio Optimization)

- Investors differ in their attitude to risk.
- Captured by the *risk-aversion* parameter ρ in the mean-variance model.
- Efficient Frontier provides a more complete understanding

solve

$$\max \mathbb{E}(X) - \rho \text{Var}(X)$$

for a range of values for ρ



Gives max achievable expected gain for a given risk, or vice versa.

Results: Capacitated MCNF by SQP

- Reported averages over 9 different network
- 4-300 nodes, Up to 600 arcs, 7021 commodities.
- up to 353.400 variables

iter	1	2	3	4	5	6	7	8	9	10
cold	12.7	11.9	13.7	15.8	16.2	15.6	14.9	14.6	14.5	15.0
warm	12.7	7.0	6.0	5.8	6.4	7.0	7.0	6.7	6.2	6.0
per	0.0	41.2	56.2	63.3	60.5	55.1	53.0	54.1	57.2	60.0

Reported are average iterations for first 10 QPs solved

Efficient Frontier (Results)

Problem: Multistage SP formulation of Mean-Variance Model

constraints	variables	ρ =0.001	0.005	0.01	0.05	0.1	0.5	1	5	10
223.321	76.881	14	14	14	14	14	13	17	16	17
		14	5	5	5	4	5	5	8	8
		0.0	64.2	64.2	64.2	71.4	61.5	70.5	50.0	52.9
533.725	198.525	14	14	14	14	14	15	18	18	17
		14	5	5	5	6	5	5	9	10
		0.0	64.3	64.3	64.3	57.1	66.7	72.2	50.0	41.2
5.982.604	16.316.191	24	23	24	23	25	22	24	23	24
		24	8	11	13	11	13	12	12	14
		0.0	65.2	54.2	43.5	56.0	40.9	50.0	47.8	41.7
70.575.308	192.478.111	52	53	45	43	44	42	44	46	46
		52	13	13	15	15	16	16	23	25
		0.0	75.5	71.1	65.1	65.9	61.9	63.6	50.0	45.6

Conclusions

- IPM **can** be warmstarted
- We have reviewed different warmstarting techniques
  - Modifications/Recentering
  - Unblocking
- Each have some benefits
- Combinations leads to very efficient warmstart
- Generalizes to large problems
- Further improvement ?