Low regularity well-posedness of the modified and the generalized Korteweg-de Vries equations

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or professional qualification.

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Abstract

In this thesis, we study the well-posedness of the modified and generalized Korteweg-de Vries equations on the one-dimensional torus. We first consider the complex-valued modified Korteweg-de Vries equation (mKdV). We observe that the momentum, a formally conserved quantity of the equation, plays a crucial role in the well-posedness theory. In particular, following the method by Guo-Oh (2018), we show the ill-posedness of the complex-valued mKdV, in the sense of non-existence of solutions, when the momentum is infinite. This result motivates the introduction of a novel renormalization of the equation, which we propose as the correct model to study at low regularity. Moreover, we establish the global well-posedness of the renormalized equation in the Fourier-Lebesgue spaces following two approaches: the Fourier restriction norm method and the recent method by Deng-Nahmod-Yue (2020). Lastly, by imposing a new notion of finite momentum at low regularity, we show the existence of distributional solutions to the original equation, with the nonlinearity interpreted in a limiting sense. Regarding the generalized Korteweg-de Vries equations (gKdV), we present a joint work with N. Kishimoto (RIMS, Kyoto University) on the well-posedness with Gibbs initial data. To bypass the analytical ill-posedness of gKdV in the Sobolev support of the Gibbs measure, we prove local well-posedness in the Fourier-Lebesgue spaces. Key ingredients are novel bilinear and trilinear Strichartz estimates adapted to the Fourier-Lebesgue setting. Finally, by applying Bourgain's invariant measure argument (1994), we construct almost sure global-in-time dynamics and show the invariance of the Gibbs measure for gKdV.

Lay summary

In this thesis, we study the modified and generalized Korteweg-de Vries equations for periodic initial data, mKdV and gKdV, respectively. These are examples of nonlinear dispersive partial differential equations (PDEs), which are used to model wave-like phenomena in various branches of physics and engineering, such as quantum mechanics, nonlinear optics, plasma physics, water waves, and atmospheric sciences. Broadly speaking, the solutions of these equations spread out spatially (disperse) as time evolves.

We aim to answer some questions which are essential in the study of PDEs from both the theoretical and applied points of view: Does the equation always have a solution for a given initial condition (existence)? If so, is this the only solution (uniqueness)? If we perturb the initial condition, does this only lead to a small change to the solution at later times (stability under perturbations)? We hope to find the roughest set of initial data for which we can positively answer the above questions. For the complex-valued mKdV equation, we show that we can find initial data for which the equation does not have solutions (non-existence). We then suggest an alternative model for which solutions do exist for all times. Regarding the gKdV equations, we find that a very natural class of solutions exists for all times as long as we ignore a negligible set of initial data. Moreover, we obtain detailed information on how the solutions behave for very long times.

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Contents

Chapter 1

Introduction

This thesis is concerned with the study of the well-posedness and long-time behavior of nonlinear dispersive partial differential equations (PDEs). In particular, we focus our attention on the Cauchy problem for the generalized Korteweg-de Vries equations (gKdV):

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm \partial_x (u^k), \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathcal{M}, \tag{1.1}$$

where $\mathcal{M} = \mathbb{R}$ or $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and $k \geq 2$ and integer. Linear dispersive equations have wave-like solutions whose speed depends on their frequency, which translates into dispersion in the absence of boundary conditions: wave components with different frequency evolve at different speeds. This causes the wave to spread spatially as time evolves (see [107, 102, 35] for more details on dispersive equations). Another classical example of a dispersive equation is the nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u = \pm |u|^{k-1} u, \tag{1.2}$$

where $k \in 2\mathbb{Z} + 1$. The gKdV and NLS equations are two of the simplest examples of PDEs combining dispersive and nonlinear effects, and they are widely used in physics and engineering to describe wave-like phenomena.

The gKdV equations encompass two famous equations: the Korteweg-de Vries (KdV) and the modified Korteweg-de Vries (mKdV) equations, when $k = 2$ and $k = 3$, respectively, whose study dates back to the 19th century. The KdV equation was proposed by Korteweg and de Vries in [67], although it appeared earlier in the work of Boussinesq [16], to describe the traveling waves observed by Scott Russell in 1835 in the Union canal in Edinburgh. Both the KdV and mKdV equations were derived to model the propagation of gravity waves in shallow water, and they are strongly related [75, 76]. Since their derivation, the gKdV equations (1.1) have been used for a wide range of applications, such as water waves, plasma physics, and crystal lattices (see [67, 108, 30] and references therein). Both KdV and mKdV are known to be completely integrable [38, 76], and can be solved using inverse scattering/inverse spectral techniques. This structure does not extend to the gKdV equations (1.1) with $k \geq 4$, and we will instead pursue a harmonic analytic approach.

A fundamental question in the study of PDEs is that of well-posedness: existence and uniqueness of solutions, and continuous dependence of solutions on the initial data. We are first concerned with constructing solutions for short times (local well-posedness) at low regularity. Due to the wave-like nature of their solutions, Fourier analytic tools are well-suited to the study of dispersive PDEs. Consequently, the Sobolev spaces $H^s(\mathcal{M})$ are a natural choice of spaces for the initial data. These spaces can be defined through the following norms

$$\|f\|_{H^s(\mathbb{R})} = \big\| \langle \xi \rangle^s \mathcal{F}_x f(\xi) \big\|_{L^2_\xi(\mathbb{R})}, \qquad \|f\|_{H^s(\mathbb{T})} = \big\| \langle n \rangle^s \mathcal{F}_x f(n) \big\|_{\ell^2_n(\mathbb{Z})},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ denotes the Japanese brackets, and $\mathcal{F}_x$ the Fourier transform. In the pursuit of the largest possible space for the initial data, we may have to leave the realm of the L^2-based spaces. A suitable alternative used in this work is the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathcal{M})$, $s \in \mathbb{R}$, $1 \leq p \leq \infty$, defined through the following norms

$$\|f\|_{\mathcal{F}L^{s,p}(\mathbb{R})} = \big\|\langle \xi \rangle^s \mathcal{F}_x(\xi)\big\|_{L^p_\xi(\mathbb{R})}, \qquad \|f\|_{\mathcal{F}L^{s,p}(\mathbb{T})} = \big\|\langle n \rangle^s \mathcal{F}_x f(n)\big\|_{\ell^p_n(\mathbb{Z})}. \tag{1.3}$$

The main approach to study the well-posedness of dispersive PDEs is through perturbative methods, i.e., to treat the nonlinear solution as a perturbation of the linear solution. We then focus on studying the Duhamel formulation, which is better suited than the classical formulation when considering low regularity initial data,

$$u(t) = S(t)u_0 \pm \int_0^t S(t-t')\partial_x(u^k)(t')\,dt, \tag{1.4}$$

where $S(t)$ denotes the linear propagator. There are two main difficulties in the study of the periodic gKdV equations (1.1) with $\mathcal{M} = \mathbb{T}$: the lack of linear smoothing coming from dispersion (when compared with the Euclidean setting $\mathcal{M} = \mathbb{R}$), and the loss in regularity due to the derivative nonlinearity (when compared with NLS (1.2), for example). While the existence of smooth solutions of NLS follows easily from the algebra property of high regularity Sobolev spaces, this is not sufficient to control the derivative nonlinearity of gKdV. The first well-posedness results for the gKdV equation followed from the classical energy method, based on parabolic regularization; see [8, 7, 96, 59]. This method does not rely on the dispersive nature of the equation and is therefore suitable for both $\mathcal{M} = \mathbb{R}$ and $\mathbb{T}$. When pursuing low regularity well-posedness, harmonic analytic tools have been crucial to exploit dispersion. Oscillatory integral techniques were used to establish decay estimates, such as those by Strichartz [99], by exploiting the dispersion of the linear propagator on $\mathcal{M} = \mathbb{R}$. These proved useful to study equations such as NLS, but still insufficient to control the derivative loss of gKdV. This was overcome through Kato's observation of a local smoothing effect on $\mathbb{R}$ [59, 60] and the maximal function estimates of Kenig-Ponce-Vega [61]. Due to the oscillatory integral techniques, these results could not be extended to the periodic setting $\mathcal{M} = \mathbb{T}$. This last difficulty was addressed by Bourgain in [9, 10]. In these works, he introduced Strichartz estimates adapted to the periodic setting, using analytic number theory. Moreover, he introduced the Fourier restriction norm method, where the perturbative analysis uses a new class of space-time Sobolev spaces adapted to the linear solution; the $X^{s,b}(\mathbb{R} \times \mathbb{T})$-spaces. These spaces are defined through the following norm

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} = \big\|\langle \partial_x \rangle^s \langle \partial_t \rangle^b S(-t)u(t)\big\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{T})} = \big\|\langle n \rangle^s \langle \tau - n^3 \rangle^b \mathcal{F}_{t,x}u(\tau,n)\big\|_{\ell^2_n(\mathbb{Z})L^2_\tau(\mathbb{R})},$$

and they capture the dispersive nature of the solutions by accounting for the distance between the space-time Fourier transform of the linear solution (supported on the curve $\tau = n^3$) and of the nonlinear one. Analogous spaces were also used successfully when $\mathcal{M} = \mathbb{R}$. See Section 1.1.4 for further details.

After establishing local well-posedness for short times, a natural subsequent question is that of global well-posedness; can we construct unique global-in-time solutions of (1.1) at a given regularity? Globalization of solutions is often based on conservation laws. A typical contraction mapping argument, in the subcritical regime, gives a local time of existence that depends on the size of the initial data. Therefore, if the equation has conserved quantities that control the H^s-norm of the solution, they can be used to extend solutions globally-in-time. As a consequence of the completely integrable structure of KdV and mKdV, they have an infinite number of conserved quantities [76]. However, this structure no longer holds for (1.1) with $k \geq 4$. In fact, the only known conserved quantities shared by all equations of the form (1.1) are the following

- mean: $\int_\mathbb{T} u(t,x)\,dx$;
- mass: $\int_\mathbb{T} u^2(t,x)\,dx$;
- energy (Hamiltonian): $H\big(u(t)\big) = \int_\mathbb{T} \big(\partial_x u(t,x)\big)^2\,dx \pm \frac{1}{k+1}\int_\mathbb{T} u^{k+1}(t,x)\,dx$.

For example, by the conservation of the Hamiltonian, we mean that for solutions u of (1.1) with sufficient regularity, $H\big(u(t)\big) = H(u_0)$ for all $t \in \mathbb{R}$. In the absence of suitably low regularity conserved quantities, we must pursue a different strategy, such as the high-low method due to Bourgain [14], or the I-method (or method of almost conservation laws) due to Colliander-Keel-Staffilani-Takaoka-Tao, see [27, 28, 29] for example. Another alternative is to exploit the Hamiltonian structure of (1.1) and the formal invariance of the associated Gibbs measure to extend solutions globally-in-time. In fact, all the gKdV equations can be understood as infinite-dimensional Hamiltonian systems with Hamiltonian H:

$$\partial_t u = \partial_x \frac{\delta H}{\delta u},$$

where $\frac{\delta H}{\delta u}$ denotes the Fréchet derivative. As a consequence, we can pursue the study of the Gibbs measure μ formally defined by

$$d\mu \text{ “ = ” } Z^{-1} e^{-H(u)}\, du.$$

In some limited settings, the formal invariance of the Gibbs measure can be used as a substitute for a conservation law when globalizing solutions. This was the idea proposed by Bourgain in [11], where he introduced what is now known as Bourgain's invariant measure argument. The method relies on exploiting the invariance of the Gibbs measures for the associated finite-dimensional systems with truncated dynamics to conclude invariance of the Gibbs measure and global well-posedness for the original one. Moreover, the invariance of the Gibbs measure informs us of the typical behavior of solutions of (1.1) with initial data in the support of this measure.

In this thesis, we study the well-posedness of the periodic complex-valued mKdV equation and of the periodic gKdV equations.

- In Chapter 2, we prove the ill-posedness of the periodic complex-valued mKdV equation outside $H^{\frac{1}{2}}(\mathbb{T})$ and introduce a new renormalized equation. We then apply the Fourier restriction norm method to construct solutions of the renormalized equation outside $H^{\frac{1}{2}}(\mathbb{T})$ and pursue the question of recovering solutions of the original mKdV equation. This is based on the following work:

  [23] A. Chapouto, *A remark on the well-posedness of the modified KdV equation in the Fourier-Lebesgue spaces*, Discrete Contin. Dyn. Syst. 41 (2021), no. 8, 3915–3950.

- In Chapter 3, we improve the well-posedness of the new renormalized mKdV equation by applying the method due to Deng-Nahmod-Yue [34]. This is based on the following work:

  [24] A. Chapouto, *A refined well-posedness result for the modified KdV equation in the Fourier-Lebesgue spaces*, to appear in J. Dynam. Differential Equations.

- In Chapter 4, we follow Bourgain's invariant measure argument to show almost sure global well-posedness of the gKdV equations (1.1) with $k \geq 4$ and invariance of the Gibbs measure. This is based on the following joint work with Nobu Kishimoto (RIMS, Kyoto University):

  [25] A. Chapouto, N. Kishimoto, *Invariance of the Gibbs measures for the periodic generalized KdV equations*, preprint.

In the remainder of this introduction, we review the literature on the mKdV and the gKdV equations, and state our main results. Chapters 2-4 are then devoted to the proofs of these results.

1.1 The complex-valued mKdV equation

Consider the Cauchy problem for the complex-valued modified Korteweg-de Vries equation (mKdV):

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm |u|^2 \partial_x u, \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathcal{M}. \tag{1.5}$$

where $u : \mathbb{R} \times \mathcal{M} \to \mathbb{C}$, and $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$. We are interested in studying mKdV (1.5) in the periodic setting, when $\mathcal{M} = \mathbb{T}$, or equivalently, by imposing the periodic boundary condition on the initial data u_0. The complex-valued mKdV equation (1.5) appears as a model for the dynamical evolution of nonlinear lattices, fluid dynamics, and plasma physics (see [95, 51], for example). This equation, also known as the mKdV equation of Hirota [55], is a completely integrable complex-valued generalization of the usual mKdV equation

$$\partial_t u + \partial_x^3 u = \pm u^2 \partial_x u, \tag{1.6}$$

or equivalently (1.1) with $k = 3$. Indeed, for real-valued initial data u_0, the solutions of (1.5) are also solutions of (1.6).

In Chapters 2 and 3, we will pursue a harmonic analytic approach to study the well-posedness of mKdV (1.5) at low regularity. We show that $H^{\frac{1}{2}}(\mathbb{T})$ is the largest space (in terms of L^2-based Sobolev spaces) where the periodic complex-valued mKdV equation (1.5) is well-posed, opposed to its Euclidean and real-valued periodic counterparts, as we see below. The ill-posedness of the complex-valued mKdV equation outside $H^{\frac{1}{2}}(\mathbb{T})$ is closely related to the momentum, a formally conserved quantity of the equation (1.5). Exploiting the formal conservation of the momentum, we introduce a new equation, the second renormalized mKdV equation. We establish its local well-posedness in the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$ (see (1.3)) outside $H^{\frac{1}{2}}(\mathbb{T})$ and propose this equation as the correct model to study (1.5) at low regularity. Consequently, we narrow the gap between the analytical ill-posedness of (1.5) outside $H^{\frac{1}{2}}(\mathbb{T})$ and the scaling critical space $H^{-\frac{1}{2}}(\mathbb{T})$ (see Section 1.1.1 for a discussion on scaling). Lastly, by introducing a new notion of finite momentum at low regularity, we extend the conservation of the momentum to the low regularity setting and construct solutions of the original complex-valued mKdV equation (1.5).

We start by discussing the scaling symmetry of mKdV (1.5) and its heuristic implication on the well-posedness theory.

1.1.1 Scaling heuristics

The mKdV equation (1.5) when posed on the real line, $\mathcal{M} = \mathbb{R}$, satisfies the following scaling symmetry; let $T > 0$. If $u : [0, T] \times \mathbb{R} \to \mathbb{C}$ is a solution of (1.5) with initial data $u_0 : \mathbb{R} \to \mathbb{C}$ then $u_\lambda : [0, \lambda^{-3}T] \times \mathbb{R} \to \mathbb{C}$, given by

$$u_\lambda(t, x) = \lambda u(\lambda^3 t, \lambda x), \quad \lambda > 0, \tag{1.7}$$

is also a solution of (1.5) with re-scaled initial data $u_{0,\lambda}(\cdot) = \lambda u_0(\lambda\,\cdot)$.

A direct computation shows the following relation between the homogeneous $\dot{H}^s(\mathbb{R})$-norms of the scaled and original initial data

$$\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R})} = \|\lambda u_0(\lambda\,\cdot)\|_{\dot{H}^s(\mathbb{R})} = \lambda^{s+\frac{1}{2}} \|u_0\|_{\dot{H}^s(\mathbb{R})}.$$

In particular, for $s_{\text{crit}} = -\frac{1}{2}$ we see that the $\dot{H}^{s_{\text{crit}}}(\mathbb{R})$-norm is left invariant under the scaling (1.7). The index s_{crit} is called the *scaling critical Sobolev index* and it motivates the following classification of the Cauchy problem (1.5):

- subcritical: if $u_0 \in \dot{H}^s(\mathbb{R})$ with $s > s_{\text{crit}}$, the scaled solution u_λ has smaller norm and longer time of existence $\lambda^{-3}T$, as $\lambda \to 0$. We expect well-posedness in this regime, at least for short times.

- critical: for $u_0 \in \dot{H}^{s_{\text{crit}}}(\mathbb{R})$, the size of the initial data is unchanged by the scaling. It may still be possible to establish local well-posedness for small initial data, but this is a more delicate case.

- supercritical: if $u_0 \in \dot{H}^s(\mathbb{R})$ with $s < s_{\text{crit}}$, the rescaled data has a larger norm than the original data, but a longer time of existence, which is too good to be true. We expect ill-posedness in this regime.

We can conduct a similar scaling analysis on the homogeneous Fourier-Lebesgue spaces $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$ defined through the norm

$$\|f\|_{\dot{\mathcal{F}L}^{s,p}(\mathbb{R})} = \big\||\xi|^s \mathcal{F}_x f(\xi)\big\|_{L^p_\xi(\mathbb{R})}.$$

Performing an analogous computation, we have that

$$\|u_{0,\lambda}\|_{\dot{\mathcal{F}L}^{s,p}(\mathbb{R})} = \lambda^{s+\frac{1}{p}} \|u_0\|_{\dot{\mathcal{F}L}^{s,p}(\mathbb{R})}.$$

Therefore, the $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$-norm is left invariant under the scaling (1.7) if $s = -\frac{1}{p}$. Moreover, we see that $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$ scales like $\dot{H}^\sigma(\mathbb{R})$ if

$$\sigma = s + \frac{1}{p} - \frac{1}{2}. \tag{1.8}$$

From the above comparison, we will classify the scaling criticality in the Fourier-Lebesgue spaces based on the related scaling in Sobolev spaces. For instance, we say that $\dot{\mathcal{F}L}^{0,p}(\mathbb{R})$ as $p \to \infty$ is almost critical, since from (1.8) it scales like $\dot{H}^\sigma(\mathbb{R})$ with $\sigma \to -\frac{1}{2}$.

Although the scaling (1.7) does not carry to the periodic setting, $\mathcal{M} = \mathbb{T}$, the scaling heuristics is still relevant. It is commonly conjectured that the periodic mKdV equation (1.5) is well-posed in $H^\sigma(\mathbb{T})$ for $\sigma > -\frac{1}{2}$. We will see that this conjecture is indeed *false* for the complex-valued periodic mKdV equation, motivating our attempt to bridge the regularity gap in the subcritical regime by focusing our analysis on the Fourier-Lebesgue spaces.

1.1.2 Literature review

On the real line, $\mathcal{M} = \mathbb{R}$, the complex-valued mKdV equation (1.5) has been a long standing topic of interest [61, 62, 42, 45, 50] and its well-posedness is now complete. In a recent breakthrough, Harrop-Griffiths, Killip, and Vişan [50] showed the optimal global well-posedness of (1.5) in $H^s(\mathbb{R})$ for $s > -\frac{1}{2}$ by exploiting the complete integrability of the equation. In the Fourier-Lebesgue setting, Grünrock [42] showed the local well-posedness of mKdV (1.5) in $\mathcal{F}L^{s,p}(\mathbb{R})$ for $s \geq 0$ and $2 \leq p < 4$, which was extended by Grünrock-Vega [45] to $s \geq \frac{1}{2p}$ and $1 \leq p < \infty$.

In the periodic setting, $\mathcal{M} = \mathbb{T}$, the real-valued mKdV equation (1.6) has garnered more attention than its complex-valued counterpart (1.5) [10, 13, 26, 100, 58, 83, 82, 77, 57, 78, 97]. In [10], Bourgain introduced the Fourier restriction norm method and proved the local well-posedness in $H^s(\mathbb{T})$, for $s \geq \frac{1}{2}$, of the *first renormalized*[1] mKdV equation (mKdV1)

$$\partial_t u + \partial_x^3 u = \pm\big(|u|^2 - M(u)\big)\partial_x u, \tag{1.9}$$

where $M(f) = ⨍ |f(x)|^2 dx = \frac{1}{2\pi}\int_\mathbb{T} |f(x)|^2 dx$ denotes the mass. The renormalized equation (1.9) is obtained from mKdV (1.5) through the following gauge transform which exploits the conservation of mass

$$\mathcal{G}_1[u](t,x) = u\big(t, x \mp M\big(u(t)\big)t\big). \tag{1.10}$$

Note that mKdV1 (1.9) is equivalent to mKdV (1.5) in $L^2(\mathbb{T})$ in the following sense; $u \in C\big(\mathbb{R}; L^2(\mathbb{T})\big)$ is a solution of (1.5) if and only if $\mathcal{G}_1[u]$ is a solution of (1.9). Bourgain's well-posedness result also extends to the complex-valued setting. The failure of C^3-continuity of the solution map [13, 26] below $H^{\frac{1}{2}}(\mathbb{T})$ implies that we cannot use a contraction mapping argument to improve the result in [10], hence requiring more robust techniques. See also Proposition 1.1.4 for the failure of local uniform continuity of the solution map in the complex-valued setting.

For the real-valued equation (1.6), Takaoka-Tsutsumi [100] and Nakanishi-Takaoka-Tsutsumi [82] applied the energy method and proved the local well-posedness of mKdV in $H^s(\mathbb{T})$ for $s > \frac{1}{3}$. In a recent paper [78], Molinet-Pilod-Vento extended this result to the

[1] The equation (1.9) is usually referred to as the *renormalized* mKdV equation. However, we will introduce a second gauge transform and a second renormalization in Section 1.1.3 which motivates the change in notation.

end-point $s = \frac{1}{3}$. By exploiting the completely integrable structure of the equation, Kappeler-Topalov [58] used the inverse spectral method to show existence and uniqueness of solutions of the real-valued defocusing mKdV (with the + sign in (1.6)) in $H^s(\mathbb{T})$, $s \geq 0$. The solutions in [58] are defined as the (unique) limit of approximating smooth solutions. Therefore, the data-to-solution map extends continuously from smooth solutions to solutions in $H^s(\mathbb{T})$. However, these solutions are not necessarily distributional solutions. See [58, 70, 91, 92] for further details on this notion of solution.

Using the short-time Fourier restriction norm method, Molinet [77] showed that the solutions in [58] are indeed distributional solutions and proved the ill-posedness of (1.6) below $L^2(\mathbb{T})$ in the sense of failure of continuity of the solution map (see also [97]). This ill-posedness result shows the sharpness of the well-posedness theory in the L^2-based Sobolev spaces. However, the scaling analysis in Section 1.1.1 suggests that the local well-posedness should hold in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. This ill-posedness result motivated the study of (1.6) in alternative function spaces, namely in the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$. Regarding the local-in-time analysis, Kappeler-Molnar [57] proved the local well-posedness of the real-valued defocusing mKdV1 in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq 0$ and $1 \leq p < \infty$ (see also [83]). These solutions are defined to be the unique limit of classical solutions, as those in [58]. In view of the scaling critical regularity, this result is almost critical, in the scale of the Fourier-Lebesgue spaces. Unlike the $L^2(\mathbb{T})$ solutions of [58, 77], the solutions in [57] are not yet known to satisfy the equation in the distributional sense.

Now, we turn our attention to the global aspect of the well-posedness of the periodic mKdV equation. In [10], Bourgain proved global well-posedness of (1.9) in $H^s(\mathbb{T})$ for $s \geq 1$. For the real-valued mKdV (1.6), Colliander-Keel-Staffilani-Takaoka-Tao [28] showed the global well-posedness in $H^s(\mathbb{T})$, $s \geq \frac{1}{2}$, using the I-method. This result was extended to $H^s(\mathbb{T})$ for $s \geq 0$ for the real-valued defocusing mKdV by Kappeler-Topalov [58], using the complete integrability of the equation. In [57], Kappeler-Molnar proved global-in-time existence of solutions of the real-valued mKdV1, (1.9) with small real-valued initial data in $\mathcal{F}L^{s,p}(\mathbb{T})$, $s \geq 0$ and $1 \leq p < \infty$. In a recent paper [63], Killip-Vişan-Zhang exploited the completely integrable structure of the equation and established global-in-time a priori bounds, in the complex-valued setting. These a priori bounds, combined with the local well-posedness results in [10, 78], yield the global well-posedness of the real-valued mKdV equation (1.6) in $H^s(\mathbb{T})$ for $s \geq \frac{1}{3}$ and of the complex-valued mKdV equation (1.5) in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$.

In summary, the only local-in-time result which extends to the complex-valued setting on $\mathbb{T}$ is that of Bourgain in [10]. Therefore, the periodic complex-valued mKdV1 (1.9) is only known to be globally well-posed in $H^{\frac{1}{2}}(\mathbb{T})$. In the following, we will see that this result is actually sharp in the scale of the L^2-based Sobolev spaces, as the equation is ill-posed at lower regularity.

1.1.3 Main results

Our main goal is to study the low regularity Cauchy problem for the complex-valued periodic mKdV equation (1.5) with $\mathcal{M} = \mathbb{T}$. We find that there is an additional difficulty in the low regularity complex-valued setting, due to the momentum, a formally conserved quantity of (1.5).

At the level of regularity considered in our work, mKdV (1.5) and mKdV1 (1.9) are equivalent. Therefore, we focus on studying the well-posedness of mKdV1 (1.9), as the results easily extend to (1.5) through the (inverse) gauge transform $\mathcal{G}_1$ in (1.10). Omitting time dependence, the nonlinearity of mKdV1 (1.9) is given by

$$\begin{aligned}
&\mathcal{F}_x\Big(|u|^2\partial_x u - M(u)\partial_x u\Big)(n) \\
&= \sum_{\substack{n=n_1+n_2+n_3,\\ n_2+n_3\neq 0}} in_1\widehat{u}(n_1)\widehat{\overline{u}}(-n_2)\widehat{u}(n_3) \\
&= \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})\neq 0}} in_1\widehat{u}(n_1)\widehat{\overline{u}}(-n_2)\widehat{u}(n_3) - in|\widehat{u}(n)|^2\widehat{u}(n) + i\bigg(\operatorname{Im}\fint_{\mathbb{T}} \overline{u}\partial_x u\, dx\bigg)\widehat{u}(n),
\end{aligned} \tag{1.11}$$

where $\phi(\overline{n}_{123}) = 3(n_1+n_2)(n_1+n_3)(n_2+n_3)$ and $\overline{n}_{123} = (n_1, n_2, n_3)$. In the real-valued setting, we have $\mathrm{Im}(\overline{u}\partial_x u) \equiv 0$ which implies that the last term on the right-hand side of (1.11) is zero. However, in the complex-valued case, this contribution is not generally zero. We define the *momentum* $P(u)$ as follows:

$$P(u) = \mathrm{Im} \fint_{\mathbb{T}} \overline{u}\partial_x u \, dx = \sum_{n\in\mathbb{Z}} n|\widehat{u}(n)|^2,$$

and rewrite the nonlinearity of mKdV1 (1.9) as

$$\big(|u|^2 - M(u)\big)\partial_x u = \mathcal{NR}(u,\overline{u},u) + \mathcal{R}(u,u,u) + iP(u)u,$$

with each term defined on the Fourier side through (1.11). For a solution $u \in C\big(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{T})\big)$, the momentum $P\big(u(t)\big)$ is finite and conserved, but below this regularity it is not clear if it is finite, let alone conserved. Consequently, a new phenomenon arises in the complex-valued setting at low regularity, as the nonlinearity (1.11) may be ill-defined. In particular, we see that the momentum is responsible for the following ill-posedness of mKdV1 (1.9) outside $H^{\frac{1}{2}}(\mathbb{T})$.

Theorem 1.1.1. *Let $s \geq \frac{1}{2}$ and $2 < p < \infty$. Suppose that $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ has infinite momentum in the sense that*

$$\limsup_{N\to\infty} |P(\mathbf{P}_{\leq N} u_0)| = \infty,$$

where $\mathbf{P}_{\leq N}$ denotes the Dirichlet projection onto the spatial frequencies $\{|n| \leq N\}$. Then, for any $T > 0$, there exists no distributional solution $u \in C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ to the complex-valued mKdV1 equation (1.9) satisfying the following conditions:

(a) $u|_{t=0} = u_0$;

(b) *the smooth global solutions $\{u_N\}_{N\in\mathbb{N}}$ of mKdV1 (1.9), with $u_N|_{t=0} = \mathbf{P}_{\leq N}u_0$, satisfy $u_N \to u$ in $C\big([-T,T]; \mathcal{D}'(\mathbb{T})\big)$.*

Remark 1.1.2. (i) The condition (b) in Theorem 1.1.1 is a natural one to impose, as we would expect "good" solutions to have the property of being well-approximated by the smooth solutions corresponding to the truncated initial data.

(ii) The momentum is identically zero for real-valued solutions. Therefore, it does not play a role in the low regularity well-posedness of the real-valued mKdV equation (1.6). Consequently, the ill-posedness result in Theorem 1.1.1 is a phenomenon specific to the complex-valued mKdV equation (1.5).

(iii) Since $H^{\frac{1}{2}}(\mathbb{T}) \subsetneq \mathcal{F}L^{\frac{1}{2},p}(\mathbb{T})$, for $2 < p < \infty$, Theorem 1.1.1 establishes the ill-posedness of the complex-valued mKdV1 equation (1.9) outside $H^{\frac{1}{2}}(\mathbb{T})$. The proof is based on the argument for the nonlinear Schrödinger equation by Guo-Oh [48], and the result can also be extended to ill-posedness in $H^s(\mathbb{T})$ for $\frac{1}{3} < s < \frac{1}{2}$; see Remark 1.1.15 for more details. In Remark 1.2 (iii) in [82], the authors claim that local well-posedness in $H^s(\mathbb{T})$ for $s > \frac{1}{3}$ extends to the complex solutions of mKdV1 (1.9), which does not seem to be the case.

Motivated by the ill-posedness result in Theorem 1.1.1, we propose an alternative model to the complex-valued mKdV1 equation (1.9), and hence to the complex-valued mKdV equation (1.5). Analogously to the first gauge transform $\mathcal{G}_1$ (1.10), which exploited the conservation of the mass, we introduce a second gauge transform $\mathcal{G}_2$ using the conservation of the momentum to remove the singular contribution $iP(u)u$ from the nonlinearity. Given $u \in C\big(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{T})\big)$, we define the following invertible gauge transform

$$\mathcal{G}_2[u](t,x) = e^{\mp iP(u)t}u(t,x). \tag{1.12}$$

The effect of the gauge transform $\mathcal{G}_2$ is to remove certain resonant frequency interactions in the nonlinearity which are responsible for the ill-posedness result in Theorem 1.1.1. If $u \in C\big(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{T})\big)$ is a solution of mKdV (1.5), the momentum is finite and conserved $P\big(u(t)\big) =$

$P(u_0)$, and thus the gauge transform $\mathcal{G}_2$ is invertible and u solves the original mKdV (1.5) if and only if $\mathcal{G}_2 \circ \mathcal{G}_1[u]$ solves the second renormalized mKdV equation (mKdV2):

$$\partial_t u + \partial_x^3 u = \pm\Big(|u|^2 \partial_x u - M(u)\partial_x u - iP(u)u\Big). \tag{1.13}$$

Focusing on the Fourier-Lebesgue spaces, for $1 \leq p < \infty$ and $s > 1 - \frac{1}{p}$, the gauge transform $\mathcal{G}_2$ is well-defined in $C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ and the equations mKdV1 (1.9) and mKdV2 (1.13) are equivalent. However, for $2 \leq p < \infty$ and $\frac{1}{2} \leq s \leq 1-\frac{1}{p}$, we have that $\mathcal{F}L^{s,p}(\mathbb{T}) \not\hookrightarrow H^{\frac{1}{2}}(\mathbb{T})$. Since outside $H^{\frac{1}{2}}(\mathbb{T})$ the momentum may be infinite, we cannot make sense of the (inverse) gauge transform $\mathcal{G}_2$, and thus cannot, in general, convert solutions of mKdV2 (1.13) into solutions of mKdV1 (1.9).

Although any renormalization is a matter of choice to some degree, we believe that Theorem 1.1.1 provides evidence for our choice of $\mathcal{G}_2$; see also Remark 1.1.13. In particular, since the assumption of infinite momentum of the initial data u_0 can only hold if $u_0 \notin H^{\frac{1}{2}}(\mathbb{T})$, we propose mKdV2 (1.13) as the correct model to study the complex-valued mKdV equation (1.5) outside $H^{\frac{1}{2}}(\mathbb{T})$. To further our evidence, we establish the following local well-posedness result for mKdV2 (1.13) outside $H^{\frac{1}{2}}(\mathbb{T})$.

Theorem 1.1.3. *Let $s \geq \frac{1}{2}$ and $1 \leq p < \infty$. Then, mKdV2 (1.13) is locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{T})$. Moreover, the data-to-solution map is locally Lipschitz continuous.*

The restriction $s \geq \frac{1}{2}$ is necessary if we require uniform continuity of the solution map, as shown by the following proposition.

Proposition 1.1.4. *Let $s < \frac{1}{2}$ and $1 \leq p < \infty$. The data-to-solution map for mKdV2 (1.13) fails to be locally uniformly continuous in $C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$.*

Remark 1.1.5. (i) The scaling heuristics in Section 1.1.1 allows us to compare the scaling of the L^2-based Sobolev spaces with the Fourier-Lebesgue spaces. In particular, $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$ scales like $\dot{H}^\sigma(\mathbb{R})$ for $\sigma = s + \frac{1}{p} - \frac{1}{2}$. From these heuristics, we see that the results in Theorem 1.1.3 are at the scale of $L^2(\mathbb{T})$. At this time, we do not know how to prove an almost critical result for (1.13) in $\mathcal{F}L^{s,p}(\mathbb{T})$ with $s = 0$ and $1 \leq p < \infty$, since Theorem 1.1.3 is sharp with respect to the method due to Proposition 1.1.4. Without imposing uniform continuous dependence on the initial data, we expect it to be possible to lower s by combining the method introduced by Deng-Nahmod-Yue [34] and the energy method in [100, 82, 78].

(ii) To show Theorem 1.1.3 for $1 \leq p < 4$, we apply the Fourier restriction norm method and the uniqueness holds conditionally in

$$C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big) \cap X^{s,\frac{1}{2}}_{p,2}(T).$$

See Definition 1.3.2 for the definition of the $X^{s,b}$ space. For $4 \leq p < \infty$, we follow the method introduced by Deng-Nahmod-Yue [34], which is based on constructing solutions u with a particular structure. As a consequence, uniqueness holds conditionally in a sub-manifold of $X^{s,\frac{1}{2}}_{p,2-\varepsilon}$ determined by the structure imposed on u.

(iii) For the range of (s,p) in Theorem 1.1.3, the mass $M(u)$ is conserved and therefore one can still establish local well-posedness of mKdV2 (1.13) without removing the term $M(u)\partial_x u$ from the nonlinearity at the cost of losing the local Lipschitz continuity of the solution map. In contrast, this renormalization is essential in [57] when taking data in $\mathcal{F}L^{0,p}(\mathbb{T})$ with $2 < p < \infty$, for example. Analogously, the second renormalization introduced by $\mathcal{G}_2$ is crucial in Theorem 1.1.3 when $\frac{1}{2} \leq s < 1$ and $\frac{1}{1-s} \leq p < \infty$. In the remaining regimes for (s,p), since the initial data is in $H^{\frac{1}{2}}(\mathbb{T})$ and the momentum is conserved, the renormalization is only required to guarantee the local Lipschitz continuity in Theorem 1.1.3. See [35, 52] for analogous proofs, for example.

(iv) In Sobolev spaces, unconditional uniqueness holds in $H^s(\mathbb{T})$ for $s \geq \frac{1}{3}$ (see [69, 78]). It would be of interest to consider the problem of unconditional uniqueness of mKdV2 (1.13) in the Fourier-Lebesgue spaces. See Section 1.1.4 for more details.

Using the a priori bounds established by Oh-Wang [91], which generalize the result by Killip-Vişan-Zhang [63] to the Fourier-Lebesgue setting, we extend the solutions in Theorem 1.1.3 globally-in-time.

Theorem 1.1.6. *The mKdV2 equation* (1.13) *is globally well-posed in* $\mathcal{F}L^{s,p}(\mathbb{T})$ *for* $s \geq \frac{1}{2}$ *and* $1 \leq p < \infty$.

Remark 1.1.7. (i) For real-valued solutions u, the momentum $P(u) \equiv 0$ which implies that $\mathcal{G}_2[u] \equiv u$. Consequently, the previous results on the complex-valued mKdV2 equation (1.13) in Theorems 1.1.3 and 1.1.6 also apply to the real-valued mKdV1 equation (1.9).

(ii) In [57], Kappeler-Molnar established the global well-posedness of the real-valued defocusing mKdV equation (1.6) with small initial data. Corollary 1.1.6 extends this result (with limited range of s) to the focusing case (with '$-$') and to the large data setting. Furthermore, our solutions satisfy the Duhamel formulation, establishing that the solutions in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq \frac{1}{2}$ constructed in [57] are indeed distributional solutions. The solutions constructed in [57] with initial data in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $0 \leq s < \frac{1}{2}$ and $1 \leq p < \infty$, are not yet known to be distributional solutions.

(iii) The proof of Theorem 1.1.6 is based on applying the a priori bound by Oh-Wang in [91] to iterate the local well-posedness argument. However, the estimate requires a restriction on the regularity $s < 1 - \frac{1}{p}$. When $\frac{1}{2} \leq s < 1 - \frac{1}{p}$, Theorem 1.1.6 follows directly from Theorem 1.1.3 and the a priori bound in [91]. When $s \geq 1 - \frac{1}{p}$, we combine the a priori bound with a persistence of regularity argument.

It remains to answer the question of how to recover solutions of mKdV1 (1.9) from those constructed in Theorem 1.1.6 for mKdV2 (1.13). To that end, for solutions in $C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ for $2 \leq p < \infty$ and $\frac{1}{2} \leq s < 1 - \frac{1}{p}$, we must endow the momentum with a notion of conditional convergence at low regularity. Since the momentum is not a sign definite quantity, we want to exploit the possible cancellation between positive and negative frequencies. This is achieved in the following definition, by considering symmetric truncations of the momentum.

Definition 1.1.8. *Suppose that*

$$P(\mathbf{P}_{\leq N} f) \text{ converges as } N \to \infty.$$

Then, we say that f *has finite momentum and denote the limit by* $P(f)$.

The following proposition validates our notion of finite momentum as follows; consider initial data $u_0 \notin H^{\frac{1}{2}}(\mathbb{T})$ with finite momentum in the sense of Definition 1.1.8. Then, not only does the corresponding solution u of mKdV2 (1.13) have finite momentum but the momentum is also conserved.

Proposition 1.1.9. *Let* (s, p) *satisfy one of the following conditions:* (i) $\frac{1}{2} \leq s < \frac{5}{6}$, $2 \leq p < \frac{6}{5-6s}$; (ii) $s \geq \frac{5}{6}$, $2 \leq p < \infty$. *In addition, let* $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ *with finite momentum in the sense of Definition 1.1.8 and* $u \in C\big([-T, T]; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ *be the corresponding solution of mKdV2* (1.13). *Then, we have that*

$$P\big(\mathbf{P}_{\leq N} u(t)\big) \to P(u_0), \quad N \to \infty.$$

We denote the limit by $P\big(u(t)\big)$ *and have that* $P\big(u(t)\big) \equiv P(u_0)$, *for each* $t \in [-T, T]$.

Remark 1.1.10. (i) Proposition 1.1.9 gives an extended notion of conservation of momentum when (s, p) satisfy $\frac{1}{2} \leq s < \frac{5}{6}$ and $\frac{1}{1-s} \leq p < \frac{6}{5-6s}$, or $\frac{5}{6} \leq s < 1$ and $\frac{1}{1-s} \leq p < \infty$. In the remaining cases, i.e., if $\frac{1}{2} \leq s < 1$ and $1 \leq p < \frac{1}{1-s}$, or $s \geq 1$ and $1 \leq p < \infty$, then $\mathcal{F}L^{s,p}(\mathbb{T}) \subset H^{\frac{1}{2}}(\mathbb{T})$ and the momentum is finite to start with, without the need to apply Proposition 1.1.9. In this regime, our limiting notion of finite momentum agrees with the classical one.

(ii) In order to show Proposition 1.1.9, we follow the argument by Takaoka-Tsutsumi [100] and Nakanishi-Takaoka-Tsutsumi [82] (see Lemma 2.5 in [100] and Lemma 3.1 in [82]) and estimate the difference of momenta at time $t \in [-T, T]$ and at the initial time. In particular, we establish the energy estimate in Proposition 2.5.1, using Strichartz estimates and the normal form approach.

(iii) In [82], the energy estimate holds in $H^\sigma(\mathbb{T})$ for $\sigma > \frac{1}{3}$. Taking into account that the Fourier-Lebesgue spaces $\dot{\mathcal{F}L}^{s,p}(\mathbb{R})$ scale like $\dot{H}^\sigma(\mathbb{R})$ for $\sigma = s + \frac{1}{p} - \frac{1}{2}$, $2 \leq p < \infty$, the condition on (s,p) in Proposition 1.1.9 agrees with the restriction in [82]. We would like to relax the regularity constraints to $s \geq \frac{1}{2}$, to match the local well-posedness of mKdV2 (1.13) (Theorem 1.1.3). In fact, some contributions in the estimate can be controlled for $s \geq \frac{1}{2}$ and $1 \leq p < 4$. In the most difficult cases, the normal form approach assures that the estimate holds outside $H^{\frac{1}{2}}(\mathbb{T})$, but it also introduces additional resonances. Consequently, we cannot use the modulations to help estimate the multiplier, which imposes the condition $\sigma > \frac{1}{3}$. Nevertheless, these heuristics do not imply the failure of the estimate for lower regularity, $s \leq \frac{5}{6} - \frac{1}{p}$ and $\sigma \leq \frac{1}{3}$.

Before stating our last result, we introduce the following notation for the nonlinearity of mKdV2 (1.13). We define the following trilinear operator

$$\mathcal{N}(u_1, u_2, u_3) = \partial_x u_1 \cdot u_2 \cdot u_3 - \partial_x u_1 \Big(\fint_{\mathbb{T}} u_2 \cdot u_3 \, dx \Big) - \Big(i \operatorname{Im} \fint_{\mathbb{T}} \partial_x u_1 \cdot u_2 \, dx \Big) u_3.$$

Thus, the nonlinearity of mKdV2 (1.13) is given by $\mathcal{N}(u, \overline{u}, u)$ and that of mKdV1 (1.9) by $\mathcal{N}(u, \overline{u}, u) + iP(u)u$. As a consequence of the conservation of the momentum at low regularity in Proposition 1.1.9, we have the following result for mKdV1 (1.9).

Theorem 1.1.11. *Let (s,p) satisfy one of the following conditions:* (i) $\frac{1}{2} \leq s < \frac{5}{6}$, $2 \leq p < \frac{6}{5-6s}$; (ii) $s \geq \frac{5}{6}$, $2 \leq p < \infty$, *and $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ with finite momentum, in the sense of Definition 1.1.8. Then, there exist $T > 0$ and a function $u \in C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ with $u|_{t=0} = u_0$ such that u is a distributional solution of the following equation:*

$$\partial_t u + \partial_x^3 u = \pm \Big(\mathcal{N}(u, \overline{u}, u) + iP(u)u \Big), \tag{1.14}$$

where $P(u)$ is interpreted as the limit of $\{P(\mathbf{P}_{\leq N} u)\}_{N \in \mathbb{N}}$ as $N \to \infty$.

We conclude this section by stating some further remarks.

Remark 1.1.12. The solution in Theorem 1.1.11 is the unique limit of the sequence $\{u_N\}_{N\in\mathbb{N}}$ of smooth solutions of mKdV1 (1.9) with truncated initial data $u_N|_{t=0} = \mathbf{P}_{\leq N} u_0$. It is of interest to improve our notion of uniqueness to match that of [58, 57], i.e., to being the unique limit under *any* sequence of smooth approximating solutions. Since our interpretation of the nonlinearity of mKdV1 in (1.14) is tied with the notion of finite momentum in Definition 1.1.8, one could restrict the latter definition to require convergence of $|P(u_{0,n})|$ to a unique limit for any smooth approximating sequence $\{u_{0,n}\}_{n\in\mathbb{N}}$ with $u_{0,n} \to u_0$ in $\mathcal{F}L^{s,p}(\mathbb{T})$.

Remark 1.1.13. Our choice of gauge transform $\mathcal{G}_2$ in (1.12) seems to be the correct one as it subtracts the right amount of infinity. If we for instance consider the following gauge

$$\mathcal{G}[u](t,x) = e^{\pm i\gamma P(u)t},$$

for some parameter $\gamma \in \mathbb{R}$, we obtain the following renormalized equation (mKdV$_\gamma$)

$$\partial_t u + \partial_x^3 u = \pm \big(\mathcal{N}(u, \overline{u}, u) + i(1-\gamma)P(u)u \big). \tag{1.15}$$

For initial data $u_0 \in H^{\frac{1}{2}}(\mathbb{T})$, since the momentum is conserved, the two renormalized equations mKdV2 (1.9) and mKdV$_\gamma$ (1.15) are equivalent. However, for initial data $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ with $\frac{1}{2} \leq s < 1$ and $\frac{1}{1-s} < p < \infty$, we can modify the proof of Theorem 1.1.1 to show non-existence of solutions of mKdV$_\gamma$ (1.15), unless $\gamma = 1$. This supports our choice for the gauge $\mathcal{G}_2$ and the second renormalized mKdV equation (1.13).

Remark 1.1.14. In [11], Bourgain proved the invariance of the Gibbs measure under the flow of the real-valued mKdV equation (1.6),

$$d\mu \text{ ``}=\text{'' } Z^{-1}\exp\bigg(\mp\frac{1}{4}\int_{\mathbb{T}}u^4\,dx-\frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2dx\bigg)\,du, \tag{1.16}$$

by establishing the local well-posedness of mKdV (1.5) in $H^s(\mathbb{T})\cap\mathcal{F}L^{s_1,\infty}(\mathbb{T})$ for some $s<\frac{1}{2}<s_1<1$, which includes the support of (1.16). The invariance of the Gibbs measure on $\mathcal{F}L^{s,p}(\mathbb{T})$ follows from the global well-posedness result in the real-valued setting in Theorem 1.1.6, as $\mathcal{F}L^{s,p}(\mathbb{T})$ with $(s-1)p<-1$ includes the support of μ (1.16). In the complex-valued setting, we can consider the question of invariance of the Gibbs measure for (1.5) and the well-posedness of this equation with randomized initial data. In particular, initial data of the following form

$$u_0(x;\omega)=\sum_{n\neq 0}\frac{g_n(\omega)}{|n|}e^{inx},$$

where $\{g_n\}_{n\in\mathbb{N}}$ is a family of independent complex-valued standard Gaussian random variables, i.e., real and imaginary parts are independent Gaussian random variables, with mean 0 and variance 1. It is known that $u_0\in\bigcap_{s<\frac{1}{2}}H^s(\mathbb{T})$ almost surely. Therefore, it is unclear if the corresponding solutions would satisfy the conservation of the momentum. However, we can actually show that it is finite. We can rewrite the momentum as follows

$$P(u_0(\omega))=\sum_{n\geq 1}\frac{|g_n(\omega)|^2-|g_{-n}(\omega)|^2}{n}=\sum_{n\neq 0}\frac{|g_n(\omega)|^2}{n}.$$

Therefore, using Isserlis' Theorem, we have

$$\mathbb{E}\big[(P(u_0))^2\big]=\sum_{n,m\neq 0}\frac{\mathbb{E}\big[g_n\overline{g_n}g_m\overline{g_m}\big]}{nm}=\sum_{n\neq 0}\frac{2\mathbb{E}\big[|g_n|^2\big]^2}{n^2}\lesssim\sum_{n\geq 1}\frac{1}{n^2}<\infty.$$

Hence, the momentum $P(u_0)$ is finite, almost surely, and we have global well-posedness of mKdV (1.5) for data $u_0(\omega)$ in the support of the Gibbs measure. Consequently, from Bourgain's invariant measure argument, we obtain invariance of the Gibbs measure in (1.16).

Remark 1.1.15. The non-existence result in Theorem 1.1.1 is not particular to the Fourier-Lebesgue setting and can be extended to other spaces outside $H^{\frac{1}{2}}(\mathbb{T})$. In particular, the same result holds for initial data in $H^s(\mathbb{T})$, $\frac{1}{3}<s<\frac{1}{2}$. By adapting the energy method in [82] to the complex-valued setting, we can show that local well-posedness of mKdV2 (1.13) holds in $H^s(\mathbb{T})$ for $\frac{1}{3}<s<\frac{1}{2}$. This is due to the similarity of the nonlinearity of the real-valued mKdV1 equation (1.9) and of the complex-valued mKdV2 equation (1.13). The same estimates hold, with some additional care required to handle the conjugate term and the lack of symmetry in the latter nonlinearity. In addition, for any sequence of smooth functions $\{u_{0,n}\}_{n\in\mathbb{N}}$ with $u_{0,n}\to u_0$ in $H^s(\mathbb{T})$, the corresponding smooth global solutions $\{u_n\}_{n\in\mathbb{N}}$ of mKdV2 (1.13) converge to the solution u of mKdV2 (1.13) in $C\big([-T,T];H^s(\mathbb{T})\big)$, for some $T>0$. If we focus on the initial data $u_0\in H^s(\mathbb{T})\setminus H^{\frac{1}{2}}(\mathbb{T})$ with infinite momentum in the following sense

$$\limsup_{N\to\infty}|P\big(\mathbf{P}_{\leq N}u_0\big)|=\infty,$$

we can show that there exists no distributional solution of the complex-valued mKdV1 equation (1.9) with initial data u_0 and with a good approximation property as in Theorem 1.1.1. This follows the same argument as in the proof of Theorem 1.1.1, using the local well-posedness of mKdV2 (1.13) in $H^s(\mathbb{T})$, $\frac{1}{3}<s<\frac{1}{2}$.

Remark 1.1.16. The question of local well-posedness in the Fourier-Lebesgue spaces has also been pursued for the derivative nonlinear Schrödinger equation (DNLS):

$$i\partial_t u+\partial_x^2 u=\partial_x(|u|^2u),\qquad (t,x)\in\mathbb{R}\times\mathbb{T}. \tag{1.17}$$

This study was initiated by Grünrock-Herr in [44] where they established local well-posedness of DNLS in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq \frac{1}{2}$ and $1 \leq p < 4$, using the Fourier restriction norm method. See also the work of Grürock for (1.17) posed on the real-line [43]. In the periodic setting, an optimal result was later established by Deng-Nahmod-Yue in [34] through a new method inspired by the paracontrolled approach due to Gubinelli-Imkeller-Perkowski [46]. As in the case of mKdV (1.5), the main difficulty in the low regularity well-posedness theory for DNLS is handling the derivative loss from the nonlinearity. In order to overcome this problem, Herr [53] introduced the following gauge transform

$$\mathcal{G}[u](t,x) = e^{-i\mathcal{I}(u)(t,x)}u(t,x),$$

where $\mathcal{I}(u)$ is the mean zero anti-derivative of $|u|^2 - \fint_{\mathbb{T}} |u|^2 dx$. The gauge transform $\mathcal{G}$ removes the following singular contribution in the nonlinearity

$$2\bigg(\operatorname{Im} \fint_{\mathbb{T}} u\partial_x \overline{u}\, dx\bigg)u. \tag{1.18}$$

In $\mathcal{F}L^{\frac{1}{2},p}(\mathbb{T})$, $2 < p < \infty$, the quantity (1.18) is not well-defined, but the gauge transform $\mathcal{G}$ is continuous and invertible, which allows for the recovery of solutions of DNLS from solutions of the gauged equation. In our work, to overcome the derivative loss, we introduced a gauge transform $\mathcal{G}_2$ which removes the following contribution

$$-i\bigg(\operatorname{Im} \fint_{\mathbb{T}} u\partial_x \overline{u}\, dx\bigg)u.$$

However, in our case, the gauge transform $\mathcal{G}_2$ depends explicitly on the momentum, which is not well-defined outside $H^{\frac{1}{2}}(\mathbb{T})$. Thus, we cannot freely convert solutions of mKdV2 (1.13) to solutions of mKdV1 (1.9), a problem which is new to the complex-valued mKdV equation, when compared to DNLS. This additional difficulty, not present for DNLS, lead us to the introduction of a new notion of finite momentum (Definition 1.1.8) and its conservation at low regularity (Proposition 1.1.9). Only then could we prove existence of solutions of mKdV1 (1.9) in Theorem 1.1.11.

Remark 1.1.17. In [65], Kishimoto-Tsutsumi studied the nonlinear Schrödinger equation with third order dispersion and Raman scattering term:

$$\partial_t u = \alpha_1 \partial_x^3 u + i\alpha_2 \partial_x^2 u + i\gamma_1 |u|^2 u + \gamma_2 \partial_x(|u|^2 u) - i\Gamma u \partial_x(|u|^2), \quad (t,x) \in \mathbb{R} \times \mathbb{T},$$

for $\alpha_j, \gamma_j, \Gamma \in \mathbb{R}$, $j = 1, 2$, satisfying $\Gamma > 0$, $\alpha_1 \neq 0$ and $\frac{2\alpha_2}{3\alpha_1} \notin \mathbb{Z}$. Note that for $\alpha_2 = \gamma_1 = 0$, the equation resembles mKdV (1.5), however, this regime is not covered in their analysis. The last term, the Raman scattering term, is responsible for the ill-posedness of this equation and can be rewritten as follows

$$\begin{aligned}\mathcal{F}\big(u\partial_x(|u|^2)\big)(n) = -&\sum_{\substack{n = n_1+n_2+n_3 \\ (n_1+n_2)(n_2+n_3)\neq 0}} i(n_1+n_2)\widehat{u}(n_1)\overline{\widehat{u}}(-n_2)\widehat{u}(n_3) \\ &- in\bigg(\sum_{n_2} |\widehat{u}(n_2)|^2\bigg)\widehat{u}(n) + \bigg(\sum_{n_2} in_2|\widehat{u}(n_2)|^2\bigg)\widehat{u}(n),\end{aligned} \tag{1.19}$$

where $\widehat{u}(n)$ denotes the n-th Fourier coefficient of u. The phase function for this equation is given by

$$\begin{aligned}\phi(\overline{n}_{123}) &= \alpha_1(n_1+n_2+n_3)^3 + \alpha_2(n_1+n_2+n_3)^2 - (\alpha_1 n_1^3 + \alpha_2 n_1^3) \\ &\qquad + (\alpha_1(-n_2)^3 + \alpha_2(n_2)^2) - (\alpha_1 n_3^3 + \alpha_2 n_3^2) \\ &= 3\alpha_1(n_1+n_2)(n_2+n_3)\bigg(n_3 + n_1 + \frac{2\alpha_2}{3\alpha_1}\bigg),\end{aligned}$$

where the last equality holds under the assumption $n = n_1 + n_2 + n_3$. Therefore, under the non-resonant condition $\frac{2\alpha_2}{3\alpha_1} \notin \mathbb{Z}$, it follows that $\phi(\overline{n}_{123}) = 0$ if and only if $(n_1+n_2)(n_2+n_3) = 0$. Consequently, the first term on the right-hand side of (1.19) corresponds to the non-resonant contribution, analogous to $\mathcal{NR}(u, \overline{u}, u)$ in our case (see (1.11)). Delving deeper into the Raman scattering term, note that the last two contributions on the right-hand side of (1.19) can be written on the physical side as $\big(\fint_{\mathbb{T}} |u|^2 dx\big)\partial_x u$ and $iP(u)u$, respectively. In [65], it is this Raman scattering term that is responsible for the ill-posedness. However, since the momentum is not a conserved quantity of the equation, it is not possible to remove it by applying a gauge transform. In our work on mKdV (1.5), the ill-posedness comes only from the momentum term, which introduces higher and higher oscillations through the gauge transform $\mathcal{G}_2$. The momentum plays a role in the ill-posedness of both equations, albeit through different mechanisms.

1.1.4 Methods

The Fourier restriction norm method

In Chapter 2, we prove Theorem 1.1.3 using the Fourier restriction norm method for a restricted range of (s,p): (i) $\frac{1}{2} \leq s < \frac{3}{4}$ and $1 \leq p < \frac{4}{3-4s}$; or (ii) $s \geq \frac{3}{4}$ and $1 \leq p < \infty$. Since for the endpoint $s = \frac{1}{2}$, these conditions impose $1 \leq p < 4$, we will sometimes refer only to the range $s \geq \frac{1}{2}$ and $1 \leq p < 4$ in later discussions.

This method was introduced by Bourgain in [9, 10] to establish the local well-posedness of dispersive PDEs on the torus, in the low regularity setting. The solutions are constructed through a contraction mapping argument on a suitably chosen Banach space. Let us focus on mKdV1 (1.9). We look for solutions of the equivalent integral equation, the Duhamel formulation:

$$u(t) = S(t)u_0 \pm \int_0^t S(t-t')\mathcal{N}(u,\overline{u},u)(t')\,dt' =: S(t)u_0 \pm D\mathcal{N}(u,\overline{u},u)(t), \tag{1.20}$$

where $S(t)$ denotes the linear propagator and D the Duhamel operator. In [9, 10], Bourgain proposed the use of the so-called Fourier restriction spaces or $X^{s,b}(\mathbb{R}\times\mathbb{T})$-spaces defined through the following norm

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} = \big\|\langle n\rangle^s \langle \tau - n^3\rangle^b \mathcal{F}_{t,x}(u)(\tau,n)\big\|_{\ell^2_n(\mathbb{Z})L^2_\tau(\mathbb{R})}. \tag{1.21}$$

These spaces appeared earlier in the work of Rauch-Reed [93] and Beals [4] on the wave equation, but they were first used in the context of local well-posedness by Bourgain in [9, 10] for the nonlinear Schrödinger equation and the gKdV equations, and by Klainerman-Machedon [66] for the wave equation. For a survey of these spaces and applications see [39, 101, 102]. In [42], Grünrock introduced analogous spaces adapted to the Fourier-Lebesgue setting and to the Euclidean setting. Here, we focus on the corresponding spaces on $\mathbb{T}$ introduced by Grünrock-Herr in [44]: the $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})$-spaces defined through the following norm

$$\|u\|_{X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})} = \big\|\langle n\rangle^s \langle \tau - n^3\rangle \mathcal{F}_{t,x}(u)(\tau,n)\big\|_{\ell^p_n(\mathbb{Z})L^q_\tau(\mathbb{R})}, \tag{1.22}$$

for $1 \leq p,q \leq \infty$ and where the iterated norm is understood as $\|\cdot\|_{\ell^p_n(\mathbb{Z})L^q_\tau(\mathbb{R})} = \|\|\cdot\|_{L^q_\tau(\mathbb{R})}\|_{\ell^p_n(\mathbb{Z})}$. Note that if $p = q = 2$ we have $X^{s,b}_{2,2}(\mathbb{R}\times\mathbb{T}) = X^{s,b}(\mathbb{R}\times\mathbb{T})$ defined in (1.21).

These spaces, defined through (1.21) and (1.22), are well-adapted to the dispersive nature of the equation, in particular, to the linear equation

$$\partial_t u + \partial_x^3 u = 0,$$

known as the Airy equation. Note that if we take a space-time Fourier transform of the linear solution, we see that $\mathcal{F}_{t,x}\big(S(t)u_0\big)(\tau,n)$ is supported on the curve $\tau = n^3$. The twisted temporal weight of the norm in (1.21) is directly related to this curve and heuristically, for $b > 0$, it *penalizes* functions which are far from being a linear solution. Consequently, the $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})$-spaces are particularly suited to look for solutions of (1.20) as nonlinear perturbations of their linear counterparts.

Since the linear solution can be easily estimated in $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})$-spaces, after a suitable time localization (see Lemma 2.1.1), constructing unique local-in-time solutions of mKdV2 reduces to establishing a nonlinear estimate of the form

$$\big\|D\mathcal{N}(u_1,u_2,u_3)\big\|_{X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})}. \tag{1.23}$$

The main difficulty resides in controlling the derivative in the nonlinearity. Since the Duhamel operator D has smoothing in time but not in space, we want to exploit the multilinear dispersion by using the modulations, i.e., the weights $\langle \tau - n^3\rangle$ in the norm (1.22).

The existence and uniqueness of solutions of mKdV2 (1.13), as well as the local Lipschitz continuity of the solution map, follow easily from a contraction mapping argument once we establish (1.23). The restriction on (s,p) in Chapter 2 is imposed by this nonlinear estimate.

The "paracontrolled" approach in [34]

In Chapter 3, we prove Theorem 1.1.3 for $s \geq \frac{1}{2}$ and $4 \leq p < \infty$ using the method introduced by Deng-Nahmod-Yue in [34] in the context of DNLS (1.17). Due to the failure of the main nonlinear estimate (1.23) for DNLS when $s = \frac{1}{2}$ and $p > 4$ [44], Deng-Nahmod-Yue instead proposed looking for solutions centered around a suitably chosen object. This structure was chosen in order to avoid the bad frequency interaction for which the analogue of (1.23) fails. For mKdV2 (1.13), although we do not know if the nonlinear estimate (1.23) fails, we are unable to prove it for $s = \frac{1}{2}$ and $4 \leq p < \infty$, leading us to pursue the method in [34].

Motivated by the paracontrolled approach introduced by Gubinelli-Imkeller-Perkowski [46], we construct solutions u centered around a smoother-in-time function w. Instead of solving the Duhamel formulation, we will solve a system of equations

$$\begin{cases} u = w + F(u,w), \\ w = S(t)u_0 \pm D\mathcal{N}(u,\overline{u},u) - F(u,w), \end{cases} \tag{1.24}$$

where $F(u,w)$ is a nonlinear functional to be determined. Centering solutions around a suitably chosen function was seen in the context of probabilistic PDEs (with random initial data or stochastic forcing), for example, in the works of Bourgain [12], Da Prato-Debussche [31], and Gubinelli-Imkeller-Perkowski [46]. In general, the center w is an explicitly known random object that introduces smoothing in space in the remainder pieces. The lack of randomness in our setting forces us to consider a moving center and solve the system (1.24). In particular, the first equation in (1.24) imposes structure to $u = u[w]$ parametrized by w, while the second finds the correct w for which u solves the Duhamel formulation (1.20).

The main difficulty in this method is choosing the correct structure for u, or equivalently, the correct nonlinear functional $F(u,w)$. This choice, which is not prescribed by the method, must allow us to solve the first equation for u while avoiding the bad frequency regions for the nonlinear estimate (1.23). There are three main points in establishing and solving the system (1.24): (i) choosing the frequency regions of the nonlinear terms in $F(u,w)$; (ii) modifying the Duhamel operator to induce smoothing in space; (iii) using the second iteration of the Duhamel formulation to solve the equation for w.

We first consider (i). In certain regions of the frequency space, the nonlinear estimate (1.23) holds for any fixed $2 \leq p < \infty$ and some $b = 1-$ and $q = \infty-$ (see Remark 3.5.4 for more details). These frequency regions will be included in the equation for w in (1.24), which we hope to solve for $w \in X^{s,1-}_{p,\infty-}(\mathbb{R}\times\mathbb{T}) \subset X^{s,\frac{1}{2}}_{p,2-}(\mathbb{R}\times\mathbb{T})$ and $u \in X^{s,\frac{1}{2}}_{p,2-}(\mathbb{R}\times\mathbb{T}) \subset C\big(\mathbb{R};\mathcal{F}L^{s,p}(\mathbb{T})\big)$. For the remaining frequency regions, we cannot show the trilinear estimate (1.23) in $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T}) \subset C\big(\mathbb{R};\mathcal{F}L^{s,p}(\mathbb{T})\big)$, regardless of the choice of b and q. These contributions should then appear in $F(u,w)$, in order to be estimated in the weaker $X^{s,\frac{1}{2}}_{p,2-}$-norm. In addition, we require the terms in $F(u,w)$ to have a smoother w term associated with the derivative and the largest frequency. Consequently, $F(u,w)$ includes terms that essentially look like the following

$$\sum_N \mathbf{P}_N\partial_x w \cdot \mathbf{P}_{\ll N}\overline{u} \cdot \mathbf{P}_{\ll N}u, \qquad \sum_N \mathbf{P}_N\partial_x w \cdot (\mathbf{P}_N\overline{w} \cdot \mathbf{P}_{\ll N}u + \mathbf{P}_N w \cdot \mathbf{P}_{\ll N}\overline{u}), \tag{1.25}$$

where $\mathbf{P}_N$ and $\mathbf{P}_{\ll N}$ denote the Dirichlet projections onto the spatial frequencies $\{|n| \sim N\}$ and $\{|n| \ll N\}$, respectively. These terms can roughly be seen as 'paracontrolled' by w (see [46] for details on paracontrolled distributions).

Unfortunately, the assumptions imposed on $F(u, w)$ and the terms in (1.25) are not yet enough to show the estimate (1.23) for any $p \geq 4$. This leads us to (ii) and to the introduction of a modified Duhamel operator which not only has smoothing in time but also in space. The modification is introduced through a time convolution with a smooth function η parameterized by the resonance relation $\phi(\overline{n}_{123})$:

$$\int_0^t S(t-t')\eta\big(\phi(\overline{n}_{123})(t-t')\big)F(t')\,dt', \tag{1.26}$$

where $\phi(\overline{n}_{123}) = n^3 - n_1^3 - n_2^3 - n_3^3$. We then choose $F(u, w)$ by applying the modified Duhamel operator in (1.26) to the terms in (1.25). In the frequency support of $F(u, w)$, we have that $|\phi(\overline{n}_{123})| \gtrsim \max(|n_1|, |n_2|, |n_3|)^2$. Therefore, the convolution with η in (1.26) introduces negative powers of $|\phi(\overline{n}_{123})|$ and consequently smoothing in space, at the cost of reduced smoothing in time (see Section 3.2 for more details). This smoothing effect allows us to solve the equation for u through a fixed point argument, for each fixed $w \in X^{s,1-}_{p,\infty-}(\mathbb{R}\times\mathbb{T})$. Consequently, we obtain a function $u = u[w]$ parameterized by w that is not yet a solution of the mKdV2 equation (1.13). This will only follow after we have found the correct center w.

Lastly, we address (iii). To solve the equation for w, we use a 'partial' iteration of the Duhamel formulation. For the terms that cannot be estimated directly in the $X^{s,1-}_{p,\infty-}(\mathbb{R}\times\mathbb{T})$-norm, we replace $u = u[w]$ by its equation $w + F(u, w)$ first in the entries associated with the derivative and then the largest frequencies. This strategy induces smoothing, by introducing terms that depend on the modified Duhamel operator (1.26) and more w terms, at the cost of increasing the multilinearity of the terms being estimated. This strategy resembles the second iteration method used by Bourgain [13], Oh [87], and Richards [94], for example. In particular, this leads to new cubic, quintic, and septic terms that we can estimate in the stronger norm (see Sections 3.3 and 3.5).

In summary, the choice of $F(u, w)$ requires a delicate balance between being able to solve the first equation for u, but also inducing sufficient spatial smoothing when using second iteration to solve the equation for w. This choice allows us to show the relevant estimates for any $2 \leq p < \infty$.

1.2 The gKdV equations

We now consider the Cauchy problem for the generalized Korteweg-de Vries equation (gKdV):

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm\partial_x(u^k), \\ u|_{t=0} = u_0, \end{cases} \qquad (t, x) \in \mathbb{R}\times\mathbb{T}, \tag{1.27}$$

where $k \geq 2$ is an integer and $u : \mathbb{R}\times\mathbb{T} \to \mathbb{R}$. When $k = 2$ and $k = 3$, (1.27) corresponds to the well-known KdV and mKdV equations, respectively. These two equations are known to be completely integrable, therefore possessing infinitely many conservation laws, which is no longer true for (1.27) with $k \geq 4$. Our goal is to construct global-in-time solutions for gKdV (1.27) at low regularity, where there are no suitable conservation laws. We instead pursue a probabilistic approach introduced by Bourgain in [11], which exploits the Hamiltonian structure of the equation and the associated Gibbs measure.

The gKdV equation (1.27) can be reformulated as a Hamiltonian system

$$\partial_t u = \partial_x \frac{\delta H}{\delta u},$$

where $\frac{\delta H}{\delta u}$ denotes the Fréchet derivative and the Hamiltonian is given by

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} (\partial_x u)^2\,dx \pm \frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}\,dx.$$

In particular, $H(u)$ is conserved under the dynamics of (1.27). Note that the mean $\int_{\mathbb{T}} u\,dx$ and the mass $\int_{\mathbb{T}} u^2\,dx$ are also conserved. Due to the conservation of the mean, we will restrict our analysis to mean zero initial data.

In view of the Hamiltonian structure of gKdV (1.27), we expect the Gibbs measure μ formally defined by

$$d\mu = Z^{-1}e^{-H(u)}\,du = Z^{-1}e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx}e^{-\frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2 dx}du, \tag{1.28}$$

to be invariant under the dynamics of gKdV (1.27). Note that in the above expression, Z is a normalizing constant. By the invariance of the Gibbs measure, we mean that

$$\mu\big(\Psi(-t)A\big) = \mu(A), \tag{1.29}$$

for all $t \in \mathbb{R}$ and measurable $A \subset L^2(\mathbb{T})$, where $\Psi(t) : u_0 \mapsto u(t)$ denotes the data-to-solution map of gKdV (1.27). In [11], Bourgain introduced the idea of exploiting the invariance of the Gibbs measure μ as a substitute for a conservation law. This globalization procedure is now known as *Bourgain's invariant measure argument*. Another interesting consequence of the invariance property is that it informs of the typical behavior of solutions, such as recurrence properties, as opposed to that of individual trajectories.

In Chapter 4, we establish the invariance of the Gibbs measure μ (1.28) (under suitable normalization) for the gKdV equations with $k \geq 4$, by following Bourgain's invariant measure argument. The main difficulty resides in establishing local well-posedness of gKdV (1.27) in the support of the Gibbs measure μ, which is not readily available in the literature. In fact, due to the mild ill-posedness of (1.27) in the L^2-based Sobolev spaces which include the support of μ [29], we instead establish local well-posedness in suitable Fourier-Lebesgue spaces. This completes the program initiated by Bourgain in [11] on the invariance of the Gibbs measure μ in (1.28) (under suitable normalization) for the gKdV equations.

1.2.1 Literature review

The construction of Gibbs measures for Hamiltonian PDEs was initiated by Lebowitz-Rose-Speer [71] in the context of the nonlinear Schrödinger equation; see also the work of Friedlander for the wave equation [37]. Since then, this construction has been successfully pursued for other equations; see [109, 11, 110, 12, 13, 104, 20, 105, 21, 85, 86, 106, 103, 80, 18, 15, 6, 33, 94, 88, 90] and references therein. The expression in (1.28) is only formal, but it can be made rigorous by interpreting the Gibbs measure μ as a probability measure which is absolutely continuous with respect to the Gaussian measure ρ

$$d\rho = Z_0^{-1}e^{-\frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2 dx}du, \tag{1.30}$$

with Z_0 a normalizing constant. The measure ρ can be seen as the induced probability measure under the map

$$\omega \mapsto u^\omega(x) = \sum_{n\in\mathbb{Z}^*}\frac{g_n(\omega)}{|n|}e^{inx}, \tag{1.31}$$

where $\{g_n\}_{n\in\mathbb{Z}^*}$, $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, is a sequence of complex-valued independent Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying $g_{-n} = \overline{g_n}$. The series in (1.31) is essentially the Fourier-Wiener series for the Wiener measure ρ in (1.30); see [5] for further details. Note that u defined in (1.31) lies in $\bigcap_{s<\frac{1}{2}} H^s(\mathbb{T})$ and in $\bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T})$ almost surely. Consequently, the support of ρ and of μ (when well-defined) is included in these sets.

Before discussing its invariance, we first need to construct the Gibbs measure μ as a well-defined probability measure on $\mathcal{F}L^{s,p}(\mathbb{T})$. In particular, we need the weight $e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx}$ to be integrable with respect to the base Gaussian measure ρ in (1.30). In the defocusing case, with the '+' sign and when $k \geq 3$ is odd in (1.27), it follows from the Sobolev embedding that μ is a well-defined probability measure on $\mathcal{F}L^{s,p}(\mathbb{T})$ for $1 \leq p \leq \infty$ and $s < 1 - \frac{1}{p}$. However, in the non-defocusing case, with the '−' sign or when $k \geq 2$ is even in (1.27), the quantity $e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx}$ is unbounded in $\mathcal{F}L^{s,p}(\mathbb{T})$ and the measure (1.28) is not normalizable.

To bypass this difficulty, Lebowitz-Rose-Speer [71] introduced a mass cutoff and proposed to study the following Gibbs measure

$$d\mu = Z^{-1}\mathbb{1}_{\{\|u\|_{L^2}\leq R\}}e^{-H(u)}du. \tag{1.32}$$

Lebowitz-Rose-Speer [71] and Bourgain [11] showed that the measure μ in (1.32) is normalizable for $2 \leq k \leq 5$ and an appropriate choice of $R > 0$; see Theorem 1.2.1. The normalizability at the optimal threshold, for $k = 5$, was recently shown by Oh-Sosoe-Tolomeo [89]. The following theorem summarizes their findings. See Section 4.3 for further details on the construction of the non-defocusing Gibbs measure.

Theorem 1.2.1 ([71, 11, 89]). *Let $k \geq 2$, $R > 0$, and define $F(u)$ by*

$$F(u) = e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx}\mathbb{1}_{\{\|u\|_{L^2}<R\}}, \tag{1.33}$$

where '$\mp$' above corresponds to '$\pm$' in the equation (1.27). Then, for $1 \leq q < \infty$, we have that $F(u) \in L^q(d\rho)$ if one of the following assumptions hold:

(a) $2 \leq k \leq 4$ *and any finite $R > 0$;*

(b) $k = 5$ *and $0 < R < \|Q\|_{L^2(\mathbb{R})}$, where Q is the (unique) optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ with $\|Q\|^6_{L^6(\mathbb{R})} = 3\|Q'\|^2_{L^2(\mathbb{R})}$. If $R = \|Q\|_{L^2(\mathbb{R})}$, then we further impose $q = 1$.*

Lastly, in order to discuss the invariance of the Gibbs measure μ, we must first construct a (globally-in-time) well-defined flow for gKdV (1.27) on the support of μ. Before proceeding further, we recall some known well-posedness results of (1.27). In [10], Bourgain introduced the Fourier restriction norm method and proved local well-posedness of KdV in $L^2(\mathbb{T})$ which contains the support of the measure μ. Global well-posedness followed immediately due to the conservation of the mass. Following the same method for mKdV, Bourgain [11] established its local well-posedness in $H^{s_1}(\mathbb{T}) \cap \mathcal{F}L^{s_2,\infty}(\mathbb{T})$ for some $s_1 < \frac{1}{2} < s_2 < 1$ which also include the support of the measure μ. Here, $\mathcal{F}L^{s,p}(\mathbb{T})$ denotes the Fourier-Lebesgue space defined in (1.3). Unfortunately, the conservation laws of mKdV were not sufficient to globalize solutions. Instead, Bourgain used a probabilistic argument to construct global-in-time solutions of mKdV. In the seminal work [11], he exploited the invariance of the finite-dimensional Gibbs measures corresponding to the truncated dynamics to globalize solutions of mKdV. Moreover, he rigorously established the invariance of the Gibbs measure μ for KdV and mKdV. Here, with the solution map $\Psi(t)$ of (1.27) given at least almost surely with respect to μ, invariance of μ is understood as

$$\mu\big(\Psi(-t)A\big) = \mu(A), \tag{1.34}$$

for any measurable set $A \subset L^2(\mathbb{T})$ and $t \in \mathbb{R}$. This approach is known as Bourgain's invariant measure argument. The main breakthrough in [11] was the globalization argument, in particular, using the formal invariance of the Gibbs measure μ as a substitute for a conservation law.

Regarding (1.27) with $k \geq 4$, Bourgain [10] proved small data global well-posedness in $H^s(\mathbb{T})$ for $s > \frac{3}{2}$ and local existence of solutions (without uniqueness) for $s \geq 1$. These results were extended to global well-posedness in $H^s(\mathbb{T})$ for $s \geq 1$ by Staffilani in [98]. In particular, they studied the following gauged gKdV equation ($\mathcal{G}$-gKdV):

$$\partial_t u + \partial_x^3 u = \pm\partial_x\big(u^k - k\mathbf{P}_0(u^{k-1})u\big), \tag{1.35}$$

where $\mathbf{P}_0$ denotes the mean $\mathbf{P}_0(f) = \fint_{\mathbb{T}} f\,dx$. The two equations (1.35) and (1.27) are equivalent in the following sense: u is a solution of (1.27) if and only if $v = \mathcal{G}[u]$ is a solution of (1.35), where the gauge transform[2] $\mathcal{G} = \mathcal{G}_{0,t}$ is given by

$$\mathcal{G}_{0,t}[u](t,x) = u\Big(t, x \mp k\int_0^t \mathbf{P}_0\big(u^{k-1}(t')\big)\,dt'\Big). \tag{1.36}$$

In [29], Colliander-Keel-Staffilani-Takaoka-Tao established local well-posedness of (1.35) in $H^s(\mathbb{T})$ for $s \geq \frac{1}{2}$ and used the I-method to construct global solutions for $s > \frac{5}{6}$ and $k = 4$ (see also [56, 3]). In addition, they showed that (1.27) and (1.35) are analytically ill-posed in $H^s(\mathbb{T})$ for $s < \frac{1}{2}$, which contains the support of μ. In fact, the data-to-solution map fails to be C^k-continuous [13, 26, 29]. As a consequence, one cannot use a contraction mapping argument to extend the local well-posedness of (1.27) below $H^{\frac{1}{2}}(\mathbb{T})$ in the L^2-based Sobolev scale. To bypass this difficulty, when $k = 4$, Richards [94] showed almost sure local well-posedness of (1.27) with the random initial data in (1.31), and proved invariance of the Gibbs measure under the flow of (1.35). However, this approach is not suitable to treat the cases $k \geq 5$ in a unified manner. Regarding the Gibbs measure for gKdV with $k \geq 5$, Oh-Richards-Thomann [88] constructed almost sure global dynamics of gKdV (1.27) (without uniqueness) and proved the following weaker notion of invariance of μ: for every $t \in \mathbb{R}$, the law of the random function $u(t)$ which solves gKdV is given by the Gibbs measure μ. The lack of uniqueness of solutions in [88] is due to the use of a compactness argument; see [1, 19]. Note that the notion of invariance in [88] is weaker than (1.34) in the sense that the dynamics constructed there do not satisfy the group property in (1.38).

1.2.2 Main results

Our main goal is to apply Bourgain's invariant measure argument to construct global-in-time dynamics of (1.27) and establish the invariance of the Gibbs measure for any $k \geq 4$. The main difficulty lies in proving local well-posedness in the support of the Gibbs measure. The lack of analyticity of the data-to-solution map in $\bigcap_{s<\frac{1}{2}} H^s(\mathbb{T})$ leads us to pursue the question of well-posedness in the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$ in (1.3). In particular, we establish deterministic local well-posedness of $\mathcal{G}$-gKdV (1.35) in the Fourier-Lebesgue spaces containing the support of the Gibbs measure μ, i.e., $\mathcal{F}L^{s,p}(\mathbb{T})$ with $(s-1)p < -1$.

Theorem 1.2.2. *For an integer $k \geq 4$ and $2 < p < \infty$, there exists $\frac{1}{2} < s_*(p) < 1 - \frac{1}{p}$ such that $\mathcal{G}$-gKdV (1.35) is locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{T})$ for any $s > s_*(p)$. Moreover, by inverting the gauge transform, we also obtain local well-posedness of the gKdV equation (1.27) in $\mathcal{F}L^{s,p}(\mathbb{T})$.*

Remark 1.2.3. We start by clarifying our notion of local well-posedness of $\mathcal{G}$-gKdV (1.35) in $\mathcal{F}L^{s,p}(\mathbb{T})$; for any $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, there exists $T = T(\|u_0\|_{\mathcal{F}L^{s,p}}) > 0$ and a unique solution u in $X^{s,\frac{1}{2}}_{p,2}(T) \cap X^{s,0}_{p,1}(T) \hookrightarrow C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ (see Definition 1.3.2) which satisfies the Duhamel formulation of (1.35):

$$u(t) = S(t)u_0 \pm \int_0^t S(t-t')\partial_x\big(u^k - k\mathbf{P}_0(u^{k-1})u\big)(t')\,dt', \quad t \in [-T,T],$$

where $S(t)$ denotes the linear propagator. Moreover, the data-to-solution map Φ is (locally Lipschitz) continuous. Note that $\mathcal{G}_{0,t}$ is a bijection on $C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ with inverse given by

$$\mathcal{G}_{0,t}^{-1}(u)(t,x) = u\Big(t, x \pm k\int_0^t \mathbf{P}_0\big(u^{k-1}(t')\big)\,dt'\Big).$$

Consequently, Theorem 1.2.2 asserts the following notion of local well-posedness for the original gKdV equation (1.27); for any $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, there exist $T = T(\|u_0\|_{\mathcal{F}L^{s,p}}) > 0$ and a unique solution $u \in \mathcal{G}_{0,t}^{-1}\big(X^{s,\frac{1}{2}}_{p,2}(T) \cap X^{s,0}_{p,1}(T)\big) \subset C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ which satisfies the Duhamel formulation (1.4) of (1.27). The same conditional uniqueness applies for the global-in-time

[2] Typically, a gauge transform is a transform on the phase space, i.e., acting on spatial functions. However, $\mathcal{G}$ defined in (1.36) is an action on space-time functions. We follow the literature [29, 94, 88] and abuse notation by referring to it as a gauge transform. We also refer to the transformed equation (1.35) as the gauged equation. See Remark 1.2.7 (ii) for further discussion.

results in Theorems 1.2.4 and 1.2.5. The data-to-solution map Ψ of gKdV (1.27) can be defined as $\Psi(t) = \mathcal{R}_t \circ \mathcal{G}_{0,t}^{-1} \circ \Phi$, where $\mathcal{R}_t$ denotes the evaluation map at time t. The map $\Psi(t)$ is defined on a neighborhood of the origin in $\mathcal{F}L^{s,p}(\mathbb{T})$ and it is continuous, but not Lipschitz or uniformly continuous in the $\mathcal{F}L^{s,p}$-topology due to the properties of $\mathcal{G}_{0,t}^{-1}$. Moreover, it satisfies the group property $\Psi(t_1 + t_2) = \Psi(t_2)\Psi(t_1)$ for any $t_1, t_2 \in \mathbb{R}$. See Section 4.4 for more details on this map.

We prove Theorem 1.2.2 by applying the Fourier restriction norm method adapted to the Fourier-Lebesgue setting (see Definition 1.3.1). The method reduces to establishing a fundamental nonlinear estimate, where the main difficulty lies in controlling the derivative in the nonlinearity. To overcome this derivative loss, we want to exploit the multilinear dispersion by analyzing the phase function

$$\phi_k(n, n_1, \ldots, n_k) = n^3 - n_1^3 - \ldots - n_k^3$$

on the hyperplane $n = n_1 + \ldots + n_k$. For KdV ($k = 2$) and mKdV ($k = 3$), the corresponding phase functions ϕ_2 and ϕ_3 are known to factorize, providing an explicit characterization of the *resonant* and *nearly-resonant sets*, where

$$\phi_k(n, n_1, \ldots, n_k) = 0 \qquad \text{and} \qquad 0 < |\phi_k(n, n_1, \ldots, n_k)| \ll \max(|n_1|, \ldots, |n_k|),$$

respectively. Unfortunately, such factorizations are no longer available for ϕ_k when $k \geq 4$, complicating the study of the resonant and nearly-resonant frequency regions. In fact, the failure of analyticity of the solution map in $H^s(\mathbb{T})$ with $s < \frac{1}{2}$ in [29] is due to the failure of the corresponding nonlinear estimate in these regions where $|\phi_k(n, n_1, \ldots, n_k)| \ll \max(|n_1|, \ldots, |n_k|)$.

To overcome this difficulty, our approach is inspired by the "bilinear + multilinear" strategy in the work of Colliander-Keel-Staffilani-Takaoka-Tao [28, 29]. Instead of starting by showing a bilinear estimate, we first pursue a more careful description of the frequency space by comparing $\phi_k(n, n_1, \ldots, n_k)$ with the phase functions $\phi_2(n, n_1, n - n_1)$ and $\phi_3(n, n_1, n_2, n - n_1 - n_2)$ associated with KdV and mKdV, respectively. Moreover, we further exploit the multilinear dispersion in the form of bilinear and trilinear Strichartz estimates, which are the Fourier-Lebesgue analogues of the periodic L^4 and L^6-Strichartz estimates, respectively. The idea of multilinearizing periodic Strichartz estimates has also been used in L^2-based Sobolev spaces; see [29] for gKdV, [54] for NLS equation on $\mathbb{T}^3$, and [44] for DNLS on $\mathbb{T}$, for example.

Now, let us turn our attention to the global aspect of the well-posedness. Following the strategy in [11], we start by proving the invariance of the Gibbs measures associated with the following truncated dynamics

$$\begin{cases} \partial_t u_N + \partial_x^3 u_N = \pm \mathbf{P}_{\leq N} \partial_x \big((\mathbf{P}_{\leq N} u_N)^k - k \mathbf{P}_0 \big((\mathbf{P}_{\leq N} u_N)^{k-1} \big) \mathbf{P}_{\leq N} u_N \big), \\ u_N|_{t=0} = u_0, \end{cases} \tag{1.37}$$

where $\mathbf{P}_{\leq N}$ denotes the Dirichlet projection onto frequencies $\{|n| \leq N\}$. Unfortunately, the Hamiltonian structure of (1.37) is disrupted by the gauge transform. Therefore, the invariance of the corresponding Gibbs measures does not follow immediately from Liouville's Theorem. A similar difficulty was found by Nahmod, Oh, Rey-Bellet, and Staffilani when studying the Gibbs measure for DNLS in [80]. See Remark 1.2.7 for additional details. As a consequence, we must establish the conservation of the mass and of the Hamiltonian for (1.37) as well as the invariance of the finite dimensional Lebesgue measures on $\mathbf{P}_{\leq N} L^2(\mathbb{T})$ under the flow of (1.37). Then, using the invariance of the finite-dimensional Gibbs measures for (1.37), we extend solutions of (1.35) globally-in-time and also establish the invariance of μ under its flow.

Theorem 1.2.4. *Assume one of the following conditions:*

(a) *defocusing case: '+' sign in* (1.27) *and k odd;*

(b) *non-defocusing case: '+' sign in* (1.27) *and $k = 4$, or '−' sign in* (1.27) *and $3 \leq k \leq 5$, with mass $0 < R \leq \|Q\|_{L^2(\mathbb{R})}$ if $k = 5$. Here, Q denotes the (unique) optimizer of the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ with $\|Q\|_{L^6(\mathbb{R})}^6 = 3\|Q'\|_{L^2(\mathbb{R})}^2$.*

Then, the $\mathcal{G}$-gKdV equation (1.35) is almost surely globally well-posed with respect to the Gibbs measure. In particular, for $2 < p < \infty$, there exists a μ-measurable set $\Sigma \subset \bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T})$ of full μ-measure such that for every $u_0 \in \Sigma$, the $\mathcal{G}$-gKdV equation (1.35) with initial data u_0 has a uniquely defined global-in-time solution $u \in \bigcap_{s<1-\frac{1}{p}} C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$. The obtained solution map $\Phi(t) : u_0 \mapsto u(t)$ of $\mathcal{G}$-gKdV defined on Σ is μ-measurable and satisfies the flow property

$$\Phi(t)\Sigma = \Sigma \quad \text{for all } t \in \mathbb{R}, \quad \Phi(t_1 + t_2) = \Phi(t_2)\Phi(t_1) \quad \text{for all } t_1, t_2 \in \mathbb{R}. \tag{1.38}$$

Moreover, the Gibbs measure μ is invariant under the flow of $\mathcal{G}$-gKdV (1.35) in the sense that $\mu\big(\Phi(-t)A\big) = \mu(A)$ for any μ-measurable set $A \subset \Sigma$ and $t \in \mathbb{R}$.

By inverting the gauge transform and exploiting the invariance of the Gibbs measure under spatial translations, we obtain our main result.

Theorem 1.2.5. *Under the assumptions of Theorem 1.2.4, the (original) gKdV equation (1.27) is almost surely globally well-posed with respect to the Gibbs measure. In particular, for every u_0 in the set Σ of full μ-measure given in Theorem 1.2.4, the gKdV equation (1.27) with initial data u_0 has a uniquely defined global-in-time solution $u \in \bigcap_{s<1-\frac{1}{p}} C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$. Moreover, the obtained solution map $\Psi(t)$ has the same flow property as (1.38), and the Gibbs measure μ is invariant under $\Psi(t)$ in the sense that (1.34) holds for any μ-measurable set $A \subset \Sigma$ and $t \in \mathbb{R}$.*

We complete this section with some further remarks.

Remark 1.2.6. (i) Theorem 1.2.5 extends the result of Richards [94] for $k = 4$ by showing invariance of the Gibbs measure under the original dynamics (1.27). Moreover, our work establishes the first result on the invariance of the Gibbs measure μ in the sense of (1.34) for arbitrarily large values of $k \geq 5$, in the defocusing case.

(ii) A weaker notion of invariance of μ for $k \geq 5$ was established by Oh-Richards-Thomann in [88]. They constructed almost sure global dynamics for gKdV (1.27), without uniqueness, and established invariance in the following sense: for any $t \in \mathbb{R}$, the law $\mathcal{L}\big(u(t)\big)$ of the random variable $u(t)$ which solves (1.27) is given by the Gibbs measure μ. They used the compactness argument in [1, 19], exploiting the invariance of the truncated measures to construct a tight sequence of space-time measures. Although their result can be easily extended to the Fourier-Lebesgue spaces in Theorem 1.2.4, we do not know if our solutions coincide with those in [88]. Due to the lack of uniqueness of solutions in [88] and the conditional uniqueness of our result, we cannot directly compare these solutions.

Remark 1.2.7. (i) In [80], Nahmod, Oh, Rey-Bellet, and Staffilani studied DNLS on the one-dimensional torus. In particular, they constructed a weighted Wiener measure, invariant under the gauged dynamics, and established almost sure global well-posedness of DNLS in the support of said measure. Unlike for gKdV (1.27), local well-posedness in the support of the measure was already available in [44]. Consequently, the main difficulty arose in the globalization process. The energy associated to the gauged dynamics was no longer conserved for truncated solutions, which required an approach reminiscent of the I-method to instead establish almost invariance of the truncated measures. In our case, the main difficulty lies in establishing the local well-posedness of $\mathcal{G}$-gKdV (1.35) in the Fourier-Lebesgue spaces that include the support of the measure, which was readily available for DNLS. Similarly to [80], we also have to prove the invariance of the finite-dimensional Lebesgue measure with respect to the truncated dynamics in (1.37). However, unlike in [80], the Hamiltonian is still conserved and we can easily show invariance of the Gibbs measures associated to (1.37).

(ii) One additional difficulty in establishing invariance of the Gibbs measure μ under the flow of (1.27) was due to the gauge transform. The map $\mathcal{G}_{0,t}$ for $k \geq 4$ is a map on space-time functions. This is a sharp contrast with DNLS, whose more involved gauge transform is well defined as a map on $\mathcal{F}L^{s,p}(\mathbb{T})$, allowing the authors in [80] to consider the push-forward of the measure μ by the gauge transform. This topic was further explored for DNLS in a subsequent work [81]. Here, we bypass the difficulty associated with the gauge transform by exploiting the invariance of the Gibbs measure under spatial translations.

1.3 Notations

Before proceeding to the proof of our main results, we introduce some relevant notation. Let $A \lesssim B$ denote an estimate of the form $A \leq CB$ for some constant $C > 0$. In addition, $A \sim B$ denotes $A \lesssim B$ and $B \lesssim A$, while $A \ll B$ denotes $A \leq \varepsilon B$ for some small constant $0 < \varepsilon < 1$. We sometimes use $\lesssim_\alpha$, $\sim_\alpha$, $\ll_\alpha$ to indicate that the implicit constants depend on a parameter α. The notations $a+$ and $a-$ represent $a + \varepsilon$ and $a - \varepsilon$ for arbitrarily small $\varepsilon > 0$, respectively.

Our conventions for the Fourier transform are as follows. The Fourier transform of $u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ with respect to the space variable is given by

$$\mathcal{F}_x u(t, n) = \frac{1}{2\pi} \int_{\mathbb{T}} u(t, x) e^{-inx} \, dx.$$

The Fourier transform of u with respect to the time variable is given by

$$\mathcal{F}_t u(\tau, x) = \frac{1}{2\pi} \int_{\mathbb{R}} u(t, x) e^{-it\tau} \, dt.$$

The space-time Fourier transform is denoted by $\mathcal{F}_{t,x} = \mathcal{F}_t \mathcal{F}_x$. For simplicity, we sometimes drop the harmless factors of 2π. We often use $\widehat{u}$ to denote $\mathcal{F}_x u$, $\mathcal{F}_t u$ and $\mathcal{F}_{t,x} u$, but it is clear which one it refers to from context, namely from the use of the spatial and time Fourier variables n and τ, respectively.

Now, we define the spaces of functions mentioned in the previous sections and used throughout the thesis. Let $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ denote the space of functions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, with $u \in C^\infty(\mathbb{R} \times \mathbb{T})$ which satisfy

$$u(t, x + 2\pi) = u(t, x), \qquad \sup_{(t,x) \in \mathbb{R} \times \mathbb{T}} |t^\alpha \partial_t^\beta \partial_x^\gamma u(t, x)| < \infty, \quad \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}.$$

We introduce the $X^{s,b}(\mathbb{R} \times \mathbb{T})$ spaces adapted to the Fourier-Lebesgue setting in [42, 44].

Definition 1.3.1. *Let $s, b \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The space $X^{s,b}_{p,q}(\mathbb{R} \times \mathbb{T})$, abbreviated $X^{s,b}_{p,q}$, is defined as the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ with respect to the norm*

$$\|u\|_{X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})} = \big\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(\tau, n) \big\|_{\ell^p_n(\mathbb{Z}) L^q_\tau(\mathbb{R})} = \Big\| \langle n \rangle^s \big\| \langle \tau - n^3 \rangle^b \widehat{u}(\tau, n) \big\|_{L^q_\tau(\mathbb{R})} \Big\|_{\ell^p_n(\mathbb{Z})}. \tag{1.39}$$

When $p = q = 2$, the $X^{s,b}_{p,q}$-spaces defined above reduce to the standard $X^{s,b}$-spaces introduced by Bourgain in [9, 10] (see also [93, 4, 66]). These spaces satisfy the following embedding for any $1 \leq p < \infty$,

$$X^{s,b}_{p,q}(\mathbb{R} \times \mathbb{T}) \hookrightarrow C(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})) \quad \text{for } b > \frac{1}{q'} = 1 - \frac{1}{q}.$$

When $b = \frac{1}{q'}$, this embedding fails. For this reason, we introduce the following space

$$Z^{s,b}_p = X^{s,b}_{p,2} \cap X^{s,b-\frac{1}{2}}_{p,1}.$$

Note that $Z^{s,\frac{1}{2}}_p \hookrightarrow C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$. We will further require the time localized version of the Fourier restriction spaces.

Definition 1.3.2. *Let $s, b \in \mathbb{R}$, $1 \leq p, q < \infty$, and $I \subset \mathbb{R}$ be an interval. We define the restriction space $X^{s,b}_{p,q}(I)$ of all functions u which satisfy*

$$\|u\|_{X^{s,b}_{p,q}(I)} := \inf \Big\{ \|v\|_{X^{s,b}_{p,q}} : \ v \in X^{s,b}_{p,q}(\mathbb{R} \times \mathbb{T}), \ v|_{t \in I} = u \Big\} < \infty,$$

with the infimum taken over all extensions v of u. If $I = [-T, T]$ for some $0 < T \leq 1$, we denote the space by $X^{s,b}_{p,q}(T)$. The spaces $Z^{s,b}_p(I)$ are defined analogously.

We introduce the linear propagator for the Airy equation $S(t)$, defined by

$$\mathcal{F}_x\big(S(t)u\big)(t,n) = e^{itn^3}\widehat{u}(t,n).$$

The norm in (1.39) can be rewritten using the interaction representation

$$\begin{aligned}\|u\|_{X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})} &= \big\|\langle\partial_x\rangle^s\langle\partial_t\rangle^b S(-t)u(t)\big\|_{\mathcal{F}L^{0,p}_x(\mathbb{T})\mathcal{F}L^{0,q}_t(\mathbb{R})} \\ &= \big\|\|\langle\partial_x\rangle^s\langle\partial_t\rangle^b S(-t)u(t)\|_{\mathcal{F}L^{0,q}_t(\mathbb{R})}\big\|_{\mathcal{F}L^{0,p}_x(\mathbb{T})}.\end{aligned}$$

For simplicity, we may drop the domains of integration/summation of the norms for $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})$, $H^s(\mathcal{M})$, $\mathcal{F}L^{s,p}(\mathcal{M})$, $\ell^p(\mathbb{Z})$, and $L^q(\mathbb{T})$, $\mathcal{M}=\mathbb{R}$ or $\mathbb{T}$. Also, we often use a subscript to indicate the variable associated with the norm. Lastly, we use $\mathbb{1}_A$ to denote the characteristic function on the set A.

Chapter 2

The modified Korteweg-de Vries equation

In this chapter, we study the first renormalized mKdV equation (mKdV1)

$$\partial_t u + \partial_x^3 u = \big(|u|^2 - M(u)\big)\partial_x u, \tag{2.1}$$

and the second renormalized mKdV equation (mKdV2)

$$\partial_t u + \partial_x^3 u = |u|^2 \partial_x u - M(u)\partial_x u - iP(u)u, \tag{2.2}$$

where the mass and the momentum are formally conserved quantities given by

$$\begin{aligned} M(f) &= \fint_{\mathbb{T}} |f(x)|^2 \, dx, \\ P(f) &= \operatorname{Im} \fint_{\mathbb{T}} \big(\overline{f}\, \partial_x f\big)(x)\, dx = \sum_{n\in\mathbb{Z}} n|\widehat{f}(n)|^2. \end{aligned}$$

We consider the defocusing equation ('+' in (1.5)), as the sign will not play a role in the analysis. Recall that these two equations are related by the gauge transform

$$\mathcal{G}_2[u](t,x) = e^{-iP(u)t} u(t,x), \tag{2.3}$$

where the momentum $P\big(u(t)\big)$ is not well-defined outside $H^{\frac{1}{2}}(\mathbb{T})$. The first part of this chapter is devoted to showing the ill-posedness of mKdV1 (2.1) (Theorem 1.1.1) and the well-posedness of mKdV2 (2.2) (Theorems 1.1.3 and 1.1.6) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq \frac{1}{2}$ and a restricted range of p.

In order to prove Theorem 1.1.3, we recall known linear estimates and auxiliary results in Section 2.1, and show the main nonlinear estimate (Proposition 2.2.1) in Section 2.2. As a consequence, we establish the local well-posedness of mKdV2 (2.2) for $1 \leq p < 4$ using the Fourier restriction norm method in Section 2.3 and the ill-posedness of mKdV1 (2.1) for this restricted range of p. Lastly, combining the a priori bounds of Oh-Wang [91] and a persistence of regularity argument, we extend the solutions of mKdV2 globally-in-time (Theorem 1.1.6) in Section 2.4.

In the second part of this chapter, we turn our attention to the problem of recovering solutions of mKdV1 (2.1) from those of mKdV2 (2.2) outside $H^{\frac{1}{2}}(\mathbb{T})$. We first establish the conservation of momentum at low regularity (Proposition 1.1.9) in Section 2.5, by using the normal form approach to show a useful energy estimate. Lastly, in Section 2.6, we exploit the conservation of momentum at low regularity to obtain distributional solutions of mKdV1 (Theorem 1.1.11).

Lastly, in Section 2.7, we prove a lemma from which the mild ill-posedness of (1.5) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s < \frac{1}{2}$ and $1 \leq p < \infty$ follows, i.e., Proposition 1.1.4.

2.1 Linear estimates and auxiliary results

In order to establish the local well-posedness of mKdV2 (2.2), we will apply the Fourier restriction norm method in $Z_p^{s,\frac{1}{2}}(T)$, for some $0 < T \leq 1$, given by

$$Z_p^{s,\frac{1}{2}}(T) = X_{p,2}^{s,\frac{1}{2}}(T) \cap X_{p,1}^{s,0}(T).$$

See Definition 1.3.2 for the $X_{p,q}^{s,b}(T)$ spaces. Recall that $Z_p^{s,\frac{1}{2}}(T) \hookrightarrow C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big)$. We require the following linear estimates (see [44, Lemma 7.1] for the proof).

Lemma 2.1.1. (i) (Homogeneous linear estimate) *Let $1 \leq p, q \leq \infty$ and $s, b \in \mathbb{R}$, then*

$$\big\|S(t)u_0\big\|_{X_{p,q}^{s,b}(T)} \lesssim \|u_0\|_{\mathcal{F}L^{s,p}},$$

for any $0 < T \leq 1$.
(ii) (Inhomogeneous linear estimate) *Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, then*

$$\bigg\| \int_0^t S(t-t')F(t')\,dt' \bigg\|_{Z_p^{s,\frac{1}{2}}(T)} \lesssim \|F\|_{Z_p^{s,-\frac{1}{2}}(T)},$$

for any $0 < T \leq 1$.

The following estimate allows us to gain a small power of the time of existence T, needed to close the contraction mapping argument (see [102, Lemma 2.11] for the proof).

Lemma 2.1.2. *Let $-\frac{1}{2} < b' \leq b < \frac{1}{2}$ and $1 \leq p < \infty$. The following holds:*

$$\|u\|_{X_{p,2}^{s,b'}(T)} \lesssim T^{b-b'}\|u\|_{X_{p,2}^{s,b}(T)},$$

for any $0 < T \leq 1$.

We will also need the fact that multiplication by a sharp cut-off is a bounded operation in $X_{2,2}^{s,b}$ (see [32, Lemma 2.1], for example).

Lemma 2.1.3. *Let $s \geq 0$, $0 \leq b < \frac{1}{2}$ and fix $T > 0$. Then, the following estimate holds*

$$\|\mathbb{1}_{[0,T]}(t)u\|_{X_{2,2}^{s,b}} \lesssim \|u\|_{X_{2,2}^{s,b}}.$$

We also recall the following well-known tools (see [40, Lemma 4.2] and [79, Lemma 5], respectively).

Lemma 2.1.4. *Let $0 \leq \alpha \leq \beta$ such that $\alpha + \beta > 1$ and $\varepsilon > 0$. Then, we have*

$$\int_{\mathbb{R}} \frac{1}{\langle x-a\rangle^{\alpha}\langle x-b\rangle^{\beta}}dx \lesssim \frac{1}{\langle a-b\rangle^{\gamma}},$$

where

$$\gamma = \begin{cases} \alpha+\beta-1, & \beta < 1, \\ \alpha - \varepsilon, & \beta = 1, \\ \alpha, & \beta > 1. \end{cases}$$

Lemma 2.1.5. *Let $0 \leq \alpha, \beta < 1$ such that $\alpha + \beta > 1$. Then, we have*

$$\sum_{\substack{n_1,n_2\in\mathbb{Z} \\ n_1+n_2=n}} \frac{1}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta}} \lesssim \frac{1}{\langle n\rangle^{\alpha+\beta-1}},$$

uniformly over $n \in \mathbb{Z}$.

Lastly, we include the periodic L^6-Strichartz estimates due to Bourgain [10]

$$\|\varphi(t)u\|_{L^6_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}+}_{2,2}},$$

where φ denotes a smooth time cutoff. Interpolating the estimate above with the Sobolev inequality $X^{\frac{1}{3}+,\frac{1}{3}+}_{2,2} \subset L^6_{t,x}$, we have the following

$$\|\varphi(t)u\|_{L^6_{t,x}} \lesssim \|u\|_{X^{0+,\frac{1}{2}-}_{2,2}}. \tag{2.4}$$

2.2 Nonlinear estimate

In this section, we establish the fundamental trilinear estimates required to show Theorem 1.1.3. We start by introducing the following multilinear operators, defined on the Fourier side and omitting time dependence,

$$\mathcal{F}_x\big(\mathcal{NR}(u_1,u_2,u_3)\big)(n) = \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})\neq 0}} in_1\,\widehat{u}_1(n_1)\widehat{u}_2(n_2)\widehat{u}_3(n_3),$$
$$\mathcal{F}_x\big(\mathcal{R}(u_1,u_2,u_3)\big)(n) = -in\,\widehat{u}_1(n)\overline{\widehat{u}}_2(n)\widehat{u}_3(n),$$

where $\overline{n}_{123} = (n_1,n_2,n_3)$ and the phase function is given by $\phi(\overline{n}_{123}) = 3(n_1+n_2)(n_1+n_3)(n_2+n_3)$. When there is no ambiguity, we will use $\overline{n}$ to denote $\overline{n}_{123}$. Recall from (1.11) that the nonlinearity of the real-valued mKdV1 equation (2.1) and that of mKdV2 (2.2) can be written as

$$\mathcal{N}(u,\overline{u},u) = \mathcal{NR}(u,\overline{u},u) + \mathcal{R}(u,u,u).$$

We then establish estimates for the resonant and non-resonant contributions $\mathcal{NR}$ and $\mathcal{R}$, respectively.

Proposition 2.2.1. *Let (s,p) satisfy one of the following conditions:* (i) $\frac{1}{2} \leq s < \frac{3}{4}$, $1 \leq p < \frac{4}{3-4s}$; (ii) $s \geq \frac{3}{4}$, $1 \leq p < \infty$. *For $u_j : \mathbb{R}\times\mathbb{T} \to \mathbb{C}$, $j = 1,2,3$, the following estimates hold:*

$$\|\mathcal{NR}(u_1,u_2,u_3)\|_{Z^{s,-\frac{1}{2}}_p(T)} \lesssim T^\delta \prod_{j=1}^{3} \|u_j\|_{X^{s,\frac{1}{2}}_{p,2}(T)}, \tag{2.5}$$

$$\|\mathcal{R}(u_1,u_2,u_3)\|_{Z^{s,-\frac{1}{2}}_p(T)} \lesssim T^\delta \prod_{j=1}^{3} \|u_j\|_{X^{s,\frac{1}{2}}_{p,2}(T)}, \tag{2.6}$$

for some $0 < \delta \ll 1$ and any $0 < T \leq 1$.

Proof. It suffices to show

$$\|\mathcal{NR}(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\|_{Z^{s,-\frac{1}{2}}_p} \lesssim \max_{k=1,2,3}\left(\|\tilde{u}_k\|_{X^{s,\frac{1}{2}}_{p,2}} \prod_{\substack{j=1,\\ j\neq k}}^{3} \|\tilde{u}_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}}\right),$$
$$\|\mathcal{R}(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)\|_{Z^{s,-\frac{1}{2}}_p} \lesssim \prod_{j=1}^{3} \|\tilde{u}_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}}, \tag{2.7}$$

for any $\tilde{u}_j$ an extension of u_j in $[-T,T]$, $j = 1,2,3$, and some $\nu > 0$. Then, taking an infimum over all extensions and using Lemma 2.1.2, we get (2.5) and (2.6). For simplicity, denote the extensions $\tilde{u}_j$ by u_j, $j = 1,2,3$, in the remaining of the proof. Let $\sigma_0 = \tau - n^3$, $\mu = (\tau,n)$, $\sigma_j = \tau_j - n_j^3$, and $\mu_j = (\tau_j,n_j)$, $j = 1,2,3$.

We start by estimating the $X^{s,-\frac{1}{2}}_{p,2}$-norm of the non-resonant contribution $\mathcal{NR}$

$$\|\mathcal{NR}(u_1,u_2,u_3)\|_{X^{s,-\frac{1}{2}}_{p,2}} \lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{\langle n\rangle^s |n_1|}{\langle \tau-n^3\rangle^{\frac{1}{2}}} \prod_{j=1}^{3} |\widehat{u}_j(\tau_j,n_j)| \Bigg\|_{\ell^p_n L^2_\tau}. \tag{2.8}$$

Note that $\sigma_0-\sigma_1-\sigma_2-\sigma_3 = -3(n_1+n_2)(n_1+n_3)(n_2+n_3) = \phi(\overline{n})$ which implies that

$$|\phi(\overline{n})| \lesssim \max_{j=0,1,2,3} |\sigma_j| =: \sigma_{\max}. \tag{2.9}$$

Let $|n_{\min}| \leq |n_{\text{med}}| \leq |n_{\max}|$ denote the increasing rearrangement of the frequencies n_1,n_2,n_3. We distinguish the following two cases for the phase function $\phi(\overline{n})$:

$$|n_1|\sim|n_2|\sim|n_3|, \quad |\phi(\overline{n})| \sim |n_{\max}|\lambda_1\lambda_2 \quad \text{and,} \tag{2.10}$$

$$|\phi(\overline{n})| \sim |n_{\max}|^2\lambda, \tag{2.11}$$

where $\lambda,\lambda_1 = \min\{|n_1+n_2|,|n_1+n_3|,|n_2+n_3|\}$ and $\lambda_2 = \text{med}\{|n_1+n_2|,|n_1+n_3|,|n_2+n_3|\}$. From (2.9), we can use the largest modulation $\sigma_{\max}$ to gain powers of the maximum frequency. Thus, we will consider different cases depending on the value of $\sigma_{\max}$ and on which of the conditions (2.10) or (2.11) holds.

Case 1.1: $\sigma_{\max} = |\sigma_0|$

Let $f_j(\tau,n) = \langle n\rangle^s\langle \tau-n^3\rangle^{\frac{1}{2}-\nu}|\widehat{u}_j(\tau,n)|$ and note that $\|f_j\|_{\ell^p_n L^2_\tau} = \|u_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}}$, $j=1,2,3$. Then, we have

$$\begin{aligned}(2.8) &\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s\langle\sigma_j\rangle^{\frac{1}{2}-\nu}} \prod_{j=1}^{3} f_j(\tau_j,n_j)\Bigg\|_{\ell^p_n L^2_\tau}\\
&\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s} J_1(\tau,\overline{n}) \bigg(\int_{\tau=\tau_1+\tau_2+\tau_3}\prod_{j=1}^3|f_j(\tau_j,n_j)|^2\bigg)^{\frac{1}{2}}\Bigg\|_{\ell^p_n L^2_\tau},\end{aligned}$$

using Hölder's inequality, where

$$J_1(\tau,\overline{n}) := \bigg(\int_{\tau=\tau_1+\tau_2+\tau_3}\frac{1}{(\langle\sigma_1\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle)^{1-2\nu}}\bigg)^{\frac{1}{2}} \lesssim 1$$

from using Lemma 2.1.4 twice and with $0<\nu<\frac{1}{6}$. Using Minkowski's and Hölder's inequalities, it follows that

$$\begin{aligned}(2.8) &\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s} \prod_{j=1}^3\|f_j(n_j)\|_{L^2_\tau}\Bigg\|_{\ell^p_n}\\
&\lesssim \sup_n \big(J_1'(n)\big)^{\frac{1}{p'}} \prod_{j=1}^3 \|f_j\|_{\ell^p_n L^2_\tau},\end{aligned}$$

where

$$J_1'(n) = \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \bigg(\frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s}\bigg)^{p'}. \tag{2.12}$$

Since $\|f_j\|_{\ell^p_n L^2_\tau} = \|u_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}}$, it only remains to estimate J_1'. If (2.10) holds, then

$$\frac{\langle n\rangle^s |n_1|}{|\phi(\overline{n})|^{\frac{1}{2}} \prod_{j=1}^3 \langle n_j\rangle^s} \lesssim \frac{1}{\langle n_1\rangle^{2s-\frac{1}{2}} \langle \lambda_1\rangle^{\frac{1}{2}} \langle \lambda_2\rangle^{\frac{1}{2}}},$$

for distinct $\lambda_1, \lambda_2 \in \{|n_1+n_2|, |n_1+n_3|, |n_2+n_3|\}$. We can write $\lambda_j = |n - n_j'|$, $j = 1,2$, where n_1', n_2' are distinct frequencies in $\{n_1, n_2, n_3\}$. Since $\lambda_1, \lambda_2 \lesssim |n_1|$, we have

$$\begin{aligned} J_1'(n) &\lesssim \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{1}{\langle n_1\rangle^{(2s-\frac{1}{2})p'} \langle n-n_1'\rangle^{\frac{p'}{2}} \langle n-n_2'\rangle^{\frac{p'}{2}}} \\ &\lesssim \sum_{n_1', n_2'} \frac{1}{\langle n-n_1'\rangle^{(s+\frac{1}{4})p'} \langle n-n_2'\rangle^{(s+\frac{1}{4})p'}} \lesssim 1, \end{aligned}$$

for $s \geq \frac{1}{4}$, $1 \leq p < 2$ or $s > \frac{3}{4} - \frac{1}{p}$, $2 \leq p < \infty$. If (2.11) holds, then

$$\frac{\langle n\rangle^s |n_1|}{|\phi(\overline{n})|^{\frac{1}{2}} \prod_{j=1}^3 \langle n_j\rangle^s} \lesssim \frac{1}{\langle n_{\min}\rangle^s \langle n_{\text{med}}\rangle^s \lambda^{\frac{1}{2}}},$$

where $\lambda \in \{|n_{\min} + n_{\text{med}}|, |n - n_{\min}|\}$. If $\lambda = |n_{\min} + n_{\text{med}}|$, since $|n_{\min}|, |n_{\min} + n_{\text{med}}| \lesssim |n_{\text{med}}|$, we have

$$\begin{aligned} J_1'(n) &\lesssim \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{1}{\langle n_{\min}\rangle^{sp'} \langle n_{\text{med}}\rangle^{sp'} \langle n_{\min} + n_{\text{med}}\rangle^{\frac{p'}{2}}} \\ &\lesssim \sum_{n_{\min}, n_{\text{med}}} \frac{1}{\langle n_{\min}\rangle^{(s+\frac{1}{4})p'} \langle n_{\min} + n_{\text{med}}\rangle^{(s+\frac{1}{4})p'}} \lesssim 1 \end{aligned}$$

given that $s \geq \frac{1}{4}$, $1 \leq p < 2$ or $s > \frac{3}{4} - \frac{1}{p}$, $2 \leq p < \infty$. If $\lambda = |n - n_{\min}|$, since $|n - n_{\min}|, |n_{\min}| \lesssim |n_{\text{med}}|$, the same estimate follows from using Lemma 2.1.5.

Case 1.2: $\sigma_{\max} = |\sigma_j|$, $j \in \{1,2,3\}$

Assume that $\sigma_{\max} = |\sigma_1|$, as a similar argument holds in the remaining cases. Let $g_1(\tau, n) = \langle n\rangle^s \langle \tau - n^3\rangle^{\frac{1}{2}} |\widehat{u}_1(\tau, n)|$, $g_j(\tau, n) = \langle n\rangle^s \langle \tau - n^3\rangle^{\frac{1}{2}-\nu} |\widehat{u}_j(\tau, n)|$, $j = 2,3$, and note that $\|g_1\|_{\ell^p_n L^2_\tau} = \|u_1\|_{X^{s,\frac{1}{2}}_{p,2}}$ and $\|g_j\|_{\ell^p_n L^2_\tau} = \|u_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}}$, $j = 2,3$. Using duality, for $g_0 \in \ell^{p'}_n L^2_\tau$, and Hölder's inequality, we have

$$\begin{aligned} (2.8) &\lesssim \sum_n \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \int_\tau \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{\langle n\rangle^s |n_1|}{|\phi(\overline{n})|^{\frac{1}{2}} \langle \sigma_0\rangle^{\frac{1}{2}-\nu} \langle n_1\rangle^s \prod_{j=2}^3 \langle n_j\rangle^s \langle \sigma_j\rangle^{\frac{1}{2}-\nu}} \\ &\qquad\qquad \times g_0(\tau, n) \prod_{j=1}^3 g_j(\tau_j, n_j) \\ &\lesssim \sum_n \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{\langle n\rangle^s |n_1|}{|\phi(\overline{n})|^{\frac{1}{2}} \prod_{j=1}^3 \langle n_j\rangle^s} \|g_0(n)\|_{L^2_\tau} \prod_{j=1}^3 \|g_j(n_j)\|_{L^2_\tau} \times J_2(\tau_1, n, \overline{n}), \end{aligned}$$

where

$$J_2(\tau_1, n, \overline{n}) = \left(\int_{\tau_1 = \tau - \tau_2 - \tau_3} \frac{1}{(\langle \sigma_0\rangle \langle \sigma_2\rangle \langle \sigma_3\rangle)^{1-2\nu}} \right)^{\frac{1}{2}} \lesssim 1,$$

by two applications of Lemma 2.1.4 with $0 < \nu < \frac{1}{6}$. Using Hölder's inequality, we obtain

$$(2.8) \lesssim \big(\sup_n J_1'(n)\big)\|g_0\|_{\ell_n^{p'}L_\tau^2}\prod_{j=1}^3\|g_j\|_{\ell_n^pL_\tau^2},$$

with $J_1'(n)$ defined in (2.12), which is uniformly bounded by following the same arguments in the previous case. This concludes the estimate for $\|\mathcal{NR}(u_1,u_2,u_3)\|_{X^{s,-\frac{1}{2}}_{p,2}}$.

Next, we consider the $X^{s,-1}_{p,1}$-norm of $\mathcal{NR}$,

$$\|\mathcal{NR}(u_1,u_2,u_3)\|_{X^{s,-1}_{p,1}} \lesssim \Bigg\|\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\int_{\tau=\tau_1+\tau_2+\tau_3}\frac{\langle n\rangle^s|n_1|}{\langle\sigma_0\rangle}\prod_{j=1}^3|\widehat{u}_j(\tau_j,n_j)|\Bigg\|_{\ell_n^pL_\tau^1}. \tag{2.13}$$

As when estimating (2.8), we will consider different cases depending on the value of $\sigma_{\max}$. If $\sigma_{\max}=|\sigma_j|$, $j\in\{1,2,3\}$, then using Cauchy-Schwarz inequality in τ gives

$$(2.13) \lesssim \Bigg\|\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\int_{\tau=\tau_1+\tau_2+\tau_3}\frac{\langle n\rangle^s|n_1|}{\langle\sigma_0\rangle^{\frac{1}{2}-\nu}}\prod_{j=1}^3|\widehat{u}_j(\tau_j,n_j)|\Bigg\|_{\ell_n^pL_\tau^2}$$

and the estimate follows from Case 1.2. Hence, we can assume that $|\sigma_0|\gg|\sigma_j|$, $j=1,2,3$, which implies that $|\sigma_0|\sim|\sigma_0-\sigma_1-\sigma_2-\sigma_3|$. Let $h_j(\tau,n)=\langle n\rangle^s\langle\tau-n^3\rangle^{\frac{1}{2}-2\nu}|\widehat{u}_j(\tau,n)|$, $j=1,2,3$. Then, using Hölder's inequality with $1=\frac{1}{q}+\frac{1}{q'}$ and $q<2$ and Minkowski's inequality, we have

$$\begin{aligned}(2.13) &\lesssim \Bigg\|\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\int_{\tau=\tau_1+\tau_2+\tau_3}\frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\langle\sigma_0\rangle^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s\langle\sigma_j\rangle^{\frac{1}{2}-2\nu}}\prod_{j=1}^3 h_j(\tau_j,n_j)\Bigg\|_{\ell_n^pL_\tau^1}\\
&\lesssim \Bigg\|\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\int_{\tau=\tau_1+\tau_2+\tau_3}\frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s\langle\sigma_j\rangle^{\frac{1}{2}-\nu}}\prod_{j=1}^3 h_j(\tau_j,n_j)\Bigg\|_{\ell_n^pL_\tau^q}\\
&\lesssim \Bigg\|\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\frac{\langle n\rangle^s|n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}\prod_{j=1}^3\langle n_j\rangle^s}J_3(\tau,\overline{n})\prod_{j=1}^3\|h_j(n_j)\|_{L_\tau^q}\Bigg\|_{\ell_n^p},\end{aligned}$$

where

$$J_3(\tau,\overline{n})=\Bigg(\int_{\tau=\tau_1+\tau_2+\tau_3}\frac{1}{(\langle\sigma_1\rangle\langle\sigma_2\rangle\langle\sigma_3\rangle)^{(\frac{1}{2}-2\nu)q'}}\Bigg)^{\frac{1}{q'}}\lesssim 1,$$

from two applications of Lemma 2.1.4, for q satisfying $\frac{1}{q}>\max\big(4\nu,\frac{1}{4}+3\nu\big)$. Using Hölder's inequality, we have

$$(2.13)\lesssim\big(\sup_n J_1'(n)\big)^{\frac{1}{p'}}\prod_{j=1}^3\|h_j\|_{\ell_n^pL_\tau^q},$$

for J_1' defined in (2.12). We know that J_1' is uniformly bounded in n from Case 1.1 and the intended estimate follows from Hölder's inequality

$$\|h_j\|_{\ell_n^pL_\tau^q}=\|u_j\|_{X^{s,\frac{1}{2}-2\nu}_{p,q}}\lesssim\|u_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}},$$

given that $\frac{1}{q}<\frac{1}{2}+\nu$. For fixed $0<\nu<\frac{1}{8}$, we choose $q=q(\nu)<2$ satisfying $\max(4\nu,\frac{1}{4}+3\nu)<\frac{1}{q}<\frac{1}{2}+\nu$. This completes the estimate of $\|\mathcal{NR}(u_1,u_2,u_3)\|_{X^{s,-1}_{p,1}}$.

Next, we consider the resonant part $\mathcal{R}$. Since by Cauchy-Schwarz inequality we have

$$\|\mathcal{R}(u_1,u_2,u_3)\|_{X^{s,-1}_{p,1}} \lesssim \|\mathcal{R}(u_1,u_2,u_3)\|_{X^{s,-\frac{1}{2}+\nu}_{p,2}},$$

for any $\nu>0$, (2.7) follows once we show the following estimate

$$\|\mathcal{R}(u_1,u_2,u_3)\|_{X^{s,-\frac{1}{2}+\nu}_{p,2}} \lesssim \prod_{j=1}^{3}\|u_j\|_{X^{s,\frac{1}{2}}_{p,2}}.$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|\mathcal{R}(u_1,u_2,u_3)\|_{X^{s,-\frac{1}{2}+\nu}_{p,2}} &\lesssim \Bigg\| \int_{\tau=\tau_1-\tau_2+\tau_3} \frac{\langle n\rangle^s|n|}{\langle \tau-n^3\rangle^{\frac{1}{2}-\nu}} \prod_{j=1}^{3}|\widehat{u}_j(\tau_j,n)|\Bigg\|_{\ell^p_n L^2_\tau} \\ &\lesssim \big(\sup_{\tau,n} J_4(\tau,n)\big)\Bigg\|\langle n\rangle^s|n|\prod_{j=1}^{3}\|\langle\tau-n^3\rangle^{\frac{1}{2}-\nu}\widehat{u}_j(\tau,n)\|_{L^2_\tau}\Bigg\|_{\ell^p_n},\end{aligned}$$

where

$$J_4(\tau,n)=\Bigg(\int_{\tau=\tau_1-\tau_2+\tau_3}\frac{1}{(\langle\tau-n^3\rangle\langle\tau_1-n^3\rangle\langle\tau_2-n^3\rangle\langle\tau_3-n^3\rangle)^{1-2\nu}}\Bigg)^{\frac{1}{2}} \lesssim 1,$$

by two applications of Lemma 2.1.4, with $0<\nu<\frac{1}{4}$. Since we want $\langle n\rangle^s|n|\lesssim\langle n\rangle^{3s}$, we must impose the condition $s\geq\frac{1}{2}$. Thus, using Hölder's inequality we get

$$\|\mathcal{R}(u_1,u_2,u_3)\|_{X^{s,-\frac{1}{2}+\nu}_{p,2}} \lesssim \prod_{j=1}^{3}\big\|\langle n\rangle^s\langle\tau-n^3\rangle^{\frac{1}{2}-\nu}\widehat{u}_j(\tau,n)\big\|_{\ell^{3p}_n} \lesssim \prod_{j=1}^{3}\|u_j\|_{X^{s,\frac{1}{2}-\nu}_{p,2}},$$

completing the estimate for the resonant contribution.

□

2.3 Proof of Theorems 1.1.3 and 1.1.1 with $1\leq p<4$

In order to prove Theorem 1.1.3, we establish that the right-hand side of the Duhamel formulation for mKdV2 (2.2)

$$u(t)=S(t)u_0+\int_0^t S(t-t')\mathcal{N}(u,\overline{u},u)(t')\,dt'$$

is a contraction in $Z^{s,\frac{1}{2}}_p(T)$, for some $0<T\leq 1$. We can then show local well-posedness of (2.2) for the range of (s,p) where the nonlinear estimate holds. In particular, for (s,p) satisfying one of the following conditions:

$$\frac{1}{2}\leq s<\frac{3}{4},\quad 1\leq p<\frac{4}{3-4s} \qquad \text{or} \qquad s\geq\frac{3}{4},\quad 1\leq p<\infty. \tag{2.14}$$

Note that at the endpoint $s=\frac{1}{2}$, we can only cover the range $1\leq p<4$.

Proof of Theorem 1.1.3 with (2.14). Let (s,p) satisfying the assumptions in (2.14). Given $u_0\in\mathcal{F}L^{s,p}(\mathbb{T})$, define the solution map Γ_{u_0} as follows

$$\Gamma_{u_0}[u](t):=S(t)u_0+\int_0^t S(t-t')\mathcal{N}(u,\overline{u},u)(t')\,dt'.$$

Let $R>0$ and $B_R:=\big\{u\in Z^{s,\frac{1}{2}}_p(T):\ \|u\|_{Z^{s,\frac{1}{2}}_p(T)}\leq R\big\}$. Using Lemma 2.1.1 and Proposition

2.2.1, for some $0 < \delta \ll 1$, we have

$$\begin{aligned}\|\Gamma_{u_0}(u)\|_{Z_p^{s,\frac{1}{2}}(T)} &\leq C_1\|u_0\|_{\mathcal{F}L^{s,p}} + C_2\|\mathcal{N}(u,\overline{u},u)\|_{Z_p^{s,-\frac{1}{2}}(T)} \\ &\leq C_1\|u_0\|_{\mathcal{F}L^{s,p}} + C_3T^\delta\|u\|^3_{X_{p,2}^{s,\frac{1}{2}}(T)}\end{aligned} \tag{2.15}$$

for constants $C_1, C_2, C_3 > 0$ and $0 < T \leq 1$. Similarly, since $\mathcal{N}(u,\overline{u},u) - \mathcal{N}(v,\overline{v},v) = \mathcal{N}(u - v,\overline{u},u) + \mathcal{N}(v,\overline{u}-\overline{v},v) + \mathcal{N}(v,\overline{v},u-v)$, we have

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{Z_p^{s,\frac{1}{2}}(T)} \leq C_4T^\delta\Big(\|u\|^2_{X_{p,2}^{s,\frac{1}{2}}(T)} + \|v\|^2_{X_{p,2}^{s,\frac{1}{2}}(T)}\Big)\|u-v\|_{X_{p,2}^{s,\frac{1}{2}}(T)} \tag{2.16}$$

for a constant $C_4 >$ and $0 < T \leq 1$. Choosing $R := 2C_1\|u_0\|_{\mathcal{F}L^{s,p}}$ and $0 < T = T(R) \leq 1$ such that

$$C_3T^\delta R^3 \leq \frac{1}{2} \quad \text{and} \quad C_4T^\delta R^2 \leq \frac{1}{4},$$

it follows from (2.15) and (2.16) that Γ_{u_0} is a contraction on the closed ball $B_R \subset Z_p^{s,\frac{1}{2}}(T)$. Consequently, Γ_{u_0} has a unique fixed point $u = \Gamma_{u_0}(u) \in Z_p^{s,\frac{1}{2}}(T)$.

It only remains to show that Γ_{u_0} is locally uniformly continuous with respect to the initial data u_0. Let $u_0, v_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ and u, v be the respective solutions. Following the same strategy as for (2.15) and (2.16), with the above assumptions on T, we have that

$$\|u-v\|_{Z_p^{s,\frac{1}{2}}(T)} = \|\Gamma_{u_0}(u) - \Gamma_{v_0}(v)\|_{Z_p^{s,\frac{1}{2}}(T)} \leq C_1\|u_0 - v_0\|_{\mathcal{F}L^{s,p}} + \frac{1}{2}\|u-v\|_{X_{p,2}^{s,\frac{1}{2}}(T)}.$$

Using the embedding $Z_p^{s,\frac{1}{2}}(T) \hookrightarrow C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$, we conclude that

$$\sup_{t\in[-T,T]}\|u(t) - v(t)\|_{\mathcal{F}L^{s,p}} \leq 2C_1\|u_0 - v_0\|_{\mathcal{F}L^{s,p}}.$$

Therefore, the data-to-solution map is locally uniformly continuous. This completes the proof of Theorem 1.1.3 for (s,p) satisfying (2.14).

□

From the local well-posedness of mKdV2 (2.2), we can show the non-existence of solution of mKdV1 (2.1) for initial data with infinite momentum. The ill-posedness result in Theorem 1.1.1 follows an argument by Guo-Oh [48]. The proof combines the local well-posedness of mKdV2 (Theorem 1.1.3) and the rapid oscillation of the phase depending on the momentum in the gauge transform $\mathcal{G}_2$ in (2.3).

Proof of Theorem 1.1.1 with (2.14). Consider $u_{0,N} := \mathbf{P}_{\leq N}u_0$ and $\{u_N\}_{N\in\mathbb{N}}$ the sequence of smooth global solutions of mKdV1 (2.1) with $u_N|_{t=0} = u_{0,N}$ for $N \in \mathbb{N}$. Suppose that there exist $T > 0$ and a solution $u \in C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ of mKdV1 (2.1) such that:

(a) $u|_{t=0} = u_0$;

(b) $u_N \to u$ in $C\big([-T,T];\mathcal{D}'(\mathbb{T})\big)$ as $N \to \infty$.

For the smooth solutions u_N, we have conservation of momentum: $P(u_N(t)) = P(u_{0,N})$, $t \in [-T,T]$, $N \in \mathbb{N}$. Thus, the gauge transform $\mathcal{G}_2$ is well-defined and invertible. Let $v_N := \mathcal{G}_2(u_N)$, which is a smooth global solution of mKdV2 (2.2) with initial data $u_{0,N}$. Then, by the local well-posedness of mKdV2 (2.2), there exists $T' = T'(\|u_0\|_{\mathcal{F}L^{s,p}}) > 0$ such that $v_N \in Z_p^{s,\frac{1}{2}}(T')$, for some $T \geq T' = T'(\|u_0\|_{\mathcal{F}L^{s,p}}) > 0$[1]. Now, we want to show that $\{v_N\}_{N\in\mathbb{N}}$ converges

[1]From unconditional uniqueness of mKdV2 (2.2) at high regularity, the solutions v_N coincide with the solutions constructed in Theorem 1.1.3 with initial data $u_{0,N}$. Moreover, there exists $T' = T'(\|u_0\|_{\mathcal{F}L^{s,p}}) > 0$ such that $v_N \in C\big([-T',T'];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ for every $N \in \mathbb{N}$.

in $C\big([-T',T'];\mathcal{F}L^{s,p}(\mathbb{T})\big)$. From the local Lipschitz property of the data-to-solution map of mKdV2 (2.2) in Theorem 1.1.3, it follows that

$$\|v_N - v_M\|_{C_T\mathcal{F}L^{s,p}} \lesssim \|v_N - v_M\|_{Z_p^{s,\frac{1}{2}}(T)} \lesssim \|u_{0,N} - u_{0,M}\|_{\mathcal{F}L^{s,p}} \to 0$$

as $N, M \to \infty$, since $\{u_{0,N}\}_{N\in\mathbb{N}}$ converges in $\mathcal{F}L^{s,p}(\mathbb{T})$. Consequently, there exists $v \in C\big([-T',T'];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ such that $v_N \to v$.

Now, we want to exploit the rapid oscillation of the phase introduced by $\mathcal{G}_2$ to arrive at a contradiction. Let $\varphi \in \mathcal{D}\big([-T',T']\times\mathbb{T}\big)$ be any test function. From the convergence of $u_N \to u$ in $C\big([-T,T];\mathcal{D}'(\mathbb{T})\big)$, we have that

$$\langle u_N(t,\cdot), \varphi(t,\cdot)\rangle_{L^2_x} \to \langle u(t,\cdot), \varphi(t,\cdot)\rangle_{L^2_x} \quad \text{as } N\to\infty.$$

Let $F(t) := \langle u(t,\cdot), \varphi(t,\cdot)\rangle_{L^2_x}$, which is a continuous function supported on $[-T',T']$. Then, $F \in L^1(\mathbb{R})$ and by the Riemann-Lebesgue lemma

$$|\widehat{F}(\tau)| \to 0 \text{ as } |\tau|\to\infty. \tag{2.17}$$

Since $\limsup_{N\to\infty}|P(u_{0,N})| = \infty$, there exists a subsequence u_{0,N_j} such that $|P(u_{0,N_j})| \to \infty$. We therefore focus on the corresponding subsequence $\{v_{N_j}\}_{j\in\mathbb{N}}$ and in its convergence in the sense of distributions. Namely, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{\mathbb{T}} v_{N_j}(t,x)\varphi(t,x)\,dx\,dt\right| &= \left|\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-iP(u_{0,N_j})t}u_N(t,x)\varphi(t,x)\,dx\,dt\right| \\ &\le \big|\widehat{F}\big(P(u_{0,N_j})\big)\big| + \int_{-T'}^{T'}\big|\langle u_{N_j}(t,\cdot) - u(t,\cdot), \varphi(t,\cdot)\rangle_{L^2_x}\big|dt \to 0\end{aligned}$$

as $j\to\infty$. The first term converges to zero as a consequence of (2.17) and the assumption that $|P(u_{0,N_j})|\to\infty$, while the second is a consequence of $u_N \to u$ in $C\big([-T',T'];\mathcal{D}'(\mathbb{T})\big)$. Hence, $\{v_{N_j}\}_{j\in\mathbb{N}}$ converges to zero in the sense of distributions and to v in $C\big([-T',T'];\mathcal{F}L^{s,p}(\mathbb{T})\big)$. Therefore, $v\equiv 0$. However, $0 = v(0) = u_0$, which means that $P(u_0)$ must be finite, i.e., $|P(\mathbf{P}_{\le N}u_0)| = |P(u_{0,N})|$ converges as $N\to\infty$, which contradicts the assumption on the initial data. □

Remark 2.3.1. Note that at this point, we have only established the local well-posedness of mKdV2 (2.2) in $\mathcal{F}L^{s,p}(\mathbb{T})$ with (s,p) satisfying (2.14), and therefore must impose the same regularity restriction on the ill-posedness of mKdV1 (2.1). The same proof holds for $s \ge \frac{1}{2}$ and $4 \le p < \infty$ after we have extended Theorem 1.1.3 to this range (see Chapter 3). We therefore omit the proof of Theorem 1.1.1 for the remaining choices of (s,p).

2.4 Proof of Theorem 1.1.6 for $1 \le p < 4$

In this section, we focus on showing the global well-posedness of mKdV2 (2.2). The following a priori bounds due to Killip-Vişan-Zhang [63] and Oh-Wang [91] are essential to extending local-in-time solutions to global ones.

Theorem 2.4.1 ([63, 91])**.** *Let $2 < p < \infty$ and $0 < s < 1 - \frac{1}{p}$ or $1 \le p \le 2$ and $0 < s < 1$. Then, there exist $C = C(p) > 0$, and $\gamma = \gamma(s,p) > 0$ such that*

$$\|u(t)\|_{\mathcal{F}L^{s,p}} \le C(1 + \|u(0)\|_{\mathcal{F}L^{s,p}})^{\gamma}\|u(0)\|_{\mathcal{F}L^{s,p}}, \tag{2.18}$$

for any smooth solutions u to the complex-valued mKdV1 equation (2.1), for any $t\in\mathbb{R}$.

We can easily obtain the equivalent a priori bound for solutions of mKdV2 (2.2).

Corollary 2.4.2. *Let $2 < p < 4$ and $\frac{1}{2} \leq s < 1 - \frac{1}{p}$ or $1 \leq p \leq 2$ and $\frac{1}{2} \leq s < 1$. Then, there exist $C = C(p) > 0$ and $\gamma = \gamma(s,p) > 0$ such that for any $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ we have*

$$\|u\|_{L^\infty_T \mathcal{F}L^{s,p}} \leq C(1 + \|u_0\|_{\mathcal{F}L^{s,p}})^\gamma \|u_0\|_{\mathcal{F}L^{s,p}}, \tag{2.19}$$

where $u \in C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ is the corresponding solution of the complex-valued mKdV2 equation (2.2).

Proof. Let $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ and consider the corresponding solution u of mKdV2 (2.2) obtained by Theorem 1.1.3. Consider a smooth approximating sequence $\{u_{0,n}\}_{n\in\mathbb{N}}$ such that $u_{0,n} \to u_0$ in $\mathcal{F}L^{s,p}(\mathbb{T})$. Therefore, the smooth solutions of mKdV2 $\{u_n\}_{n\in\mathbb{N}}$ with initial data $u_n(0) = u_{0,n}$ satisfy $u_n \to u$ in $C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ from the Lipschitz property of the solution map in Theorem 1.1.3. Consequently, for fixed $t \in [-T,T]$, we have the following estimate

$$\begin{aligned}
\|u\|_{L^\infty_T \mathcal{F}L^{s,p}} &\leq \|u - u_n\|_{L^\infty_T \mathcal{F}L^{s,p}} + \|u_n\|_{L^\infty_T \mathcal{F}L^{s,p}} \\
&\leq \|u - u_n\|_{L^\infty_T \mathcal{F}L^{s,p}} + \|\mathcal{G}_2^{-1}[u_n]\|_{L^\infty_T \mathcal{F}L^{s,p}} \\
&\leq \|u - u_n\|_{L^\infty_T \mathcal{F}L^{s,p}} + C(1 + \|\mathcal{G}_2^{-1}[u_{0,n}]\|_{\mathcal{F}L^{s,p}})^\gamma \|\mathcal{G}_2^{-1}[u_{0,n}]\|_{\mathcal{F}L^{s,p}},
\end{aligned}$$

using the fact that $\mathcal{G}_2$ is well-defined for smooth functions and an isometry in $L^\infty\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$, and the a priori bound (2.18). Using the convergence $u_n \to u$ in $C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ and the isometry property of $\mathcal{G}_2$, we conclude that

$$\|u\|_{L^\infty_T \mathcal{F}L^{s,p}} \leq C(1 + \|u_0\|_{\mathcal{F}L^{s,p}})^\gamma \|u_0\|_{\mathcal{F}L^{s,p}},$$

as intended.

□

When $2 < p < \infty$ and $\frac{1}{2} \leq s < 1 - \frac{1}{p}$ or $1 \leq p \leq 2$ and $\frac{1}{2} \leq s < 1$, the global well-posedness immediately follows from the local well-posedness in Theorem 1.1.3 and the global-in-time bound (2.19) in Corollary 2.4.2, by iterating the local argument. However, we want to remove the upper bound on s, using a persistence of regularity argument. Before proving Theorem 1.1.6 for $1 \leq p < 4$, we need to modify the nonlinear estimate in Section 2.2 accordingly.

Corollary 2.4.3. *Let $s \geq \frac{1}{2}$ and $1 \leq p < 4$. Then, the following estimate holds*

$$\|\mathcal{N}(u,\overline{u},u)\|_{Z_p^{s,-\frac{1}{2}}(T)} \lesssim T^\delta \|u\|_{X_{p,2}^{s,\frac{1}{2}}(T)} \|u\|^2_{X_{p,2}^{\frac{1}{2},\frac{1}{2}}(T)}$$

for some $0 < \delta \ll 1$ and any $0 < T \leq 1$.

The above estimate follows from Proposition 2.2.1 with $s = \frac{1}{2}$ and by placing the remaining $s - \frac{1}{2}$ derivatives on the factor with the largest frequency. The above corollary also holds if (s,p) satisfy (2.14), but we are mostly concerned with the endpoint case and will focus only on the regime $s \geq \frac{1}{2}$ and $1 \leq p < 4$. It is now possible to prove Theorem 1.1.6 under these assumptions.

Proof of Theorem 1.1.6 when $1 \leq p < 4$. If $2 < p < 4$ and $\frac{1}{2} \leq s < 1 - \frac{1}{p}$ or $1 \leq p \leq 2$ and $\frac{1}{2} \leq s < 1$, the result follows from iterating Theorem 1.1.3. Now, consider the case when $2 < p < 4$ and $s \geq 1 - \frac{1}{p}$ or $1 \leq p \leq 2$ and $s \geq 1$. Then, $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T}) \subset \mathcal{F}L^{\frac{1}{2},p}(\mathbb{T})$ and there exists a unique global solution $u \in C\big(\mathbb{R};\mathcal{F}L^{\frac{1}{2},p}(\mathbb{T})\big)$ of mKdV2 (2.2). Using the a priori bound in Corollary 2.4.2 when running a contraction mapping argument in $Z_p^{\frac{1}{2},\frac{1}{2}}(I)$, for any interval I of length $T > 0$, imposes a local time of existence

$$T \sim (1 + \|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}})^{-\theta} > 0, \tag{2.20}$$

for the resulting solution, for some $\theta > 0$. Moreover, by choosing $I = [t_0, t_0 + T]$, we get

$$\|u\|_{Z_p^{\frac{1}{2},\frac{1}{2}}(I)} \leq C\|u(t_0)\|_{\mathcal{F}L^{\frac{1}{2},p}}, \tag{2.21}$$

for some $C > 0$. Note that by using the a priori bound, the bounds (2.20) and (2.21) hold uniformly in t_0. Using Corollary 2.4.3 and (2.21), it follow that

$$\|u\|_{Z_p^{s,\frac{1}{2}}(I)} \le C_1 \|u(t_0)\|_{\mathcal{F}L^{s,p}} + C_2 T^\delta \|u(t_0)\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \|u\|_{X_{p,2}^{s,\frac{1}{2}}(I)}$$

for constants $C_1, C_2 > 0$. Using the a priori bound, we have

$$C_2 T^\delta \|u(t_0)\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \le C_3 T^\delta \big(1 + \|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}}\big)^{2\gamma} \|u_0\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \le \frac{1}{2},$$

where the last inequality holds by possibly refining the choice of θ in (2.20), for some $C_3 > 0$. Using the embedding $Z_p^{s,\frac{1}{2}}(I) \hookrightarrow C\big(I; \mathcal{F}L^{s,p}(\mathbb{T})\big)$, it follows that

$$\sup_{t \in I} \|u(t)\|_{\mathcal{F}L^{s,p}} \le 2C_1 \|u(t_0)\|_{\mathcal{F}L^{s,p}}.$$

Iterating this argument, we obtain

$$\sup_{t \in [-T^*, T^*]} \|u(t)\|_{\mathcal{F}L^{s,p}} \le (2C_1)^{\left(1 + \|u_0\|_{\mathcal{F}L^{1/2,p}}\right)^\theta T^*} \|u_0\|_{\mathcal{F}L^{s,p}},$$

for any $T^* > 0$. This shows the global well-posedness of mKdV2 (2.2) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $2 < p < 4$ and $s \ge 1 - \frac{1}{p}$ or $1 \le p \le 2$ and $s \ge 1$.

□

2.5 Conservation of momentum outside $H^{\frac{1}{2}}(\mathbb{T})$

In order to show the conservation of momentum at low regularity (Proposition 1.1.9), we establish an energy estimate on smooth solutions of the mKdV2 equation (2.2). The main idea is to use the normal form approach ('integration by parts in time') to estimate the difference of the momentum at time t and at the initial time. The normal form approach was first introduced by Babin-Ilyin-Titi to study the periodic KdV equation [2], and further developed and applied to many aspects of the well-posedness theory of dispersive equations, see [69, 35, 47, 70, 64] for example. Here, we closely follow the argument in [82].

Proposition 2.5.1. *Let (s,p) satisfy one of the following conditions:* (i) $\frac{1}{2} \le s < \frac{5}{6}$, $1 \le p < \frac{6}{5-6s}$; (ii) $s \ge \frac{5}{6}$, $1 \le p < \infty$, *and* $u_0 \in H^\infty(\mathbb{T})$. *Let u be a smooth solution of* (2.2) *with* $u|_{t=0} = u_0$. *Then, the following estimate holds*

$$\big|P(\mathbf{P}_{>N} u(t)) - P(\mathbf{P}_{>N} u(0))\big| \lesssim \frac{1}{N^\varepsilon} \bigg(\sup_{t' \in [0,t]} \|u(t')\|^4_{\mathcal{F}L^{s,p}} + \|u\|^4_{X_{p,2}^{s,\frac{1}{2}}} + \|u\|^6_{X_{p,2}^{s,\frac{1}{2}}} \bigg),$$

for $t > 0$, any $N \in \mathbb{N}$ and $0 < \varepsilon \ll 1$ small enough, where $\mathbf{P}_{>N} = \mathrm{Id} - \mathbf{P}_{\le N}$.

Proof. Using the Fundamental Theorem of Calculus and the mKdV2 equation (2.2) on the Fourier side, we have the following

$$\begin{aligned}
&\big|P\big(\mathbf{P}_{>N} u(t)\big) - P\big(\mathbf{P}_{>N} u(0)\big)\big| \\
&\qquad = \bigg| \sum_{|n|>N} n \big(|\widehat{u}(t,n)|^2 - |\widehat{u}(0,n)|^2 \big) \bigg| \\
&\qquad = \bigg| 2 \sum_{|n|>N} n \,\mathrm{Re} \int_0^t (\partial_t \widehat{u}(t',n)) \overline{\widehat{u}}(t',n)\, dt' \bigg| \\
&\qquad = \bigg| 2\,\mathrm{Im} \int_0^t \sum_{|n|>N} \sum_{\substack{n = n_1+n_2+n_3, \\ \phi(\overline{n}) \ne 0}} n n_1 \widehat{u}(t',n_1) \overline{\widehat{u}}(t',-n_2) \widehat{u}(t',n_3) \overline{\widehat{u}}(t',n)\, dt' \bigg|.
\end{aligned}$$

Let $|n_{\min}| \leq |n_{\text{med}}| \leq |n_{\max}|$ denote the increasing rearrangement of n_1, n_2, n_3. We will consider the following 6 cases depending on the relative size of the frequencies:

- Case 1: (i) $|n_{\max}| \gg |n_{\text{med}}| \gtrsim |n_1|$ or (ii) $|n_{\max}| \sim |n_{\text{med}}| \gg |n_1|$
- Case 2: $|n_{\max}| \gg |n_1| \gg |n_{\min}|$
- Case 3: $|n_1| \sim |n_{\text{med}}| \gg |n_{\min}|$
- Case 4: (i) $|n_1| \gg |n_{\text{med}}| \geq |n_{\min}| \gtrsim |n_1|^{\frac{1}{2}}$ or (ii) $|n_1| \gg |n_{\text{med}}| \gtrsim |n_1|^{\frac{1}{2}} \gg |n_{\min}|$
- Case 5: $|n_1| \sim |n_2| \sim |n_3|$
- Case 6: $|n_1|^{\frac{1}{2}} \gg |n_{\text{med}}| \gtrsim |n_{\min}|$

In Cases 1–4, the difference can be estimated directly, while in Cases 5–6 we will require the normal form approach.

Cases 1–4

Let $\sigma_j := \tau_j - n_j^3$, $j = 1, 2, 3$, and $\sigma_0 := \tau - n^3$ denote the modulations. The following relation holds

$$-\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 = n^3 - n_1^3 - n_2^3 - n_3^3 = \phi(\overline{n}).$$

In Cases 1–4, the resonance relation $\phi(\overline{n})$ satisfies the following

$$|n_{\max}|^2 \lambda \sim |\phi(\overline{n})| \lesssim \sigma_{\max} := \max_{j=0,\ldots,3} |\sigma_j|,$$

where $\lambda \in \{|n_1+n_2|, |n_1+n_3|, |n_2+n_3|\}$. Let $\mu_j = (\tau_j, n_j)$, $j = 1, \ldots, 3$, $\mu = (\tau, n)$ and assume that $\sigma_{\max} = |\sigma_0|$, as the remaining cases can be handled analogously. In order to extend the integral from $[0, t]$ to the whole real line, we must associate the time-cutoff with one of the factors. We can always choose one of the three factors which does not have the largest modulation $\sigma_{\max}$, for example $\widehat{u}(t, n_1)$. Using Parseval's identity, we have that

$$\begin{aligned}
&\big|P\big(\mathbf{P}_{>N} u(t)\big) - P\big(\mathbf{P}_{>N} u(0)\big)\big| \\
&\sim \left| 2\,\mathrm{Im} \int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} n n_1 \mathcal{F}_{t,x}\big(\mathbb{1}_{[0,t]} u\big)(\mu_1) \overline{\widehat{u}}(-\mu_2) \widehat{u}(\mu_3) \overline{\widehat{u}}(\mu)\, d\tau_1\, d\tau_2\, d\tau_3 \right| \\
&\lesssim \int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{|n n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}} \widehat{f}_1(\mu_1) \widehat{f}_2(\mu_2) \widehat{f}_2(\mu_3) \widehat{f}_3(\mu)\, d\tau_1\, d\tau_2\, d\tau_3, \qquad (2.22)
\end{aligned}$$

where $\widehat{f}_1(\tau, n) = \big|\mathcal{F}_{t,x}(\mathbb{1}_{[0,t]} u)(\tau, n)\big|$, $\widehat{f}_2(\tau, n) = |\widehat{u}(\tau, n)|$, $\widehat{f}_3(\tau, n) = \langle \tau - n^3 \rangle^{\frac{1}{2}} |\widehat{u}(\tau, n)|$. We focus on estimating the spatial multiplier in (2.22).

In Case 1 (i), $|\phi(\overline{n})| \sim |n_{\max}|^2 |n_1 + n_{\text{med}}|$, while in Case 1 (ii) we have $|\phi(\overline{n})| \sim |n_{\max}|^2 |n_{\text{med}} + n_{\max}|$. Therefore,

$$\frac{|n n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}} \lesssim \frac{|n n_1|}{|n_{\max}|} \lesssim |n|^{\frac{1}{4}} |n_1|^{\frac{3}{4}} \lesssim N^{0-} |n n_1 n_2 n_3|^{\frac{1}{4}+}.$$

In Case 2, we see that $|\phi(\overline{n})| \sim |n_{\max}|^2 |n_1|$, which implies that

$$\frac{|n n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}} \lesssim \frac{|n n_1|}{|n_{\max}| |n_1|^{\frac{1}{2}}} \lesssim |n_1|^{\frac{1}{2}} \lesssim N^{0-} |n n_1 n_{\max}|^{\frac{1}{6}+}.$$

In Case 3, we have $|\phi(\overline{n})| \sim |n_1|^2 |n_1 + n_{\text{med}}|$, from which we get

$$\frac{|n n_1|}{|\phi(\overline{n})|^{\frac{1}{2}}} \lesssim \frac{|n n_1|}{|n_1|} \lesssim |n| \lesssim N^{0-} |n n_1 n_{\text{med}}|^{\frac{1}{3}+}.$$

In Case 4 (i), $|\phi(\overline{n})| \sim |n_1|^2|n_{\min}+n_{\text{med}}|$ and we can estimate the multiplier as

$$\frac{|nn_1|}{|\phi(\overline{n})|^{\frac{1}{2}}} \lesssim \frac{|nn_1|}{|n_1|} \lesssim |n_1| \lesssim N^{0-}|nn_1n_2n_3|^{\frac{1}{3}+}. \tag{2.23}$$

In Case 4 (ii), $|\phi(\overline{n})| \sim |n_1|^2|n_{\text{med}}| \gtrsim |n_1|^{\frac{5}{2}}$. Thus,

$$\frac{|nn_1|}{|\phi(\overline{n})|^{\frac{1}{2}}} \sim \frac{|nn_1|}{|n_1|^{\frac{5}{4}}} \lesssim |n_1|^{\frac{3}{4}} \lesssim N^{0-}|nn_1n_{\text{med}}|^{\frac{3}{10}+}.$$

The worst estimate for the multiplier comes from Case 4 (i) in (2.23), where we must associate $\frac{1}{3}$ spatial derivatives to each function. Consequently, we can estimate Cases 1–4 by using Hölder's inequality, L^6-Strichartz (2.4) and Lemma 2.1.3, as follows

$$\begin{aligned}
(2.22) &\lesssim \frac{1}{N^{0+}}\|D^{\frac{1}{3}+}f_1\cdot(D^{\frac{1}{3}+}f_2)^2\cdot D^{\frac{1}{3}+}f_3\|_{L^1_{t,x}}\\
&\lesssim \frac{1}{N^{0+}}\|D^{\frac{1}{3}+}f_1\|_{L^6_{t,x}}\|D^{\frac{1}{3}+}f_2\|^2_{L^6_{t,x}}\|D^{\frac{1}{3}+}f_3\|_{L^2_{t,x}}\\
&\lesssim \frac{1}{N^{0+}}\|\mathbb{1}_{[0,t]}u\|_{X^{\frac{1}{3}+,\frac{1}{2}-}_{2,2}}\|u\|^2_{X^{\frac{1}{3}+,\frac{1}{2}-}_{2,2}}\|u\|_{X^{\frac{1}{3}+,\frac{1}{2}}_{2,2}}\\
&\lesssim \frac{1}{N^{0+}}\|u\|^4_{X^{s,\frac{1}{2}}_{p,2}},
\end{aligned}$$

for $1 \le p < \infty$ and $s > \max\left(\frac{1}{3}, \frac{5}{6}-\frac{1}{p}\right)$.

Cases 5–6

Since $P(\mathbf{P}_{>N}u(t)) = P(\mathbf{P}_{>N}v(t))$, where $v(t) = S(-t)u(t)$ stands for the interaction representation, the difference of momenta can be written as follows, in terms of v,

$$\begin{aligned}
&P\big(\mathbf{P}_{>N}v(t)\big) - P\big(\mathbf{P}_{>N}(v(0)\big)\\
&\qquad = -2\operatorname{Im}\int_0^t \sum_{|n|>N}\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} nn_1e^{-it'\phi(\overline{n})}\widehat{v}(t',n_1)\overline{\widehat{v}}(t',-n_2)\widehat{v}(t',n_3)\overline{\widehat{v}}(t',n)\,dt'.
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
&\operatorname{Im}\int_0^t \sum_{|n|>N}\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} nn_1\frac{d}{dt}\left(\frac{e^{-it'\phi(\overline{n})}}{-i\phi(\overline{n})}\right)\widehat{v}(t',n_1)\overline{\widehat{v}}(t',-n_2)\widehat{v}(t',n_3)\overline{\widehat{v}}(t',n)\,dt'\\
&\quad = -\operatorname{Re}\sum_{|n|>N}\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\frac{nn_1}{\phi(\overline{n})}\Big(e^{-it\phi(\overline{n})}\widehat{v}(t,n_1)\overline{\widehat{v}}(t,-n_2)\widehat{v}(t,n_3)\overline{\widehat{v}}(t,n)\\
&\qquad\qquad - \widehat{v}(0,n_1)\overline{\widehat{v}}(0,-n_2)\widehat{v}(0,n_3)\overline{\widehat{v}}(0,n)\Big)\\
&\quad + \operatorname{Re}\int_0^t\sum_{|n|>N}\sum_{\substack{n=n_1+n_2+n_3\\ \phi(\overline{n})\neq 0}}\frac{nn_1}{\phi(\overline{n})}e^{-it'\phi(\overline{n})}\partial_t\big\{\widehat{v}(t',n_1)\overline{\widehat{v}}(t',-n_2)\widehat{v}(t',n_3)\overline{\widehat{v}}(t',n)\big\}\,dt'.
\end{aligned}$$

In order to estimate the last term on the right-hand side, we use the equation for v again, substituting the time derivative by the corresponding resonant and non-resonant nonlinear terms. Therefore, writing the terms depending on u, we are interested in estimating the following quantities, omitting the time dependence within the integrals,

$$\mathcal{B}(t) = \operatorname{Re}\sum_{|n|>N}\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}}\frac{nn_1}{\phi(\overline{n})}\widehat{u}(t,n_1)\overline{\widehat{u}}(t,-n_2)\widehat{u}(t,n_3)\overline{\widehat{u}}(t,n),$$

$$\mathcal{R}_0 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{n^2 n_1}{\phi(\overline{n})} \widehat{u}(n_1)\overline{\widehat{u}}(-n_2)\widehat{u}(n_3)\overline{\widehat{u}}(n)|\widehat{u}(n)|^2\, dt',$$

$$\mathcal{R}_1 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{n n_1^2}{\phi(\overline{n})} \widehat{u}(n_1)|\widehat{u}(n_1)|^2\overline{\widehat{u}}(-n_2)\widehat{u}(n_3)\overline{\widehat{u}}(n)\, dt',$$

$$\mathcal{R}_2 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{n n_1 n_2}{\phi(\overline{n})} \widehat{u}(n_1)\overline{\widehat{u}}(-n_2)|\widehat{u}(-n_2)|^2\widehat{u}(n_3)\overline{\widehat{u}}(n)\, dt',$$

$$\mathcal{R}_3 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{n n_1 n_3}{\phi(\overline{n})} \widehat{u}(n_1)\overline{\widehat{u}}(-n_2)\widehat{u}(n_3)|\widehat{u}(n_3)|^2\overline{\widehat{u}}(n)\, dt',$$

$$\mathcal{NR}_0 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{-n=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{n n_1 m_1}{\phi(\overline{n})} \widehat{u}(n_1)\overline{\widehat{u}}(-n_2)\widehat{u}(n_3)\overline{\widehat{u}}(-m_1)\widehat{u}(m_2)\overline{\widehat{u}}(-m_3)\, dt',$$

$$\mathcal{NR}_1 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{n_1=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{n n_1 m_1}{\phi(\overline{n})} \overline{\widehat{u}}(-n_2)\widehat{u}(n_3)\overline{\widehat{u}}(n)\widehat{u}(m_1)\overline{\widehat{u}}(-m_2)\widehat{u}(m_3)\, dt',$$

$$\mathcal{NR}_2 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{n_2=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{n n_1 m_1}{\phi(\overline{n})} \widehat{u}(n_1)\widehat{u}(n_3)\overline{\widehat{u}}(n)\overline{\widehat{u}}(-m_1)\widehat{u}(m_2)\overline{\widehat{u}}(-m_3)\, dt',$$

$$\mathcal{NR}_3 = \operatorname{Im}\int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{n_3=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{n n_1 m_1}{\phi(\overline{n})} \widehat{u}(n_1)\overline{\widehat{u}}(-n_2)\overline{\widehat{u}}(n)\widehat{u}(m_1)\overline{\widehat{u}}(-m_2)\widehat{u}(m_3)\, dt',$$

where $\overline{m} = (m_1, m_2, m_3)$ and $\phi(\overline{m}) = 3(m_1+m_2)(m_1+m_3)(m_2+m_3)$.

• **Estimate for $\mathcal{B}(t)$**

Case 5: $|n_1| \sim |n_2| \sim |n_3|$

Note that $|\phi(\overline{n})| \sim |n_1|\lambda_1\lambda_2$, where $\lambda_1, \lambda_2 \in \{|n_1+n_2|, |n_1+n_3|, |n_2+n_3|\}$, $\lambda_1 \neq \lambda_2$. Assume that $\lambda_1 = |n_1+n_3|, \lambda_2 = |n_2+n_3|$. We will omit the estimate for the remaining choices of λ_1, λ_2, as it follows an analogous approach. Therefore, we have that

$$\frac{|nn_1|}{|\phi(\overline{n})|(\langle n\rangle\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle)^{\frac14+}} \lesssim \frac{1}{N^{0+}\langle n_1+n_3\rangle\langle n_2+n_3\rangle}.$$

Hence, with $g(t,n) = \langle n\rangle^s|\widehat{u}(t,n)|$, using Hölder's inequality and the fact that $|n| \lesssim |n_j|$, $j = 1,2,3$, it follows that

$$\begin{aligned}
|\mathcal{B}(t)| &\lesssim \frac{1}{N^{0+}} \sum_{n,n_1,n_2} \frac{g(t,n_1)g(t,-n_2)g(t,n-n_1-n_2)g(t,n)}{\langle n-n_2\rangle\langle n-n_1\rangle\langle n\rangle^{4(s-\frac14-)}} \\
&\lesssim \frac{1}{N^{0+}} \left(\sum_{n,n_1,n_2} \frac{g(t,n)^{p'}}{\langle n-n_2\rangle^{p'}\langle n-n_1\rangle^{p'}\langle n\rangle^{4(s-\frac14-)p'}} \right)^{\frac{1}{p'}} \|g(t)\|^3_{\ell^p_n} \\
&\lesssim \frac{1}{N^{0+}} \|u(t)\|^4_{\mathcal{F}L^{s,p}},
\end{aligned}$$

for $1 \leq p < \infty$ and $s > \max\left(\frac12 - \frac{1}{2p}, \frac14\right)$.

Case 6: $|n_1|^{\frac12} \gg |n_{\text{med}}| \gtrsim |n_{\min}|$

Assume that $n_{\text{med}} = n_2, n_{\min} = n_3$, as the estimate is analogous otherwise. Since $|\phi(\overline{n})| \sim |n_1|^2|n_2+n_3|$, we control the multiplier as follows

$$\frac{|nn_1|}{|\phi(\overline{n})|} \lesssim \frac{1}{\langle n_2+n_3\rangle}.$$

Using the fact that $|n_2| \gtrsim |n_3|$, Hölder's inequality, and Lemma 2.1.5, we have

$$\begin{aligned}
|\mathcal{B}(t)| &\lesssim \frac{1}{N^{0+}} \Bigg(\sum_{n_1,n_2,n_3} \frac{g(t,n_3)^{p'}}{\langle n_2+n_3\rangle^{p'} \langle n_1\rangle^{sp'} \langle n_3\rangle^{2sp'} \langle n_1+n_2+n_3\rangle^{sp'-}} \Bigg)^{\frac{1}{p'}} \|g(t)\|^3_{\ell^p_n} \\
&\lesssim \frac{1}{N^{0+}} \Bigg(\sum_{n_2,n_3} \frac{g(t,n_3)^{p'}}{\langle n_2+n_3\rangle^{p'} \langle n_3\rangle^{2sp'}} \Bigg)^{\frac{1}{p'}} \|g(t)\|^3_{\ell^p_n} \\
&\lesssim \frac{1}{N^{0+}} \|u(t)\|^4_{\mathcal{F}L^{s,p}},
\end{aligned}$$

for $1 \leq p < \infty$ and $s > \frac{1}{2} - \frac{1}{2p}$.

• **Estimate for $\mathcal{R}_j$, $j = 0,1,2,3$**

We now focus on estimating $\mathcal{R}_0$. The estimate for the remaining contributions follows by a similar approach. Let the following notation denote the modulations of the 6 factors

$$\begin{aligned}
&\sigma_j = \tau_j - n_j^3, \ j = 1,2,3, \\
&\sigma_4 = \tau_4 + n^3, \quad \sigma_5 = \tau_5 - n^3, \quad \sigma_6 = \tau_6 + n^3,
\end{aligned}$$

which implies that $|\phi(\overline{n})| = |\sigma_1 + \ldots + \sigma_6| \lesssim \max_{j=1,\ldots,6} |\sigma_j|$. Assume that $|\sigma_1|$ is the largest modulation. Then, we can associate the time cut-off with the second factor. If another $|\sigma_j|$ is the largest modulation, we can associate the cut-off with the first factor, for example, and the estimate follows analogously. Note that we can rewrite $\mathcal{R}_0$ as follows

$$\begin{aligned}
\mathcal{R}_0 &= \operatorname{Im} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{n^2 n_1}{\phi(\overline{n})} \mathcal{F}_t \Big(\widehat{u}(n_1) (\mathbb{1}_{[0,t]} \overline{\widehat{u}})(-n_2) \widehat{u}(n_3) \overline{\widehat{u}}(n) \widehat{u}(n) \overline{\widehat{u}}(n) \Big)(0) \\
&= \operatorname{Im} \int_{\tau_1+\ldots+\tau_6=0} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{n^2 n_1}{\phi(\overline{n})} \widehat{u}(\tau_1,n_1) \overline{\mathcal{F}\big(\mathbb{1}_{[0,t]} u\big)}(-\tau_2,-n_2) \\
&\qquad\qquad \times \widehat{u}(\tau_3,n_3) \overline{\widehat{u}}(-\tau_4,n) \widehat{u}(\tau_5,n) \overline{\widehat{u}}(-\tau_6,n) \ d\tau_1 \cdots d\tau_5.
\end{aligned}$$

Using the following notation

$$\begin{aligned}
&g_1(\tau,n) = \langle n\rangle^s \langle \tau - n^3\rangle^{\frac{1}{2}} |\widehat{u}(\tau,n)|, \\
&g_2(\tau,n) = \langle n\rangle^s \langle \tau - n^3\rangle^{\frac{1}{2}-} \big|\mathcal{F}\big(\mathbb{1}_{[0,t]} u\big)(\tau,n)\big|,
\end{aligned}$$

we apply Cauchy-Schwarz inequality to obtain the following estimate

$$\begin{aligned}
|\mathcal{R}_0| \lesssim \frac{1}{N^{0+}} &\sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{|n|^{2+}|n_1|}{|\phi(\overline{n})|^{\frac{3}{2}} (\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle)^s \langle n\rangle^{3s}} \|g_2(-n_2)\|_{L^2_\tau} \\
&\times \|g_1(n_3)\|_{L^2_\tau} \|g_1(n)\|^3_{L^2_\tau} \Bigg(\int_{\tau_1+\ldots+\tau_6=0} \frac{|g_1(\tau_1,n_1)|^2}{\langle\sigma_2\rangle^{1-}\langle\sigma_3\rangle \cdots \langle\sigma_6\rangle} \ d\tau_1 \ldots d\tau_5 \Bigg)^{\frac{1}{2}}.
\end{aligned}$$

By applying Lemma 2.1.4, we estimate the last factor on the right-hand side by $\|g_2(n_1)\|_{L^2_\tau}$ and the problem reduces to showing

$$\begin{aligned}
&\sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{|n|^{2+}|n_1|}{|\phi(\overline{n})|^{\frac{3}{2}} (\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle)^s \langle n\rangle^{3s}} \\
&\qquad \times \|g_1(n_1)\|_{L^2_\tau} \|g_2(-n_2)\|_{L^2_\tau} \|g_1(n_3)\|_{L^2_\tau} \|g_1(n)\|^3_{L^2_\tau} \lesssim \|g_1\|^5_{\ell^p_n L^2_\tau} \|g_2\|_{\ell^p_n L^2_\tau}, \qquad (2.24)
\end{aligned}$$

since $\|g_1\|_{\ell^p_n L^2_\tau} \lesssim \|u\|_{X^{s,\frac{1}{2}}_{p,2}}$ and $\|g_2\|_{\ell^p_n L^2_\tau} = \|u\|_{X^{s,\frac{1}{2}-}_{p,2}}$, from Lemma 2.1.3 for the second term.

Case 5: $|n_1| \sim |n_2| \sim |n_3|$

Since $|\phi(\overline{n})| \gtrsim |n_1|\lambda_1\lambda_2$, for $\lambda_j = |n - n'_j|$, $j = 1, 2$, and $n'_1, n'_2 \in \{n_1, n_2, n_3\}$ distinct, we have the following

$$\frac{|n|^{2+}|n_1|}{|\phi(\overline{n})|^{\frac{3}{2}}(\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle)^{\frac{1}{4}+}\langle n\rangle^{\frac{3}{4}+}} \lesssim \frac{1}{\langle n - n'_1\rangle^{\frac{3}{2}}\langle n - n'_2\rangle^{\frac{3}{2}}}.$$

Then, since $|n| \lesssim |n_j|$, $j = 1, 2, 3$, using Hölder's inequality gives

$$\begin{aligned}\text{LHS of (2.24)} &\lesssim \Bigg(\sum_{n,n'_1,n'_2} \frac{\|g_1(n)\|^{3p'}_{L^2_\tau}}{\langle n - n'_1\rangle^{1+}\langle n - n'_2\rangle^{1+}\langle n\rangle^{6(s-\frac{1}{4}-)p'}}\Bigg)^{\frac{1}{p'}} \|g_1\|^2_{\ell^p_n L^2_\tau}\|g_2\|_{\ell^p_n L^2_\tau} \\ &\lesssim \Bigg(\sum_{n} \frac{\|g_1(n)\|^{3p'}_{L^2_\tau}}{\langle n\rangle^{6(s-\frac{1}{4}-)p'}}\Bigg)^{\frac{1}{p'}} \|g_1\|^2_{\ell^p_n L^2_\tau}\|g_2\|_{\ell^p_n L^2_\tau}\end{aligned}$$

where the last inequality follows if $1 \leq p < \infty$ and $s > \max\left(\frac{5}{12} - \frac{2}{3p}, \frac{1}{4}\right)$.

Case 6: $|n_1|^{\frac{1}{2}} \gg |n_{\text{med}}| \gtrsim |n_{\text{min}}|$

As before, we can assume without loss of generality that $n_{\text{med}} = n_2$ and $n_{\text{min}} = n_3$. Since $|\phi(\overline{n})| \sim |n_1|^2|n_2 + n_3|$ and $|n_1| \sim |n| \gg |n_2| \gtrsim |n_3|$, we have

$$\frac{|n|^{2+}|n_1|}{|\phi(\overline{n})|^{\frac{3}{2}}} \lesssim \frac{|n|^{0+}}{\langle n_2 + n_3\rangle^{\frac{3}{2}}}.$$

Using Holder's inequality, it follows that

$$\begin{aligned}\text{LHS of (2.24)} &\lesssim \Bigg(\sum_{n_2,n_3,n} \frac{\|g_1(n)\|^{3p'}_{L^2_\tau}}{\langle n_2 + n_3\rangle^{1+}\langle n_2\rangle^{sp'}\langle n_3\rangle^{sp'}\langle n\rangle^{4sp'-}}\Bigg)^{\frac{1}{p'}} \|g_1\|^2_{\ell^p_n L^2_\tau}\|g_2\|_{\ell^p_n L^2_\tau} \\ &\lesssim \Bigg(\sum_{n_3} \frac{1}{\langle n_3\rangle^{2sp'-}} \sum_{n} \frac{\|g_1(n)\|^{3p'}_{L^2_\tau}}{\langle n\rangle^{4sp'-}}\Bigg)^{\frac{1}{p'}} \|g_1\|^2_{\ell^p_n L^2_\tau}\|g_2\|_{\ell^p_n L^2_\tau}\end{aligned}$$

and the estimate follows if $1 \leq p < \infty$ and $s > \frac{1}{2} - \frac{1}{2p}$.

• **Estimate for $\mathcal{NR}_0$, $\mathcal{NR}_1$**

We will omit the estimate for $\mathcal{NR}_1$ and focus on $\mathcal{NR}_0$. Let the following denote the modulations of the 6 factors

$$\begin{aligned}&\sigma_j = \tau_j - n_j^3,\ j = 1, 2, 3, \\ &\sigma_4 = \tau_4 - m_1^3 \quad \sigma_5 = \tau_5 - m_2^3, \quad \sigma_6 = \tau_6 - m_3^3,\end{aligned}$$

which implies that $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 = \phi(\overline{n}) + \phi(\overline{m})$. Thus, we will consider two regions:

$$|\phi(\overline{m})| \lesssim |\phi(\overline{n}) + \phi(\overline{m})|, \tag{2.25}$$
$$|\phi(\overline{m})| \gg |\phi(\overline{n}) + \phi(\overline{m})|. \tag{2.26}$$

If (2.25) holds, we can use the largest modulation to gain a power of $|\phi(\overline{m})|^{\frac{1}{2}}$. For (2.26), we have no gain from the largest modulation, so we will use Strichartz estimates and the fact that $|\phi(\overline{n})| \sim |\phi(\overline{m})|$. Note that we can rewrite $\mathcal{NR}_0$ as follows

$$\begin{aligned}\mathcal{NR}_0 = \text{Im} \int_{\tau_1+\ldots+\tau_6=0} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{-n=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} &\frac{nn_1m_1}{\phi(\overline{n})}\widehat{u}(\tau_1, n_1)\overline{\mathcal{F}(\mathbb{1}_{[0,t]}u)}(-\tau_2, -n_2) \\ &\times \widehat{u}(\tau_3, n_3)\overline{\widehat{u}}(-\tau_4, -m_1)\widehat{u}(\tau_5, m_2)\overline{\widehat{u}}(-\tau_6, -m_3)\, d\tau_1 \cdots d\tau_5.\end{aligned}$$

Consider the case (2.25) and proceed as in the estimate for $\mathcal{R}_0$. Assuming that we can associate the time cut-off with the second factor, we have

$$|\mathcal{N}\mathcal{R}_0| \lesssim \frac{1}{N^{0+}} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{-n=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{|n|^{1+}|n_1 m_1|}{|\phi(\overline{n})||\phi(\overline{m})|^{\frac{1}{2}} \prod_{j=1}^{3} \langle n_j\rangle^s \langle m_j\rangle^s} \|g_1(n_1)\|_{L^2_\tau}$$
$$\times \|g_2(-n_2)\|_{L^2_\tau} \|g_1(n_3)\|_{L^2_\tau} \|g_1(-m_1)\|_{L^2_\tau} \|g_1(m_2)\|_{L^2_\tau} \|g_1(-m_3)\|_{L^2_\tau}. \quad (2.27)$$

For simplicity, we can apply Lemma 2.1.3 to obtain $\|g_2\|_{L^2_\tau} \lesssim \|g_1\|_{L^2_\tau}$. In order to control the multiplier in (2.27), we must take into account the value of $\phi(\overline{m})$ and the relation between the frequencies of the first generation n_1, n_2, n_3.

Case 5 and (2.25): $|n_1| \sim |n_2| \sim |n_3|$

If $|m_1| \sim |m_2| \sim |m_3|$ and $|\phi(\overline{m})| \gtrsim |m_1||n+m_1'||n+m_2'|$, for some distinct $m_1', m_2' \in \{m_1, m_2, m_3\}$, we have

$$\frac{|n|^{1+}|n_1 m_1|}{|\phi(\overline{n})||\phi(\overline{m})|^{\frac{1}{2}}} \lesssim \frac{|n_1 n_2 n_3|^{\frac{1}{3}+}|m_1 m_2|^{\frac{1}{4}+}}{\langle n-n_1'\rangle\langle n-n_2'\rangle\langle n+m_1'\rangle^{\frac{1}{2}+}\langle n+m_2'\rangle^{\frac{1}{2}+}},$$

for some distinct $n_1', n_2' \in \{n_1, n_2, n_3\}$. Using Hölder's inequality, we get

$$\begin{aligned}
(2.27) &\lesssim \frac{1}{N^{0+}} \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \frac{\|g_1(n_1)\|_{L^2_\tau}\|g_1(-n_2)\|_{L^2_\tau}\|g_1(n_3)\|_{L^2_\tau}}{\langle n-n_1'\rangle\langle n-n_2'\rangle(\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle)^{s-\frac{1}{3}-}} \Bigg\|_{\ell^2_n} \\
&\quad \times \Bigg\| \sum_{\substack{-n=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{\|g_1(-m_1)\|_{L^2_\tau}\|g_1(m_2)\|_{L^2_\tau}\|g_1(-m_3)\|_{L^2_\tau}}{\langle n+m_1'\rangle^{\frac{1}{2}+}\langle n+m_2'\rangle^{\frac{1}{2}+}(\langle m_1\rangle\langle m_2\rangle\langle m_3\rangle)^{s-\frac{1}{3}-}} \Bigg\|_{\ell^2_n} \\
&\lesssim \frac{1}{N^{0+}} \sup_n \left(\sum_{n_1',n_2',m_1',m_2'} \frac{1}{\langle n-n_1'\rangle^{1+}\langle n-n_2'\rangle^{1+}\langle n+m_1'\rangle^{1+}\langle n+m_2'\rangle^{1+}} \right)^{\frac{1}{2}} \left\| \frac{g_1}{\langle n\rangle^{s-\frac{1}{3}-}} \right\|^6_{\ell^2_n L^2_\tau} \\
&\lesssim \frac{1}{N^{0+}} \|g_1\|^6_{\ell^p_n L^2_\tau} = \frac{1}{N^{0+}} \|u\|^6_{X^{s,\frac{1}{2}}_{p,2}},
\end{aligned}$$

for $1 \leq p < \infty$ and $s > \max\left(\frac{1}{3}, \frac{5}{6} - \frac{1}{p}\right)$. In the remaining regions of frequency space for m_1, m_2, m_3, we have $|\phi(\overline{m})| \gtrsim |m_{\max}|^2 \lambda'$, for $\lambda' \in \{|m_{\max}+m_{\text{med}}|, |m_{\text{med}}+m_{\min}|\}$. Thus,

$$\frac{|n|^{1+}|n_1 m_1|}{|\phi(\overline{n})||\phi(\overline{m})|^{\frac{1}{2}}} \lesssim \frac{|n_1 n_2 n_3|^{\frac{1}{3}+}}{\langle n-n_1'\rangle\langle n-n_2'\rangle\langle \lambda'\rangle^{\frac{1}{2}}}.$$

Since $(\langle m_{\max}\rangle\langle m_{\text{med}}\rangle)^{-\frac{1}{3}+} \lesssim \langle m_{\text{med}}\rangle^{-\frac{2}{3}-}$, we can proceed as in the previous case, with $\langle\lambda'\rangle^{\frac{1}{2}+}\langle m_{\text{med}}\rangle^{\frac{2}{3}+}$ instead of $\langle n+m_1'\rangle^{\frac{1}{2}+}\langle n+m_2'\rangle^{\frac{1}{2}+}$.

Case 6 and (2.25): $|n_1|^{\frac{1}{2}} \gg |n_{\text{med}}| \gtrsim |n_{\min}|$

Since we have

$$\frac{|n|^{1+}|n_1|}{|\phi(\overline{n})|} \lesssim \frac{|n_2 n_3|^{\frac{1}{4}+}}{\langle n_2+n_3\rangle\langle n_{\min}\rangle^{\frac{1}{2}+}},$$

we can follow the same argument in the previous case, substituting $\langle n-n_1'\rangle\langle n-n_2'\rangle$ by $\langle n_2 + n_3\rangle\langle n_{\min}\rangle^{\frac{1}{2}+}$, and proceeding as before when estimating $|\phi(\overline{m})|$.

Now, we must consider (2.26). Since we have $|\phi(\overline{n})+\phi(\overline{m})| \ll |\phi(\overline{m})|$, we can no longer trade the largest modulation by a $\frac{1}{2}$ power of $|\phi(\overline{m})|$. However, we know that $|\phi(\overline{n})| \sim |\phi(\overline{m})|$, which allows us to trade powers of $|\phi(\overline{n})|$ by powers of $|\phi(\overline{m})|$. Consequently, we focus on estimating the following multiplier

$$\frac{|n|^{1+}|n_1 m_1|}{|\phi(\overline{n})|^{\alpha}|\phi(\overline{m})|^{1-\alpha}}, \quad (2.28)$$

for some $0 \leq \alpha \leq 1$.

Case 5 and (2.26): $|n_1| \sim |n_2| \sim |n_3|$

Choosing $\alpha = 0$, we have

$$(2.28) \sim \frac{|n|^{1+}|n_1 m_1|}{|\phi(\overline{m})|} \lesssim \begin{cases} |n_1 n_2 n_3 m_1 m_2 m_3|^{\frac{1}{3}+}, & \text{if} \quad |m_1| \sim |m_2| \sim |m_3| \\ |n_1 n_2 n_3|^{\frac{1}{3}+}, & \text{if} \quad |\phi(\overline{m})| \gtrsim |m_{\max}|^2 \end{cases}.$$

Let $\widehat{h}_1(\tau, n) = \langle n \rangle^{\frac{1}{3}+} \mathcal{F}_{t,x}\big(\mathbb{1}_{[0,t]} u\big)(\tau, n)$, $\widehat{h}_2(\tau, n) = \langle n \rangle^{\frac{1}{3}+} |\widehat{u}(\tau, n)|$ and note that we can associate the cut-off with any factor. Using Hölder's inequality, the Strichartz estimate (2.4) and Lemma 2.1.3, we get

$$|\mathcal{NR}_0| \lesssim \frac{1}{N^{0+}} \|h_1 h_2^5\|_{L^1_{t,x}} \lesssim \frac{1}{N^{0+}} \|h_1\|_{L^6_{t,x}} \|h_2\|^5_{L^6_{t,x}} \lesssim \frac{1}{N^{0+}} \|u\|^6_{X^{s,\frac{1}{2}}_{p,2}},$$

for $1 \leq p < \infty$ and $s > \max\big(\frac{1}{3}, \frac{5}{6} - \frac{1}{p}\big)$.

Case 6 and (2.26): $|n_1|^{\frac{1}{2}} \gg |n_{\mathrm{med}}| \gtrsim |n_{\min}|$

If $|m_1| \sim |m_2| \sim |m_3|$, choosing $\alpha = 1$, gives

$$(2.28) \lesssim \frac{|n|^{1+}|n_1 m_1|}{|n_1|^2} \lesssim |m_1 m_2 m_3|^{\frac{1}{3}+}$$

and the result follows from the previous case. Now, assume that $|\phi(\overline{m})| \sim |m_{\max}|^2 \lambda'$ where $\lambda' \in \{|m_{\max} + m_{\mathrm{med}}|, |m_{\mathrm{med}} + m_{\min}|\}$. We must consider a finer case separation for the second generation of frequencies. For $\alpha = 0$, we can estimate the multiplier as follows

$$(2.28) \lesssim \frac{|n|^{1+}|n_1 m_1|}{|m_{\max}|^2 |\lambda'|} \lesssim \begin{cases} |n_1 m_1|^{\frac{1}{3}+} \max(|m_2|, |m_3|)^{\frac{1}{3}+}, & \text{if} \quad |m_1| \lesssim \max(|m_2|, |m_3|) \\ |n_1 m_1 m_2 m_3|^{\frac{1}{3}+}, & \text{if} \quad |m_1|^{\frac{1}{2}} \lesssim |m_{\min}| \leq |m_{\mathrm{med}}| \ll |m_1| \\ |n_1 m_1|^{\frac{1}{4}+}, & \text{if} \quad |m_{\min}| \ll |m_1|^{\frac{1}{2}} \lesssim |m_{\mathrm{med}}| \ll |m_1| \end{cases}$$

and use the strategy in the previous case. It only remains to consider the case when $|m_1|^{\frac{1}{2}} \gg |m_2|, |m_3|$. Consider the following decomposition

$$\begin{aligned} \mathcal{NR}_0 = \operatorname{Im} \int_0^t \sum_{|n|>N} \sum_{\substack{n = n_1 + n_2 + n_3, \\ \phi(\overline{n}) \neq 0}} \frac{m_1}{n_2 + n_3} \sum_{\substack{-n = m_1 + m_2 + m_3, \\ \phi(\overline{m}) \neq 0}} \left(\frac{n n_1}{(n_1 + n_3)(n_1 + n_2)} - 1 \right) \\ \times \widehat{u}(n_1) \widehat{\overline{u}}(-n_2) \widehat{u}(n_3) \widehat{\overline{u}}(-m_1) \widehat{u}(m_2) \widehat{\overline{u}}(-m_3) \, dt' \\ + \operatorname{Im} \int_0^t \sum_{|n|>N} \sum_{\substack{n = n_1 + n_2 + n_3, \\ \phi(\overline{n}) \neq 0}} \sum_{\substack{-n = m_1 + m_2 + m_3, \\ \phi(\overline{m}) \neq 0}} \frac{m_1}{n_2 + n_3} \\ \times \widehat{u}(n_1) \widehat{\overline{u}}(-n_2) \widehat{u}(n_3) \widehat{\overline{u}}(-m_1) \widehat{u}(m_2) \widehat{\overline{u}}(-m_3) \, dt' \\ =: \mathrm{I}_0 + \mathrm{II}_0. \end{aligned}$$

In order to estimate I_0, note that

$$n n_1 - (n_1 + n_3)(n_1 + n_2) = n_1^2 + (n_2 + n_3) n_1 - n_1^2 - n_1(n_2 + n_3) - n_2 n_3 = -n_2 n_3,$$

which implies that

$$\left| \frac{n n_1}{(n_1 + n_3)(n_1 + n_2)} - 1 \right| = \frac{|n_2 n_3|}{|(n_1 + n_3)(n_1 + n_2)|} \lesssim \frac{|n_1|}{|n_1|^2} \lesssim \frac{1}{|n_1|}.$$

Hence, using Hölder's inequality and the L^6-Strichartz estimate (2.4), we have

$$|\mathrm{I}_0| \lesssim \frac{1}{N^{0+}} \|\mathbb{1}_{[0,t]} u^6\|_{L^1_{t,x}} \lesssim \frac{1}{N^{0+}} \|\mathbb{1}_{[0,t]} u\|_{X^{0+,\frac{1}{2}-}_{2,2}} \|u\|^5_{X^{0+,\frac{1}{2}-}_{2,2}} \lesssim \frac{1}{N^{0+}} \|u\|^6_{X^{s,\frac{1}{2}}_{p,2}}$$

for $1 \le p < \infty$ and $s > \max\big(\frac{1}{2} - \frac{1}{p}, 0\big)$.

Now, we focus on estimating $\mathrm{I\!I}_0$. First, assume that $n_1 + m_1 \neq 0$. Then,

$$\begin{aligned}
|\phi(\overline{n}) + \phi(\overline{m})| &= |n^3 - n_1^3 - n_2^3 - n_3^3 - n^3 - m_1^3 - m_2^3 - m_3^3| \\
&= |(n_1 + n_3 + m_1)^3 - n_1^3 - n_3^3 - m_1^3 - (n_1 + n_3 + m_1)^3 - n_2^3 - m_2^3 - m_3^3| \\
&= |3(n_1 + m_1)(n_1 + n_3)(n_3 + m_1) + 3(n_2 + m_2)(n_2 + m_3)(m_2 + m_3)| \\
&\gtrsim |n_1|^2,
\end{aligned}$$

since $|(n_1+m_1)(n_1+n_3)(n_3+m_1)| \gtrsim |n_1|^2$ and $|(n_2+m_2)(n_2+m_3)(m_2+m_3)| \ll |n_1|^{\frac{3}{2}}$. Then, using the largest modulation, we have

$$\frac{|m_1|}{|\phi(\overline{n}) + \phi(\overline{m})|^{\frac{1}{2}}} \lesssim 1.$$

Proceeding as in (2.27), we first focus on estimating $\mathrm{I\!I}_0$ with respect to time

$$\begin{aligned}
|\mathrm{I\!I}_0| \lesssim \frac{1}{N^{0+}} \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n})\neq 0}} \sum_{\substack{-n=m_1+m_2+m_3,\\ \phi(\overline{m})\neq 0}} \frac{1}{\langle n_2+n_3\rangle \prod_{j=1}^3 (\langle n_j\rangle\langle m_j\rangle)^{s-}} \|g_1(n_1)\|_{L^2_\tau} \\
\times \|g_1(-n_2)\|_{L^2_\tau} \|g_1(n_3)\|_{L^2_\tau} \|g_1(-m_1)\|_{L^2_\tau} \|g_1(m_2)\|_{L^2_\tau} \|g_1(-m_3)\|_{L^2_\tau}. \qquad (2.29)
\end{aligned}$$

The estimate follows from the approach in Case 5 and (2.25), since

$$\frac{1}{\langle n_2+n_3\rangle \prod_j^3 (\langle n_j\rangle\langle m_j\rangle)^{\frac{1}{3}+}} \lesssim \frac{1}{\langle n_2+n_3\rangle \langle n_{\min}\rangle^{\frac{1}{2}+} \langle m_{\min}\rangle^{\frac{1}{2}+} \langle m_{\text{med}}\rangle^{\frac{1}{2}+}}.$$

On the other hand, if $n_1 + m_1 = 0$, we focus on the following quantity

$$\begin{aligned}
\mathrm{I\!I}_0 = \int_0^t \sum_{|n|>N} \sum_{\substack{n=n_1+n_2+n_3,\\ |n_2|,|n_3|\ll|n_1|^{\frac{1}{2}},\\ \phi(\overline{n})\neq 0}} \sum_{\substack{-n=m_1+m_2+m_3,\\ |m_2|,|m_3|\ll|n_1|^{\frac{1}{2}},\\ \phi(\overline{m})\neq 0}} \frac{-n_1}{n_2+n_3} \\
\times |\widehat{u}(n_1)|^2 \overline{\widehat{u}}(-n_2)\widehat{u}(n_3)\widehat{u}(-n_2-n_3-m_3)\overline{\widehat{u}}(-m_3)\,dt'.
\end{aligned}$$

In order to estimate this quantity, we need further assumptions on the frequencies. Let $0 < \varepsilon < 1$ denote the constant such that $|n_2|, |n_3|, |m_2|, |m_3| \le \varepsilon |n_1|^{\frac{1}{2}}$. We will consider two distinct cases: (i) $|n_2+n_3| > \varepsilon^2 |n_1|^{\frac{1}{2}}$; (ii) $|n_2+n_3| \le \varepsilon^2 |n_1|^{\frac{1}{2}}$.

If $|n_2+n_3| > \varepsilon^2 |n_1|^{\frac{1}{2}}$, then

$$\frac{|n_1|}{|n_2+n_3|\langle n_1\rangle^{\frac{1}{2}+}} \lesssim \frac{1}{N^{0+}}.$$

For simplicity, assume that $|n_3| \le |n_2|$ and $|m_3| \le |m_2|$. Note that to estimate the multiplier we only used $\frac{1}{2}+$ power of $|n_1|$, which leaves us with $\langle n_1\rangle^{-\frac{1}{6}-} \lesssim (\langle n_3\rangle\langle m_3\rangle)^{-\frac{1}{6}-}$ from the relation between the frequencies. Consequently, following a similar approach to (2.29) to handle the time integral, with $h(\tau,n) = \langle n\rangle^{\frac{1}{3}+}\langle \tau - n^3\rangle^{\frac{1}{2}-}|\widehat{u}(\tau,n)|$, and using Hölder's inequality, we obtain

$$\begin{aligned}
|\mathrm{I\!I}_0| &\lesssim \frac{1}{N^{0+}} \|h\|^2_{\ell^2_n L^2_\tau} \\
&\quad \times \sum_{n_2,n_3,m_3} \frac{\|h(-n_2)\|_{L^2_\tau}\|h(n_3)\|_{L^2_\tau}\|h(-m_3)\|_{L^2_\tau}\|h(-n_2-n_3-m_3)\|_{L^2_\tau}}{N^{0+}(\langle n_2\rangle\langle n_3\rangle\langle m_3\rangle\langle n_2+n_3+m_3\rangle)^{\frac{1}{3}+}(\langle n_3\rangle\langle m_3\rangle)^{\frac{1}{6}+}} \\
&\lesssim \frac{1}{N^{0+}} \Bigg(\sum_{n_3,n_2,m_3} \frac{\|h(-n_2)\|^2_{L^2_\tau}}{\langle n_3\rangle^{1+}\langle m_3\rangle^{1+}} \Bigg)^{\frac{1}{2}} \|u\|^5_{X^{s,\frac{1}{2}}_{p,2}}
\end{aligned}$$

$$\lesssim \frac{1}{N^{0+}}\|u\|^6_{X^{s,\frac12}_{p,2}},$$

for $1 \le p < \infty$ and $s > \max\left(\frac13, \frac56 - \frac1p\right)$.

It remains to estimate the case when (ii) $|n_2+n_3| \le \varepsilon^2|n_1|^{\frac12}$. Under this assumption and $|n_2|, |n_3| \le \varepsilon|n_1|^{\frac12}$, it follows that $|n_j| \le \varepsilon|n_1|^{\frac12} - |n_2+n_3|$ or $\varepsilon|n_1|^{\frac12} - |n_2+n_3| < |n_j| < \varepsilon|n_1|^{\frac12}$, $j = 1, 2$. For simplicity, let $|n_3| \le |n_2|$ and $|m_3| \le |n_2+n_3+m_3|$, as the result follows from an analogous approach for the remaining cases. We consider the following two regions of summation

$$\begin{aligned}
H_1 := \Big\{(n_2,n_3,m_3) \in \mathbb{Z}^3 :\ & |n_3|, |m_3| < \varepsilon|n_1|^{\frac12} - |n_2+n_3|,\\
& |n_2|, |n_2+n_3+m_3| < \varepsilon|n_1|^{\frac12},\ |n_2+n_3| < \varepsilon^2|n_1|^{\frac12}\Big\},\\
H_2 := \Big\{(n_2,n_3,m_3) \in \mathbb{Z}^3 :\ & |n_2|, |n_3|, |m_2|, |n_2+n_3+m_3| \le \varepsilon|n_1|^{\frac12},\\
& |n_3| \text{ or } |m_3| \ge \varepsilon|n_1|^{\frac12} - |n_2+n_3|,\ |n_2+n_3| < \varepsilon^2|n_1|^{\frac12}\Big\}.
\end{aligned}$$

We first consider the contribution restricted to the region H_2, when $|n_3| \ge \varepsilon|n_1|^{\frac12} - |n_2+n_3|$. Note that the following holds

$$|n_3| \ge \varepsilon|n_1|^{\frac12} - |n_2+n_3| \ge (\varepsilon - \varepsilon^2)|n_1|^{\frac12}.$$

Therefore, the multiplier can be controlled as follows

$$\frac{|n_1|}{|n_2+n_3|\langle n_1\rangle^{\frac23+}\langle n_2\rangle^{\frac13+}\langle n_3\rangle^{\frac13+}} \lesssim \frac{1}{N^{0+}|n_2+n_3|^{1+}}.$$

The estimate follows from the previous case for $s > \max\left(\frac13, \frac56 - \frac1p\right)$, $1 \le p < \infty$, using $\langle n_2 + n_3\rangle^{-1-}\langle m_3\rangle^{-\frac23-}$ to sum.

Now, consider the contribution localized on the region H_1, with the change of variables $n_2' = n_2 + n_3$,

$$\begin{aligned}
&\int_0^t \sum_{\substack{|n|>N,\\ |n_2'|<\varepsilon^2|n-n_2'|^{\frac12}}} \frac{n-n_2'}{n_2'}|\widehat{u}(n-n_2')|^2\\
&\qquad\qquad \times \left(\operatorname{Im} \sum_{\substack{|n_3|,|m_3|\\ <\varepsilon|n-n_2'|^{\frac12}-|n_2'|}} \widehat{u}(n_3)\overline{\widehat{u}}(n_3-n_2')\overline{\widehat{u}}(-m_3)\widehat{u}(-n_2'-m_3)\right) dt'.
\end{aligned}$$

Use J to denote the two inner sums. We can decompose J as follows

$$\begin{aligned}
J = \operatorname{Im}\Bigg(& \sum_{0<n_3,m_3<\varepsilon|n-n_2'|^{\frac12}-|n_2'|} \widehat{u}(n_3)\overline{\widehat{u}}(n_3-n_2')\overline{\widehat{u}}(-m_3)\widehat{u}(-n_2'-m_3)\\
&+ \sum_{0<n_3,m_3<\varepsilon|n-n_2'|^{\frac12}-|n_2'|} \widehat{u}(-n_3)\overline{\widehat{u}}(-n_3-n_2')\overline{\widehat{u}}(-m_3)\widehat{u}(-n_2'-m_3)\\
&+ \sum_{0<n_3,m_3<\varepsilon|n-n_2'|^{\frac12}-|n_2'|} \widehat{u}(n_3)\overline{\widehat{u}}(n_3-n_2')\overline{\widehat{u}}(m_3)\widehat{u}(-n_2'+m_3)\\
&+ \sum_{0<n_3,m_3<\varepsilon|n-n_2'|^{\frac12}-|n_2'|} \widehat{u}(-n_3)\overline{\widehat{u}}(-n_3-n_2')\overline{\widehat{u}}(m_3)\widehat{u}(-n_2'+m_3)\\
&+ \sum_{0<|n_3|<\varepsilon|n-n_2'|^{\frac12}-|n_2'|} \widehat{u}(0)\overline{\widehat{u}}(-n_2')\overline{\widehat{u}}(-n_3)\widehat{u}(-n_2'-n_3)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0<|n_3|<\varepsilon|n-n_2'|^{\frac{1}{2}}-|n_2'|} \widehat{u}(-n_3)\overline{\widehat{u}}(-n_3-n_2')\overline{\widehat{u}}(0)\widehat{u}(-n_2') \\
&+ \widehat{u}(0)\overline{\widehat{u}}(-n_2')\overline{\widehat{u}}(0)\widehat{u}(-n_2')\Big) \\
= \mathrm{Im} &\sum_{\substack{0<|n_3|,|m_3|<\varepsilon|n-n_2'|^{\frac{1}{2}}-|n_2'|,\\ n_3m_3>0}} \widehat{u}(n_3)\overline{\widehat{u}}(n_3-n_2')\overline{\widehat{u}}(m_3)\widehat{u}(-n_2'+m_3) \\
= \mathrm{Im} &\bigg(\frac{1}{2} \sum_{\substack{0<|n_3|,|m_3|<\varepsilon|n-n_2'|^{\frac{1}{2}}-|n_2'|,\\ n_3m_3>0,n_3\neq m_3}} \widehat{u}(n_3)\overline{\widehat{u}}(n_3-n_2')\overline{\widehat{u}}(m_3)\widehat{u}(-n_2'+m_3) \\
&+ \frac{1}{2} \sum_{\substack{0<|n_3|,|m_3|<\varepsilon|n-n_2'|^{\frac{1}{2}}-|n_2'|,\\ n_3m_3>0,n_3\neq m_3}} \widehat{u}(m_3)\overline{\widehat{u}}(m_3-n_2')\overline{\widehat{u}}(n_3)\widehat{u}(-n_2'+n_3) \\
&- \sum_{0<|n_3|<\varepsilon|n-n_2'|^{\frac{1}{2}}-|n_2'|} \widehat{u}(n_3)\overline{\widehat{u}}(n_3-n_2')\overline{\widehat{u}}(n_3)\widehat{u}(-n_2'+n_3)\bigg) = 0.
\end{aligned}
$$

This completes the estimate for the contribution $\mathcal{NR}_0$.

• **Estimate for $\mathcal{NR}_2, \mathcal{NR}_3$**
In order to control the contributions $\mathcal{NR}_2, \mathcal{NR}_3$, we will follow a similar approach to that of $\mathcal{NR}_0$. Most cases follow an analogous approach, but the estimate is significantly different in Case 6, when $|\phi(\overline{m})| \gtrsim |m_{\max}|^2$ and (2.26) hold.

In this case, we cannot use the maximum modulation to help estimate the multiplier. However, we can use the fact that $|\phi(\overline{n})| \sim |\phi(\overline{m})|$ to obtain the following

$$
\frac{|nn_1m_1|}{|\phi(\overline{n})|^{\alpha}|\phi(\overline{m})|^{1-\alpha}} \lesssim \frac{|n|^{1+}|n_1m_1|}{N^{0+}|n_1|^{2\alpha}|m_{\max}|^{2(1-\alpha)}}, \tag{2.30}
$$

for some $0 \leq \alpha \leq 1$. Estimating this multiplier requires more care than for the $\mathcal{NR}_0$ contribution since we cannot directly compare the sizes of $|n| \sim |n_1|$ and $|m_{\max}|$. We can estimate the multiplier as follows

$$
(2.30) \lesssim \begin{cases} |n_1m_1m_2m_3|^{\frac{1}{4}+}, & \text{if} \quad |m_1| \lesssim |m_2|, |m_3| \text{ and } \alpha = \frac{7}{8}, \\ |nn_1m_1m_{\max}|^{\frac{1}{4}+}, & \text{if} \quad |m_{\min}| \ll |m_1| \lesssim |m_{\max}| \text{ and } \alpha = \frac{3}{4}, \end{cases}
$$

and following the previous arguments, using Hölder's inequality and the L^6-Strichartz estimate (2.4). If $|m_1| \gg |m_2|, |m_3|$, then

$$
|\phi(\overline{n})| \gtrsim |n_1|^2 \gg |m_1|^2|m_2+m_3| \sim |\phi(\overline{m})|
$$

and from our assumption (2.26), we have $|\phi(\overline{m})| \gg |\phi(\overline{n}) + \phi(\overline{m})| \sim |\phi(\overline{n})|$, which cannot happen.

□

We can now use the energy estimate in Proposition 2.5.1 to show the conservation of momentum at low regularity.

Proof of Proposition 1.1.9. Let $u_{0,M} = \mathbf{P}_{\leq M}u_0$ and u_M be the corresponding smooth global solution of mKdV2 (2.2). Then, using Theorem 1.1.3, there exist a time $T = T\big(\|u_0\|_{\mathcal{F}L^{s,p}}\big) > 0$ and a solution $u \in C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big)$ of mKdV2 (2.2) such that

$$
u_M \to u \quad \text{in} \quad C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big), \tag{2.31}
$$

as $M \to \infty$. In order to show convergence of $\{P(\mathbf{P}_{\leq N}u(t))\}_{N\in\mathbb{N}}$, $t \in [-T,T]$, and its conservation, we will fix $t \in [-T, T]$ and prove the following

$$P(\mathbf{P}_{\le N} u(t)) = \lim_{M\to\infty} P(\mathbf{P}_{\le N} u_M(t)), \tag{2.32}$$

$$\lim_{N\to\infty} \lim_{M\to\infty} P(\mathbf{P}_{\le N} u_M(t)) = \lim_{M\to\infty} P(u_M(t)). \tag{2.33}$$

If the two equalities hold, we have

$$\lim_{N\to\infty} P(\mathbf{P}_{\le N} u(t)) = \lim_{M\to\infty} P(u_M(t)) = \lim_{M\to\infty} P(u_{0,M}) = \lim_{M\to\infty} P(\mathbf{P}_{\le M} u_0) = P(u_0),$$

using the conservation of momentum for smooth solutions u_M and the assumption of finite momentum of u_0, in the sense of Definition 1.1.8.

We start by showing (2.32). Note that, for each fixed $N \in \mathbb{N}$,

$$\begin{aligned}\big|P(\mathbf{P}_{\le N} u)(t) - P(\mathbf{P}_{\le N} u_M)(t)\big| &\le \sum_{|n|\le N} |n| \big|\widehat{u}(t,n) - \widehat{u}_M(t,n)\big| \big(|\widehat{u}(t,n)| + |\widehat{u}_M(t,n)|\big) \\ &\lesssim N^{\frac{p-2}{p}} \|u - u_M\|_{C_T \mathcal{F}L^{s,p}} \big(\|u\|_{C_T \mathcal{F}L^{s,p}} + \|u_M\|_{C_T \mathcal{F}L^{s,p}}\big),\end{aligned}$$

which implies (2.32) due to (2.31). Now, we want to show (2.33). Since $P(\mathbf{P}_{\le N} u_M(t)) = P(u_M(t)) - P(\mathbf{P}_{>N} u_M(t))$, we will focus on showing that the second term goes to zero. Note that

$$|P(\mathbf{P}_{>N} u_M(t))| \le |P(\mathbf{P}_{>N} u_M(t)) - P(\mathbf{P}_{>N} u_{0,M})| + |P(\mathbf{P}_{>N} u_{0,M})|. \tag{2.34}$$

Using Proposition 2.5.1, for some $0 < \varepsilon \ll 1$, we have

$$\begin{aligned}|P(\mathbf{P}_{>N} u_M(t)) - P(\mathbf{P}_{>N} u_{0,M})| &\lesssim N^{-\varepsilon} \big(\|u_M\|^4_{C_T \mathcal{F}L^{s,p}} + \|u_M\|^4_{X^{s,\frac{1}{2}}_{p,2}} + \|u_M\|^6_{X^{s,\frac{1}{2}}_{p,2}}\big) \\ &\lesssim N^{-\varepsilon} \big(\|u_0\|^4_{\mathcal{F}L^{s,p}} + \|u_0\|^6_{\mathcal{F}L^{s,p}}\big),\end{aligned}$$

which shows that $\lim_{N\to\infty} \lim_{M\to\infty} \big(P(\mathbf{P}_{>N} u_M(t)) - P(\mathbf{P}_{>N} u_{0,M})\big) = 0$. Focusing on the last term of (2.34), we have

$$P(\mathbf{P}_{>N} u_{0,M}) = P(\mathbf{P}_{>N} \mathbf{P}_{\le M} u_0) = P(\mathbf{P}_{\le M} u_0) - P(\mathbf{P}_{\le N} \mathbf{P}_{\le M} u_0).$$

For $M \ge N$, taking a limit as $M \to \infty$ first and then $N \to \infty$, both terms converge to $P(u_0)$ and the result follows.

□

2.6 Construction of solutions of mKdV1 outside $H^{\frac{1}{2}}(\mathbb{T})$

Proposition 1.1.9 gives a new interpretation of finite momentum and its conservation at low regularity. Exploiting this conservation, we can make sense of the nonlinearity of the complex-valued mKdV1 equation (2.1) and show the existence of solutions, outside $H^{\frac{1}{2}}(\mathbb{T})$.

Proof of Theorem 1.1.11. Let $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ with finite momentum in the sense of Definition 1.1.8. Given $N \in \mathbb{N}$, let $u_{0,N} = \mathbf{P}_{\le N} u_0$ and v_N be the corresponding smooth global solution of mKdV2 (2.2). From Theorem 1.1.3 and a persistence of regularity argument, we can show that there exists $T = T\big(\|u_0\|_{\mathcal{F}L^{s,p}(\mathbb{T})}\big) > 0$ and a solution $v \in Z^{s,\frac{1}{2}}_p(T)$ of mKdV2 (2.2) such that

$$v_N \to v \quad \text{in} \quad Z^{s,\frac{1}{2}}_p(T).$$

Since $\{v_N\}_{N\in\mathbb{N}}$ are smooth solutions, the conservation of momentum holds and $P(v_N(t)) = P(u_{0,N})$ for all $t \in \mathbb{R}$. Let $u_N := \mathcal{G}_2^{-1}[v_N] = e^{iP(u_{0,N})t} v_N$, which is a smooth global solution of mKdV1 (2.1) with initial data $u_{0,N}$, $N \in \mathbb{N}$. We want to show that the sequence $\{u_N\}_{N\in\mathbb{N}}$ converges to $u := e^{iP(u_0)t} v$ in $Z^{s,\frac{1}{2}}_p(T)$. The limit u will be our candidate solution in

$C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$. First, we have that

$$\begin{aligned}\|u_N-u\|_{C_T\mathcal{F}L^{s,p}} &\leq \big\|e^{iP(u_{0,N})t}(v_N-v)\big\|_{C_T\mathcal{F}L^{s,p}}+\big\|\big(e^{iP(u_{0,N})t}-e^{iP(u_0)t}\big)v\big\|_{C_T\mathcal{F}L^{s,p}}\\ &\leq \|v_N-v\|_{C_T\mathcal{F}L^{s,p}}+T\big|P(u_{0,N})-P(u_0)\big|\|v\|_{C_T\mathcal{F}L^{s,p}}\to 0,\end{aligned}$$

from the mean value theorem, the assumption on the momentum, and the convergence of $\{v_N\}_{N\in\mathbb{N}}$. Moreover, $u\in Z_p^{s,\frac12}(T)$, since

$$\|u\|_{Z_p^{s,\frac12}(T)}\lesssim \langle P(u_0)\rangle^{\frac12}\|v\|_{X_{p,2}^{s,\frac12}(T)}+\|v\|_{X_{p,1}^{s,0}(T)}<\infty.$$

If we show that the sequence $\{u_N\}_{N\in\mathbb{N}}$ is Cauchy in $Z_p^{s,\frac12}(T_*)$ for some $0<T_*\leq T$, the convergence to u in this space will follow. For $N,M\in\mathbb{N}$, u_N and u_M are smooth solutions of mKdV1 (2.1), thus using Lemma 2.1.1 and Proposition 2.2.1, we have

$$\begin{aligned}\|u_N-u_M\|_{Z_p^{s,\frac12}(T)} &\leq C_1\|u_{0,N}-u_{0,M}\|_{\mathcal{F}L^{s,p}}+C_2\Big(\big\|P(u_{0,N})u_N-P(u_{0,M})u_M\big\|_{Z_p^{s,-\frac12}(T)}\\ &\quad+\big\|\mathcal{N}(u_N,\overline{u}_N,u_N)-\mathcal{N}(u_M,\overline{u}_M,u_M)\big\|_{Z_p^{s,-\frac12}(T)}\Big)\\ &\leq C_1\|u_{0,N}-u_{0,M}\|_{\mathcal{F}L^{s,p}}+C_3T^\delta|P(u_{0,N})-P(u_{0,M})|\|u_N\|_{Z_p^{s,\frac12}(T)}\\ &\quad+C_4T^\delta\Big(\|u_N\|^2_{X_{p,2}^{s,\frac12}(T)}+\|u_M\|^2_{X_{p,2}^{s,\frac12}(T)}+|P(u_{0,M})|\Big)\|u_N-u_M\|_{X_{p,2}^{s,\frac12}(T)},\end{aligned}$$

for some constants $C_1,C_2,C_3,C_4>0$. By the definition of u_N and the continuous dependence on the initial data for mKdV2 (2.2), for large enough N, we have $\|u_N\|_{Z_p^{s,\frac12}(T)}\leq C(\|u_0\|_{\mathcal{F}L^{s,p}}+1)$, for some $C>0$. Analogously, for large enough N, $|P(u_{0,N})|\leq|P(u_0)|+1$. Consequently,

$$\begin{aligned}\|u_N-u_M\|_{Z_p^{s,\frac12}(T)} &\leq C_1\|u_{0,N}-u_{0,M}\|_{\mathcal{F}L^{s,p}}+CC_2T^\delta(\|u_0\|_{\mathcal{F}L^{s,p}}+1)|P(u_{0,N})-P(u_{0,M})|\\ &\quad+C_3T^\delta\Big(4C^2\big(\|u_0\|_{\mathcal{F}L^{s,p}}+1\big)^2+\big(|P(u_0)|+1\big)\Big)\|u_N-u_M\|_{X_{p,2}^{s,\frac12}(T)},\end{aligned}$$

for N,M large enough. Choosing $0<T_0\leq T$ such that

$$C_3T_0^\delta\Big(4C^2\big(\|u_0\|_{\mathcal{F}L^{s,p}}+1\big)^2+\big(|P(u_0)|+1\big)\Big)<\frac12,$$

it follows that

$$\begin{aligned}\|u_N-u_M\|_{Z_p^{s,\frac12}(T_0)} &\leq 2C_1\|u_{0,N}-u_{0,M}\|_{\mathcal{F}L^{s,p}}\\ &\quad+2CC_2T_0^\delta(\|u_0\|_{\mathcal{F}L^{s,p}}+1)|P(u_{0,N})-P(u_{0,M})|. \quad (2.35)\end{aligned}$$

By iterating this approach, we can cover the whole interval $[-T,T]$ and the estimate (2.35) holds with T instead of T_0. Thus, $\{u_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $Z_p^{s,\frac12}(T)$ and $u_N\to u$ in $Z_p^{s,\frac12}(T)$.

Now, we want to show that u satisfies mKdV1 (2.1) in the sense of distributions, with the nonlinearity interpreted as

$$\mathbf{N}(u):=\mathcal{N}(u,\overline{u},u)+iP(u_0)u.$$

Considering the linear part and any test function $\varphi\in C_c^\infty([-T,T]\times\mathbb{T})$, it follows that

$$\big|\langle u-u_N,(\partial_t+\partial_x^3)\varphi\rangle_{t,x}\big|\lesssim\|u-u_N\|_{X_{p,2}^{s,\frac12}(T)}\to 0,$$

as $N\to\infty$, which implies that $(\partial_t+\partial_x^3)u_N\to(\partial_t+\partial_x^3)u$ in the sense of distributions. For the nonlinearity, using the fact that $\mathbf{N}(u_N) = \mathcal{N}(u_N, \overline{u}_N, u_N) + iP(u_N)u_N$, it follows that

$$\left|\langle \mathbf{N}(u_N) - \mathbf{N}(u), \varphi\rangle_{t,x}\right| \lesssim \|\mathcal{N}(u_N, \overline{u}_N, u_N) - \mathcal{N}(u, \overline{u}, u)\|_{X^{s,-\frac{1}{2}}_{p,2}(T)} + |P(u_{0,N}) - P(u_0)| \|u_N\|_{X^{s,\frac{1}{2}}_{p,2}(T)} + |P(u_0)| \|u_N - u\|_{X^{s,\frac{1}{2}}_{p,2}(T)}.$$

Using the convergence of momentum $P(u_{0,N}) \to P(u_0)$ and of $\{u_N\}_{N\in\mathbb{N}}$, it suffices to estimate the first term on the right-hand side. We can write $\mathcal{N}(u_N, \overline{u}_N, u_N) - \mathcal{N}(u, \overline{u}, u) = \mathcal{N}(u_N - u, \overline{u}_N, u_N) + \mathcal{N}(u, \overline{u}_N - \overline{u}, u_N) + \mathcal{N}(u, \overline{u}, u_N - u)$ and using the nonlinear estimate in Proposition 2.2.1, we have that

$$\|\mathcal{N}(u_N, \overline{u}_N, u_N) - \mathcal{N}(u, \overline{u}, u)\|_{X^{s,-\frac{1}{2}}_{p,2}(T)} \lesssim \|u_N - u\|_{X^{s,\frac{1}{2}}_{p,2}(T)} \left(\|u_N\|_{X^{s,\frac{1}{2}}_{p,2}(T)} + \|u\|_{X^{s,\frac{1}{2}}_{p,2}(T)}\right)^2,$$

and the convergence follows from that of $\{u_N\}_{N\in\mathbb{N}}$. The limit u satisfies the following equation

$$\partial_t u + \partial_x^3 u = \mathcal{N}(u, \overline{u}, u) + iP(u_0)u,$$

in the sense of distributions, where $P(u_0)$ is interpreted in the sense of Definition 1.1.8.

□

2.7 Mild ill-posedness in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s < \frac{1}{2}$

In the following, we show the failure of local uniform continuity of the data-to-solution map of the complex-valued mKdV (1.5) on bounded sets of $\mathcal{F}L^{s,p}(\mathbb{T})$, for $1 \leq p \leq \infty$ and $s < \frac{1}{2}$. Proposition 1.1.4 follows once we estalish the following lemma. The proof follows an argument by Burq-Gérard-Tzvetkov [17] and Christ-Colliander-Tao [26].

Lemma 2.7.1. *Let $s < \frac{1}{2}$ and $1 \leq p \leq \infty$. Then, there exist two sequences $\{u_{0,n}\}_{n\in\mathbb{N}}$, $\{\tilde{u}_{0,n}\}_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{T})$ satisfying the following conditions:*

1. *$\{u_{0,n}\}_{n\in\mathbb{N}}$, $\{\tilde{u}_{0,n}\}_{n\in\mathbb{N}}$ are uniformly bounded in $\mathcal{F}L^{s,p}(\mathbb{T})$;*
2. *$\lim_{n\to\infty} \|u_{0,n} - \tilde{u}_{0,n}\|_{\mathcal{F}L^{s,p}} = 0$;*
3. *Let u_n, $\tilde{u}_n$ be the solutions of (2.2) with initial data $u_{0,n}$, $\tilde{u}_{0,n}$, respectively. Then, there exists $C > 0$ such that*

$$\liminf_{n\to\infty} \sup_{t\in[-T,T]} \|u_n(t) - \tilde{u}_n(t)\|_{\mathcal{F}L^{s,p}} \geq C,$$

for any $T > 0$.

Proof. Let $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Define $u^{N,a}$ as follows

$$u^{N,a}(t,x) := N^{-s} a e^{i(Nx + N^3 t \pm |a|^2 N^{1-2s} t)},$$

a smooth solution of (2.2). Given $n \in \mathbb{N}$, let $u_{0,n} = u^{N_n,1}(0)$ and $\tilde{u}_{0,n} = u^{N_n, 1+\frac{1}{n}}(0)$, for some $N_n \in \mathbb{N}$ to be chosen later. Then,

$$\|u_{0,n}\|_{\mathcal{F}L^{s,p}}, \|\tilde{u}_{0,n}\|_{\mathcal{F}L^{s,p}} \lesssim 1,$$

uniformly in $n \in \mathbb{N}$. Moreover,

$$\|u_{0,n} - \tilde{u}_{0,n}\|_{\mathcal{F}L^{s,p}} \sim \frac{1}{n}.$$

Let $u_n = u^{N_n,1}$, $\tilde{u}_n = u^{N_n,1+\frac{1}{n}}$ be the solutions corresponding to initial data $u_{0,n}$, $\tilde{u}_{0,n}$, respectively. Now, considering the difference between the two solutions at time $t \in \mathbb{R}$, we have

$$\|u_n(t) - \tilde{u}_n(t)\|_{\mathcal{F}L^{s,p}} \sim \left| e^{\pm i N^{1-2s}\left(1-(1+\frac{1}{n})^2\right)t} - \left(1 + \frac{1}{n}\right) \right|.$$

Therefore, the solutions have opposite phases at time $t_n > 0$ defined as follows

$$t_n = \frac{\pi N_n^{2s-1}}{\left(1+\frac{1}{n}\right)^2 - 1}.$$

Since $s < \frac{1}{2}$, we can choose N_n large enough, such that $t_n \leq \frac{1}{n}$. Consequently, we have

$$\|u_n(t_n) - \tilde{u}_n(t_n)\|_{\mathcal{F}L^{s,p}} \sim 2 + \frac{1}{n} \geq 2.$$

Since $t_n \to 0$ as $n \to \infty$, the functions constructed satisfy the intended conditions and the result follows.

□

Chapter 3

Further study of the modified Korteweg-de Vries equation

In this chapter, we continue the study of the second renormalized mKdV equation (mKdV2):

$$\partial_t u + \partial_x^3 u = \pm\Big(|u|^2\partial_x u - M(u)\partial_x u - iP(u)u\Big). \tag{3.1}$$

As in the previous chapter, we focus on the defocusing equation ('+' in (3.1)), as the sign will not play a role in the analysis. The main goal of this chapter is to complete the proof of Theorem 1.1.3, by establishing the local well-posedness of mKdV2 (3.1) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq \frac{1}{2}$ and $4 \leq p < \infty$. In Chapter 2, we established the local well-posedness under the restriction $1 \leq p < 4$, which was imposed by the main nonlinear estimate in Section 2.2, needed to apply the Fourier restriction norm method. Here, we apply the method introduced by Deng-Nahmod-Yue [34] to remove this restriction on p. Moreover, we extend the solutions globally-in-time by combining the a priori estimates of Oh-Wang [91] and a persistence of regularity argument, completing the proof of Theorem 1.1.6.

In Section 3.1, we start by decomposing the nonlinearity through localization in the frequency space. Moreover, we choose the $X^{s,b}_{p,q}$-spaces used to conduct the analysis, and relevant auxiliary estimates. One of the key points of this new method is the introduction of a modified Duhamel operator for which we can explicitly trade smoothing in time for smoothing in space, without using the time modulations. This new operator $\mathbf{G}$ is introduced in Section 3.2, alongside the remainder part of the Duhamel operator $\mathbf{B}$. Kernel estimates for the operators $\mathcal{D}$, $\mathbf{G}$, and $\mathbf{B}$ are also established here.

After introducing the needed notation, in Section 3.3, we establish a system of equations for u and w, and prove local well-posedness of (3.1) from solving the equations for u and w, namely Propositions 3.3.1 and 3.3.2. In Section 3.4, we establish the main nonlinear estimates needed to solve the equation for u. These are used in Secion 3.6 to obtain $u = u[w]$. The process of finding the correct w is more involved. We describe the partial second iteration procedure and show the relevant estimates in Section 3.5. Lastly, in Section 3.6, we extend these solutions of mKdV2 (3.1) globally-in-time.

3.1 Nonlinearity, function spaces, and auxiliary results

We start by rewriting the nonlinearity of mKdV2 (3.1). Recall that, omitting time dependence, the nonlinearity $\mathcal{N}(u, \overline{u}, u)$ has the following spatial Fourier transform

$$\mathcal{F}_x\big(\mathcal{N}(u,\overline{u},u)\big)(n) = \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})\neq 0}} in_1\widehat{u}(n_1)\widehat{\overline{u}}(n_2)\widehat{u}(n_3) - in|\widehat{u}(n)|^2\widehat{u}(n),$$

where $\overline{n}_{123} = (n_1, n_2, n_3)$ and $\phi(\overline{n}_{123})$ denotes the phase function

$$\phi(\overline{n}_{123}) = n^3 - n_1^3 - n_2^3 - n_3^3 = 3(n_1 + n_2)(n_1 + n_3)(n_2 + n_3),$$

where the factorization holds if $n = n_1 + n_2 + n_3$. We will use $\phi = \phi(\overline{n}_{123})$ if the dependence on $\overline{n}_{123}$ is clear from context. In Chapter 2, we wrote $\mathcal{N}(u, \overline{u}, u) = \mathcal{NR}(u, \overline{u}, u) + \mathcal{R}(u, u, u)$. Here, we refine this decomposition to have

$$\mathcal{N}(u, \overline{u}, u) = \mathcal{NR}_{\geq}(u, \overline{u}, u) + \mathcal{NR}_{>}(u, u, \overline{u}) + \mathcal{R}(u, u, u),$$

where

$$\mathcal{F}_x\big(\mathcal{NR}_{\geq}(u_1, u_2, u_3)\big)(n) = \sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})\neq 0,\\ |n_2|\geq|n_3|}} in_1\widehat{u}_1(n_1)\widehat{u}_2(n_2)\widehat{u}_3(n_3), \tag{3.2}$$

$$\mathcal{F}_x\big(\mathcal{R}(u_1, u_2, u_3)\big)(n) = -in\widehat{u}_1(n)\overline{\widehat{u}}_2(n)\widehat{u}_3(n),$$

and for $\mathcal{NR}_{>}$ we impose $|n_2| > |n_3|$ on the right-hand side of (3.2). Note that $\mathcal{NR} = \mathcal{NR}_{\geq} + \mathcal{NR}_{>}$. We want to further decompose these operators to introduce additional frequency assumptions. Let n_j denote the spatial frequency corresponding to $\widehat{u}_j$, $j = 1, 2, 3$, in (3.2), then the sum is taken over the following set

$$\mathbb{X}(n) = \big\{(n_1, n_2, n_3) \in \mathbb{Z}^3 : \; n = n_1 + n_2 + n_3, \; |n_2| \geq |n_3|, \; \phi(\overline{n}_{123}) \neq 0\big\},$$

with the stronger assumption $|n_2| > |n_3|$ for $\mathcal{NR}_{>}$. We can therefore consider the following subregions of $\mathbb{X}(n)$:

$$\begin{aligned}
\mathbb{X}_A(n) &= \big\{(n_1, n_2, n_3) \in \mathbb{X}(n) : \; |n_2| \ll |n_1|\big\},\\
\mathbb{X}_B(n) &= \big\{(n_1, n_2, n_3) \in \mathbb{X}(n) : \; |n_3| \ll \min(|n|, |n_1|) \leq \max(|n|, |n_1|) \sim |n_2|\big\},\\
\mathbb{X}_C(n) &= \big\{(n_1, n_2, n_3) \in \mathbb{X}(n) : \; |n| \lesssim |n_3| \ll |n_1|\big\},\\
\mathbb{X}_D(n) &= \big\{(n_1, n_2, n_3) \in \mathbb{X}(n) : \; |n_1| \lesssim |n_3|\big\}.
\end{aligned}$$

For $* \in \{A, B, C, D\}$, let $\mathcal{NR}_{*,\geq}, \mathcal{NR}_{*,>}$ denote the restriction of the operators to $\mathbb{X}_*(n)$. We can write the non-resonant contributions of the nonlinearity as

$$\mathcal{NR}_{\geq} = \mathcal{NR}_{A,\geq} + \mathcal{NR}_{B,\geq} + \mathcal{NR}_{C,\geq} + \mathcal{NR}_{D,\geq}$$

and equivalently for $\mathcal{NR}_{>}$. We also introduce the following notation

$$\mathbb{X}_*^{\mu}(n) := \big\{\overline{n}_{123} \in \mathbb{X}_*(n) : \; \phi(\overline{n}_{123}) = \mu\big\}. \tag{3.3}$$

The following lemma clarifies the relation between the frequencies in the subregions introduced.

Lemma 3.1.1. *The sets $\mathbb{X}_*(n)$, where $* \in \{A, B, C, D\}$, satisfy the following properties:*

(i) $(n_1, n_2, n_3) \in \mathbb{X}_A(n) \implies |n_3| \leq |n_2| \ll |n_1| \sim |n|$;

(ii) $(n_1, n_2, n_3) \in \mathbb{X}_B(n) \implies |n_3| \ll |n| \lesssim |n_1| \sim |n_2|$ *or* $|n_3| \ll |n_1| \ll |n| \sim |n_2|$;

(iii) $(n_1, n_2, n_3) \in \mathbb{X}_C(n) \implies |n| \lesssim |n_3| \ll |n_2| \sim |n_1|$;

(iv) $(n_1, n_2, n_3) \in \mathbb{X}_D(n) \implies |n_1| \lesssim |n_3| \leq |n_2|$;

(v) $(n_1, n_2, n_3) \in \mathbb{X}_A(n) \cup \mathbb{X}_B(n) \cup \mathbb{X}_C(n) \implies |\phi(\overline{n}_{123})| \gtrsim \max(|n_1|, |n_2|)^2$;

(vi) $(n_1, n_2, n_3) \in \mathbb{X}_D(n) \implies |\phi(\overline{n}_{123})| \gtrsim |n_2|$ *and* $\langle n \rangle^{\frac{1}{2}} |n_1| \lesssim (\langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle)^{\frac{1}{2}}$.

Remark 3.1.2. Note that the nearly-resonant case when $|n_1| \sim |n_2| \sim |n_3|$ and $|\phi(\overline{n}_{123})| \gtrsim \max(|n_1|, |n_2|, |n_3|)$ is included in $\mathbb{X}_D(n)$. There are other frequency interactions allowed in this region which are fully non-resonant, i.e., $|\phi(\overline{n}_{123})| \gtrsim \max(|n_1|, |n_2|, |n_3|)^2$ holds. However, due to the condition in (vi), the phase function will not play a crucial role when estimating this contribution.

We want to construct solutions of mKdV2 (3.1) which satisfy the Duhamel formulation

$$u(t) = S(t)u_0 + D\mathcal{NR}(u, \overline{u}, u)(t) + D\mathcal{R}(u, u, u)(t), \tag{3.4}$$

for some $T > 0$ and $|t| \leq T$. Let φ be a smooth time cutoff with $\varphi \equiv 1$ on $[-1, 1]$ and $\varphi \equiv 0$ outside $[-2, 2]$, and $\varphi_T(\cdot) = \varphi(T^{-1}\cdot)$ for $T > 0$. Recall that D denotes the Duhamel operator, $DF(t, x) = \int_0^t S(t - t')F(t')\, dt'$ and let $\mathcal{D}$ denote its truncated version

$$\mathcal{D}F(t, x) = \varphi(t) \cdot D\big(\varphi(t') \cdot F(t', x)\big)(t) = \varphi(t) \int_0^t S(t - t')\varphi(t')F(t', x)\, dt'.$$

Equivalently, locally-in-time, we can focus on solving the truncated Duhamel formulation

$$u(t) = \varphi \cdot S(t)u_0 + \varphi_T \cdot \mathcal{D}\mathcal{NR}(u, \overline{u}, u)(t) + \varphi_T \cdot \mathcal{D}\mathcal{R}(u, u, u)(t), \tag{3.5}$$

for some $0 < T \leq 1$.

In order to construct solutions of mKdV2 (3.1), we will run a suitable contraction mapping argument in $X^{s,b}$-spaces adapted to the Fourier-Lebesgue setting (see Definition 1.3.1). In the following, we introduce the relevant parameters and the spaces involved in the argument. Let $0 < \delta \ll 1$ be a small parameter to be chosen later, depending on $2 < p < \infty$. We introduce the following parameters

$$\begin{aligned}
&b_0 = 1 - 2\delta, && b_1 = 1 - \delta, && \\
&q_0 = \frac{1}{4\delta}, && q_1 = \frac{1}{4.5\delta}, && \\
&\frac{1}{r_0} = \frac{1}{2} + \delta, && \frac{1}{r_1} = \frac{1}{2} + 2\delta, && \frac{1}{r_2} = \frac{1}{2} + 3\delta.
\end{aligned}$$

Note that $b_0 < b_1$, $q_1 < q_0$, and $r_2 < r_1 < r_0$. We will conduct our analysis in the following $X^{s,b}_{p,q}$-spaces:

$$\begin{aligned}
Y_0^s &= X^{s,\frac{1}{2}}_{p,r_0}(\mathbb{R} \times \mathbb{T}), & Y_1^s &= X^{s,\frac{1}{2}}_{p,r_1}(\mathbb{R} \times \mathbb{T}), \\
Z_0^s &= X^{s,b_0}_{p,q_0}(\mathbb{R} \times \mathbb{T}), & Z_1^s &= X^{s,b_1}_{p,q_0}(\mathbb{R} \times \mathbb{T}).
\end{aligned}$$

Note that $Z_0^s \subset Y_0^s \subset C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$.

Lastly, we introduce auxiliary results needed for the analysis.

Lemma 3.1.3 (Schur's test)**.** *Let X, Y be measurable spaces, $K : X \times Y \to \mathbb{R}$ a non-negative Schwartz kernel, and $1 \leq p, q, r \leq \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. If for some $C > 0$ we have*

$$\sup_{x \in X} \int_Y |K(x, y)|^r dy + \sup_{y \in Y} \int_X |K(x, y)|^r dx \leq C^r,$$

then

$$\bigg\| \int_Y K(x, y)f(y)\ dy \bigg\|_{L^q(X)} \leq C\|f\|_{L^p(Y)}.$$

The following lemma allows us to gain a small power of the time of existence T, needed to close the contraction mapping argument. Note that the second estimate follows from the same proof of Lemma 2.1.2.

Lemma 3.1.4. *Suppose that F is a smooth space-time function such that $F|_{t=0} = 0$. Then, we have the following estimates*

$$\|\varphi_T \cdot F\|_{Y_0^s} \lesssim T^\theta \|F\|_{Y_1^s}, \qquad \|\varphi_T \cdot F\|_{Z_0^s} \lesssim T^\theta \|F\|_{Z_1^s}, \tag{3.6}$$

for any $0 < \theta \leq \frac{\delta}{2}$ *and* $0 < T \leq 1$.

Proof. We want to estimate the following quantity

$$\|\varphi_T \cdot F\|_{X^{s,b}_{p,q}} = \Big\| \langle n\rangle^s \langle \tau\rangle^b \big(\widehat{\varphi}_T *_\tau \widehat{F}(\cdot + n^3, n)\big)(\tau) \Big\|_{\ell^p_n L^q_\tau}.$$

Both estimates follow once we show

$$\|\langle \tau\rangle^b \widehat{\varphi}_T * f(\tau)\|_{L^q_\tau} \lesssim T^{\frac{1}{\tilde q} - \frac{1}{q}} \|\langle \tau\rangle^b f(\tau)\|_{L^{\tilde q}_\tau}, \tag{3.7}$$

for f satisfying $\int_{\mathbb{R}} f(\tau)\, d\tau = 0$, $1 < \tilde q < q < \infty$, and $0 < b < 1 < b + \frac{1}{\tilde q}$. In fact, from (3.7), we have

$$\|\varphi_T \cdot F\|_{X^{s,b}_{p,q}} \lesssim T^{\frac{1}{\tilde q} - \frac{1}{q}} \|F\|_{X^{s,b}_{p,\tilde q}}, \tag{3.8}$$

by choosing $f(\tau) = \widehat{F}(\tau + n^3, n)$ which satisfies $\int_{\mathbb{R}} \widehat{F}(\tau + n^3, n)\, d\tau = \int_{\mathbb{R}} \widehat{F}(\tau, n)\, d\tau = 2\pi F(0,n) = 0$, from the assumption on F at time zero. We get the first estimate in (3.6) by setting $q = r_0 > r_1 = \tilde q$ and $b = \frac{1}{2} < 1 < 1 + 2\delta$ in (3.8), and the second estimate in (3.6) with $q = q_0 > q_1 = \tilde q$, $b = b_0 = 1 - 2\delta < 1 < 1 + 2.5\delta$, and Hölder's inequality.

The estimate (3.7) follows once we prove the following

$$\begin{aligned} \big\| \langle \tau\rangle^b \big(\widehat{\varphi}_T * (\mathbb{1}_{|\tau| \geq T^{-1}} f)\big)(\tau) \big\|_{L^q_\tau} &\lesssim T^{\frac{1}{\tilde q} - \frac{1}{q}} \|\langle \tau\rangle^b f(\tau)\|_{L^{\tilde q}_\tau}, \\ \big\| \langle \tau\rangle^b \big(\widehat{\varphi}_T * (\mathbb{1}_{|\tau| < T^{-1}} f)\big)(\tau) \big\|_{L^q_\tau} &\lesssim T^{\frac{1}{\tilde q} - \frac{1}{q}} \|\langle \tau\rangle^b f(\tau)\|_{L^{\tilde q}_\tau}. \end{aligned} \tag{3.9}$$

Note that the first inequality is equivalent to the following result

$$\Big\| \int_{\mathbb{R}} \frac{\langle \tau\rangle^b}{\langle \lambda\rangle^b} \mathbb{1}_{|\lambda| \geq T^{-1}} \widehat{\varphi}_T(\tau - \lambda) f(\lambda)\, d\lambda \Big\|_{L^q_\tau} \lesssim T^{\frac{1}{\tilde q} - \frac{1}{q}} \|f\|_{L^{\tilde q}_\tau}. \tag{3.10}$$

Since $T \leq 1$ and $|T\lambda| \geq 1$, we have

$$\frac{\langle \tau\rangle^b}{\langle \lambda\rangle^b} \lesssim \frac{\langle T\tau\rangle^b}{\langle T\lambda\rangle^b} \lesssim \frac{\langle T(\tau-\lambda)\rangle^b \langle T\lambda\rangle^b}{\langle T\lambda\rangle^b} \lesssim \langle T(\tau - \lambda)\rangle^b.$$

Using Young's inequality with $1 + \frac{1}{q} = \frac{1}{\tilde q} + \frac{1}{r}$ gives

$$\text{LHS of (3.10)} \lesssim \Big\| \int_{\mathbb{R}} \langle T(\tau - \lambda)\rangle^b T \widehat{\varphi}(T(\tau-\lambda)) f(\lambda)\ d\lambda \Big\|_{L^q_\tau} \lesssim T \|\langle T\tau\rangle^b \widehat{\varphi}(T\tau)\|_{L^r_\tau} \|f\|_{L^{\tilde q}_\tau}.$$

The estimate follows from $T\|\langle T\tau\rangle^b \widehat{\varphi}(T\tau)\|_{L^r_\tau} \lesssim_\varphi T^{1-\frac{1}{r}} = T^{\frac{1}{\tilde q} - \frac{1}{q}}$.

To prove (3.9), using the fact that $\int_{\mathbb{R}} f(\tau)\, d\tau = 0$, we note that

$$\widehat{\varphi}_T * \big(\mathbb{1}_{|\tau|<T^{-1}} f\big)(\tau) = \int\limits_{|\lambda| < T^{-1}} f(\lambda) T \big[\widehat{\varphi}(T(\tau - \lambda)) - \widehat{\varphi}(T\tau)\big]\, d\lambda - T\widehat{\varphi}(T\tau) \int\limits_{|\lambda| \geq T^{-1}} f(\lambda)\, d\lambda. \tag{3.11}$$

For the first contribution in (3.11), we distinguish between the regions $\{\tau : |\tau| \lesssim T^{-1}\}$ and $\{\tau : |\tau| \gg T^{-1}\}$. If $|\tau| \lesssim T^{-1}$, then $\langle T\tau\rangle \sim 1$ and we apply the mean value theorem to obtain

$$\begin{aligned} \int\limits_{|\lambda| < T^{-1}} T|\widehat{f}(\lambda)||\widehat{\varphi}(T(\tau-\lambda)) - \widehat{\varphi}(T\tau)|\ d\lambda &\lesssim \int_{\mathbb{R}} \frac{T}{\langle T\tau\rangle^\alpha} |\widehat{f}(\lambda)||T\lambda||\partial\widehat{\varphi}(\xi)|\ d\lambda \\ &\lesssim \|\partial \widehat{\varphi}\|_{L^\infty_\tau} \int_{\mathbb{R}} \frac{T|T\lambda|}{\langle T\tau\rangle^\alpha} |\widehat{f}(\lambda)|\ d\lambda, \end{aligned}$$

for some ξ between $T(\tau - \lambda)$ and $T\tau$ and any $\alpha > 0$. In the remaining region, if $|\tau| \gg T^{-1}$, then $|T(\tau - \lambda)| \sim |T\tau| \sim |\xi|$, for any ξ between $T(\tau-\lambda)$ and $T\tau$, and from the mean value theorem, we have

$$\begin{aligned}\int\limits_{|\lambda|<T^{-1}} T|\widehat{f}(\lambda)||\widehat{\varphi}(T(\tau-\lambda))-\widehat{\varphi}(T\tau)|\,d\lambda &\lesssim \int_{\mathbb{R}} T|T\lambda||\widehat{f}(\lambda)||\partial\widehat{\varphi}(\xi)|\,d\lambda\\ &\lesssim \int_{\mathbb{R}} T|T\lambda||\widehat{f}(\lambda)|\frac{\langle\xi\rangle^\alpha}{\langle T\tau\rangle^\alpha}|\partial\widehat{\varphi}(\xi)|\,d\lambda\\ &\lesssim \|\langle\tau\rangle^\alpha\partial\widehat{\varphi}\|_{L^\infty_\tau}\int_{\mathbb{R}}\frac{T|T\lambda|}{\langle T\tau\rangle^\alpha}|\widehat{f}(\lambda)|\,d\lambda,\end{aligned}$$

for any $\alpha>0$. Combining the two estimates, we obtain the following, for any $\alpha>0$,

$$\int\limits_{|\lambda|<T^{-1}} T|f(\lambda)||\widehat{\varphi}(T(\tau-\lambda))-\widehat{\varphi}(T\tau)|\,d\lambda \lesssim_\varphi \frac{T}{\langle T\tau\rangle^\alpha}\int\limits_{|\lambda|<T^{-1}}|T\lambda|\,|f(\lambda)|\,d\lambda.$$

For the second contribution in (3.11), we have

$$\int\limits_{|\lambda|\geq T^{-1}} |f(\lambda)|T|\widehat{\varphi}(T\tau)|d\lambda \lesssim_\varphi \frac{T}{\langle T\tau\rangle^\alpha}\int\limits_{|\lambda|\geq T^{-1}}|f(\lambda)|\,d\lambda.$$

Combining the estimates for the two contributions in (3.11), we obtain

$$\left\|\langle\tau\rangle^b\widehat{\varphi}_T*\left(\mathbb{1}_{|\tau|<T^{-1}}f\right)(\tau)\right\|_{L^q_\tau}\lesssim\left\|\frac{T\langle\tau\rangle^b}{\langle T\tau\rangle^\alpha}\right\|_{L^q_\tau}\|\min(1,|T\lambda|)\langle\lambda\rangle^{-b}\|_{L^r_\lambda}\|\langle\tau\rangle^b f(\tau)\|_{L^{\tilde{q}}_\tau},$$

by using Hölder's inequality with $1=\frac{1}{r}+\frac{1}{\tilde{q}}$. Thus, we have

$$\left\|\frac{T\langle\tau\rangle^b}{\langle T\tau\rangle^\alpha}\right\|_{L^q_\tau}\lesssim T^{1-b}\left(\int_{\mathbb{R}}\langle T\tau\rangle^{-(\alpha-b)q}\,d\tau\right)^{\frac{1}{q}}\lesssim T^{1-b-\frac{1}{q}}\left(\int_{\mathbb{R}}\langle\tau\rangle^{-(\alpha-b)q}\,d\tau\right)^{\frac{1}{q}}\lesssim T^{1-b-\frac{1}{q}},$$

by choosing $\alpha>0$ such that $(\alpha-b)q>1$, and

$$\begin{aligned}\|\min(1,|T\lambda|)\langle\lambda\rangle^{-b}\|^r_{L^r_\lambda}&=\int\limits_{T^{-1}\leq|\lambda|}\frac{1}{\langle\lambda\rangle^{br}}\,d\lambda+\int\limits_{T^{-1}>|\lambda|}\frac{|T\lambda|^r}{\langle\lambda\rangle^{br}}\,d\lambda\lesssim T^{br-1}+T^r\int\limits_{T^{-1}>|\lambda|}|\lambda|^{(1-b)r}d\lambda\\ &\lesssim T^{br-1},\end{aligned}$$

given that $b>1-\frac{1}{\tilde{q}}$ and $b<1$. Combining the two bounds, we obtain the intended power of T. □

Remark 3.1.5. Lemma 3.1.4 will only be applied to functions of the form $F(t,x)=\int_0^t G(t',x)\,dt'$ which satisfy the assumption $F|_{t=0}=0$, namely the Duhamel operator $\mathcal{D}$ and the operators $\mathbf{G},\mathbf{B}$ defined in Section 3.2.

We state the well-known divisor counting estimate (see [49, Lemma 315] for the proof).

Lemma 3.1.6. *Let $n\in\mathbb{Z}$ and let $d(n)$ denote the set of divisors of n. Then,*

$$|d(n)|\lesssim_\varepsilon |n|^\varepsilon,$$

for any $\varepsilon>0$.

We also require the following refined divisor counting estimate.

Lemma 3.1.7 ([34, Lemma 3.4])**.** (i) *Fix $0<\varepsilon<1$, $\rho\geq1$ and let $k,q\in\mathbb{Z}$ such that $|q|\gtrsim|k|^\varepsilon>0$. Then, the number of divisors $r\in\mathbb{Z}$ of k that satisfy $|r-q|\lesssim\rho$ is at most $\mathcal{O}_\varepsilon(\rho^\varepsilon)$.*

(ii) Let $N_1 \geq N_2 \geq N_3$ be dyadic numbers, μ, and m be positive integers, and consider the following set

$$A(m,\mu,N_1,N_2,N_3) = \big\{(m_1,m_2,m_3) \in \mathbb{Z}^3 : \ m_1+m_2+m_3 = m, \ m_1^3+m_2^3+m_3^3 = \mu, \\ (m_1+m_2)(m_1+m_3)(m_2+m_3) \neq 0, \ |m_j| \sim N_j, \ j=1,2,3\big\}.$$

Then, $|A(n,\mu,N_1,N_2,N_3)| \lesssim_\varepsilon N_2^\varepsilon$, for some $0<\varepsilon \ll 1$.

Proof. For the proof of (i), see [34]. To prove (ii), we start by noticing that if $(m_1,m_2,m_3) \in A(m,\mu,N_1,N_2,N_3)$, then

$$m^3 - \mu = m^3 - m_1^3 - m_2^3 - m_3^3 = 3(m_2+m_3)(m-m_2)(m-m_3).$$

If $N_1 \sim N_2$, then $|m| = |m_1+m_2+m_3| \lesssim N_2$ and $|\mu| = |m_1^3+m_2^3+m_3^3| \lesssim N_2^3$, from which we conclude that $1 \leq |m^3-\mu| \lesssim N_2^3$. Using Lemma 3.1.6, we conclude that the number of divisors $m-m_2$ and $m-m_3$ of $m^3-\mu$ is bounded by N_2^ε, for any $\varepsilon>0$. Since choosing m_2,m_3 fixes m_1, we obtain

$$|A(m,\mu,N_1,N_2,N_3)| \lesssim_\varepsilon N_2^\varepsilon$$

under the assumption $N_1 \sim N_2$. If instead $N_1 \gg N_2$, then $|m| \sim |m_1| \gg |m_2|$, $|m^3-\mu| \lesssim |m|^3$, which implies that $1 \leq |m^3-\mu|^\varepsilon \leq |m^3-\mu|^{\frac{1}{3}} \lesssim |m|$ for any $0<\varepsilon\leq \frac{1}{3}$. Since $|(m-m_2)-m| \sim N_2$ and $|(m-m_3)-m| \sim N_3$, from (i), we conclude that

$$|A(m,\mu,N_1,N_2,N_3)| \lesssim_\varepsilon (N_2N_3)^\varepsilon \sim N_2^{2\varepsilon}.$$

This concludes the proof.

□

Lastly, we fix a Schwartz function η satisfying

$$\widehat{\eta}(-1) = 0, \quad \mathcal{H}\widehat{\eta}(-1) = -1, \tag{3.12}$$

where $\mathcal{H}$ denotes the Hilbert transform, i.e., principal value convolution with $\frac{1}{\tau}$. See Appendix A.1 for a possible choice of η.

3.2 Splitting the Duhamel operator and kernel estimates

In this section, we explicitly establish the smoothing-in-time of the Duhamel operator by estimating its kernel. Moreover, we introduce the modified version of the Duhamel operator and the kernel estimate for the nonlinear contributions localized to $\mathbb{X}_A(n), \mathbb{X}_B(n)$.

Proposition 3.2.1. *The truncated Duhamel operator $\mathcal{D}$ has the following space-time Fourier transform*

$$\mathcal{F}_{t,x}\big(\mathcal{D}F\big)(\tau,n) = \int_{\mathbb{R}} K(\tau-n^3,\lambda-n^3)\widehat{F}(\lambda,n)\,d\lambda$$

where the kernel K is given by the following expression

$$K(\tau,\lambda) = -i\int_{\mathbb{R}} \widehat{\varphi}(\mu-\lambda)\frac{\widehat{\varphi}(\tau-\mu)-\widehat{\varphi}(\tau)}{\mu}\,d\mu$$

and satisfies the following estimates

$$|K(\tau,\lambda)| \lesssim \left(\frac{1}{\langle\tau-\lambda\rangle^\alpha} + \frac{1}{\langle\tau\rangle^\alpha}\right)\frac{1}{\langle\lambda\rangle} \lesssim \frac{1}{\langle\tau\rangle\langle\tau-\lambda\rangle} \tag{3.13}$$

for any $\alpha>0$ large enough.

Proof. We start by calculating the space-time Fourier transform of $\mathcal{D}F$,

$$\mathcal{F}_{t,x}\big(\mathcal{D}F\big)(\tau,n) = \int_{\mathbb{R}} \widehat{\varphi}(\tau-\mu)\mathcal{F}_t\bigg(\int_0^t e^{i(t-t')n^3}\varphi(t')\widehat{F}(t',n)\ dt'\bigg)(\mu)\,d\mu.$$

Using the fact that $\int_0^t f(t')\ dt' = \frac{1}{2}\int_{\mathbb{R}} f(t')\big(\operatorname{sgn}(t-t')+\operatorname{sgn}(t')\big)\ dt'$, we have

$$\begin{aligned}\mathcal{F}_t\bigg(\int_0^t e^{i(t-t')n^3}\varphi(t')\widehat{F}(t',n)\ dt'\bigg)(\mu) &= \frac{1}{4\pi}\int_{\mathbb{R}} e^{-it'\mu}\varphi(t')\widehat{F}(t',n)\,dt' \int_{\mathbb{R}} e^{-it(\mu-n^3)}\operatorname{sgn}(t)dt \\ &\quad + \frac{1}{4\pi}\int_{\mathbb{R}} e^{-it'n^3}\varphi(t')\widehat{F}(t',n)\operatorname{sgn}(t')\,dt'\int_{\mathbb{R}} e^{-it(\mu-n^3)}\ dt.\end{aligned}$$

Consequently, since $\mathcal{F}_t(\operatorname{sgn})(\tau) = \frac{1}{i\pi\tau}$, we have

$$\begin{aligned}\mathcal{F}_t\bigg(\int_0^t e^{i(t-t')n^3}\varphi(t')\widehat{F}(t',n)\ dt'\bigg)(\mu) &= \frac{-i}{\mu-n^3}\int_{\mathbb{R}}\widehat{\varphi}(\mu-\lambda)\widehat{F}(\lambda,n)\,d\lambda \\ &\quad - i\delta_0(\mu-n^3)\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{\varphi}(n^3-\lambda-\mu')\frac{1}{\mu'}\widehat{F}(\lambda,n)\,d\mu'\,d\lambda,\end{aligned}$$

where δ_0 denotes the Dirac delta function. Calculating the convolution with $\widehat{\varphi}$, we get

$$\mathcal{F}_{t,x}\big(\mathcal{D}F\big)(\tau,n) = -i\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\widehat{\varphi}(\mu+n^3-\lambda)\frac{\widehat{\varphi}(\tau-\mu-n^3)-\widehat{\varphi}(\tau-n^3)}{\mu}d\mu\bigg)\widehat{F}(\lambda,n)\ d\lambda,$$

as intended

It remains to show the estimate on the kernel. In the region $\{|\mu|>1\}$, using Cauchy-Schwarz inequality and Lemma 2.1.4, we have

$$\begin{aligned}\int\limits_{|\mu|>1}\frac{|\widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)|}{|\mu|}\,d\mu &\lesssim \bigg(\int_{\mathbb{R}}\frac{d\mu}{\langle\tau-\mu\rangle^{2\alpha}\langle\mu-\lambda\rangle^{1+2\alpha}}\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}}\frac{d\mu}{\langle\mu\rangle^2\langle\mu-\lambda\rangle^2}\bigg)^{\frac{1}{2}} \\ &\lesssim \frac{1}{\langle\lambda\rangle\langle\tau-\lambda\rangle^{\alpha}}, \\ \int\limits_{|\mu|>1}\frac{|\widehat{\varphi}(\tau)\widehat{\varphi}(\mu-\lambda)|}{|\mu|}d\mu &\lesssim \frac{1}{\langle\tau\rangle^{\alpha}}\int_{\mathbb{R}}\frac{1}{\langle\mu-\lambda\rangle^{\alpha}\langle\mu\rangle}\,d\mu \lesssim \frac{1}{\langle\lambda\rangle\langle\tau\rangle^{\alpha}},\end{aligned}$$

for any $\alpha>0$. In the region $\{|\mu|\le 1\}$, we consider two subregions: $\{|\tau|\lesssim 1\}$ or $\{|\tau|\gg 1\}$. If $|\tau|\lesssim 1$, then $\langle\tau\rangle\sim 1$ and using mean value theorem, where ξ is a number between $\tau-\mu$ and τ, we have

$$\begin{aligned}\int\limits_{|\mu|\le 1}|\widehat{\varphi}(\mu-\lambda)|\frac{|\widehat{\varphi}(\tau-\mu)-\widehat{\varphi}(\tau)|}{|\mu|}d\mu &= \int\limits_{|\mu|\le 1}|\widehat{\varphi}(\mu-\lambda)|\frac{|\mu||\partial\widehat{\varphi}(\xi)|}{|\mu|}d\mu \\ &\lesssim \frac{1}{\langle\tau\rangle^{\alpha}}\int\limits_{|\mu|\le 1}\frac{1}{\langle\mu-\lambda\rangle^{1+}\langle\mu\rangle}d\mu \\ &\lesssim \frac{1}{\langle\lambda\rangle\langle\tau\rangle^{\alpha}},\end{aligned}$$

for any $\alpha>0$. If $|\tau|\gg 1$, we once again apply mean value theorem, where ξ is between $\tau-\mu$ and τ. Since $|\mu|\le 1$, we must have $|\xi|\sim|\tau|$ and $\langle\mu\rangle\sim 1$. Therefore, applying mean value theorem and Lemma 2.1.4, we get

$$\int\limits_{|\mu|\le 1}|\widehat{\varphi}(\mu-\lambda)|\frac{|\widehat{\varphi}(\tau-\mu)-\widehat{\varphi}(\tau)|}{|\mu|}d\mu \sim \int\limits_{|\mu|\le 1}|\widehat{\varphi}(\mu-\lambda)|\frac{\langle\xi\rangle^{\alpha}|\partial\widehat{\varphi}(\xi)|}{\langle\tau\rangle^{\alpha}}d\mu \lesssim \frac{1}{\langle\lambda\rangle^{\alpha}\langle\tau\rangle^{\alpha}},$$

for any $\alpha > 0$. From the estimates for the regions $\{|\mu| > 1\}$ and $\{|\mu| \le 1\}$, we get

$$|K(\tau,\lambda)| \lesssim \Big(\frac{1}{\langle \tau-\lambda\rangle^{\alpha}} + \frac{1}{\langle \tau\rangle^{\alpha}}\Big)\frac{1}{\langle\lambda\rangle}.$$

To show (3.13), note that $\langle\tau\rangle \lesssim \langle\tau-\lambda\rangle\langle\lambda\rangle$ and $\langle\tau-\lambda\rangle \lesssim \langle\tau\rangle\langle\lambda\rangle$. Thus, for $\alpha \ge 2$,

$$\Big(\frac{1}{\langle \tau-\lambda\rangle^{\alpha}} + \frac{1}{\langle \tau\rangle^{\alpha}}\Big)\frac{1}{\langle\lambda\rangle} \lesssim \frac{1}{\langle\tau\rangle\langle\tau-\lambda\rangle^{\alpha-1}} + \frac{1}{\langle\tau-\lambda\rangle\langle\tau\rangle^{\alpha-1}} \lesssim \frac{1}{\langle\tau\rangle\langle\tau-\lambda\rangle}.$$

□

We want to split each of the non-resonant contributions $\mathcal{DNR}_{*,\ge}, \mathcal{DNR}_{*,>}$ for $* \in \{A,B\}$ into two components:

$$\mathcal{DNR}_{*,\ge} = \mathbf{G}_{*,\ge} + \mathbf{B}_{*,\ge}, \qquad \mathcal{DNR}_{*,>} = \mathbf{G}_{*,>} + \mathbf{B}_{*,>}.$$

The 'good' contributions $\mathbf{G}$ will depend on the modified Duhamel operator. By introducing a convolution with a smooth function η parameterized by the resonance relation $\phi(\overline{n}_{123})$, we induce sufficient smoothing in space to control the derivative nonlinearity. Consider a Schwartz function η satisfying

$$\widehat{\eta}(-1) = 0, \quad \mathcal{H}\widehat{\eta}(-1) = -1,$$

where $\mathcal{H}$ denotes the Hilbert transform, i.e., principal value convolution with $\frac{1}{\tau}$, as chosen in Section 3.1. We first define the operators $\mathbf{G}_{*,\ge}$, $\mathbf{B}_{*,\ge}$ through their spatial Fourier transform

$$\begin{aligned}&\mathcal{F}_x\big(\mathbf{G}_{*,\ge}(u_1,u_2,u_3)\big)(t,n)\\ &\quad= \varphi(t)\sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n),\\ |n_2|\ge|n_3|}} in_1\int_0^t e^{i(t-t')n^3}\eta\big(\phi(\overline{n}_{123})(t-t')\big)\varphi(t')\prod_{j=1}^{3}\widehat{u}_j(t',n_j)\,dt',\\ &\mathcal{F}_x\big(\mathbf{B}_{*,\ge}(u_1,u_2,u_3)\big)(t,n)\\ &\quad= \varphi(t)\sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n),\\ |n_2|\ge|n_3|}} in_1\int_0^t e^{i(t-t')n^3}\big[1-\eta\big(\phi(\overline{n}_{123})(t-t')\big)\big]\varphi(t')\prod_{j=1}^{3}\widehat{u}_j(t',n_j)\,dt',\end{aligned}$$

with equivalent definitions for $\mathbf{G}_{*,>}, \mathbf{B}_{*,>}$ imposing the condition $|n_2| > |n_3|$ to the sum.

In the following, we estimate the kernels for the truncated operators when $* \in \{A,B\}$.

Proposition 3.2.2. *Let $* \in \{A,B\}$. Then, the convolution operators $\mathbf{G}_{*,\ge}, \mathbf{G}_{*,>}$ have the following space-time Fourier transform*

$$\begin{aligned}&\mathcal{F}_{t,x}\big(\mathbf{G}_{*,\underset{(>)}{\ge}}(u_1,u_2,u_3)\big)(\tau,n)\\ &\quad= \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n),\\ |n_2|\underset{(>)}{\ge}|n_3|}} n_1\int_{\mathbb{R}} K_G\big(\tau-n^3,\lambda-n^3,\phi(\overline{n}_{123})\big)\int_{\lambda=\tau_1+\tau_2+\tau_3}\prod_{j=1}^{3}\widehat{u}_j(\tau_j,n_j)\,d\tau_1\,d\tau_2\,d\lambda,\end{aligned}$$

where the kernel K_G is given by the following expression

$$K_G(\tau,\lambda,\phi) = \int_{\mathbb{R}}\Big(\widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big) + \widehat{\varphi}(\tau-\mu)\mathcal{H}\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\Big)\,d\mu,$$

and satisfies the following estimates

$$
\begin{aligned}
|K_G(\tau,\lambda,\phi)| &\lesssim \frac{1}{\langle \tau-\lambda\rangle^{\alpha}} \min\left(\frac{1}{\langle \phi\rangle},\frac{1}{\langle \lambda\rangle}\right) + \frac{1}{\langle \tau-\lambda\rangle} \min\left(\frac{1}{\langle \phi\rangle},\frac{1}{\langle \tau\rangle}\right),\\
&\lesssim \frac{1}{\langle \tau-\lambda\rangle} \min\left(\frac{1}{\langle \phi\rangle},\frac{1}{\langle \tau\rangle}\right),
\end{aligned} \tag{3.14}
$$

where α is a large enough positive number and $|\phi| \geq 1$.

Proof. Since the relation between $|n_2|$ and $|n_3|$ will not play an important role in the proof, we will use $\mathbf{G}_*$ to denote both $\mathbf{G}_{*,\geq}, \mathbf{G}_{*,>}$. We want to calculate the following

$$
\begin{aligned}
&\mathcal{F}_{t,x}\big(\mathbf{G}_*(u_1,u_2,u_3)\big)(\tau,n)\\
&\quad= \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} i n_1 \int_{\mathbb{R}} \widehat{\varphi}(\tau-\mu)\mathcal{F}_t\bigg(\int_0^t e^{i(t-t')n^3}\eta\big(\phi(\overline{n}_{123})(t-t')\big)F(t')\,dt'\bigg)(\mu)\,d\mu,
\end{aligned}
$$

where $F(t)=\varphi(t)\prod_{j=1}^3 \widehat{u}_j(t,n_j)$. Note that $|\phi(\overline{n}_{123})| \geq 1$ for $\overline{n}_{123}\in\mathbb{X}_*(n)$ and denote it by ϕ, for simplicity. Proceeding as in the proof of Proposition 3.2.1, we have

$$
\begin{aligned}
&\mathcal{F}_t\bigg(\int_0^t e^{i(t-t')n^3}\eta\big(\phi(t-t')\big)F(t')\,dt'\bigg)(\mu)\\
&\qquad= \frac{1}{4\pi}\int_{\mathbb{R}} e^{-it'n^3}F(t')\int_{\mathbb{R}} e^{it(n^3-\mu)}\eta(\phi(t-t'))\operatorname{sgn}(t-t')\,dt\,dt'\\
&\qquad+ \frac{1}{4\pi}\int_{\mathbb{R}} e^{-it'n^3}F(t')\operatorname{sgn}(t')\int_{\mathbb{R}} e^{it(n^3-\mu)}\eta(\phi(t-t'))\,dt\,dt'.
\end{aligned} \tag{3.15}
$$

The first contribution in (3.15) equals

$$
\frac{1}{2}\int_{\mathbb{R}} e^{-it'\mu'}F(t')\,dt'\Big[\mathcal{F}_t\big(\eta(\phi\cdot)\big)*\mathcal{F}_t\big(\operatorname{sgn}(\cdot)\big)(\mu-n^3)\Big] = -i\widehat{F}(\mu)\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu-n^3}{\phi}\Big),
$$

while the second gives

$$
\frac{1}{2}\int_{\mathbb{R}} e^{-it'\mu}F(t')\operatorname{sgn}(t')\,dt'\Big[\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu-n^3}{\phi}\Big)\Big] = -i\mathcal{H}\widehat{F}(\mu)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu-n^3}{\phi}\Big).
$$

Consequently, we obtain

$$
\begin{aligned}
&\mathcal{F}_{t,x}\big(\mathbf{G}_*(u_1,u_2,u_3)\big)(\tau,n)\\
&= \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} n_1\int_{\mathbb{R}}\widehat{\varphi}(\tau-\mu)\Big[\widehat{F}(\mu)\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu-n^3}{\phi}\Big)+\mathcal{H}\widehat{F}(\mu)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu-n^3}{\phi}\Big)\Big]d\mu.
\end{aligned}
$$

Since

$$
\begin{aligned}
\widehat{F}(\tau) &= \int_{\mathbb{R}}\widehat{\varphi}(\tau-\lambda)\int_{\lambda=\tau_1+\tau_2+\tau_3}\prod_{j=1}^3\widehat{u}_j(\tau_j,n_j)\,d\lambda,\\
\mathcal{H}\widehat{F}(\tau) &= \int_{\mathbb{R}}\mathcal{H}\widehat{\varphi}(\tau-\lambda)\int_{\lambda=\tau_1+\tau_2+\tau_3}\prod_{j=1}^3\widehat{u}_j(\tau_j,n_j)\,d\lambda,
\end{aligned}
$$

we obtain the intended expression by substituting $\widehat{F}(\mu)$ and $\mathcal{H}\widehat{F}(\mu)$.

It remains to show the kernel estimate. First, note that for a Schwartz function f, we have

$$|\mathcal{H}f(\xi)| \leq \lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|\mu|<1} \left|\frac{f(\xi-\mu)-f(\xi)}{\mu}\right| d\mu + \left|\lim_{\varepsilon\to 0} \int\limits_{\varepsilon<|\mu|<1} \frac{f(\xi)}{\mu} d\mu\right| + \int\limits_{|\mu|\geq 1} \left|\frac{f(\xi-\mu)}{\mu}\right| d\mu.$$

Considering the first contribution, using mean value theorem and distinguishing between the cases $|\xi| \lesssim 1$ and $|\xi| \gg 1$ gives

$$\int\limits_{\varepsilon<|\mu|<1} \frac{|f(\xi-\mu)-f(\xi)|}{|\mu|} d\mu \lesssim \frac{1}{\langle\xi\rangle^\alpha},$$

for any $\alpha > 0$. The second contribution is equal to zero, so it only remains to control the third one. Using Lemma 2.1.4, it follows that

$$\int\limits_{|\mu|\geq 1} \frac{|f(\xi-\mu)|}{|\mu|} d\mu \lesssim \frac{1}{\langle\xi\rangle}.$$

Consequently, the following holds for any Schwartz function f

$$|\mathcal{H}f(\xi)| \lesssim \frac{1}{\langle\xi\rangle}. \tag{3.16}$$

Since $\widehat{\varphi}$ is a Schwartz function, using (3.16),

$$\frac{1}{\langle\phi\rangle}\left|\mathcal{H}\widehat{\eta}\left(\frac{\mu}{\phi}\right)\right| \lesssim \frac{1}{\langle\phi\rangle}\left(\mathbb{1}_{|\mu|\leq|\phi|} + \mathbb{1}_{|\mu|\geq|\phi|\geq 1}\frac{\langle\phi\rangle}{\langle\mu\rangle}\right) \lesssim \min\left(\frac{1}{\langle\phi\rangle}, \frac{1}{\langle\mu\rangle}\right).$$

Now, considering the kernel and the estimates for $\mathcal{H}\widehat{\eta}, \mathcal{H}\widehat{\varphi}$, we have the following

$$\begin{aligned}|K_G(\tau,\lambda,\phi)| \lesssim &\int_{\mathbb{R}} |\widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)| \min\left(\frac{1}{\langle\phi\rangle}, \frac{1}{\langle\mu\rangle}\right) d\mu \\ &+ \int_{\mathbb{R}} \frac{1}{\langle\phi\rangle\langle\mu-\lambda\rangle}\left|\widehat{\varphi}(\tau-\mu)\widehat{\eta}\left(\frac{\mu}{\phi}\right)\right| d\mu =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Applying Lemma 2.1.4 and Cauchy-Schwarz inequality, we have

$$\mathrm{I} \lesssim \int_{\mathbb{R}} \frac{1}{\langle\tau-\mu\rangle^{\alpha+1}\langle\mu-\lambda\rangle^{\alpha+1}} \min\left(\frac{1}{\langle\phi\rangle}, \frac{1}{\langle\mu\rangle}\right) d\mu \lesssim \frac{1}{\langle\tau-\lambda\rangle^\alpha} \min\left(\frac{1}{\langle\phi\rangle}, \frac{1}{\langle\lambda\rangle}\right).$$

For II, applying Lemma 2.1.4 and Cauchy-Schwarz inequality gives the following estimates

$$\mathrm{II} \lesssim \int_{\mathbb{R}} \frac{1}{\langle\phi\rangle\langle\mu-\lambda\rangle\langle\tau-\mu\rangle^{1+}} d\mu \lesssim \frac{1}{\langle\phi\rangle\langle\tau-\lambda\rangle},$$

$$\mathrm{II} \lesssim \left(\int_{\mathbb{R}} \frac{d\mu}{\langle\tau-\mu\rangle^2\langle\mu\rangle^2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{d\mu}{\langle\tau-\mu\rangle^2\langle\mu-\lambda\rangle^2}\right)^{\frac{1}{2}} \lesssim \frac{1}{\langle\tau\rangle\langle\tau-\lambda\rangle}.$$

Consequently, $\mathrm{II} \lesssim \frac{1}{\langle\tau-\lambda\rangle} \min\left(\frac{1}{\langle\phi\rangle}, \frac{1}{\langle\tau\rangle}\right)$. For (3.14), we consider different cases $\max(\langle\phi\rangle, \langle\lambda\rangle) \gtrsim \max(\langle\phi\rangle, \langle\tau\rangle)$ or $\max(\langle\phi\rangle, \langle\lambda\rangle) \ll \max(\langle\phi\rangle, \langle\tau\rangle)$. Note that for the latter, $\max(\langle\phi\rangle, \langle\tau\rangle) = \langle\tau\rangle$ and $\langle\tau-\lambda\rangle \sim \langle\tau\rangle$. The estimate follows by choosing $\alpha \geq 2$.

□

Remark 3.2.3. For $* \in \{A, B\}$, consider the operators $\mathcal{DNR}_{*,\geq}(u_1, u_2, u_3)$ and

$\mathbf{G}_{*,\geq}(u_1,u_2,u_3)$, and the kernel estimates in Propositions 3.2.1 and 3.2.2. Then,

$$\big|\mathcal{F}_{t,x}\big(\mathcal{DNR}_{*,\geq}(u_1,u_2,u_3)\big)(\tau,n)\big| \lesssim \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n),\\ |n_2|\geq|n_3|}} \int_{\mathbb{R}} \frac{|n_1|}{\langle\tau-\lambda\rangle\langle\tau-n^3\rangle}\widehat{F}(\lambda,\overline{n}_{123})\,d\lambda,$$

$$\big|\mathcal{F}_{t,x}\big(\mathbf{G}_{*,\geq}(u_1,u_2,u_3)\big)(\tau,n)\big| \lesssim \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n),\\ |n_2|\geq|n_3|}} \int_{\mathbb{R}} \frac{|n_1|}{\langle\tau-\lambda\rangle}\min\left(\frac{1}{\langle\phi(\overline{n}_{123})\rangle},\frac{1}{\langle\tau-n^3\rangle}\right) \times \widehat{F}(\lambda,\overline{n}_{123})\,d\lambda,$$

where $\widehat{F}(\lambda,\overline{n}_{123}) = \big(|\widehat{u}_1(\cdot,n_1)| * |\widehat{u}_2(\cdot,n_2)| * |\widehat{u}_3(\cdot,n_3)|\big)(\lambda)$. Thus, for the modified Duhamel operators $\mathbf{G}_{*,\geq}$ we can 'exchange' the smoothing in time through $\langle\tau-n^3\rangle$ for smoothing in space through $\langle\phi(\overline{n}_{123})\rangle$, unlike the usual Duhamel operator.

Now we estimate the kernel of the remainder 'bad' operators $\mathbf{B}_{*,\geq},\mathbf{B}_{*,>}$, $*\in\{A,B\}$. The assumptions on η in (3.12) play an important role in establishing the following kernel estimates.

Proposition 3.2.4. *Let $*\in\{A,B\}$. Then, the convolution operators $\mathbf{B}_{*,\geq},\mathbf{B}_{*,>}$ have the following space-time Fourier transform*

$$\begin{aligned}&\mathcal{F}_{t,x}\big(\mathbf{B}_{*,\underset{(>)}{\geq}}(u_1,u_2,u_3)\big)(\tau,n)\\ &\quad= \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n),\\ |n_2|\underset{(>)}{\geq}|n_3|}} n_1\int_{\mathbb{R}} K_B(\tau-n^3,\lambda-n^3,\phi(\overline{n}_{123}))\int_{\lambda=\tau_1+\tau_2+\tau_3}\prod_{j=1}^3 \widehat{u}_j(\tau_j,n_j)\,d\tau_1\,d\tau_2\,d\lambda,\end{aligned}$$

where the kernel K_B is given by

$$\begin{aligned}K_B(\tau,\lambda,\phi) = \int_{\mathbb{R}}\frac{\widehat{\varphi}(\tau-\mu)-\widehat{\varphi}(\tau)}{\mu}\widehat{\varphi}(\mu-\lambda)\,d\mu - \int_{\mathbb{R}}\widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)d\mu\\ + \int_{\mathbb{R}}\widehat{\varphi}(\tau-\mu)\mathcal{H}\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\,d\mu,\end{aligned}$$

and satisfies the following estimate

$$|K_B(\tau,\lambda,\phi)| \lesssim \frac{1}{\langle\tau\rangle^\alpha\langle\lambda\rangle} + \frac{\langle\lambda+\phi\rangle}{\langle\tau-\lambda\rangle^\alpha\langle\lambda\rangle}\min\left(\frac{1}{\langle\phi\rangle},\frac{1}{\langle\lambda\rangle}\right) + \frac{\langle\tau+\phi\rangle}{\langle\tau-\lambda\rangle}\min\left(\frac{1}{\langle\phi\rangle},\frac{1}{\langle\tau\rangle}\right)^2,$$

for any $\alpha>0$ and $|\phi|\geq1$.

Proof. Let $*\in\{A,B\}$ and let $\mathbf{B}_*$ denote both $\mathbf{B}_{*,\geq}$ and $\mathbf{B}_{*,>}$. By definition, we have that $\mathbf{B}_* = \mathcal{DNR}_* - \mathbf{G}_*$. Therefore, the kernel K_B is given by $K_B(\tau,\lambda,\phi) = -iK(\tau,\lambda) - K_G(\tau,\lambda,\phi)$, and the intended expression follows from Propositions 3.2.1 and 3.2.2.

We now focus on estimating the kernel, by first rewriting $K_B(\tau,\lambda,\phi)$ as

$$\begin{aligned}&\int_{|\mu|\leq1}\big(\widehat{\varphi}(\tau-\mu)-\widehat{\varphi}(\tau)\big)\widehat{\varphi}(\mu-\lambda)\frac{1}{\mu}\,d\mu - \int_{|\mu|\leq1}\widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\,d\mu\\ &+\int_{|\mu|>1}\widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\Big\{\frac{1}{\mu}-\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\Big\}\,d\mu + \int_{\mathbb{R}}\widehat{\varphi}(\tau-\mu)\mathcal{H}\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\,d\mu\\ &-\int_{|\mu|>1}\widehat{\varphi}(\tau)\widehat{\varphi}(\mu-\lambda)\frac{1}{\mu}\,d\mu\\ &=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5.\end{aligned}$$

For I_1, mean value theorem gives, for any $\alpha > 0$,

$$|\mathrm{I}_1| \lesssim \int\limits_{|\mu|\le 1} \mathbb{1}_{|\tau|\lesssim 1} \frac{d\mu}{\langle\tau\rangle^\alpha\langle\mu\rangle^\alpha\langle\mu-\lambda\rangle^{1+\alpha}} + \int\limits_{|\mu|\le 1} \mathbb{1}_{|\tau|\gg 1} \frac{d\mu}{\langle\tau\rangle^\alpha\langle\mu\rangle^\alpha\langle\mu-\lambda\rangle^{1+\alpha}} \lesssim \frac{1}{\langle\tau\rangle^\alpha\langle\lambda\rangle^\alpha}.$$

Using (3.16) and Cauchy-Schwarz inequality gives

$$|\mathrm{I}_2| \lesssim \int\limits_{|\mu|\le 1} \frac{d\mu}{\langle\phi\rangle\langle\tau-\mu\rangle^\alpha\langle\mu-\lambda\rangle^{\alpha+1}\langle\mu\rangle^\alpha} \lesssim \frac{1}{\langle\phi\rangle\langle\tau-\lambda\rangle^\alpha\langle\lambda\rangle^\alpha}.$$

Before estimating I_3, note that since $\mathcal{H}\widehat{\eta}(-1) = -1$ and using mean value theorem, we get

$$\begin{aligned}
\left|\frac{1}{\mu} - \frac{1}{\phi}\mathcal{H}\widehat{\eta}\left(\frac{\mu}{\phi}\right)\right| &\sim \frac{1}{\langle\mu\rangle}\left|\mathcal{H}\widehat{\eta}(-1) - \frac{\mu}{\phi}\mathcal{H}\widehat{\eta}\left(\frac{\mu}{\phi}\right)\right| \\
&\lesssim \mathbb{1}_{\langle\phi\rangle\gtrsim\langle\mu\rangle}\frac{1}{\langle\mu\rangle}\left|-1-\frac{\mu}{\phi}\right| + \mathbb{1}_{\langle\phi\rangle\ll\langle\mu\rangle}\frac{\langle\mu+\phi\rangle}{\langle\mu\rangle^2} \\
&\lesssim \frac{\langle\mu+\phi\rangle}{\langle\mu\rangle}\min\left(\frac{1}{\langle\phi\rangle},\frac{1}{\langle\mu\rangle}\right).
\end{aligned}$$

Using the above estimate, it follows from previous arguments that, for any $\alpha > 0$,

$$|\mathrm{I}_3| \lesssim \mathbb{1}_{\langle\phi\rangle\gtrsim\langle\lambda\rangle}\frac{\langle\lambda+\phi\rangle}{\langle\tau-\lambda\rangle^\alpha\langle\lambda\rangle\langle\phi\rangle} + \mathbb{1}_{\langle\phi\rangle\ll\langle\lambda\rangle}\frac{\langle\lambda+\phi\rangle}{\langle\tau-\lambda\rangle^\alpha\langle\lambda\rangle^2} \lesssim \frac{\langle\lambda+\phi\rangle}{\langle\tau-\lambda\rangle^\alpha\langle\lambda\rangle}\min\left(\frac{1}{\langle\phi\rangle},\frac{1}{\langle\lambda\rangle}\right).$$

In order to estimate I_4, we start by showing a bound for $\frac{1}{\phi}\widehat{\eta}\big(\frac{\mu}{\phi}\big)$. If $|\phi| \gtrsim |\mu|$, we use the fact that $\widehat{\eta}(-1) = 0$ and mean value theorem. Otherwise, $|\phi| \ll |\mu|$ and $\langle\mu+\phi\rangle \sim \langle\mu\rangle$. It follows that

$$\begin{aligned}
\left|\frac{1}{\phi}\widehat{\eta}\left(\frac{\mu}{\phi}\right)\right| &\lesssim \mathbb{1}_{|\phi|\gtrsim|\mu|}\frac{1}{\langle\phi\rangle}\left|\widehat{\eta}\left(\frac{\mu}{\phi}\right) - \widehat{\eta}(-1)\right| + \mathbb{1}_{|\phi|\ll|\mu|}\frac{1}{\langle\mu\rangle}\left|\frac{\mu}{\phi}\widehat{\eta}\left(\frac{\mu}{\phi}\right)\right| \\
&\lesssim \langle\mu+\phi\rangle\min\left(\frac{1}{\langle\phi\rangle},\frac{1}{\langle\mu\rangle}\right)^2.
\end{aligned}$$

Using the above estimate and the fact that $|\mathcal{H}\widehat{\varphi}(\mu-\lambda)| \lesssim \langle\mu-\lambda\rangle^{-1}$ in (3.16), we have

$$|\mathrm{I}_4| \lesssim \mathbb{1}_{\langle\phi\rangle\gtrsim\langle\tau\rangle}\frac{\langle\tau+\phi\rangle}{\langle\phi\rangle^2\langle\tau-\lambda\rangle} + \mathbb{1}_{\langle\phi\rangle\ll\langle\tau\rangle}\frac{\langle\tau+\phi\rangle}{\langle\tau-\lambda\rangle\langle\tau\rangle^2} \lesssim \frac{\langle\tau+\phi\rangle}{\langle\tau-\lambda\rangle}\min\left(\frac{1}{\langle\phi\rangle},\frac{1}{\langle\tau\rangle}\right)^2.$$

For the last contribution, for any $\alpha > 0$, we have

$$|\mathrm{I}_5| \lesssim \int\limits_{|\mu|>1} \frac{\langle\tau\rangle^\alpha|\widehat{\varphi}(\tau)|\langle\mu-\lambda\rangle|\widehat{\varphi}(\mu-\lambda)|}{\langle\tau\rangle^\alpha\langle\mu-\lambda\rangle\langle\mu\rangle}\, d\mu \lesssim \frac{1}{\langle\tau\rangle^\alpha\langle\lambda\rangle}.$$

This completes the estimate.

□

We want to further split the operators $\mathbf{B}_{*,\ge}, \mathbf{B}_{*,>}$, $* \in \{A, B\}$, to obtain better kernel estimates. We will split the kernel K_B in two pieces: when we can estimate the multiplier directly, and when we also need to use $\sigma_{\max} = \max\limits_{j=1,2,3}|\tau_j - n_j^3|$. Let $K_B = K_0 + K_+$ where the kernels are defined below

$$K_0(\tau,\lambda,\phi) = \mathbb{1}_{\langle\lambda\rangle\gtrsim\langle\phi\rangle}\Bigg(\int\limits_{|\mu|\le 1}\big(\widehat{\varphi}(\tau-\mu) - \widehat{\varphi}(\tau)\big)\widehat{\varphi}(\mu-\lambda)\frac{1}{\mu}\, d\mu - \int\limits_{|\mu|>1}\widehat{\varphi}(\tau)\widehat{\varphi}(\mu-\lambda)\frac{1}{\mu}\, d\mu\Bigg)$$

$$
\begin{aligned}
&+ \mathbb{1}_{\langle \lambda+\phi\rangle \lesssim \langle \tau-\lambda\rangle} \bigg(\int\limits_{|\mu|>1} \widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\bigg\{ \frac{1}{\mu} - \frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\bigg\}\, d\mu \bigg) \\
&+ \mathbb{1}_{\langle \tau+\phi\rangle \lesssim \langle \tau-\lambda\rangle} \bigg(\int_{\mathbb{R}} \widehat{\varphi}(\tau-\mu)\mathcal{H}\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\, d\mu \bigg) \\
&- \int\limits_{|\mu|\le 1} \widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\, d\mu,
\end{aligned}
$$

$$
\begin{aligned}
K_+(\tau,\lambda,\phi) &= \mathbb{1}_{\langle \lambda\rangle \ll \langle \phi\rangle} \bigg(\int\limits_{|\mu|\le 1} \big(\widehat{\varphi}(\tau-\mu)-\widehat{\varphi}(\tau)\big)\widehat{\varphi}(\mu-\lambda)\frac{1}{\mu}\, d\mu - \int\limits_{|\mu|>1} \widehat{\varphi}(\tau)\widehat{\varphi}(\mu-\lambda)\frac{1}{\mu}\, d\mu \bigg) \\
&+ \mathbb{1}_{\langle \lambda+\phi\rangle \gg \langle \tau-\lambda\rangle} \int\limits_{|\mu|>1} \widehat{\varphi}(\tau-\mu)\widehat{\varphi}(\mu-\lambda)\bigg\{ \frac{1}{\mu} - \frac{1}{\phi}\mathcal{H}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\bigg\}\, d\mu \\
&+ \mathbb{1}_{\langle \tau+\phi\rangle \gg \langle \tau-\lambda\rangle} \bigg(\int_{\mathbb{R}} \widehat{\varphi}(\tau-\mu)\mathcal{H}\widehat{\varphi}(\mu-\lambda)\frac{1}{\phi}\widehat{\eta}\Big(\frac{\mu}{\phi}\Big)\, d\mu \bigg).
\end{aligned}
$$

Thus, we have the following estimates for the kernels, for any $0 \le \alpha \le 1$,

$$
|K_0(\tau,\lambda,\phi)| \lesssim \frac{1}{\langle \tau\rangle^{1+\alpha}\langle \phi\rangle^{1-\alpha}}, \tag{3.17}
$$

$$
|K_+(\tau,\lambda,\phi)| \lesssim \mathbb{1}_{\langle \lambda\rangle \ll \langle \phi\rangle} \frac{1}{\langle \tau-\lambda\rangle\langle \tau\rangle} + \frac{\langle \lambda+\phi\rangle^{1-\alpha}}{\langle \tau-\lambda\rangle\langle \tau\rangle} \min\bigg(\frac{1}{\langle \phi\rangle}, \frac{1}{\langle \tau\rangle}\bigg)^{1-\alpha}. \tag{3.18}
$$

In Section 3.5.1, we will see that the contribution corresponding to the kernel K_0 in $\mathbf{B}_{*,\ge}, \mathbf{B}_{*,>}$ can be easily estimated, due to the explicit smoothing in space (i.e., the negative power of $\langle \phi\rangle$). However, in order to estimate the one corresponding to K_+, we need the largest modulation $\sigma_{\max}$. In particular, from (3.18), we see that

$$
\big|K_+\big(\tau-n^3, \lambda-n^3, \phi(\overline{n}_{123})\big)\big| \lesssim \frac{\langle \lambda - n^3 + \phi(\overline{n}_{123})\rangle^{1-\alpha}}{\langle \tau-\lambda\rangle\langle \tau-n^3\rangle\langle \phi(\overline{n}_{123})\rangle^{1-\alpha}}, \tag{3.19}
$$

for any $0 \le \alpha \le 1$, since $\lambda = \tau_1+\tau_2+\tau_3$ and

$$
|\lambda - n^3 + \phi(\overline{n}_{123})| = |\tau_1 - n_1^3 + \tau_2 - n_2^3 + \tau_3 - n_3^3| \lesssim \max_{j=1,2,3} |\tau_j - n_j^3| = \sigma_{\max}.
$$

Thus, we can use $\sigma_{\max}$ in order to estimate the numerator of the second contribution in (3.18), which motivates splitting the operators depending on which modulation is the largest. In particular, we have

$$
\mathbf{B}_{*,\ge} = \mathbf{B}^0_{*,\ge} + \mathbf{B}^1_{*,\ge} + \mathbf{B}^2_{*,\ge} + \mathbf{B}^3_{*,\ge}, \tag{3.20}
$$

where $\mathbf{B}^0_{*,\ge}$ has kernel K_0 and $\mathbf{B}^j_{*,\ge}$ has kernel K_+ localized to the region where $\sigma_{\max} = |\sigma_j|$, $j=1,2,3$. An analogous decomposition holds for $\mathbf{B}_{*,>}$.

3.3 System of equations and proof of Theorem 1.1.3 for $4 \le p < \infty$

Instead of running a contraction mapping argument on the integral equation (3.5), we will solve an ordered system of equations. In this section, we establish the relevant equations for u and w and the main results needed to show Theorem 1.1.3 for $4 \le p < \infty$. For a fixed p with $2 \le p < \infty$, we will first focus on showing local well-posedness in $\mathcal{F}L^{s,p}(\mathbb{T})$.

Let $T > 0$ and fix $w \in Z^s_0$. Then, we consider the following equation for u

$$
u = w + \varphi_T\big[\mathbf{G}_{A,\ge}(w,\overline{u},u) + \mathbf{G}_{A,>}(w,u,\overline{u}) + \mathbf{G}_{B,\ge}(w,\overline{w},u) + \mathbf{G}_{B,>}(w,w,\overline{u})\big]. \tag{3.21}
$$

We first solve the equation (3.21) obtaining $u = u[w]$, i.e., u parameterized by w.

Proposition 3.3.1. *Let $s \geq \frac{1}{2}$ and $2 \leq p < \infty$. There exist $T = T(A_2) > 0$ and $A_3 = A_3(A_2) > 0$ such that for any $w \in Z_0^s$ satisfying $\|w\|_{Z_0^s} \leq A_2$, there exists a unique $u \in Y_0^s$ with $\|u\|_{Y_0^s} \leq A_3$ satisfying (3.21). The mapping $w \mapsto u[w]$ is Lipschitz from the A_2-ball of Z_0 to the A_3-ball of Y_0.*

We establish the above proposition by running a contraction mapping argument in Y_0^s. The argument then reduces to establishing the nonlinear estimates in Section 3.4. We postpone the proof of Proposition 3.3.1 until Section 3.6. To guarantee that $u = u[w]$, the solution of (3.21), satisfies the Duhamel formulation (3.5), then w must satisfy the following equation

$$\begin{aligned}
w = \varphi(t)S(t)u_0 &+ \varphi_T \cdot \mathcal{DR}(u,u,u) \\
&+ \varphi_T \big[\mathcal{DNR}_{C,\geq}(u,\overline{u},u) + \mathcal{DNR}_{C,>}(u,u,\overline{u})\big] \\
&+ \varphi_T \big[\mathcal{DNR}_{D,\geq}(u,\overline{u},u) + \mathcal{DNR}_{D,>}(u,u,\overline{u})\big] \\
&+ \varphi_T \big[\mathbf{B}_{A,\geq}(w,\overline{u},u) + \mathbf{B}_{A,>}(w,u,\overline{u})\big] \\
&+ \varphi_T \big[\mathbf{B}_{B,\geq}(w,\overline{w},u) + \mathbf{B}_{B,>}(w,w,\overline{u})\big] \\
&+ \varphi_T \big[\mathcal{DNR}_{A,\geq}(u,\overline{u},u) - \mathcal{DNR}_{A,\geq}(w,\overline{u},u)\big] \\
&+ \varphi_T \big[\mathcal{DNR}_{A,>}(u,u,\overline{u}) - \mathcal{DNR}_{A,>}(w,u,\overline{u})\big] \\
&+ \varphi_T \big[\mathcal{DNR}_{B,\geq}(u,\overline{u},u) - \mathcal{DNR}_{B,\geq}(w,\overline{w},u)\big] \\
&+ \varphi_T \big[\mathcal{DNR}_{B,>}(u,u,\overline{u}) - \mathcal{DNR}_{B,>}(w,w,\overline{u})\big].
\end{aligned} \tag{3.22}$$

In order to solve the above equation, we use a *partial* second iteration by replacing $u = u[w]$ by its equation (3.21). The decomposition on the operators $\mathcal{DNR}$ and $\mathbf{B}$, introduced in Sections 3.1 and 3.2, explicitly identifies which entries have the largest frequencies and the largest modulations, respectively. This information will guide the second iteration process.

For the terms $\mathcal{DNR}_{*,\geq}, \mathcal{DNR}_{*,>}$, $* \in \{C, D\}$, we replace the equation for u (3.21) from left to right to obtain only cubic and quintic terms, as in the following example

$$\begin{aligned}
\mathcal{DNR}_{C,\geq}(u,\overline{u},u) &= \mathcal{DNR}_{C,\geq}(w,\overline{u},u) \\
&+ \mathcal{DNR}_{C,\geq}\big(\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],\overline{u},u\big) + \mathcal{DNR}_{C,\geq}\big(\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],\overline{u},u\big) \\
&+ \mathcal{DNR}_{C,\geq}\big(\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u],\overline{u},u\big) + \mathcal{DNR}_{C,\geq}\big(\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}],\overline{u},u\big), \\
\mathcal{DNR}_{C,\geq}(w,\overline{u},u) &= \mathcal{DNR}_{C,\geq}(w,\overline{w},u) \\
&+ \mathcal{DNR}_{C,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]},u\big) + \mathcal{DNR}_{C,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]},u\big) \\
&+ \mathcal{DNR}_{C,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]},u\big) + \mathcal{DNR}_{C,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]},u\big), \\
\mathcal{DNR}_{C,\geq}(w,\overline{w},u) &= \mathcal{DNR}_{C,\geq}(w,\overline{w},w) \\
&+ \mathcal{DNR}_{C,\geq}\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]\big) + \mathcal{DNR}_{C,\geq}\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]\big), \\
&+ \mathcal{DNR}_{C,\geq}\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]\big) + \mathcal{DNR}_{C,\geq}\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]\big).
\end{aligned}$$

This strategy prioritizes the entry with the derivative followed by the one with the largest frequency between the remaining two factors. For $\mathcal{DNR}_{*,\geq}, \mathcal{DNR}_{*,>}$ with $* \in \{A, B\}$, there will be no cubic terms after second iteration, due to the differences in (3.22), as seen in the following

$$\begin{aligned}
&\mathcal{DNR}_{A,\geq}(u,\overline{u},u) - \mathcal{DNR}_{A,\geq}(w,\overline{u},u) \\
&\quad = \mathcal{DNR}_{A,\geq}\big(\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],\overline{u},u\big) + \mathcal{DNR}_{A,\geq}\big(\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],\overline{u},u\big) \\
&\quad + \mathcal{DNR}_{A,\geq}\big(\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u],\overline{u},u\big) + \mathcal{DNR}_{A,\geq}\big(\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}],\overline{u},u\big), \\
&\mathcal{DNR}_{B,\geq}(u,\overline{u},u) - \mathcal{DNR}_{B,\geq}(w,\overline{w},u) \\
&\quad = \mathcal{DNR}_{B,\geq}\big(\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],\overline{u},u\big) + \mathcal{DNR}_{B,\geq}\big(\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],\overline{u},u\big) \\
&\quad + \mathcal{DNR}_{B,\geq}\big(\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u],\overline{u},u\big) + \mathcal{DNR}_{B,\geq}\big(\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}],\overline{u},u\big) \\
&\quad + \mathcal{DNR}_{B,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]},u\big) + \mathcal{DNR}_{B,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]},u\big)
\end{aligned}$$

$$+ \mathcal{DNR}_{B,\geq}\big(w, \overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]}, u\big) + \mathcal{DNR}_{B,\geq}\big(w, \overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]}, u\big).$$

For the terms $\mathbf{B}_{*,\geq}, \mathbf{B}_{*,>}$, with $* \in \{A, B\}$, we split the operators into four pieces $\mathbf{B}^j_{*,\geq}, \mathbf{B}^j_{*,>}$, $j = 0,1,2,3$, as defined in (3.20). The contributions corresponding to $j = 0$ are easily estimated, but for $j = 1,2,3$ the largest modulation plays an important role in estimating the kernel. If the j-th entry corresponds to a u or $\overline{u}$ term, we replace it with the equation for u (3.21). For example, we have

$$\begin{aligned}\mathbf{B}_{B,\geq}(w,\overline{w},u) = {} & \mathbf{B}^0_{B,\geq}(w,\overline{w},w) + \mathbf{B}^1_{B,\geq}(w,\overline{w},u) + \mathbf{B}^2_{B,\geq}(w,\overline{w},u) + \mathbf{B}^3_{B,\geq}(w,\overline{w},w) \\ & + \mathbf{B}^3_{B,\geq}\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]\big) + \mathbf{B}^3_{B,\geq}\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]\big) \\ & + \mathbf{B}^3_{B,\geq}\big(w,\overline{u},\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]\big) + \mathbf{B}^3_{B,\geq}\big(w,\overline{u},\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]\big).\end{aligned}$$

Proceeding as detailed above, we obtain a new equation for w. Due to its length, we have decided to only include it in Appendix A.2.

Proposition 3.3.2. *Let $s \geq \frac{1}{2}$ and $2 \leq p < \infty$. Then, for any $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ satisfying $\|u_0\|_{\mathcal{F}L^{s,p}} \leq A_1$, there exist $T = T(A_1) > 0$ and a unique $w \in Z^s_0$ with $\|w\|_{Z^s_0} \leq A_2$ satisfying (3.22), for some $A_2 = A_2(A_1) > 0$. The mapping $u_0 \mapsto w$ is Lipschitz from the A_1-ball of $\mathcal{F}L^{s,p}(\mathbb{T})$ to the A_2-ball of Z_0.*

Remark 3.3.3. (i) In the above proposition, $u = u[w]$ is always understood as being the solution in Proposition 3.3.1 of equation (3.21).

(ii) In order to show Proposition 3.3.2, we will not run a contraction mapping argument for the map defined by the right-hand side of (3.22) nor the equation (A.1) included in Appendix A.2. Some quintic terms in (A.1) require the use of the equation for u (3.21) once again, introducing new quintic terms but also new septic terms. Given the considerable number of new terms that this additional step introduces, we have decided to omit them when presenting the equation for w. The strategy for obtaining the new contributions is described in Section 3.5.2 along with the estimates needed for both the quintic and septic terms. The terms are given in detail in Appendix A.3.

Combining Propositions 3.3.1 and 3.3.2, we can now complete the proof of Theorem 1.1.3 by establishing the local well-posedness of mKdV2 (3.1) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq \frac{1}{2}$ and $4 \leq p < \infty$.

Proof of Theorem 1.1.3 for $4 \leq p < \infty$. Let $s \geq \frac{1}{2}$, $4 \leq p < \infty$, and $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$. From Proposition 3.3.1, for any $A_2 > 0$ and $w \in Z^s_0$ with $\|w\|_{Z^s_0} \leq A_2$, there exist $T_1 = T_1(A_2) > 0$, $A_3 = A_3(A_2) > 0$, and a unique $u = \Phi_1[w]$ which satisfies

$$\begin{aligned}\Phi_1[w] = w + \varphi_{T_1}\big[&\mathbf{G}_{A,\geq}(w, \overline{\Phi_1[w]}, \Phi_1[w]) + \mathbf{G}_{A,>}(w, \Phi_1[w], \overline{\Phi_1[w]}) \\ & + \mathbf{G}_{B,\geq}(w,\overline{w},\Phi_1[w]) + \mathbf{G}_{B,>}(w,w,\overline{\Phi_1[w]})\big].\end{aligned}$$

For simplicity, let F denote a multilinear operator such that we can rewrite the above equation as

$$\Phi_1[w] = w + \varphi_{T_1} \cdot F(w, \Phi_1[w]). \tag{3.23}$$

From Proposition 3.3.1, we also know that the map Φ_1 is Lipschitz continuous from the A_2-ball of Z^s_0 to the A_3-ball of Y^s_0. We now consider the equation for w. First, we can rewrite (3.22) as

$$w = \varphi(t)S(t)u_0 + \varphi_{T_1}\Big[\mathcal{DN}\big(\Phi_1[w], \overline{\Phi_1[w]}, \Phi_1[w]\big) - F\big(w, \Phi_1[w]\big)\Big].$$

The operations needed to reach the full equation for w, correspond to substituting certain instances of $\Phi_1[w]$ above by the right-hand side of (3.23). Therefore, we get that

$$w = \varphi(t)S(t)u_0 + \varphi_{T_1} \cdot G\Big(w, \Phi_1[w], F\big(w,\Phi_1[w]\big), F(w,w), F\big(w, F(\Phi_1[w])\big)\Big), \tag{3.24}$$

where G is a multilinear operator with cubic, quintic, and septic terms. Note that the two equations for w above are equivalent.

Now let $A_1 > 0$ such that $\|u_0\|_{\mathcal{F}L^{s,p}} \le A_1$. From Proposition 3.3.2, there exists $T_2 = T_2(A_1) > 0$, $A_2 = A_2(A_1) > 0$, and a unique $w = \Phi_2[u_0] \in Z_0^s$ with $\|w\|_{Z_0^s} \le A_2$ solving (3.24). Moreover, the map Φ_2 is Lipschitz continuous from the A_1-ball of $\mathcal{F}L^{s,p}(\mathbb{T})$ to the A_2-ball of Z_0^s.

Let $T = \min(T_1, T_2)$. We want to show that $\Phi_1 \circ \Phi_2[u_0] \in Y_0^s$ is a solution of mKdV2 (3.1) in the sense of solving the Duhamel formulation in (3.4) for $|t| \lesssim T$. Using the equivalence between the w equations above and the fact that $\Phi_2[u_0]$ is a solution, we have that

$$
\begin{aligned}
\Phi_2[u_0] &= \varphi(t)S(t)u_0 + \varphi_T \cdot G\Big(w, \Phi_1[w], F\big(w, \Phi_1[w]\big), F(w,w), F\big(w, F(\Phi_1[w])\big)\Big) \\
&= \varphi(t)S(t)u_0 + \varphi_T\Big[\mathcal{DN}\big(\Phi_1\circ\Phi_2[u_0], \overline{\Phi_1\circ\Phi_2[u_0]}, \Phi_1\circ\Phi_2[u_0]\big) - F\big(\Phi_2[u_0], \Phi_1\circ\Phi_2[u_0]\big)\Big].
\end{aligned}
$$

Since $\Phi_1 \circ \Phi_2[u_0]$ solves (3.23) with $w = \Phi_2[u_0]$, rearranging the above equation, we have

$$
\begin{aligned}
\Phi_1\circ\Phi_2[u_0] &= \Phi_2[u_0] + \varphi_T \cdot F\big(\Phi_2[u_0], \Phi_1\circ\Phi_2[u_0]\big) \\
&= \varphi(t)S(t)u_0 + \varphi_T \cdot \mathcal{DN}\big(\Phi_1\circ\Phi_2[u_0], \overline{\Phi_1\circ\Phi_2[u_0]}, \Phi_1\circ\Phi_2[u_0]\big)
\end{aligned}
$$

which shows that $\Phi_1 \circ \Phi_2[u_0]$ satisfies the Duhamel formulation (3.5) for mKdV2 (3.1), or equivalently, (3.4) for $|t| \le T$. Lastly, since Φ_2 is Lipschitz continuous from the A_1-ball of $\mathcal{F}L^{s,p}(\mathbb{T})$ to the A_2-ball of Z_0^s, and Φ_1 is Lipschitz continuous from the A_2-ball of Z_0^s to the A_3-ball of Y_0^s, we conclude that $\Phi_1 \circ \Phi_2$ is locally Lipschitz continuous from $\mathcal{F}L^{s,p}(\mathbb{T})$ to $C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$. This completes the proof.

□

In the remaining sections, we will use the following notation for simplicity: $\mathcal{DNR}_*$ to denote $\mathcal{DNR}_{*,\ge}, \mathcal{DNR}_{*,>}$, for $* \in \{A,B,C,D\}$, $\mathbf{G}_*$ to denote $\mathbf{G}_{*,\ge}, \mathbf{G}_{*,>}$ for $* \in \{A,B\}$, and $\mathbf{B}_*^j$ for $\mathbf{B}_{*,\ge}^j, \mathbf{B}_{*,>}^j$ for $* \in \{A,B\}$ and $j = 0,1,2,3$. In the estimates, there is no distinction between the frequency regions where $|n_2| \ge |n_3|$ and $|n_2| > |n_3|$, motivating this simplified notation.

3.4 Nonlinear estimates for u

From Lemma 3.1.4, it suffices to estimate the terms in the equation (3.21) in Y_1^s, dropping the factor of φ_T.

Lemma 3.4.1. *The following estimates hold*

$$
\begin{aligned}
\|\mathbf{G}_A(u_1,u_2,u_3)\|_{Y_1^s} &\lesssim \|u_1\|_{Z_0^s}\|u_2\|_{Y_0^{\frac12}}\|u_3\|_{Y_0^{\frac12}}. \\
\|\mathbf{G}_B(u_1,u_2,u_3)\|_{Y_1^s} &\lesssim \|u_1\|_{Z_0^{\frac12}}\|u_2\|_{Z_0^s}\|u_3\|_{Y_0^{\frac12}}.
\end{aligned}
$$

Proof. We will prove the above estimates for $s = \frac12$, since the additional factor $\langle n\rangle^{s-\frac12}$ on the left-hand side can be controlled by $\langle n_1\rangle^{s-\frac12}$ if $* = A$ and $\langle n_2\rangle^{s-\frac12}$ if $* = B$, due to the restrictions on the frequencies.

Using (3.14) and the change of variables $\tau_j - n_j^3 = \sigma_j$, $j = 1,2,3$, it follows that

$$
\begin{aligned}
&\|\mathbf{G}_*(u_1,u_2,u_3)\|_{Y_1^{\frac12}} \\
&\lesssim \Bigg\| \sum_{\substack{n = n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \int_{\mathbb{R}} \frac{\langle n\rangle^{\frac12}|n_1|}{\langle \tau-\lambda\rangle\langle\phi(\overline{n}_{123})\rangle^{\frac12}} \int_{\lambda=\tau_1+\tau_2+\tau_3} \prod_{j=1}^3 |\widehat{u}_j(\tau_j,n_j)|\, d\lambda \Bigg\|_{\ell_n^p L_\tau^{r_1}} \\
&\lesssim \Bigg\| \sum_{\substack{n = n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \int_{\sigma_1,\sigma_2,\sigma_3} \frac{\langle n\rangle^{\frac12}|n_1|}{\langle \tau-n^3-\bar\sigma+\phi(\overline{n}_{123})\rangle\langle\phi(\overline{n}_{123})\rangle^{\frac12}} \prod_{j=1}^3 |\widehat{u}_j(\sigma_j+n_j^3,n_j)| \Bigg\|_{\ell_n^p L_\tau^{r_1}}, \quad (3.25)
\end{aligned}
$$

where $\bar{\sigma} = \sigma_1 + \sigma_2 + \sigma_3$. Note that

$$\frac{\langle n\rangle^{\frac{1}{2}}|n_1|}{\langle \phi(\overline{n}_{123})\rangle^{\frac{1}{2}}} \lesssim \frac{\langle n\rangle^{\frac{1}{2}}|n_1|}{\max\limits_{j=1,2,3}\langle n_j\rangle} \lesssim \langle n_1\rangle^{\frac{1}{2}}.$$

Minkowski's inequality gives

$$\|\mathbf{G}_*(u_1,u_2,u_3)\|_{Y_1^{\frac{1}{2}}} \lesssim \int_{\sigma_1,\sigma_2,\sigma_3} \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \frac{\langle n_1\rangle^{\frac{1}{2}}}{\langle \tau - n^3 - \bar{\sigma} + \phi(\overline{n}_{123})\rangle} \prod_{j=1}^{3} |\widehat{u}_j(\sigma_j + n_j^3, n_j)| \Bigg\|_{\ell^p_n L^{r_1}_\tau}.$$

Denoting the inner norm by I, we can rewrite the sum as follows

$$\mathrm{I} \lesssim \Bigg\| \sum_{\mu} \frac{1}{\langle \tau - n^3 - \bar{\sigma} + \mu\rangle} \Bigg(\sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}^\mu_*(n)}} \langle n_1\rangle^{\frac{1}{2}} \prod_{j=1}^{3} |\widehat{u}_j(\sigma_j + n_j^3, n_j)| \Bigg) \Bigg\|_{\ell^p_n L^{r_1}_\tau},$$

for $\mathbb{X}^\mu_*(n)$ in (3.3). Since the following bounds hold uniformly in $\bar{\sigma}$ and n, for any $\tilde{r} > 1$,

$$\sum_{\mu} \frac{1}{\langle \tau - n^3 - \bar{\sigma} + \mu\rangle^{\tilde{r}}} \lesssim 1, \qquad \int_{\mathbb{R}} \frac{1}{\langle \tau - n^3 - \bar{\sigma} + \mu\rangle^{\tilde{r}}} d\tau \lesssim 1,$$

choosing $\tilde{r} = \frac{1}{1-\delta}$, we have $\frac{1}{r_1} + 1 = \frac{1}{\tilde{r}} + \frac{1}{r_2}$ and we can apply Schur's test (Lemma 3.1.3) to obtain

$$\mathrm{I} \lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}^\mu_*(n)}} \langle n_1\rangle^{\frac{1}{2}} \prod_{j=1}^{3} |\widehat{u}_j(\sigma_j + n_j^3, n_j)| \Bigg\|_{\ell^p_n \ell^{r_2}_\mu}. \tag{3.26}$$

Let $\mathbf{P}_{N_j}$ denote the projection onto $\langle n\rangle \sim N_j$ and let $f_1(\sigma,n) = \langle n\rangle^{\frac{1}{2}}|\widehat{u}_1(\sigma+n^3,n)|$, $f_j(\sigma,n) = |\widehat{\mathbf{P}_{N_j}u_j}(\sigma+n^3,n)|$, $j=2,3$. Then, using Minkowski and Hölder's inequalities, we get

$$\mathrm{I} \lesssim \sum_{N_2,N_3} \Bigg\| |\mathbb{X}^\mu_*(n)|^{\frac{1}{r_2'}} \Bigg(\sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}^\mu_*(n)}} \prod_{j=1}^{3} |f_j(\sigma_j,n_j)|^{r_2} \Bigg)^{\frac{1}{r_2}} \Bigg\|_{\ell^p_n \ell^{r_2}_\mu}.$$

If $* = A$, we have $|\phi(\overline{n}_{123})| \lesssim |n_1|^3 \sim |n|^3$, so we use Lemma 3.1.7 to count the divisors $d_2 = n - n_2, d_3 = n - n_3$ of μ. Since

$$|d_2 - n| = |n_2| \le N_2, \quad |d_3 - n| = |n_3| \le N_3$$

and $1 \le |\mu|^\varepsilon \le |\mu|^{\frac{1}{3}} \lesssim |n|$, for any $0 < \varepsilon \le \frac{1}{3}$, we conclude that there are at most $\mathcal{O}(N_j^\varepsilon)$ choices for d_j, $j = 2,3$. Since n is fixed, this determines the choices of n_2, n_3 and consequently of n_1. If $* = B$, then $|\phi(\overline{n}_{123})| \lesssim |n_2|^3$ and we can use the standard divisor counting estimate in Lemma 3.1.6 to conclude that there are at most $\mathcal{O}(N_2^\varepsilon)$ choices for n_2, n_3. Consequently, $|\mathbb{X}^\mu_*(n)| \lesssim (N_2N_3)^\varepsilon$ and we have

$$\begin{aligned}
\mathrm{I} &\lesssim \sum_{N_2,N_3} (N_2N_3)^\varepsilon \Bigg\| \Bigg(\sum_{\mu} \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}^\mu_*(n)}} \prod_{j=1}^{3} |f_j(\sigma_j,n_j)|^{r_2} \Bigg)^{\frac{1}{r_2}} \Bigg\|_{\ell^p_n} \\
&\lesssim \sum_{N_2,N_3} (N_2N_3)^\varepsilon \Bigg\| \Bigg(\sum_{n=n_1+n_2+n_3} \Big(\sum_{\mu} \mathbb{1}_{\phi(\overline{n}_{123})=\mu} \Big) \prod_{j=1}^{3} |f_j(\sigma_j,n_j)|^{r_2} \Bigg)^{\frac{1}{r_2}} \Bigg\|_{\ell^p_n} \\
&\lesssim \sum_{N_2,N_3} (N_2N_3)^\varepsilon \|f_1\|_{\ell^p_n} \|f_2\|_{\ell^{r_2}_n} \|f_3\|_{\ell^{r_2}_n},
\end{aligned}$$

where we apply Minkowski's inequality and the fact that $r_2 < 2 \le p$ in the last inequality. Choosing $\varepsilon < \delta$, we obtain

$$\mathrm{I} \lesssim \|\langle n\rangle^{\frac{1}{2}}\widehat{u}_1(\sigma_1+n^3,n)\|_{\ell^p_n}\|\langle n\rangle^{\delta}\widehat{u}_2(\sigma_2+n^3,n)\|_{\ell^{r_2}_n}\|\langle n\rangle^{\delta}\widehat{u}_3(\sigma_3+n^3,n)\|_{\ell^{r_2}_n}.$$

Applying this estimate to (3.26) gives

$$\|\mathbf{G}_*(u_1,u_2,u_3)\|_{Y_1^{\frac{1}{2}}} \lesssim \|\langle n\rangle^{\frac{1}{2}}\widehat{u}_1(\sigma+n^3,n)\|_{L^1_\sigma\ell^p_n}\prod_{j=2}^{3}\|\langle n\rangle^{\delta}\widehat{u}_j(\sigma+n^3,n)\|_{L^1_\sigma\ell^{r_2}_n}.$$

The estimate follows from Hölder and Minkowski's inequalities, imposing $\delta < \frac{1}{4p}$.

□

Remark 3.4.2. The change of variables from τ_j to the modulation $\sigma_j = \tau_j - n_j^3$, $j = 1,2,3$, in (3.25) is needed to guarantee that the quantity

$$\frac{1}{\langle\tau-\lambda\rangle^{1-\delta}} = \frac{1}{\langle\tau-n^3-\sigma_1-\sigma_2-\sigma_3+\phi(\overline{n}_{123})\rangle^{1-\delta}}$$

has an explicit dependence on the phase function $\phi(\overline{n}_{123})$ and that when fixing its value, $\phi(\overline{n}_{123}) = \mu$, there is no longer dependence on the variables n_1, n_2, n_3. Thus, one can consider the quantity inside the norm as a convolution operator in μ, depending on τ:

$$\sum_{\mu}\frac{1}{\langle\tau-n^3-\sigma_1-\sigma_2-\sigma_3+\mu\rangle^{1-\delta}}F(\mu,n_1,n_2,n_3).$$

This trick allows us to estimate the norm in τ and introduce a restriction on the value of the phase function. This strategy will be used in other estimates.

3.5 Nonlinear estimates for w

Analogously to the previous section, from Lemma 3.1.4, it suffices to estimate the terms in the equation (3.22) in Z_1^s, dropping the factor of φ_T. In this section, we show the multilinear estimates needed to prove Proposition 3.3.2 by a contraction mapping argument. In particular, we estimate the trilinear and quintilinear operators on the right-hand side of (A.1).

In Section 3.5.1, we focus on the cubic terms in (A.1), namely

$$\begin{array}{llll}\mathcal{DR}(u_1,u_2,u_3), & \mathcal{DNR}_C(w_1,w_2,w_3), & \mathcal{DNR}_D(w_1,w_2,w_3), & \\ \mathbf{B}^0_A(w_1,u_2,u_3), & \mathbf{B}^1_A(w_1,u_2,u_3), & \mathbf{B}^2_A(w_1,w_2,u_3), & \mathbf{B}^3_A(w_1,u_2,w_3), \\ \mathbf{B}^0_B(w_1,w_2,u_3), & \mathbf{B}^1_B(w_1,w_2,u_3), & \mathbf{B}^2_B(w_1,w_2,u_3), & \mathbf{B}^3_B(w_1,w_2,w_3),\end{array} \tag{3.27}$$

where $u_j \in \{u,\overline{u}\}$, $w_j \in \{w,\overline{w}\}$, $j = 1,2,3$.

The quintic terms in (A.1) arise from substituting a u entry by a $\mathbf{G}_\#$-operator, for $\# \in \{A,B\}$. First, note that

$$\begin{aligned}&\big|\mathcal{F}_{t,x}\big(\varphi_T\cdot\mathbf{G}_\#(u_1,u_2,u_3)\big)(\tau,n)\big|\\ &\quad\lesssim \sum_{\overline{n}_{123}\in\mathbb{X}_\#(n)}\int_{\mathbb{R}^2}\frac{T|n_1||\widehat{\varphi}(T(\tau-\mu))|}{\langle\mu-\mu'\rangle\langle\phi(\overline{n}_{123})\rangle}\int_{\mu'=\tau_1+\tau_2+\tau_3}\prod_{j=1}^{3}|\widehat{u}_j(\tau_j,n_j)|\,d\mu'\,d\mu\\ &\quad\lesssim \sum_{\overline{n}_{123}\in\mathbb{X}_\#(n)}\int_{\mathbb{R}^2}\frac{|n_1|\|\langle\cdot\rangle\widehat{\varphi}\|_{L^\infty}}{\langle\mu-\mu'\rangle\langle\tau-\mu\rangle\langle\phi(\overline{n}_{123})\rangle}\int_{\mu'=\tau_1+\tau_2+\tau_3}\prod_{j=1}^{3}|\widehat{u}_j(\tau_j,n_j)|\,d\mu'\,d\mu\\ &\quad\lesssim \sum_{\overline{n}_{123}\in\mathbb{X}_\#(n)}\int_{\mathbb{R}}\frac{|n_1|}{\langle\tau-\mu'\rangle^{1-\theta}\langle\phi(\overline{n}_{123})\rangle}\int_{\mu'=\tau_1+\tau_2+\tau_3}\prod_{j=1}^{3}|\widehat{u}_j(\tau_j,n_j)|\,d\mu',\end{aligned}$$

$$
\begin{aligned}
&\big|\mathcal{F}_{t,x}\overline{\big(\varphi_T\cdot\mathbf{G}_\#(u_1,u_2,u_3)\big)}(\tau,n)\big|\\
&\qquad\lesssim \sum_{\overline{n}_{123}\in\mathbb{X}_\#(n)}\int_{\mathbb{R}^2}\frac{T|n_1||\widehat{\varphi}(T(-\tau-\mu))|}{\langle\mu+\mu'\rangle\langle\phi(\overline{n}_{123})\rangle}\int_{\mu'=\tau_1+\tau_2+\tau_3}\prod_{j=1}^3|\widehat{\overline{u}}_j(\tau_j,n_j)|\,d\mu'\,d\mu\\
&\qquad\lesssim \sum_{\overline{n}_{123}\in\mathbb{X}_\#(n)}\int_{\mathbb{R}}\frac{|n_1|}{\langle\tau-\mu'\rangle^{1-\theta}\langle\phi(\overline{n}_{123})\rangle}\int_{\mu'=\tau_1+\tau_2+\tau_3}\prod_{j=1}^3|\widehat{\overline{u}}_j(\tau_j,n_j)|\,d\mu',
\end{aligned}
$$

for any $0<\theta\ll 1$, by using Lemma 2.1.4. Since $\|\overline{u}\|_{X^{s,b}_{p,q}}=\|u\|_{X^{s,b}_{p,q}}$ for any choice of s,b,p,q, we will omit the contributions that depend on $\overline{\mathbf{G}_\#}$, as they can be estimated analogously. We first calculate the space-time Fourier transform of the quintic contributions arising from the $\mathcal{DN}$ terms. For example, for $*\in\{A,B,C,D\}$ and $\#\in\{A,B\}$, we have the following estimate

$$
\begin{aligned}
&\big|\mathcal{F}_{t,x}\mathcal{DNR}_*\big(\varphi_T\cdot\mathbf{G}_\#[u_1,u_2,u_3],u_4,u_5\big)(\tau,n)\big|\\
&\qquad\lesssim \sum_{\substack{\overline{n}_{045}\in\mathbb{X}_*(n),\\ \overline{n}_{123}\in\mathbb{X}_\#(n_0)}}\int_{\mathbb{R}}\int_{\lambda=\tau_1+\ldots+\tau_5}\frac{|n_0n_1|}{\langle\tau-\lambda\rangle^{1-\theta}\langle\tau-n^3\rangle\langle\phi(\overline{n}_{123})\rangle}\prod_{j=1}^5|\widehat{u}_j(\tau_j,n_j)|\,d\lambda,\\
&\big|\mathcal{F}_{t,x}\mathcal{DNR}_*\big(u_1,\varphi_T\cdot\mathbf{G}_\#[u_2,u_3,u_4],u_5\big)(\tau,n)\big|\\
&\qquad\lesssim \sum_{\substack{\overline{n}_{105}\in\mathbb{X}_*(n),\\ \overline{n}_{234}\in\mathbb{X}_\#(n_0)}}\int_{\mathbb{R}}\int_{\lambda=\tau_1+\ldots+\tau_5}\frac{|n_1n_2|}{\langle\tau-\lambda\rangle^{1-\theta}\langle\tau-n^3\rangle\langle\phi(\overline{n}_{234})\rangle}\prod_{j=1}^5|\widehat{u}_j(\tau_j,n_j)|\,d\lambda,\\
&\big|\mathcal{F}_{t,x}\mathcal{DNR}_*\big(u_1,u_2,\varphi_T\cdot\mathbf{G}_\#[u_3,u_4,u_5]\big)(\tau,n)\big|\\
&\qquad\lesssim \sum_{\substack{\overline{n}_{120}\in\mathbb{X}_*(n),\\ \overline{n}_{345}\in\mathbb{X}_\#(n_0)}}\int_{\mathbb{R}}\int_{\lambda=\tau_1+\ldots+\tau_5}\frac{|n_1n_3|}{\langle\tau-\lambda\rangle^{1-\theta}\langle\tau-n^3\rangle\langle\phi(\overline{n}_{345})\rangle}\prod_{j=1}^5|\widehat{u}_j(\tau_j,n_j)|\,d\lambda,
\end{aligned}
\tag{3.28}
$$

for any $0<\theta<1$. The main difficulty is controlling the spatial multiplier defined as follows

$$
\alpha(n,\overline{n}_{0\ldots5})=\begin{cases}\dfrac{|n_0n_1|}{|\phi(\overline{n}_{123})|}, & \text{if } \overline{n}_{045}\in\mathbb{X}_*(n),\ \overline{n}_{123}\in\mathbb{X}_\#(n_0),\\ \dfrac{|n_1n_2|}{|\phi(\overline{n}_{234})|}, & \text{if } \overline{n}_{105}\in\mathbb{X}_*(n),\ \overline{n}_{234}\in\mathbb{X}_\#(n_0),\\ \dfrac{|n_1n_3|}{|\phi(\overline{n}_{345})|}, & \text{if } \overline{n}_{120}\in\mathbb{X}_*(n),\ \overline{n}_{345}\in\mathbb{X}_\#(n_0),\end{cases}
$$

where $\overline{n}_{0\ldots5}=(n_0,n_1,\ldots,n_5)$. We will refer to the frequencies in $\mathbb{X}_*(n)$ as the first generation of frequencies and those in $\mathbb{X}_\#(n_0)$ as the second when discussing the quintic terms.

In Section 3.5.2, we estimate the contributions for which $\alpha(n,\overline{n}_{0\ldots5})\lesssim 1$, namely

$$
\begin{aligned}
&\mathcal{DNR}_*\big(\varphi_T\cdot\mathbf{G}_A[w_1,u_2,u_3],u_4,u_5\big), && \mathcal{DNR}_*\big(\varphi_T\cdot\mathbf{G}_B[w_1,w_2,u_3],u_4,u_5\big),\\
&\mathcal{DNR}_\#\big(w_1,\varphi_T\cdot\mathbf{G}_A[w_2,u_3,u_4],u_5\big), && \mathcal{DNR}_\#\big(w_1,\varphi_T\cdot\mathbf{G}_B[w_2,w_3,u_4],u_5\big),\\
&\mathcal{DNR}_D\big(w_1,w_2,\varphi_T\cdot\mathbf{G}_A[w_3,u_4,u_5]\big), && \mathcal{DNR}_D\big(w_1,w_2,\varphi_T\cdot\mathbf{G}_B[w_3,w_4,u_5]\big),
\end{aligned}
\tag{3.29}
$$

where $*\in\{A,B,C,D\}$, $\#\in\{B,C,D\}$ and $u_j\in\{u,\overline{u}\}$, $w_j\in\{w,\overline{w}\}$, $j=1,\ldots,5$. The estimate for these contributions follows once we control $\mathcal{Q}(u_1,\ldots,u_5)$ defined by its space-time Fourier transform

$$
\begin{aligned}
&\mathcal{F}_{t,x}\big(\mathcal{Q}(u_1,\ldots,u_5)\big)(\tau,n)\\
&\qquad=\sum_{n=n_1+\ldots+n_5}\int_{\mathbb{R}}\int_{\lambda=\tau_1+\ldots+\tau_5}\frac{1}{\langle\tau-\lambda\rangle^{1-\theta}\langle\tau-n^3\rangle^{1-\theta}}\prod_{j=1}^5|\widehat{u}_j(\tau_j,n_j)|\,d\lambda.
\end{aligned}
\tag{3.30}
$$

In Section 3.5.2, we establish an estimate for the standard quintic contribution (3.30) under particular assumptions on the frequencies. Not all the quintic contributions in (3.29) are of the form (3.30), which forces us to use the equation for u once again, introducing new septic terms. In particular, the following quintic contributions will not be estimated directly

$$\begin{aligned}
\mathcal{DNR}_C\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_A[w_3, u_4, u_5]\big), &\quad \mathcal{DNR}_C\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5]\big), \\
\mathbf{B}_A^2\big(w_1, \varphi_T \cdot \mathbf{G}_A[w_2, u_3, u_4], u_4\big), &\quad \mathbf{B}_A^2\big(w_1, \varphi_T \cdot \mathbf{G}_B[w_2, w_3, u_4], u_4\big), \\
\mathbf{B}_A^3\big(w_1, u_2, \varphi_T \cdot \mathbf{G}_A[w_3, u_4, u_5]\big), &\quad \mathbf{B}_A^3\big(w_1, u_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5]\big), \\
\mathbf{B}_B^3\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_A[w_3, u_4, u_5]\big), &\quad \mathbf{B}_B^3\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5]\big),
\end{aligned} \tag{3.31}$$

where $u_j \in \{u, \overline{u}\}$, $w_j \in \{w, \overline{w}\}$, $j = 1, \ldots, 5$. The $\mathcal{DNR}_C$ contributions are not controlled by (3.30) and thus need a more refined approach. For the $\mathbf{B}_*^j$ contributions, not only does the j-th modulation play an important role, but also the largest modulation of the new functions in $\mathbf{G}_\#$. This is detailed in Section 3.5.3.

3.5.1 Cubic terms

We start by estimating the cubic terms in (3.27).

Lemma 3.5.1. *The following estimate holds*

$$\|\mathcal{DR}(u_1, u_2, u_3)\|_{Z_1^s} \lesssim \|u_1\|_{Y_0^s} \|u_2\|_{Y_0^{\frac{1}{2}}} \|u_3\|_{Y_0^{\frac{1}{2}}}.$$

Proof. Using the kernel estimate for $\mathcal{D}$ in (3.13) and Young's inequality, we have

$$\begin{aligned}
\|\mathcal{DR}(u_1, u_2, u_3)\|_{Z_1^s} &\lesssim \Big\| \int_{\mathbb{R}} \frac{1}{\langle \tau - \lambda \rangle} \int_{\lambda = \tau_1 - \tau_2 + \tau_3} \langle n \rangle^s |\widehat{u}_1(\tau_1, n)| \prod_{j=2}^3 \langle n \rangle^{\frac{1}{2}} |\widehat{u}_j(\tau_j, n)| \, d\lambda \Big\|_{\ell_n^p L_\tau^{q_0}} \\
&\lesssim \Big\| \int_{\tau = \tau_1 - \tau_2 + \tau_3} \langle n \rangle^s |\widehat{u}_1(\tau_1, n)| \prod_{j=2}^3 \langle n \rangle^{\frac{1}{2}} |\widehat{u}_j(\tau_j, n)| \Big\|_{\ell_n^p L_\tau^{r_0}},
\end{aligned}$$

for $\delta < \frac{1}{6}$. Applying Hölder's inequality gives

$$\begin{aligned}
&\|\mathcal{DR}(u_1, u_2, u_3)\|_{Z_1^s} \\
&\quad \lesssim \sup_{\tau, n} J(\tau, n) \Big\| \|\langle n \rangle^s \langle \tau - n^3 \rangle^{\frac{1}{2}} \widehat{u}_1(\tau, n)\|_{L_\tau^{r_0}} \prod_{j=2}^3 \|\langle n \rangle^{\frac{1}{2}} \langle \tau - n^3 \rangle^{\frac{1}{2}} \widehat{u}_j(\tau, n)\|_{L_\tau^{r_0}} \Big\|_{\ell_n^p},
\end{aligned}$$

where

$$J(\tau, n)^{r_0'} = \int_{\mathbb{R}^2} \frac{d\tau_1 \, d\tau_2}{\langle \tau_1 - n^3 \rangle^{\frac{r_0'}{2}} \langle \tau_2 - n^3 \rangle^{\frac{r_0'}{2}} \langle \tau - \tau_1 + \tau_2 - n^3 \rangle^{\frac{r_0'}{2}}} \lesssim 1$$

from Lemma 2.1.4. The result follows from Hölder's inequality. □

Remark 3.5.2. *As in the estimates in Chapter 2, the resonant contribution is responsible for the regularity restriction $s \geq \frac{1}{2}$, since we require $s + 1 \leq 3s \iff s \geq \frac{1}{2}$.*

Lemma 3.5.3. *Let $* \in \{C, D\}$. Then, the following estimate holds*

$$\|\mathcal{DNR}_*(u_1, u_2, u_3)\|_{Z_1^s} \lesssim \|u_1\|_{Z_0^{\frac{1}{2}}} \|u_2\|_{Z_0^s} \|u_3\|_{Z_0^{\frac{1}{2}}}.$$

Proof. Let $* \in \{C, D\}$, then $\overline{n}_{123} \in \mathbb{X}_*(n)$ implies that $\langle n \rangle^s |n_1| \lesssim \langle n_1 \rangle^{\frac{1}{2}} \langle n_2 \rangle^s \langle n_3 \rangle^{\frac{1}{2}}$. Using

(3.13), we have

$$\begin{aligned}\|\mathcal{DNR}_*(u_1,u_2,u_3)\|_{Z_1^s}&\\ &\lesssim \Big\| \int_{\mathbb{R}} \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \frac{\langle n_1\rangle^{\frac12}\langle n_2\rangle^s\langle n_3\rangle^{\frac12}}{\langle \tau-n^3\rangle^{\delta}\langle\tau-\lambda\rangle} \int_{\lambda=\tau_1+\tau_2+\tau_3} \prod_{j=1}^3 |\widehat{u}_j(\tau_j,n_j)| d\lambda \Big\|_{\ell^p_n L^{q_0}_\tau}.\end{aligned}$$

Let $f_j(\sigma,n) = \langle n\rangle^{\frac12}\langle\sigma\rangle^{\delta}|\widehat{u}_j(\sigma+n^3,n)|$, $j\in\{1,3\}$, $f_2(\sigma,n)=\langle n\rangle^s\langle\sigma\rangle^\delta|\widehat{u}_2(\sigma+n^3,n)|$, $\bar\sigma=\sigma_1+\sigma_2+\sigma_3$ and proceed as in (3.25). Using Minkowski's and Hölder's inequalities gives

$$\begin{aligned}&\|\mathcal{DNR}_*(u_1,u_2,u_3)\|_{Z_1^s}\\ &\lesssim \int_{\sigma_1,\sigma_2,\sigma_3} \Big\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \frac{1}{\langle\phi(\overline{n}_{123})\rangle^{\delta}\langle\tau-n^3-\bar\sigma+\phi(\overline{n}_{123})\rangle^{1-\delta}} \prod_{j=1}^3 f_j(\sigma_j,n_j)\Big\|_{\ell^p_n L^{q_0}_\tau}\\ &\lesssim \int_{\sigma_1,\sigma_2,\sigma_3} \Big\| \sum_{\mu} \frac{|\mu|^{\varepsilon}}{\langle\mu\rangle^{\delta}\langle\tau-n^3-\bar\sigma+\mu\rangle^{1-\delta}} \Big(\sum_{\overline{n}_{123}\in\mathbb{X}^\mu_*(n)} \prod_{j=1}^3 |f_j(\sigma_j,n_j)|^p\Big)^{\frac1p}\Big\|_{\ell^p_n L^{q_0}_\tau},\end{aligned}$$

since from the standard divisor counting estimate (Lemma 3.1.6), we have that $|\mathbb{X}^\mu_*(n)|\lesssim_\varepsilon |\mu|^\varepsilon$, for any $\varepsilon>0$. Choosing $\varepsilon\le\delta$ and applying Schur's test with $1+\frac{1}{q_0}=\frac1p+\frac1q$, we obtain

$$\|\mathcal{DNR}_*(u_1,u_2,u_3)\|_{Z_1^s} \lesssim \int_{\sigma_1,\sigma_2,\sigma_3} \Big\| \Big(\sum_{\overline{n}_{123}\in\mathbb{X}^\mu_*(n)} \prod_{j=1}^3 |f_j(\sigma_j,n_j)|^p\Big)^{\frac1p}\Big\|_{\ell^p_n\ell^p_\mu} \lesssim \prod_{j=1}^3 \|f_j(\sigma,n)\|_{L^1_\sigma\ell^p_n},$$

for $\delta<\frac{1}{5p}$. Consequently, using Hölder's and Minkowski's inequalities, it follows that

$$\|\mathcal{DNR}_*(u_1,u_2,u_3)\|_{Z_1^s} \lesssim \prod_{j=1}^3 \|\langle\sigma\rangle^{1-4\delta+}\widehat{f}_j(\sigma,n)\|_{L^{q_0}_\sigma\ell^p_n} \lesssim \|u_1\|_{Z_0^{\frac12}}\|u_2\|_{Z_0^s}\|u_3\|_{Z_0^{\frac12}}.$$

□

Remark 3.5.4. (i) The terms $\mathcal{DNR}_A$, $\mathcal{DNR}_B$ cannot be estimated in a similar manner because $\langle n\rangle^{\frac12}|n_1|$ is not controlled by $(\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle)^{\frac12}$. This motivated the application of the modified Duhamel operator to introduce smoothing in space needed to control the loss of derivative from the nonlinearity *without* using the largest modulation.

(ii) Consider the estimate

$$\big\|\mathcal{DNR}_D(u_1,u_2,u_3)\big\|_{X^{\frac12,b}_{p,q}} \lesssim \prod_{j=1}^3 \|u_j\|_{X^{\frac12,b}_{p,q}},$$

for some $b\ge0$, $2\le q<\infty$. The region $\mathbb{X}_D(n)$ includes the case when $|n_1|\sim|n_2|\sim|n_3|$, $\max_{j=1,2,3}|n_j|\lesssim|\phi(\overline{n}_{123})|\ll\max_{j=1,2,3}|n_j|^2$. When attempting to show the above estimate under the nearly-resonant assumption, we must impose the conditions

$$\max\Big(1-\frac{1}{2q}, 1+\frac1q-\frac1p\Big)<b<1,$$

which motivate our choice of $b=1-$ and $q=\infty-$ for the definition of the Z_0^s space.

Lemma 3.5.5. *Let $*\in\{A,B\}$. The following estimates hold*

$$\|\mathbf{B}^0_A(u_1,u_2,u_3)\|_{Z_1^s} \lesssim \|u_1\|_{Y_0^s}\|u_2\|_{Y_0^{\frac12}}\|u_3\|_{Y_0^{\frac12}},$$

$$\|\mathbf{B}^0_B(u_1,u_2,u_3)\|_{Z^s_1} \lesssim \|u_1\|_{Y_0^{\frac12}}\|u_2\|_{Y_0^s}\|u_3\|_{Y_0^{\frac12}}.$$

Proof. It suffices to show

$$\|\mathbf{B}^0_*(u_1,u_2,u_3)\|_{Z^{\frac12}_1} \lesssim \|u_1\|_{Y_0^{\frac12}}\|u_2\|_{Y_0^{\frac12}}\|u_3\|_{Y_0^{\frac12}},$$

for $* \in \{A,B\}$, since the intended estimates follow from $\langle n\rangle^{s-\frac12} \lesssim \langle n_1\rangle^{s-\frac12}\mathbb{1}_{*=A} + \langle n_2\rangle^{s-\frac12}\mathbb{1}_{*=B}$.

Choosing $\alpha = 4\delta$ in the kernel estimate (3.17), gives

$$\begin{aligned}\|\mathbf{B}^0_*(u_1,u_2,u_3)\|_{Z^{\frac12}_1} &\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \int_{\tau_1,\tau_2,\tau_3} \frac{\langle n\rangle^{\frac12}|n_1|}{\langle \tau-n^3\rangle^{5\delta}\langle\phi(\overline{n}_{123})\rangle^{1-4\delta}} \prod_{j=1}^3 |\widehat{u}_j(\tau_j,n_j)| \Bigg\|_{\ell^p_n L^{q_0}_\tau} \\ &\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in\mathbb{X}_*(n)}} \frac{\langle n\rangle^{\frac12}|n_1|}{\langle\phi(\overline{n}_{123})\rangle^{1-4\delta}} \prod_{j=1}^3 \|\widehat{u}_j(n_j)\|_{L^1_\tau} \Bigg\|_{\ell^p_n},\end{aligned}$$

by applying Minkowski's inequality in the last step and integrating in τ. For $\overline{n}_{123}\in\mathbb{X}_*(n)$, we have $|\phi(\overline{n}_{123})| \sim \max\limits_{j=1,2,3}|n_j|^2 \min\limits_{\ell=1,2,3}|n-n_\ell|$, which implies

$$\frac{\langle n\rangle^{\frac12}|n_1|}{\langle\phi(\overline{n}_{123})\rangle^{1-4\delta}} \lesssim \frac{\langle n_1\rangle^{\frac12}}{\max\limits_{j=1,2,3}\langle n_j\rangle^{1-8\delta}\min\limits_{\ell=1,2,3}\langle n-n_\ell\rangle^{1-4\delta}}.$$

Applying Hölder and Minkowski's inequalities, it follows that

$$\|\mathbf{B}^0_*(u_1,u_2,u_3)\|_{Z^{\frac12}_1} \lesssim \big(\sup_n J(n)\big)^{\frac{1}{p'}} \|\langle n\rangle^{\frac12}\widehat{u}_1\|_{\ell^p_n L^1_\tau} \prod_{j=2}^3 \|\widehat{u}_j\|_{\ell^p_n L^1_\tau},$$

where $J(n)$ is defined as follows

$$J(n) := \sum_{n=n_1+n_2+n_3} \frac{1}{\max\limits_{j=1,2,3}\langle n_j\rangle^{(1-8\delta)p'}\min\limits_{\ell=1,2,3}\langle n-n_\ell\rangle^{(1-4\delta)p'}}.$$

Let $j,\ell\in\{1,2,3\}$ denote the indices at which the maximum and minimum in the definition of $J(n)$ are attained, respectively. If $j=\ell$, we can use the fact that $\langle n_j\rangle \gtrsim \langle n_i\rangle$ for $i\in\{1,2,3\}\backslash\{j\}$ and sum in n_i, n_j. If $j\neq\ell$, we sum in n_j, n_ℓ. Thus, $J(n)\lesssim 1$ uniformly in n for $\delta < \frac{1}{8p}$. The intended estimate follows from applying Hölder's inequality in time.

□

Lemma 3.5.6. *Let $*\in\{A,B\}$. Then, the following estimates hold*

$$\|\mathbf{B}^j_*(u_1,u_2,u_3)\|_{Z^s_1} \lesssim \|u_j\|_{Z_0^{s_j}} \prod_{\substack{k=1\\k\neq j}}^3 \|u_k\|_{Y_0^{s_k}}, \quad j=1,2,3,$$

for $s\geq\frac12$, $(s_1,s_2,s_3)=(s,\frac12,\frac12)$ if $*=A$ or $(s_1,s_2,s_3)=(\frac12,s,\frac12)$ if $*=B$.

Proof. We will only show the estimate for $j=1$ and $s=\frac12$, as the remaining estimates follow an analogous proof. The conditions on s_j, $j=1,2,3$, follow from the fact that $|n|\lesssim|n_1|$ when $*=A$ and $|n|\lesssim|n_2|$ when $*=B$. Fix $*\in\{A,B\}$. From (3.19) with $1-\alpha = b_0-\delta$ and for $\overline{n}_{123}\in\mathbb{X}_*(n)$ we have

$$\langle n\rangle^{\frac12}|n_1|\langle\tau-n^3\rangle^{b_1}\big|K_+\big(\tau-n^3,\lambda-n^3,\phi(\overline{n}_{123})\big)\big| \lesssim \frac{\langle n_1\rangle^{\frac12}\langle\tau_1-n_1^3\rangle^{1-3\delta}}{\langle\tau-\lambda\rangle\langle\phi(\overline{n}_{123})\rangle^{\frac12-3\delta}\langle n-n_\ell\rangle^{\frac12}},$$

where $|n-n_\ell| = \min_{j=1,2,3}|n-n_j|$. Let $f(\tau,n) = \langle n\rangle^{\frac12}\langle\tau_1-n_1^3\rangle^{b_0-\delta}|\widehat{u}_1(\tau,n)|$. Then, using

Minkowski's and Young's inequalities, we have

$$
\begin{aligned}
&\left\| \mathbf{B}^1_*(u_1,u_2,u_3) \right\|_{Z_1^{\frac{1}{2}}} \\
&\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in \mathbb{X}_*(n)}} \frac{1}{\langle \phi(\overline{n}_{123})\rangle^{\frac{1}{2}-3\delta} \langle n-n_l\rangle^{\frac{1}{2}}} \Big(\frac{1}{\langle \cdot \rangle} * f(\cdot, n_1) *_{j=2,3} |\widehat{u}_j(\cdot, n_j)| \Big)(\tau) \Bigg\|_{\ell^p_n L^{q_0}_\tau} \\
&\lesssim \Bigg\| \sum_{\substack{n=n_1+n_2+n_3,\\ \overline{n}_{123}\in \mathbb{X}_*(n)}} \frac{1}{\langle \phi(\overline{n}_{123})\rangle^{\frac{1}{2}-3\delta} \langle n-n_l\rangle^{\frac{1}{2}}} \|f(n_1)\|_{L^{q_1}_\tau} \prod_{j=2}^3 \|\widehat{u}_j(n_j)\|_{L^1_\tau} \Bigg\|_{\ell^p_n}.
\end{aligned}
$$

Using Hölder's inequality, we obtain

$$
\left\| \mathbf{B}^1_*(u_1,u_2,u_3) \right\|_{Z_1^{\frac{1}{2}}} \lesssim \big(\sup_n J(n) \big)^{\frac{1}{p'}} \|f\|_{\ell^p_n L^{q_1}_\tau} \prod_{j=2}^3 \|u_j\|_{\ell^p_n L^1_\tau},
$$

where

$$
J(n) = \sum_{n=n_1+n_2+n_3} \frac{1}{\langle n_{\max}\rangle^{p'(1-6\delta)} \langle n-n_\ell\rangle^{p'(1-3\delta)}} \lesssim \sum_{n_i,n_\ell} \frac{1}{\langle n_i\rangle^{p'(1-6\delta)} \langle n-n_\ell\rangle^{p'(1-3\delta)}} \lesssim 1,
$$

for some distinct $n_i, n_\ell \in \{n_1,n_2,n_3\}$ and $\delta < \frac{1}{6p}$. The intended estimate follows from applying Hölder's inequality.

□

3.5.2 Standard quintic term $\mathcal{Q}$ (3.30)

In this section, we focus on estimating the quintic terms in (3.29). Before doing so, we must take into account the new 'resonances' introduced by using the second iteration. For the estimates to hold, we need the largest frequency to correspond to a w term and to not be in a pairing, as defined below. Otherwise, we will use the equation for u (3.21), which introduces new septic terms.

Looking at $\mathcal{Q}$ in (3.30) in more detail, note that the sum in (3.30) over $n = n_1 + \ldots + n_5$ does not exclude all resonances, i.e., we can have $n_i + n_j = 0$ for distinct $i, j \in \{1,\ldots,5\}$. If this holds, we say that (i,j) is a pairing.

We will show a general estimate for $\mathcal{Q}$ in (3.30), given that one of the following holds:

(i) There are no pairings in $(n_1,\ldots,n_5)$ and the largest frequency corresponds to a function in Z_0^s;

(ii) There is one pairing (i,j) and the largest frequency in $\{|n_k| : 1 \le k \le 5, \ k \neq i,j\}$ corresponds to a function in Z_0^s;

(iii) There are two pairings and the remaining frequency corresponds to a function in Z_0^s.

Note that if (i), (ii) or (iii) hold, we can always use the largest frequency which is not in a pairing to control the spatial weight from the norm $\langle n\rangle^s$. If the contributions do not satisfy any of the above conditions, then the largest frequency that is not in a pairing corresponds to a function u and we want to use the equation for u again. This leads to one quintic term that satisfies the assumptions above and four septic terms, which are easily estimated.

To further clarify, let $\mathcal{Q}'(u_1,\ldots,u_5)$ denote a contribution in (3.29), $u_j \in \{u, \overline{u}, w, \overline{w}\}$. Let n_j correspond to the spatial Fourier variable of $\widehat{u}_j$, $j = 1,\ldots,5$. If n_1 is the largest frequency that is not in a pairing and $u_1 \in \{w, \overline{w}\}$, then we keep the contribution as is. Otherwise, $u_1 \in \{u, \overline{u}\}$ and we will use the equation (3.21) to replace the first entry in $\mathcal{Q}'$. For simplicity, assume that $u_1 = u$, then we have

$$\begin{aligned}\mathcal{Q}'(u, u_2, \ldots, u_5) &= \mathcal{Q}'(w, u_2, \ldots, u_5) \\ &\quad + \mathcal{Q}'\big(\varphi_T \cdot \mathbf{G}_{A,\geq}[w, \overline{u}, u], u_2, \ldots, u_5\big) + \mathcal{Q}'\big(\varphi_T \cdot \mathbf{G}_{A,>}[w, u, \overline{u}], u_2 \ldots, u_5\big) \\ &\quad + \mathcal{Q}'\big(\varphi_T \cdot \mathbf{G}_{B,\geq}[w, \overline{w}, u], u_2, \ldots, u_5\big) + \mathcal{Q}'\big(\varphi_T \cdot \mathbf{G}_{B,>}[w, w, \overline{u}], u_2, \ldots, u_5\big).\end{aligned}$$

By carefully examining the frequencies and pairings of the terms in (3.29) and applying the above modification, we obtain the final equation for w. Due to its length, we have decided to not include it in full. All the resulting quintic and septic terms arising from (3.29), can be estimated by the two following propositions. The details on how to apply these estimates are included in Appendix A.3.

Proposition 3.5.7. *Let $\mathcal{Q}$ as defined in (3.30) where the first factor has the largest spatial Fourier frequency which is not in a pairing and with $\theta < \frac{\delta}{2}$. Then, the following estimate holds*

$$\big\|\mathcal{Q}(u_1, \ldots, u_5)\big\|_{Z_1^s} \lesssim \|u_1\|_{Z_0^s} \prod_{j=2}^{5} \|u_j\|_{Y_0^{\frac{1}{2}}}.$$

Proof.

<u>Case 1: no pairing</u>

Let $\mathbf{P}_{N_j}$ denote the Dirichlet projection onto $\langle n_j \rangle \sim N_j$, $j = 1, \ldots, 5$, and assume by symmetry that $N_2 \geq \ldots \geq N_5$. Since there is no pairing we have $|n| \lesssim |n_1|$, therefore using Minkowski's inequality gives

$$\begin{aligned}\|\mathcal{Q}(u_1, \ldots, u_5)\|_{Z_1^s} \lesssim \sum_{N_2, \ldots, N_5} \int_{\tau_1, \ldots, \tau_5} \Bigg\| \sum_{n = n_1 + \ldots + n_5} & \frac{1}{\langle \tau - \tau_1 - \ldots - \tau_5 \rangle^{1-\theta}} \\ & \times \langle n_1 \rangle^s |\widehat{u}_1(\tau_1, n_1)| \prod_{j=2}^{5} |\widehat{\mathbf{P}_{N_j} u}(\tau_j, n_j)| \Bigg\|_{\ell_n^p L_\tau^{q_0}}.\end{aligned}$$

Using the change of variables $\sigma_j = \tau_j - n_j^3$, $j = 1, \ldots, 5$, and Schur's test (Lemma 3.1.3), we get

$$\begin{aligned}\|\mathcal{Q}(u_1, \ldots, u_5)\|_{Z_1^s} &\lesssim \sum_{N_2, \ldots, N_5} \int_{\sigma_1, \ldots, \sigma_5} \Bigg\| \sum_{n = n_1 + \ldots + n_5} \frac{1}{\langle \tau - n^3 - \sigma_1 - \ldots - \sigma_5 + \psi(n, \overline{n}_{1\ldots5}) \rangle^{1-\theta}} \\ &\qquad \times \langle n_1 \rangle^s |\widehat{u}_1(\sigma_1 + n_1^3, n_1)| \prod_{j=2}^{5} |\widehat{\mathbf{P}_{N_j} u}(\sigma_j + n_j^3, n_j)| \Bigg\|_{\ell_n^p L_\tau^{q_0}} d\sigma_1 \cdots d\sigma_5 \\ &\lesssim \sum_{N_2, \ldots, N_5} \int_{\sigma_1, \ldots, \sigma_5} \Bigg\| \sum_{\substack{n = n_1 + \ldots + n_5, \\ \psi(n, \overline{n}_{1\ldots5}) = \mu}} \prod_{j=1}^{5} f_j(\sigma_j, n_j) \Bigg\|_{\ell_n^p \ell_\mu^{q_1}},\end{aligned}$$

where $\psi(n, \overline{n}_{1\ldots5}) = n^3 - n_1^3 - \ldots - n_5^3$, $f_1(\sigma, n) = \langle n \rangle^s |\widehat{u}_1(\sigma + n^3, n)|$ and $f_j(\sigma, n) = |\widehat{P_{N_j} u_j}(\sigma + n^3, n)|$, $j = 2, \ldots, 5$. Note that we can trivially restrict μ to the following region

$$\begin{aligned}A(n, N_2, \ldots, N_5) = \big\{\mu \in \mathbb{Z} : \ \mu = n - (n - n_2 - \ldots - n_5)^3 - n_2^3 - \ldots - n_5^3, \\ |n_j| \sim N_j, \ j = 2, \ldots, 5\big\},\end{aligned}$$

which satisfies $|A(n, N_2, \ldots, N_5)| \lesssim N_2^4$ for fixed n. Thus, by taking a supremum inμ, it follows that

$$\|\mathcal{Q}(u_1, \ldots, u_5)\|_{Z_1^s} \lesssim \sum_{N_2, \ldots, N_5} \int_{\sigma_1, \ldots, \sigma_5} \Bigg\| \mathbb{1}_{\mu \in A(n, N_2, \ldots, N_5)} \sum_{\substack{n = n_1 + \ldots + n_5, \\ \psi(n, \overline{n}_{1\ldots5}) = \mu}} \prod_{j=1}^{5} f_j(\sigma_j, n_j) \Bigg\|_{\ell_n^p \ell_\mu^{q_1}}$$

$$\lesssim \sum_{N_2,\ldots,N_5} N_2^{\frac{4}{q_1}} \int_{\sigma_1,\ldots,\sigma_5} \Big\| \sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^{5} f_j(\sigma_j,n_j) \Big\|_{\ell^p_n \ell^\infty_\mu}. \tag{3.32}$$

Now, we consider two distinct cases depending on the size of the frequencies.

<u>**Subcase 1.1:** $N_3 \geq N_2^{4\sqrt{\delta}}$</u>

Using Cauchy's inequality with $\alpha > 0$, omitting the time dependence, we have

$$\begin{aligned}\sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^{5} f_j(n_j) &\lesssim \sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} f_1(n_1)\big(\alpha|f_2(n_2)f_3(n_3)|^2 + \alpha^{-1}|f_4(n_4)f_5(n_5)|^2\big)\\ &\lesssim \sum_{n_2,n_3} \sum_{(n_1,n_4,n_5)\in B(n,n_2,n_3,\mu)} \alpha f_1(n_1)|f_2(n_2)f_3(n_3)|^2\\ &\quad + \sum_{n_4,n_5} \sum_{(n_1,n_2,n_3)\in B(n,n_4,n_5,\mu)} \alpha^{-1} f_1(n_1)|f_4(n_4)f_5(n_5)|^2,\end{aligned}$$

where

$$\begin{aligned}B(k,k_1,k_2,\mu) := \big\{(n_1,n_2,n_3)\in\mathbb{Z}^3 :\ & n_1+n_2+n_3 = k-k_1-k_2 =: \ell,\\ & 3(n_2+n_3)(\ell-n_2)(\ell-n_3) = \mu - k^3 + k_1^3 + k_2^3 + \ell^3\big\}.\end{aligned}$$

Taking a supremum in n_1, we obtain

$$\begin{aligned}\sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^{5} f_j(n_j) &\lesssim \alpha \sup_{|n-n_1|\lesssim N_2} f_1(n_1) \sum_{n_2,n_3} |B(n,n_2,n_3,\mu)|\cdot|f_2(n_2)f_3(n_3)|^2\\ &\quad + \alpha^{-1} \sup_{|n-n_1|\lesssim N_2} f_1(n_1) \sum_{n_4,n_5} |B(n,n_4,n_5,\mu)|\cdot|f_4(n_4)f_5(n_5)|^2.\end{aligned}$$

In order to estimate $|B(n,n_2,n_3,\mu)|, |B(n,n_4,n_5,\mu)|$, we use Lemma 3.1.7 (i). For the first one, to count the choices of (n_1,n_4,n_5) it suffices to count the number of divisors $\ell-n_4, \ell-n_5$, where $\ell = n-n_2-n_3$, of $\tilde{\psi} := 3(n_4+n_5)(n-n_2-n_3-n_4)(n-n_2-n_3-n_5) = 3(n_4+n_5)(\ell-n_4)(\ell-n_5)$. If $|n| \gg |n_2|$, then

$$|\tilde{\psi}| \sim |(n_4+n_5)(n-n_2-n_3-n_4)(n-n_2-n_3-n_5)| \lesssim |n|^3 \implies |\tilde{\psi}|^\varepsilon \leq |\tilde{\psi}|^{\frac{1}{3}} \lesssim |n|,$$

for any $\varepsilon > 0$. Otherwise, $|n| \lesssim |n_2|$ and we have

$$|\tilde{\psi}| \sim |(n_4+n_5)(n-n_2-n_3-n_4)(n-n_2-n_3-n_5)| \lesssim |n_2|^3 \implies |\tilde{\psi}|^\varepsilon \leq |\tilde{\psi}|^{\frac{1}{3}} \lesssim |n_2|,$$

for any $\varepsilon > 0$. Applying the lemma, the number of divisors $d_4 = n-n_2-n_3-n_4$, $d_5 = n-n_2-n_3-n_5$ satisfying

$$\begin{cases} |d_j - n| = |n_2+n_3+n_j| \lesssim N_2, & \text{if } |n| \gg |n_2|,\\ |d_j - n_2| = |n-n_3-n_j| \lesssim N_2, & \text{if } |n| \lesssim |n_2|, \end{cases}$$

is bounded by N_2^ε, for $j=4,5$. Thus, $|B(n,n_2,n_3,\mu)| \lesssim N_2^\varepsilon$, for any $\varepsilon > 0$. An analogous approach gives $|B(n,n_4,n_5,\mu)| \lesssim N_2^\varepsilon$. Consequently, we have

$$\sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^{5} f_j(n_j) \lesssim N_2^\varepsilon \sup_{|n-n_1|\lesssim N_2} f_1(n_1) \prod_{j=2}^{5} \|f_j\|_{\ell^2_n},$$

by choosing $\alpha = (\|f_2\|_{\ell^2_n}\|f_3\|_{\ell^2_n})^{-1}\|f_4\|_{\ell^2_n}\|f_5\|_{\ell^2_n}$. Looking at (3.32), since $|n-n_1| \lesssim N_2$, taking a supremum in n_1 gives

$$\begin{aligned}\|\mathcal{Q}(u_1,\dots,u_5)\|_{Z_1^s} &\lesssim \sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\varepsilon} \int_{\sigma_1,\dots,\sigma_5} \Big\| \sup_{|n-n_1|\lesssim N_2} f_1(\sigma_1,n_1)\Big\|_{\ell^p_n} \prod_{j=2}^5 \|f_j(\sigma_j)\|_{\ell^2_n} \\ &\lesssim \sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\varepsilon} \int_{\sigma_1,\dots,\sigma_5} \Big(\sum_{n_1} |f_1(\sigma_1,n_1)|^p \sum_{|n-n_1|\lesssim N_2} 1\Big)^{\frac{1}{p}} \prod_{j=2}^5 \|f_j(\sigma_j)\|_{\ell^2_n} \\ &\lesssim \sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\frac{1}{p}+\varepsilon} \|f_1\|_{L^1_\sigma \ell^p_n} \prod_{j=2}^5 \|f_j\|_{L^1_\sigma \ell^2_n}.\end{aligned}$$

Using Hölder's and Minkowski's inequalities, we have

$$\|\mathcal{Q}(u_1,\dots,u_5)\|_{Z_1^s} \lesssim \sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\frac{1}{p}+\varepsilon} (N_2N_3N_4N_5)^{\delta-\frac{1}{p}+} \|u_1\|_{Z_0^s} \prod_{j=2}^5 \|u_j\|_{Y_0^{\frac{1}{2}}}.$$

It only remains to sum in the dyadic numbers N_j. Using the fact that $N_3 \geq N_2^{4\sqrt{\delta}}$, we have

$$\sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\delta+} (N_3N_4N_5)^{\delta-\frac{1}{p}+} \lesssim \sum_{N_2,\dots,N_5} (N_2N_3N_4N_5)^{-\delta} N_3^{2\delta+5\sqrt{\delta}-\frac{1}{p}+} (N_4N_5)^{2\delta-\frac{1}{p}+} \lesssim 1$$

for $\delta < \frac{1}{2p}$ and $2\delta + 5\sqrt{\delta} - \frac{1}{p} < 0 \implies 0 < \sqrt{\delta} < -\frac{5}{4} + \sqrt{\left(\frac{5}{4}\right)^2 + \frac{1}{2p}}$, and the estimate follows.

Subcase 1.2: $N_3 \leq N_2^{4\sqrt{\delta}}$

Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^5 f_j(n_j) \\ &= \sum_{n_3,n_4,n_5} \sum_{n_1\in C(n,n_3,n_4,n_5,\mu)} f_1(n_1)f_2(n-n_1-n_3-n_4-n_5)f_3(n_3)f_4(n_4)f_5(n_5) \\ &\lesssim N_2^\varepsilon \sum_{n_3,n_4,n_5} f_3(n_3)f_4(n_4)f_5(n_5)\big(\sup_{n_1} f_1(n_1)f_2(n-n_1-n_3-n_4-n_5)\big)\end{aligned}$$

where

$$\begin{aligned}C(n,n_3,n_4,n_5,\mu) := \{n_1\in\mathbb{Z} : \ell := n_3+n_4+n_5, |\ell| \lesssim |n-n_1-\ell| \lesssim |n_1|, \\ |n-n_1-\ell| \lesssim N_2,\ 3(n-\ell)(n_1+\ell)(n-n_1) = \mu - \ell^3 + n_3^3 + n_4^3 + n_5^3\},\end{aligned}$$

for which $|C(n,n_3,n_4,n_5,\mu)| \lesssim N_2^\varepsilon$ for any $\varepsilon > 0$ from Lemma 3.1.7 (i). Note that if $|n| \gg |n-n_1-\ell|$, then $\tilde{\psi} := 3(n-\ell)(n_1+\ell)(n-n_1)$ satisfies $|\tilde{\psi}| \lesssim |n|^3$ and $|\tilde{\psi}|^\varepsilon \leq |\tilde{\psi}|^{\frac{1}{3}} \lesssim |n|$ for any $0 < \varepsilon < \frac{1}{3}$. Counting the number of choices for n_1 is equivalent to counting the number of divisors $d = n_1 + \ell$. Since $|d-n| = |n-n_1-\ell| \lesssim N_2$, from Lemma 3.1.7, there exist at most N_2^ε values for n_1. If $|n| \lesssim |n-n_1-\ell|$, then $|\tilde{\psi}| \lesssim |n-n_1-\ell|^3 \lesssim N_2^3$, so by the standard divisor counting lemma, there are at most N_2^ε divisors $n_1+\ell$. Consequently, $|C(n,n_3,n_4,n_5,\mu)| \lesssim N_2^\varepsilon$, for any $0 < \varepsilon < \frac{1}{3}$. Minkowski's inequality gives the following

$$\Big\| \sum_{\substack{n=n_1+\ldots+n_5,\\ \psi(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^5 f_j(n_j) \Big\|_{\ell^p_n \ell^\infty_\mu} \lesssim N_2^\varepsilon \|f_1\|_{\ell^p_n} \|f_2\|_{\ell^p_n} \prod_{j=3}^5 \|f_j\|_{\ell^1_n}.$$

Applying the previous estimate to (3.32) and Hölder's inequality give

$$\|\mathcal{Q}(u_1,\dots,u_5)\|_{Z_1^s} \lesssim \sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\varepsilon}\|f_1\|_{L^1_\sigma\ell^p_n}\|f_2\|_{L^1_\sigma\ell^p_n}\prod_{j=3}^5\|f_j\|_{\ell^1_nL^1_\sigma}$$
$$\lesssim \sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\frac{1}{r_0}-\frac{1}{p}-\frac{1}{2}+}(N_3N_4N_5)^{\frac{1}{2}-\frac{1}{p}+}\|u_1\|_{Z_0^s}\prod_{j=2}^5\|u_j\|_{Y_0^{\frac{1}{2}}}.$$

It only remains to sum in the dyadics N_j:

$$\sum_{N_2,\dots,N_5} N_2^{\frac{4}{q_1}+\delta-\frac{1}{p}+}(N_3N_4N_5)^{\frac{1}{2}-\frac{1}{p}+} \lesssim \sum_{N_2,\dots,N_5}(N_3N_4N_5)^{-\sqrt{\delta}}N_2^{\frac{4}{q_1}+\delta-\frac{1}{p}+12\sqrt{\delta}(\frac{1}{2}-\frac{1}{p}+3\sqrt{\delta})} \lesssim 1$$

if $\frac{4}{q_1}+\delta-\frac{1}{p}+12\sqrt{\delta}(\frac{1}{2}-\frac{1}{p}+3\sqrt{\delta})<0 \implies 0<\sqrt{\delta}<-\frac{6}{21}\big(\frac{1}{2}-\frac{1}{p}\big)+\sqrt{\frac{6^2}{21^2}\big(\frac{1}{2}-\frac{1}{p}\big)^2+\frac{21}{p}}$, completing the proof for Case 1.

Case 2: one pairing $(4,5)$

In this case, we have $n=n_1+n_2+n_3$, $n_4+n_5=0$. Let $f_1(\sigma,n)=\langle n\rangle^s|\widehat{u}_1(\sigma+n^3,n)|$ and $f_j(\sigma,n)=|\widehat{\mathbf{P}_{N_j}u_j}(\sigma+n^3,n)|$, $j=2,3$. Using Cauchy-Schwarz inequality in n_4 and proceeding as in Case 1 gives the following

$$\|\mathcal{Q}(u_1,\dots,u_5)\|_{Z_1^s}$$
$$\lesssim \sum_{N_2,N_3}\int_{\sigma_1,\sigma_2,\sigma_3}\Bigg\|\sum_{\substack{n=n_1+n_2+n_3\\ \phi(\overline{n}_{123})=\mu}}\prod_{j=1}^3 f_j(\sigma_j,n_j)\Bigg\|_{\ell^p_n\ell^{q_1}_\mu}\prod_{k=4}^5\|\widehat{u}_k\|_{L^1_\tau\ell^2_n}$$
$$\lesssim \sum_{N_2,N_3}\int_{\sigma_1,\sigma_2,\sigma_3}\Bigg\|\mathbb{1}_{\mu\in A(n,N_2,N_3)}\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})=\mu}}\prod_{j=1}^3 f_j(\sigma_j,n_j)\Bigg\|_{\ell^p_n}\prod_{k=4}^5\|\widehat{u}_k\|_{L^1_\tau\ell^2_n}$$
$$\lesssim \sum_{N_2,N_3}N_2^{\frac{2}{q_1}}\int_{\sigma_1,\sigma_2,\sigma_3}\Bigg\|\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})=\mu}}\prod_{j=1}^3 f_j(\sigma_j,n_j)\Bigg\|_{\ell^p_n\ell^\infty_\mu}\prod_{k=4}^5\|\widehat{u}_k\|_{L^1_\tau\ell^2_n}, \tag{3.33}$$

where $A(n,N_2,N_3):=\big\{\mu\in\mathbb{Z}:\ \mu=n^3-(n-n_2-n_3)^3-n_2^3-n_3^3,\ |n_j|\sim N_j, j=2,3\big\}$, which satisfies $|A(n,N_2,N_3)|\lesssim N_2^2$ uniformly in n.

Subcase 2.1: $N_2^{4\sqrt{\delta}}\le N_3$

Focusing on the inner sum, we apply Cauchy's inequality, with $\alpha>0$, to obtain the following

$$\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})=\mu}} f_1(n_1)f_2(n_2)f_3(n_3)$$
$$\lesssim \sum_{n_2}\sum_{n_1\in B(n,n_2,\mu)}\alpha f_1(n_1)|f_2(n_2)|^2+\sum_{n_3}\sum_{n_1\in B(n,n_3,\mu)}\alpha^{-1}f_1(n_1)|f_3(n_3)|^2,$$

where $B(n,n_j,\mu)=\big\{n_1\in\mathbb{Z}:\ 3(n-n_1)(n-n_j)(n_1+n_j)=\mu\big\}$, $j=2,3$. Note that $|B(n,n_j,\mu)|\le 2$ because the given equation is quadratic in n_1, since we know that $n-n_j\neq 0$, otherwise we would have another pairing. Thus, taking a supremum in n_1 and using the fact that $|n-n_1|\lesssim N_2$, we get

$$\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})=\mu}}\prod_{j=1}^3 f_j(\sigma_j,n_j)\lesssim \sup_{|n-n_1|\lesssim N_2} f_1(\sigma_1,n_1)\big(\alpha\|f_2(\sigma_2)\|^2_{\ell^2_n}+\alpha^{-1}\|f_3(\sigma_3)\|^2_{\ell^2_n}\big)$$
$$\lesssim \sup_{|n-n_1|\lesssim N_2} f_1(\sigma_1,n_1)\prod_{j=2}^3\|f_j(\sigma_j)\|_{\ell^2_n},$$

by choosing $\alpha = \|f_2(\sigma_2)\|_{\ell^2_n}^{-1} \|f_3(\sigma_3)\|_{\ell^2_n}$. Using this estimate on $\mathcal{Q}$ gives

$$\begin{aligned}\|\mathcal{Q}(u_1, \dots, u_5)\|_{Z_1^s} &\lesssim \sum_{N_2, N_3} N_2^{\frac{2}{q_1} + \frac{1}{p}} \|f_1\|_{L^1_\sigma \ell^p_n} \bigg(\prod_{j=2}^{3} \|f_j\|_{L^1_\sigma \ell^2_n} \bigg) \bigg(\prod_{k=4}^{5} \|\widehat{u}_k\|_{L^1_\tau \ell^2_n} \bigg) \\ &\lesssim \sum_{N_2, N_3} N_2^{\frac{2}{q_1} + \frac{1}{p}} (N_2 N_3)^{\frac{1}{r_0} - \frac{1}{p} - \frac{1}{2} +} \|u_1\|_{Z_0^s} \prod_{j=2}^{5} \|u_j\|_{Y_0^{\frac{1}{2}}}.\end{aligned}$$

The estimate follows from summing in the dyadics.

Subcase 2.2: $N_2^{4\sqrt{\delta}} \geq N_3$

Focusing on the spatial norm on (3.33), we have

$$\begin{aligned}&\bigg\| \sum_{\substack{n = n_1 + n_2 + n_3, \\ \phi(\overline{n}_{123}) = \mu}} \prod_{j=1}^{3} f_j(\sigma_j, n_j) \bigg\|_{\ell^p_n \ell^\infty_\mu} \\ &\qquad \lesssim \bigg\| \sum_{n_3} \bigg(\sum_{n_1 \in C(n, n_3, \mu)} f_1(\sigma_1, n_1) f_2(\sigma_2, n - n_1 - n_3) \bigg) f_3(\sigma_3, n_3) \bigg\|_{\ell^p_n \ell^\infty_\mu} \\ &\qquad \lesssim \sum_{n_3} f_3(\sigma_3, n_3) \big\| \sup_{n_1} f_1(\sigma_1, n_1) f_2(\sigma_2, n - n_1 - n_3) \big\|_{\ell^p_n},\end{aligned}$$

where $C(n, n_3, \mu) = \big\{ n_1 \in \mathbb{Z} : 3(n - n_1)(n - n_3)(n_1 + n_3) = \mu \big\}$ satisfies $|C(n, n_3, \mu)| \leq 2$, since $n - n_3 \neq 0$. Substituting this estimate in (3.33) and using Hölder's inequality gives

$$\|\mathcal{Q}(u_1, \dots, u_5)\|_{Z_1^s} \lesssim \sum_{N_2, N_3} N_2^{\frac{2}{q_1} + \frac{1}{r_0} - \frac{1}{p} - \frac{1}{2} +} N_3^{\frac{1}{2} - \frac{1}{p} +} \|u_1\|_{Z_0^s} \prod_{j=2}^{5} \|u_j\|_{Y_0^{\frac{1}{2}}}.$$

The estimate follows from summing in the dyadics.

Case 3: two pairings $(2,3), (4,5)$

Using Minkowski's and Cauchy-Schwarz inequalities, we get the following

$$\begin{aligned}&\|\mathcal{Q}(u_1, \dots, u_5)\|_{Z_1^s} \\ &\quad \lesssim \bigg\| \|\langle n \rangle^s \widehat{u}_1(n)\|_{L^1_\tau} \sum_{n_2} \|\widehat{u}_2(n_2)\|_{L^1_\tau} \|\widehat{u}_3(-n_2)\|_{L^1_\tau} \sum_{n_4} \|\widehat{u}_4(n_4)\|_{L^1_\tau} \|\widehat{u}_5(-n_4)\|_{L^1_\tau} \bigg\|_{\ell^p_n} \\ &\quad \lesssim \|\langle n \rangle^s \widehat{u}_1\|_{\ell^p_n L^1_\tau} \prod_{j=2}^{5} \|\widehat{u}_j\|_{\ell^2_n L^1_\tau}.\end{aligned}$$

The result follows from Hölder's inequality.

□

Remark 3.5.8. Note that the above estimate still holds if we include a factor of $\langle n_j \rangle^\varepsilon$ in the multiplier, for some $j \in \{2, \dots, 5\}$ and a small $0 < \varepsilon \ll 1$.

Proposition 3.5.9. *Let $\mathcal{Q}$ be defined as in (3.30), with the highest frequency which is not associated to a pairing corresponding to the first entry, and $\theta < \frac{\delta}{2}$. Then, the following estimates hold*

$$\begin{aligned}\big\|\mathcal{Q}\big(\varphi_T \cdot \mathbf{G}_A[u_1, u_2, u_3], u_4, \dots, u_7\big)\big\|_{Z_1^s}, \big\|\mathcal{Q}\big(\overline{\varphi_T \cdot \mathbf{G}_A[u_1, u_2, u_3]}, u_4, \dots, u_7\big)\big\|_{Z_1^s} \\ \lesssim \|u_1\|_{Z_0^s} \prod_{j=2}^{7} \|u_j\|_{Y_0^{\frac{1}{2}}}, \quad (3.34)\end{aligned}$$

$$\big\|\mathcal{Q}\big(\varphi_T\cdot\mathbf{G}_B[u_1,u_2,u_3],u_4,\dots,u_7\big)\big\|_{Z_1^s},\ \big\|\mathcal{Q}\big(\overline{\varphi_T\cdot\mathbf{G}_B[u_1,u_2,u_3]},u_4,\dots,u_7\big)\big\|_{Z_1^s}$$
$$\lesssim \|u_2\|_{Z_0^s}\prod_{\substack{j=1\\ j\neq 2}}^{7}\|u_j\|_{Y_0^{\frac12}}.$$

Proof. We will only focus on establishing the estimate for the first term in (3.34), as the same approach holds for the second term. Similarly, we can establish the second set of estimates by exchanging the roles of u_1 and u_2. The first term on the left-hand side of (3.34) is controlled by the following quantity

$$\bigg\|\sum_{n=n_1+\ldots+n_7}\int_{\mathbb{R}}\frac{\langle n\rangle^s|n_1|}{\langle\phi(\overline{n}_{123})\rangle\langle\tau-\lambda\rangle^{1-\theta}}\int_{\lambda=\tau_1+\ldots+\tau_7}\prod_{j=1}^{7}|\widehat{u}_j(\tau_j,n_j)|\,d\lambda\bigg\|_{\ell_n^p L_\tau^{q_0}},\tag{3.35}$$

with $\widehat{u}_j$ substituted by $\widehat{\overline{u}}_j$, $j=1,2,3$, for the second term. It suffices to estimate (3.35). Since $(n_1,n_2,n_3)\in\mathbb{X}_A(n_0)$ and $|n_0|=\max(|n_0|,|n_4|,\dots,|n_7|)$, we have

$$\frac{\langle n\rangle^s|n_1|}{\langle\phi(\overline{n}_{123})\rangle}\lesssim\frac{1}{\max\limits_{j=1,2}\langle n_j\rangle^{1-s}}.$$

Consider the change of variables $\sigma_j=\tau_j-n_j^3$, $j=1,\dots,7$ and let $f_j(\sigma,n)=|\widehat{\mathbf{P}_{N_j}u_j}(\sigma+n^3,n)|$, $j=1,\dots,7$. Since $|n_1+n_2+n_3|\geq\max\limits_{j=4,\dots,7}|n_j|$ and $|n_1|\sim|n_1+n_2+n_3|\gg\max\limits_{j=2,\dots,7}|n_j|$, n_1 cannot be in a pairing. Moreover, $n_2+n_3\neq 0$. We will consider four cases depending on the number of pairings.

Case 1: no pairings

Since $|n_1|\geq|n_j|$, $j=2,\dots,7$, using Minkowski's inequality and Schur's test (Lemma 3.1.3), we have

$$\begin{aligned}
(3.35)&\lesssim\sum_{N_1,\dots,N_7}N_1^{-1+s}\int_{\mathbb{R}^7}\bigg\|\sum_{n=n_1+\ldots+n_7}\frac{1}{\langle\tau-\tau_1-\ldots-\tau_7+\psi(n,\overline{n}_{1\ldots7})\rangle^{1-\theta}}\prod_{j=1}^{7}f_j(\sigma_j,n_j)\bigg\|_{\ell_n^p L_\tau^{q_0}}\\
&\lesssim\sum_{N_1,\dots,N_7}\int_{\mathbb{R}^7}N_1^{-1+s}\bigg\|\sum_{\substack{n=n_1+\ldots+n_7,\\ \psi(n,\overline{n}_{1\ldots7})=\mu}}\prod_{j=1}^{7}f_j(\sigma_j,n_j)\bigg\|_{\ell_n^p\ell_\mu^{q_1}},
\end{aligned}$$

where $\psi(n,\overline{n}_{1\ldots7})=n^3-n_1^3-\ldots-n_7^3$. Using Hölder's inequality, it follows that

$$\begin{aligned}
(3.35)&\lesssim\sum_{N_1,\dots,N_7}\int_{\mathbb{R}^7}N_1^{-1+s}\bigg\|\mathbb{1}_{\mu\in A(n,N_1,\dots,N_7)}\sum_{\substack{n=n_1+\ldots+n_7,\\ \psi(n,\overline{n}_{1\ldots7})=\mu}}\prod_{j=1}^{7}f_j(\sigma_j,n_j)\bigg\|_{\ell_n^p\ell_\mu^{q_1}}\\
&\lesssim\sum_{N_1,\dots,N_7}\int_{\mathbb{R}^7}N_1^{-1+s}(N_2\cdots N_7)^{\frac{1}{q_1}}\bigg\|\sum_{\substack{n=n_1+\ldots+n_7,\\ \psi(n,\overline{n}_{1\ldots7})=\mu}}\prod_{j=1}^{7}f_j(\sigma_j,n_j)\bigg\|_{\ell_n^p\ell_\mu^{\infty}},
\end{aligned}$$

where

$$\begin{aligned}
A(n,N_1,\dots,N_7)=\big\{\mu\in\mathbb{Z}:\ \mu=n^3-(n-n_2-\ldots-n_7)^3-n_2^3-\ldots-n_7^3,&\\
|n_j|\sim N_j, j=2,\dots,7\big\}&
\end{aligned}$$

which satisfies $|A(n,N_2,\dots,N_7)|\lesssim N_2\cdots N_7$, uniformly in n. Focusing on the inner sum and omitting the time dependence, we have for $\alpha > 0$

$$\sum_{\substack{n=n_1+\ldots+n_7,\\ \psi(n,\overline{n}_{1\ldots7})=\mu}} \prod_{j=1}^{7} f_j(n_j) \lesssim \sum_{\overline{n}_{234}} \sum_{n_1} f_1(n_1) \sum_{\substack{\overline{n}_{567}\\ \in B(n,n_1,\overline{n}_{234},\mu)}} \alpha |f_2(n_2)f_3(n_3)f_4(n_4)|^2$$
$$+ \sum_{\overline{n}_{567}} \sum_{n_1} f_1(n_1) \sum_{\substack{\overline{n}_{234}\\ \in B(n,n_1,\overline{n}_{567},\mu)}} \alpha^{-1} |f_5(n_5)f_6(n_6)f_7(n_7)|^2,$$

where

$$B(n,n_1,n_2,n_3,n_4,\mu) = \big\{(n_5,n_6,n_7) \in \mathbb{Z}^3 : \; n_5+n_6+n_7 = n-n_1-n_2-n_3-n_4,$$
$$n_5^3+n_6^3+n_7^3 = n^3-n_1^3-n_2^3-n_3^3-n_4^3-\mu, \; |n_j| \sim N_j, \; j=4,5,6\big\}.$$

Using Lemma 3.1.7 (ii), we have that $|B(n,n_1,n_2,n_3,n_4,\mu)|, |B(n,n_1,n_5,n_6,n_7,\mu)| \lesssim N_{2+}^{\varepsilon}$, for any $\varepsilon > 0$ and $N_{2+} = \max(N_2,\ldots,N_7)$. In addition, we know that $|n-n_1| \lesssim N_{2+}$, giving

$$\sum_{\substack{n=n_1+\ldots+n_7,\\ \psi(n,\overline{n}_{1\ldots7})=\mu}} \prod_{j=1}^{7} f_j(n_j) \lesssim N_{2+}^{\varepsilon} \Big(\sum_{|n-n_1|\lesssim N_{2+}} f_1(n_1) \Big) \prod_{j=2}^{7} \|f_j\|_{\ell^2_n},$$

by choosing $\alpha = (\|f_2\|_{\ell^2_n}\|f_3\|_{\ell^2_n}\|f_4\|_{\ell^2_n})^{-1}\|f_5\|_{\ell^2_n}\|f_6\|_{\ell^2_n}\|f_7\|_{\ell^2_n}$. Consequently, using Hölder's and Minkowski's inequality gives the following

$$(3.35) \lesssim \sum_{N_1,\ldots,N_7} N_1^{s-\frac{1}{p}+} N_{2+}^{\frac{1}{p}+\varepsilon} (N_2\cdots N_7)^{\frac{1}{q_1}} \|f_1\|_{L^1_\sigma \ell^p_n} \prod_{j=1}^{7} \|f_j\|_{L^1_\sigma \ell^2_n}$$
$$\lesssim \sum_{N_1,\ldots,N_7} N_1^{-\frac{1}{p}+} N_{2+}^{\frac{1}{p}+\varepsilon} (N_2\cdots N_7)^{\frac{1}{q_1}+\frac{1}{r_0}-\frac{1}{p}-\frac{1}{2}+} \|u_1\|_{Z_0^s} \prod_{j=2}^{7} \|u_j\|_{Y_0^{\frac{1}{2}}}.$$

The estimate follows from summing in the dyadics.

Case 2: one pairing

In this case, there is only one pairing involving two frequencies in $n_4,\ldots,n_7$. Let $* = A$, assume without loss of generality that $n_6+n_7=0$, and $N_{2+} = \max(N_2,\ldots,N_5)$. Proceeding as in the previous case, we have

$$(3.35) \lesssim \sum_{N_1,\ldots,N_5} \int_{\mathbb{R}^7} N_1^{-1+s} \Big\| \sum_{n_6} \sum_{\substack{n=n_1+\ldots+n_5,\\ \tilde{\psi}(n,\overline{n}_{1\ldots5})=\mu}} \widehat{u}_6(\tau_6,n_6)\widehat{u}_7(\tau_7,-n_6) \prod_{j=1}^{5} f_j(\sigma_j,n_j) \Big\|_{\ell^p_n \ell^{q_1}_\mu}$$
$$\lesssim \|\widehat{u}_6\|_{\ell^2_n L^1_\tau} \|\widehat{u}_7\|_{\ell^2_n L^1_\tau} \sum_{N_1,\ldots,N_5} \int_{\mathbb{R}^5} N_1^{-1+s} N_{2+}^{\frac{4}{q_1}} \Big\| \sum_{\substack{n=n_1+\ldots+n_5,\\ \tilde{\psi}(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^{5} f_j(\sigma_j,n_j) \Big\|_{\ell^p_n \ell^\infty_\mu},$$

where $\tilde{\psi}(n,\overline{n}_{1\ldots5}) = n^3 - n_1^3 - \ldots - n_5^3$. Focusing on the inner sum, we have

$$\sum_{\substack{n=n_1+\ldots+n_5,\\ \tilde{\psi}(n,\overline{n}_{1\ldots5})=\mu}} \prod_{j=1}^{5} f_j(n_j) \lesssim \sum_{(n_4,n_5)} \sum_{\substack{\overline{n}_{123}\\ \in B(n,n_4,n_5,\mu)}} f_1(n_1)\alpha |f_4(n_4)f_5(n_5)|^2$$
$$+ \sum_{(n_2,n_3)} \sum_{\substack{\overline{n}_{145}\\ \in B(n,n_2,n_3,\mu)}} f_1(n_1)\alpha^{-1} |f_2(n_2)f_3(n_3)|^2,$$

where

$$B(n,n_4,n_5,\mu) = \big\{(n_1,n_2,n_3) \in \mathbb{Z}^3 : \; n_1+n_2+n_3 = n-n_4-n_5,$$

$$n_1^3+n_2^3+n_3^3=n^3-n_4^3-n_5^3-\mu,\ |n_j|\sim N_j,\ j=1,2,3\}.$$

Using Lemma 3.1.7 (ii), we have $|B(n,n_4,n_5,\mu)|,|B(n,n_2,n_3,\mu)|\lesssim N_{2+}^{\varepsilon}$, for $\varepsilon>0$ small enough, which implies

$$\Big\|\sum_{\substack{n=n_1+\ldots+n_5,\\ \tilde{\psi}(n,\overline{n}_{1\ldots5})=\mu}}\prod_{j=1}^5 f_j(n_j)\Big\|_{\ell^p_n\ell^\infty_\mu}\lesssim N_{2+}^{\varepsilon}\Big\|\sup_{|n-n_1|\lesssim N_{2+}} f_1(n_1)\Big\|_{\ell^p_n}\prod_{j=2}^5\|f_j\|_{\ell^2_n}$$
$$\lesssim N_{2+}^{\frac1p+\varepsilon}\|f_1\|_{\ell^p_n}\prod_{j=2}^5\|f_j\|_{\ell^2_n}.$$

Consequently,

$$\begin{aligned}(3.35)&\lesssim\|\widehat{u}_6\|_{\ell^2_nL^1_\tau}\|\widehat{u}_7\|_{\ell^2_nL^1_\tau}\sum_{N_1,\ldots,N_5}N_1^{-1+s}N_{2+}^{\frac{4}{q_1}+\frac1p+\varepsilon}\prod_{j=1}^5\|f_j\|_{L^1_\sigma\ell^2_n}\\
&\lesssim\|u_1\|_{Y_0^s}\prod_{j=2}^7\|u_j\|_{Y_0^{\frac12}}\sum_{N_1,\ldots,N_5}N_1^{-1+\frac{1}{r_0}-\frac1p+}N_{2+}^{\frac{4}{q_1}+\frac1p+\varepsilon}(N_2N_3N_4N_5)^{\frac{1}{r_0}-\frac1p-\frac12+}\\
&\lesssim\|u_1\|_{Y_0^s}\prod_{j=2}^7\|u_j\|_{Y_0^{\frac12}},\end{aligned}$$

by choosing $\delta<\frac1p$ and $\delta<\frac{1}{40}$, needed to sum in the dyadics

$$\sum_{N_1,\ldots,N_5}N_1^{-1+\frac{1}{r_0}-\frac1p+}N_2^{\frac{4}{q_1}+\frac{1}{r_0}+\varepsilon-\frac12+}(N_3N_4N_5)^{\frac{1}{r_0}-\frac1p-\frac12}$$
$$\lesssim\sum_{N_1,\ldots,N_5}(N_1N_2)^{-\frac14+10\delta-\frac{1}{2p}+}(N_3N_4N_5)^{\delta-\frac1p+}\lesssim1.$$

<u>Case 3: two pairings</u>

We can assume without loss of generality that $n_2+n_5=n_6+n_7=0$. Proceeding as before, with $N_{3+}=\max(N_3,N_4)$ we have

$$\begin{aligned}(3.35)\lesssim&\sum_{N_1,N_2,N_3}\int_{\mathbb{R}^7}N_1^{-1+s}\Big\|\sum_{n_2,n_6}|\widehat{u}_2(\tau_2,n_2)\widehat{u}_5(\tau_5,-n_2)\widehat{u}_6(\tau_6,n_6)\widehat{u}_7(\tau_7,-n_6)|\\
&\times\sum_{\substack{n=n_1+n_2+n_3,\\ \phi(\overline{n}_{123})=\mu}}\prod_{j\in\{1,3,4\}}f_j(\sigma_j,n_j)\Big\|_{\ell^p_n\ell^{q_1}_\mu}\\
\lesssim&\prod_{j\in\{2,5,6,7\}}\|\widehat{u}_j\|_{\ell^2_nL^1_\tau}\sum_{N_1,N_3,N_4}\int_{\mathbb{R}^3}N_1^{-1+s}N_{3+}^{\frac{2}{q_1}}\Big\|\sum_{\substack{n=n_1+n_3+n_4,\\ \phi(\overline{n}_{134})=\mu}}\prod_{j\in\{1,3,4\}}f_j(\sigma_j,n_j)\Big\|_{\ell^p_n\ell^\infty_\mu}.\end{aligned}$$

Applying Lemma 3.1.7 (ii), we have

$$\Big\|\sum_{\substack{n=n_1+n_3+n_4,\\ \phi(\overline{n}_{134})=\mu}}\prod_{j\in\{1,3,4\}}f_j(\sigma_j,n_j)\Big\|_{\ell^p_n\ell^\infty_\mu}\lesssim N_{3+}^{\varepsilon}\prod_{j\in\{1,3,4\}}\|f_j\|_{\ell^p_n},$$

for any $\varepsilon>0$. Consequently,

$$(3.35)\lesssim\prod_{j\in\{2,5,6,7\}}\|\widehat{u}_j\|_{\ell^2_nL^1_\tau}\sum_{N_1,N_3,N_4}N_1^{-1+s}N_{3+}^{\frac{2}{q_1}+\varepsilon}\prod_{k\in\{1,3,4\}}\|f_k\|_{L^1_\sigma\ell^p_n}$$

$$\lesssim \|u_1\|_{Y_0^s} \prod_{j=2}^{7} \|u_j\|_{Y_0^{\frac{1}{2}}} \sum_{N_1,N_3,N_4} N_1^{-1+\frac{1}{r_0}-\frac{1}{p}+} N_{3+}^{\frac{2}{q_1}+\varepsilon} (N_3N_4)^{\frac{1}{r_0}-\frac{1}{p}-\frac{1}{2}+}$$
$$\lesssim \|u_1\|_{Y_0^s} \prod_{j=2}^{7} \|u_j\|_{Y_0^{\frac{1}{2}}},$$

assuming that $\delta < \frac{1}{10p}$ to sum in the dyadics.

Case 4: three pairings

Assume that $n_2 + n_5 = n_3 + n_4 = n_6 + n_7 = 0$, then using the weight $\langle \tau - \lambda \rangle^{-1+\theta}$ for the $L_\tau^{q_0}$-norm and Hölder's inequality, it follows that

$$\begin{aligned}(3.35) &\lesssim \Bigg\| \sum_{n_2,n_3,n_6} \int_{\mathbb{R}^7} \langle n \rangle^s |\widehat{u}_1(\tau_1,n)\widehat{u}_2(\tau_2,n_2)\widehat{u}_3(\tau_3,n_3)| \\ &\qquad \times |\widehat{u}_4(\tau_4,-n_3)\widehat{u}_5(\tau_5,-n_2)\widehat{u}_6(\tau_6,n_6)\widehat{u}_7(\tau_7,-n_6)| \Bigg\|_{\ell_n^p} \\ &\lesssim \|\langle n \rangle^s \widehat{u}_1\|_{\ell_n^p L_\tau^1} \prod_{j=2}^{7} \|\widehat{u}_j\|_{\ell_n^2 L_\tau^1} \\ &\lesssim \|u_1\|_{Y_0^s} \prod_{j=2}^{7} \|u_j\|_{Y_0^{\frac{1}{2}}}.\end{aligned}$$

□

3.5.3 Remaining quintic terms

It remains to estimate the terms in (3.31). These terms cannot be written as (3.30) and thus require a finer analysis. For the $\mathbf{B}_*^j$ terms, we need to use the modulations. For example, calculating the space-time Fourier transform of the $\mathbf{B}_*^3$ terms in (3.31), $* \in \{A, B\}$, we have

$$\begin{aligned}&\big|\mathcal{F}_{t,x}\mathbf{B}_*^3\big(u_1,u_2,\varphi_T\cdot\mathbf{G}_\#[u_3,u_4,u_5]\big)(\tau,n)\big| \\ &\lesssim \sum_{\substack{\overline{n}_{120}\in\mathbb{X}_*(n),\\ \overline{n}_{345}\in\mathbb{X}_\#(n_0)}} \int_{\mathbb{R}^3} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} |n_1n_3|\big|K_+(\tau-n^3,\lambda-n^3,\phi(\overline{n}_{120}))\big|\langle\tau_0-\mu\rangle|\widehat{\varphi}_T(\tau_0-\mu)| \\ &\quad \times \frac{\mathbb{1}_{|\tau_0-n_0^3|\gtrsim|\lambda-n^3+\phi(\overline{n}_{120})|}}{\langle\tau_0-\mu\rangle\langle\mu-\sigma\rangle} \min\left(\frac{1}{\langle\phi(\overline{n}_{345})\rangle},\frac{1}{\langle\mu-n_0^3\rangle}\right)\prod_{j=1}^{5}|\widehat{u}_j(\tau_j,n_j)|\,d\sigma\,d\mu\,d\lambda \\ &\lesssim \sum_{\substack{\overline{n}_{120}\in\mathbb{X}_*(n),\\ \overline{n}_{345}\in\mathbb{X}_\#(n_0)}} \int_{\mathbb{R}^3} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} \frac{|n_1n_3|\langle\tau_0-n_0^3\rangle^{1-\alpha}}{\langle\tau-\lambda\rangle\langle\tau-n^3\rangle\langle\phi(\overline{n}_{120})\rangle^{1-\alpha}} \\ &\quad \times \frac{\mathbb{1}_{|\tau_0-n_0^3|\gtrsim|\lambda-n^3+\phi(\overline{n}_{120})|}}{\langle\tau_0-\mu\rangle\langle\mu-\sigma\rangle} \min\left(\frac{1}{\langle\phi(\overline{n}_{345})\rangle},\frac{1}{\langle\mu-n_0^3\rangle}\right)\prod_{j=1}^{5}|\widehat{u}_j(\tau_j,n_j)|\,d\sigma\,d\mu\,d\lambda\end{aligned}$$

using (3.14) and (3.19), for $0 \le \alpha \le 1$. In order to control the multiplier, we must consider two cases depending on the modulations of the second generation:

$$|\tau_0 - n_0^3| \gg |\sigma - n_0^3|, \tag{3.36}$$
$$|\tau_0 - n_0^3| \lesssim |\sigma - n_0^3|. \tag{3.37}$$

If (3.36) holds, then $|\tau_0 - \sigma| \sim |\tau_0 - n_0^3| \gtrsim |\lambda - n^3 + \phi(\overline{n}_{120})|$ and we can obtain powers of the resonance relation of the first and the second generations. Using Lemma 2.1.4, we get

$$\mathbb{1}_{|\tau_0-n_0^3|\gg|\sigma-n_0^3|}\big|\mathcal{F}_{t,x}\mathbf{B}_*^3\big(u_1,u_2,\varphi_T\cdot\mathbf{G}_\#[u_3,u_4,u_5]\big)(\tau,n)\big|$$

$$\lesssim \sum_{\substack{\overline{n}_{120}\in\mathbb{X}_*(n),\\ \overline{n}_{345}\in\mathbb{X}_\#(n_0)}} \int_{\mathbb{R}^2} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} \frac{|n_1n_3|\langle\tau_0-n_0^3\rangle^{1-\alpha}}{\langle\tau-\lambda\rangle\langle\tau-n^3\rangle\langle\phi(\overline{n}_{120})\rangle^{1-\alpha}\langle\phi(\overline{n}_{345})\rangle\langle\tau_0-\sigma\rangle^{1-\theta}} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma\,d\lambda$$

for $\theta<\alpha\leq 1$. Since $|\tau_0-\sigma|\gtrsim|\lambda-n^3+\phi(\overline{n}_{120})|$, by setting $\sigma'=\sigma+\tau_1+\tau_2$, changing variables in the integrals above, and applying Lemma 2.1.4, we have

$$\int_{\mathbb{R}^2} \frac{1}{\langle\tau-\lambda\rangle\langle\lambda-n^3+\phi(\overline{n}_{120})\rangle^{\alpha-\theta}} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma\,d\lambda$$
$$\lesssim \frac{1}{\langle\tau-n^3+\phi(\overline{n}_{120})\rangle^{\alpha-2\theta}} \int_{\mathbb{R}} \int_{\sigma'=\tau_1+\ldots+\tau_5} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma'.$$

Substituting this estimate and using the fact that $|\phi(\overline{n}_{120})|\lesssim\max_{j=1,\ldots,5}|n_j|^3$, $\langle\tau-n^3\rangle\lesssim\langle\tau-n^3+\phi(\overline{n}_{120})\rangle\langle\phi(\overline{n}_{120})\rangle$, and $\alpha=4\delta+2\theta$, gives

$$\mathbb{1}_{|\tau_0-n_0^3|\gg|\sigma-n_0^3|}\big|\mathcal{F}_{t,x}\mathbf{B}_*^3\big(u_1,u_2,\varphi_T\cdot\mathbf{G}_\#[u_3,u_4,u_5]\big)(\tau,n)\big|$$
$$\lesssim \sum_{\substack{\overline{n}_{120}\in\mathbb{X}_*(n),\\ \overline{n}_{345}\in\mathbb{X}_\#(n_0)}} \int_{\mathbb{R}} \frac{\max_{j=1,\ldots,5}\langle n_j\rangle^{9\theta}|n_1n_3|}{\langle\phi(\overline{n}_{120})\rangle\langle\phi(\overline{n}_{345})\rangle\langle\tau-n^3\rangle^{1+4\delta}} \int_{\sigma'=\tau_1+\ldots+\tau_5} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma'.$$

If (3.37) holds, we can only gain a power of the resonance relation of the first generation. We can use (3.19) to gain a power of $\langle\phi(\overline{n}_{120})\rangle$ at the cost of $\langle\tau_0-n_0^3\rangle$, but we can no longer use $\langle\tau_0-\sigma\rangle$ to help control this loss, which is why we cannot keep the power of $\langle\phi(\overline{n}_{345})\rangle$ as before:

$$\mathbb{1}_{|\tau_0-n_0^3|\lesssim|\sigma-n_0^3|}\big|\mathcal{F}_{t,x}\mathbf{B}_*^3\big(u_1,u_2,\varphi_T\cdot\mathbf{G}_\#[u_3,u_4,u_5]\big)(\tau,n)\big|$$
$$\lesssim \sum_{\substack{\overline{n}_{120}\in\mathbb{X}_*(n),\\ \overline{n}_{345}\in\mathbb{X}_\#(n_0)}} \int_{\mathbb{R}^3} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} \frac{|n_1n_3|\langle\tau_0-n_0^3\rangle^{1-\alpha}}{\langle\tau-\lambda\rangle\langle\tau-n^3\rangle\langle\phi(\overline{n}_{120})\rangle^{1-\alpha}\langle\tau_0-\mu\rangle\langle\mu-\sigma\rangle\langle\mu-n_0^3\rangle}$$
$$\times\prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\;d\sigma\,d\mu\,d\lambda.$$

Focusing on the integrals, from Cauchy-Schwarz inequality, Lemma 2.1.4, the fact that $|\lambda-n^3+\phi(\overline{n}_{120})|\lesssim|\tau_0-n_0^3|$, and the change of variables $\sigma'=\sigma+\tau_1+\tau_2$, we get

$$\int_{\mathbb{R}^3} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} \frac{\langle\tau_0-n_0^3\rangle^{1-\alpha}}{\langle\tau-\lambda\rangle\langle\tau_0-\mu\rangle\langle\mu-\sigma\rangle\langle\mu-n_0^3\rangle} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma\,d\mu\,d\lambda$$
$$\lesssim \int_{\mathbb{R}^2} \int_{\substack{\lambda=\tau_1+\tau_2+\tau_0,\\ \sigma=\tau_3+\tau_4+\tau_5}} \frac{\langle\tau_0-n_0^3\rangle^{1-\alpha}}{\langle\tau-\lambda\rangle\langle\tau_0-\sigma\rangle^{1-\theta}\langle\tau_0-n_0^3\rangle^{1-\theta}} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma\,d\lambda$$
$$\lesssim \int_{\mathbb{R}^2} \int_{\sigma'=\tau_1+\ldots+\tau_5} \frac{1}{\langle\tau-\lambda\rangle\langle\lambda-\sigma'\rangle^{1-\theta}\langle\tau_0-n_0^3\rangle^{\alpha-\theta}} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma'\,d\lambda$$
$$\lesssim \int_{\mathbb{R}} \int_{\sigma'=\tau_1+\ldots+\tau_5} \frac{\langle\phi(\overline{n}_{120})\rangle^{\theta}\langle\tau-n^3\rangle^{\theta}}{\langle\tau-\sigma'\rangle^{1-\theta}} \prod_{j=1}^5 |\widehat{u}_j(\tau_j,n_j)|\,d\sigma',$$

by choosing $\alpha = 2\theta$ and using $\langle \tau - \lambda \rangle \lesssim \langle \tau - n^3 \rangle \langle \lambda - n^3 + \phi(\overline{n}_{120}) \rangle \langle \phi(\overline{n}_{120}) \rangle$. Substituting the above estimate, we obtain

$$\mathbb{1}_{|\tau_0 - n_0^3| \lesssim |\sigma - n_0^3|} \big| \mathcal{F}_{t,x} \mathbf{B}_*^3 \big(u_1, u_2, \varphi_T \cdot \mathbf{G}_\#[u_3, u_4, u_5] \big)(\tau, n) \big| \\ \lesssim \sum_{\substack{\overline{n}_{120} \in \mathbb{X}_*(n), \\ \overline{n}_{345} \in \mathbb{X}_\#(n_0)}} \int_{\mathbb{R}} \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_1 n_3|}{\langle \phi(\overline{n}_{120}) \rangle \langle \tau - n^3 \rangle^{1-\theta} \langle \tau - \sigma' \rangle^{1-\theta}} \int_{\sigma' = \tau_1 + \ldots + \tau_5} \prod_{j=1}^{5} |\widehat{u}_j(\tau_j, n_j)| \, d\sigma'$$

From the above calculation and (3.28), the $\mathbf{B}_*^j$ and the $\mathcal{DNR}_C$ terms left to consider, can be controlled as follows

$$\big| \mathcal{F}_{t,x} \mathbf{B}_A^2 \big(u_1, \varphi_T \cdot \mathbf{G}_\#[u_2, u_3, u_4], u_5 \big)(\tau, n) \big| \\ \lesssim \sum_{\substack{\overline{n}_{105} \in \mathbb{X}_A(n), \\ \overline{n}_{234} \in \mathbb{X}_\#(n_0)}} \int_{\mathbb{R}} \frac{\alpha_1(n, n_1, \dots, n_5)}{\langle \tau - n^3 \rangle^{1+4\delta}} + \frac{\beta_1(n, n_1, \dots, n_5)}{\langle \tau - n^3 \rangle^{1-\theta} \langle \tau - \sigma' \rangle^{1-\theta}} \int_{\sigma' = \tau_1 + \ldots + \tau_5} \prod_{j=1}^{5} |\widehat{u}_j(\tau_j, n_j)| \, d\sigma',$$

$$\big| \mathcal{F}_{t,x} \mathbf{B}_*^3 \big(u_1, u_2, \varphi_T \cdot \mathbf{G}_\#[u_3, u_4, u_5] \big)(\tau, n) \big| \\ \lesssim \sum_{\substack{\overline{n}_{120} \in \mathbb{X}_*(n), \\ \overline{n}_{345} \in \mathbb{X}_\#(n_0)}} \int_{\mathbb{R}} \frac{\alpha_2(n, n_1, \dots, n_5)}{\langle \tau - n^3 \rangle^{1+4\delta}} + \frac{\beta_2(n, n_1, \dots, n_5)}{\langle \tau - n^3 \rangle^{1-\theta} \langle \tau - \sigma' \rangle^{1-\theta}} \int_{\sigma' = \tau_1 + \ldots + \tau_5} \prod_{j=1}^{5} |\widehat{u}_j(\tau_j, n_j)| \, d\sigma',$$

$$\big| \mathcal{F}_{t,x} \mathcal{DNR}_C \big(u_1, u_2, \varphi_T \cdot \mathbf{G}_\#[u_3, u_4, u_5] \big)(\tau, n) \big| \\ \lesssim \sum_{\substack{\overline{n}_{120} \in \mathbb{X}_C(n), \\ \overline{n}_{345} \in \mathbb{X}_\#(n_0)}} \int_{\mathbb{R}} \frac{\beta_3(n, n_1, \dots, n_5)}{\langle \tau - n^3 \rangle^{1-\theta} \langle \tau - \sigma' \rangle^{1-\theta}} \int_{\sigma' = \tau_1 + \ldots + \tau_5} \prod_{j=1}^{5} |\widehat{u}_j(\tau_j, n_j)| \, d\sigma', \tag{3.38}$$

where $*, \# \in \{A, B\}$ and the spatial multipliers are given by

$$\alpha_1(n, n_1, \dots, n_5) = \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_1 n_2|}{\langle \phi(\overline{n}_{105}) \rangle \langle \phi(\overline{n}_{234}) \rangle}, \qquad \beta_1(n, n_1, \dots, n_5) = \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_1 n_2|}{\langle \phi(\overline{n}_{105}) \rangle},$$

$$\alpha_2(n, n_1, \dots, n_5) = \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_1 n_3|}{\langle \phi(\overline{n}_{120}) \rangle \langle \phi(\overline{n}_{345}) \rangle}, \qquad \beta_2(n, n_1, \dots, n_5) = \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_1 n_3|}{\langle \phi(\overline{n}_{120}) \rangle},$$

$$\beta_3(n, n_1, \dots, n_5) = \frac{|n_1 n_3|}{\langle \phi(\overline{n}_{345}) \rangle}.$$

In the frequency regions where $|\beta_j(n, n_1, \dots, n_5)| \lesssim 1$, the corresponding contributions have the standard quintic form in (3.30). We therefore proceed as in Section 3.5.2. Otherwise, we can apply the following result.

Proposition 3.5.10. *Assume that the frequencies are ordered as follows* $|n_1| \geq \ldots \geq |n_5|$. *If* $|n_1| \sim |n_2| \gg |n_3| \gtrsim |n|$, $(1, 2)$ *not a pairing, and*

$$\beta(n, n_1, \dots, n_5) \lesssim \frac{|n_1|^{1+9\theta}}{|n_3|},$$

then the following estimate holds

$$\Big\| \langle n \rangle^s \langle \tau - n^3 \rangle^{b_1} \sum_{n = n_1 + \ldots + n_5} \int_{\mathbb{R}} \frac{\beta(n, n_1, \dots, n_5)}{\langle \tau - n^3 \rangle^{1-\theta} \langle \tau - \sigma' \rangle^{1-\theta}} \int_{\sigma' = \tau_1 + \ldots + \tau_5} \prod_{j=1}^{5} |\widehat{u}_j(\tau_j, n_j)| \, d\sigma' \Big\|_{\ell_n^p L_\tau^{q_0}} \\ \lesssim \|u_1\|_{Z_0^s} \|u_2\|_{Z_0^{\frac{1}{2}}} \|u_3\|_{Z_0^{\frac{1}{2}}} \|u_4\|_{Y_0^{\frac{1}{2}}} \|u_5\|_{Y_0^{\frac{1}{2}}}. \tag{3.39}$$

Lastly, the $\mathbf{B}_*^j$ terms in (3.38) with α_j multiplier can be estimated by the following proposition.

Proposition 3.5.11. *Let $\mathcal{Q}'(u_1,\dots,u_5)$ be such that*

$$\begin{aligned}&\big|\mathcal{F}_{t,x}\mathcal{Q}'(u_1,\dots,u_5)(\tau,n)\big|\\&\qquad\lesssim \sum_{n=n_1+n_2+n_0}\sum_{n_0=n_3+n_4+n_5}\int_{\mathbb{R}^5}\frac{\max_{j=1,\dots,5}\langle n_j\rangle^{9\theta}}{\langle n_1\rangle\langle n_3\rangle\langle \tau-n^3\rangle^{1+\theta}}\prod_{j=1}^5|\widehat{u}_j(\tau_j,n_j)|\,d\tau_1\cdots d\tau_5,\end{aligned}$$

where $|n_1|\geq \max(|n_0|,|n_2|)$ and $|n_3|\geq|n_4|\geq|n_5|$. Then, the following estimate holds

$$\big\|\mathcal{Q}'(u_1,\dots,u_5)\big\|_{Z_1^s}\lesssim \max\Big(\|u_1\|_{Y_0^s}\|u_3\|_{Y_0^{\frac12}},\|u_1\|_{Y_0^{\frac12}}\|u_3\|_{Y_0^s}\Big)\|u_2\|_{Y_0^{\frac12}}\|u_4\|_{Y_0^{\frac12}}\|u_5\|_{Y_0^{\frac12}}.$$

See Appendix A.4 for further detail on how to estimate these contributions. We complete this section by showing Propositions 3.5.10 and 3.5.11.

Proof of Proposition 3.5.10. Due to the θ loss in the largest frequency when estimating α, we will distinguish two cases: when $|n_1|^{\frac12}\lesssim|n_3|$ and when $|n_1|^{\frac12}\gg|n_3|$.

Case 1: $|n_1|^{\frac12}\lesssim|n_3|$

Using the notation $\psi(n,\overline{n}_{1\dots5})=n^3-n_1^3-\dots-n_5^3$ and the change of variables $\sigma_j=\tau_j-n_j^3$, $j=1,\dots,5$, we start by applying Minkowski's inequality and Schur's test (Lemma 3.1.3) to obtain

$$\begin{aligned}&\text{LHS of (3.39)}\\&\lesssim \int_{\sigma_1,\dots,\sigma_5}\Bigg\|\sum_\mu\frac{1}{\langle\tau-n^3-\bar\sigma+\mu\rangle^{1-\theta}}\Bigg(\sum_{\substack{n=n_1+\dots+n_5,\\ \psi(n,\overline{n}_{1\dots5})=\mu}}\frac{\langle n_1\rangle^{1+9\theta}\langle n\rangle^{s-\frac12}}{\max(\langle n_3\rangle,\langle n\rangle)^{\frac12}}\prod_{j=1}^5|\widehat{u}_j(\sigma_j+n_j^3,n_j)|\Bigg)\Bigg\|_{\ell^p_nL^{q_0}_\tau}\\&\lesssim\int_{\sigma_1,\dots,\sigma_5}\Bigg\|\sum_{\substack{n=n_1+\dots+n_5,\\ \psi(n,\overline{n}_{1\dots5})=\mu}}\frac{\langle n_1\rangle^{s+\frac12+9\theta}}{\max(\langle n_3\rangle,\langle n\rangle)^{\frac12}}\prod_{j=1}^5|\widehat{u}_j(\sigma_j+n_j^3,n_j)|\Bigg\|_{\ell^p_n\ell^{q_1}_\mu},\end{aligned}$$

where $\bar\sigma=\sigma_1+\dots+\sigma_5$ and $\theta<\frac{\delta}{2}$. Let $f_1(\sigma,n)=\langle n\rangle^s|\widehat{\mathbf{P}_{N_1}u_1}(\sigma+n^3,n)|$, $f_2(\sigma,n)=\langle n\rangle^{\frac12}|\widehat{\mathbf{P}_{N_2}u_2}(\sigma+n^3,n)|$, and $f_k(\sigma,n)=|\widehat{\mathbf{P}_{N_k}u_k}(\sigma+n^3,n)|$, $k=3,4,5$, where $N_1\sim N_2\gg N_3\geq N_4\geq N_5$ dyadic numbers with $N_1\lesssim N_3^2$. Omitting the time dependence, using Hölder's inequality, the standard divisor counting estimate (Lemma 3.1.6), and Minkowski's inequality, we have

$$\begin{aligned}&N_1^{9\theta}N_3^{-\frac12}\Bigg\|\sum_{\substack{n=n_1+\dots+n_5,\\ \psi(n,\overline{n}_{1\dots5})=\mu}}\prod_{j=1}^5 f_j(n_j)\Bigg\|_{\ell^p_n\ell^{q_1}_\mu}\\&\lesssim N_1^{9\theta+\varepsilon}N_3^{-\frac12}\Bigg\|\sum_{n_4,n_5}f_4(n_4)f_5(n_5)\Bigg(\sum_{\substack{n_1+n_2+n_3=n-n_4-n_5,\\ n_1^3+n_2^3+n_3^3=n^3-n_4^3-n_5^3-\mu}}\prod_{j=1}^3|f_j(n_j)|^p\Bigg)^{\frac1p}\Bigg\|_{\ell^p_n\ell^p_\mu}\\&\lesssim N_1^{9\theta+\varepsilon}N_3^{-\frac12}\|f_1\|_{\ell^p_n}\|f_2\|_{\ell^p_n}\|f_3\|_{\ell^p_n}\|f_4\|_{\ell^1_n}\|f_5\|_{\ell^1_n}.\end{aligned}$$

Note that we can use the divisor counting estimate because $(n_1+n_2)(n_1+n_3)(n_2+n_3)\neq0$, from the assumption that $(1,2)$ is not a pairing and the fact that $|n_1|\sim|n_2|\gg|n_3|$ which does not allow $(1,3),(2,3)$ to be pairings. From this estimate, we get

$$\begin{aligned}\text{LHS of (3.39)}&\lesssim\sum_{N_1,\dots,N_5}N_1^{9\theta+\varepsilon}N_3^{-\frac12}\Bigg(\prod_{j=1}^3\|f_j\|_{L^1_\sigma\ell^p_n}\Bigg)\Bigg(\prod_{k=4}^5\|f_k\|_{L^1_\sigma\ell^1_n}\Bigg)\\&\lesssim\sum_{N_1,\dots,N_5}N_1^{9\theta+\varepsilon}N_3^{-1}(N_4N_5)^{\frac12-\frac1p+}\|u_1\|_{Z_0^s}\Bigg(\prod_{j=2}^3\|u_j\|_{Z_0^{\frac12}}\Bigg)\Bigg(\prod_{k=4}^5\|u_k\|_{Y_0^{\frac12}}\Bigg).\end{aligned}\tag{3.40}$$

Using the fact that $N_1 \sim N_2 \lesssim N_3^2$, for $\varepsilon, \theta < \frac{\delta}{2}$ and δ small enough, the estimate follows from summing in the dyadic numbers.

Case 2: $|n_1|^{\frac{1}{2}} \gg |n_3|$

In this case, we need a different approach to control the small power of N_1 in the multiplier as well as the ε-loss from using the divisor counting estimate. Note that $\psi(n, \overline{n}_{1\dots5}) = 3(n_1 + n_2)(n_1 + n_3 + n_4 + n_5)(n_2 + n_3 + n_4 + n_5) + 3(n_3 + n_4)(n_3 + n_5)(n_4 + n_5)$. Since

$$|(n_3 + n_4)(n_3 + n_5)(n_4 + n_5)| \lesssim |n_3|^3 \ll |n_1|^{\frac{3}{2}},$$
$$|(n_1 + n_2)(n_1 + n_3 + n_4 + n_5)(n_2 + n_3 + n_4 + n_5)| \gtrsim |n_1|^2,$$

then $|\psi(n, \overline{n}_{1\dots5})| \gtrsim |n_1|^2$. Following the previous strategy, we have

$$\text{LHS of (3.39)} \lesssim \int_{\sigma_1,\dots,\sigma_5} \Bigg\| \sum_{n=n_1+\dots+n_5} \frac{\langle n_1 \rangle^{9\theta + \frac{1}{2} + s}}{\max(\langle n \rangle, \langle n_3 \rangle)^{\frac{1}{2}}} \times \frac{1}{\langle \tau - n^3 - \bar{\sigma} + \psi(n, \overline{n}_{1\dots5}) \rangle^{1-\theta} \langle \tau - n^3 \rangle^{\delta - \theta}} \prod_{j=1}^{5} |\widehat{u}_j(\sigma_j + n_j^3, n_j)| \Bigg\|_{\ell^p_n L^{q_0}_\tau}$$

Thus, in order to control the small powers of $\langle n_1 \rangle$, we use the following fact

$$|n_1|^2 \lesssim \langle \psi(n, \overline{n}_{1\dots5}) \rangle \lesssim \langle \tau - n^3 - \bar{\sigma} + \psi(n, \overline{n}_{1\dots5}) \rangle \langle \tau - n^3 \rangle \langle \sigma_1 \rangle \cdots \langle \sigma_5 \rangle.$$

To gain a power of $\langle \tau - n^3 \rangle$, we impose $\theta \leq \frac{\delta}{2}$. For $\langle \tau - n^3 - \bar{\sigma} + \psi(n, \overline{n}_{1\dots5}) \rangle$, when applying Schur's test (Lemma 3.1.3), we want to keep $\frac{\delta}{4}$ of this quantity. Thus, with $1 + \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{r}$, we need

$$1 - \theta - \frac{\delta}{4} > \frac{1}{r} = 1 - \frac{\delta}{2} \implies \theta < \frac{\delta}{4}.$$

We can obtain a power of $\langle \sigma_k \rangle^\alpha$ for $k = 4, 5$, given that

$$\|\langle \sigma \rangle^\alpha f_k\|_{\ell^1_n L^1_\sigma} \lesssim \|\langle \sigma \rangle^{\alpha + \frac{1}{2} - \delta +} f_k\|_{\ell^1_n L^{r_0}_\sigma} \lesssim \|\langle \sigma \rangle^{\frac{1}{2}} f_k\|_{\ell^1_n L^{r_0}_\sigma},$$

given that $\alpha + \frac{1}{2} - \delta < \frac{1}{2} \implies \alpha < \delta$, thus we can choose $\alpha = \frac{\delta}{4}$. Similarly, for $\langle \sigma_j \rangle^\beta$, $j = 1, 2, 3$, we have

$$\|\langle \sigma \rangle^\beta f_j\|_{L^1_\sigma \ell^p_n} \lesssim \|\langle \sigma \rangle^{\beta + 1 - 4\delta +} f_j\|_{L^{q_0}_\sigma \ell^p_n} \lesssim \|\langle \sigma \rangle^{1 - 2\delta} f_j\|_{L^{q_0}_\sigma \ell^p_n},$$

for $\beta = \frac{\delta}{4}$. Combining all of these powers, we get $\langle \psi(n, \overline{n}_{1\dots5}) \rangle^{-\frac{\delta}{4}} \lesssim N_1^{-\frac{\delta}{2}}$ which we use in (3.40) instead of the condition $N_1 \lesssim N_3^2$.

□

Proof of Proposition 3.5.11. We have the following estimate

$$\begin{aligned}
&\|\mathcal{Q}'(u_1, \dots, u_5)\|_{Z^s_1} \\
&\lesssim \Bigg\| \sum_{n=n_1+n_2+n_0} \sum_{n_0=n_3+n_4+n_5} \int_{\mathbb{R}^5} \frac{\max_j \langle n_j \rangle^{9\theta} \langle n \rangle^s}{\langle n_1 \rangle \langle n_3 \rangle \langle \tau - n^3 \rangle^{5\delta}} \prod_{j=1}^{5} |\widehat{u}_j(\tau_j, n_j)| \, d\tau_1 \cdots d\tau_5 \Bigg\|_{\ell^p_n L^{q_0}_\tau} \\
&\lesssim \Bigg\| \sum_{n=n_1+n_2+n_0} \sum_{n_0=n_3+n_4+n_5} \frac{\max_j \langle n_j \rangle^{9\theta} \langle n \rangle^s}{\langle n_1 \rangle \langle n_3 \rangle} \prod_{j=1}^{5} \|\widehat{u}_j(n_j)\|_{L^1_\tau} \Bigg\|_{\ell^p_n}.
\end{aligned}$$

Let $f_1(\tau, n) = \langle n \rangle^s |\widehat{\mathbf{P}_{N_1} u_1}(\tau, n)|$ and $f_j(\tau, n) = \langle n \rangle^{\frac{1}{2}} |\widehat{\mathbf{P}_{N_j} u_j}(\tau, n)|$, $j = 2, \dots, 5$, for dyadic numbers N_j, $j = 1, \dots, 5$. We have that $N_1 \geq N_2$, $N_3 \geq N_4 \geq N_5$, and we will consider two cases: $|n_1| \geq |n_3|$ or $|n_1| < |n_3|$. Assume that $|n_1| \geq |n_3|$. Using Young's and Hölder's inequality, we get

$$\|\mathcal{Q}'(u_1,\dots,u_5)\|_{Z_1^s} \lesssim \sum_{N_1,\dots,N_5} N_1^{9\theta-1}(N_2N_4N_5)^{-\frac{1}{2}}N_3^{-\frac{3}{2}}\Big\|\sum_{n=n_1+\dots+n_5}\|f_j(n_j)\|_{L_\tau^{\frac{1}{2}}}\Big\|_{\ell_n^p}$$
$$\lesssim \sum_{N_1,\dots,N_5} N_1^{9\theta-1}(N_2N_4N_5)^{\frac{1}{2}-\frac{1}{p}+}N_3^{-\frac{1}{2}-\frac{1}{p}+}\|u_1\|_{Y_0^s}\prod_{j=2}^{5}\|u_j\|_{Y_0^{\frac{1}{2}}}.$$

It only remains to sum in the dyadics

$$\sum_{N_1,\dots,N_5} N_1^{9\theta-1}(N_2N_4N_5)^{\frac{1}{2}-\frac{1}{p}+}N_3^{-\frac{1}{2}-\frac{1}{p}+} \lesssim \sum_{N_1,\dots,N_5} N_1^{-\theta}(N_2N_4)^{5\theta-\frac{1}{p}+}(N_3N_5)^{-\frac{1}{p}+} \lesssim 1$$

for $3\delta < \theta < \frac{1}{5p}$. If $|n_1| < |n_3|$, then by following the same approach we obtain

$$\|\mathcal{Q}'(u_1,\dots,u_5)\|_{Z_1^s} \sum_{N_1,\dots,N_5} N_1^{-\frac{1}{2}-\frac{1}{p}+}(N_2N_4N_5)^{\frac{1}{2}-\frac{1}{p}+}N_3^{9\theta-1}\|u_1\|_{Y_0^s}\prod_{j=2}^{5}\|u_j\|_{Y_0^{\frac{1}{2}}}$$

and we sum in the dyadics using the fact that $N_3 \geq N_1 \geq N_2$ and $N_3 \geq N_4 \geq N_5$

$$\sum_{N_1,\dots,N_5} N_1^{-\frac{1}{2}-\frac{1}{p}+}(N_2N_4N_5)^{\frac{1}{2}-\frac{1}{p}+}N_3^{9\theta-1} \lesssim \sum_{N_1,\dots,N_5}(N_1N_2)^{-\frac{1}{p}+}(N_3N_4N_5)^{3\theta-\frac{2}{3p}} \lesssim 1$$

by choosing $3\delta < \theta < \frac{2}{9p}$. □

3.6 Solving the system and extending solutions globally in time

In this section, we use the nonlinear estimates in Section 3.4 to prove Proposition 3.3.1. We can similarly combine the nonlinear estimates in Section 3.5 to prove Proposition 3.3.2, but we omit its proof due to the large number of terms involved. In addition, we show how to apply the a priori estimates in Corollary 2.4.2 to prove global well-posedness of mKdV2 (3.1) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \geq \frac{1}{2}$ and $4 \leq p < \infty$, completing the proof of Theorem 1.1.6.

We start by following a contraction mapping argument to construct the solution $u = u[w]$ of (3.21).

Proof of Proposition 3.3.1. Fix $w \in Z_0^s$ with $\|w\|_{Z_0^s} \leq A_2$ and consider the map

$$\Gamma_w(u) := w + \varphi_T\big[\mathbf{G}_{A,\geq}(w,\overline{u},u) + \mathbf{G}_{A,>}(w,u,\overline{u}) + \mathbf{G}_{B,\geq}(w,\overline{w},u) + \mathbf{G}_{B,>}(w,w,\overline{u})\big],$$

for some $0 < T \leq 1$ to be chosen later. Let $\|u\|_{Y_0^s} \leq A_3$, for some $A_3 > 0$ to be chosen later. Using Lemma 3.1.4, Lemma 3.4.1, and the embedding $Z_0^s \hookrightarrow Y_0^s$, we have

$$\begin{aligned}\|\Gamma_w(u)\|_{Y_0^s} &\leq \|w\|_{Y_0^s} + C_1T^\theta\Big(\|\mathbf{G}_{A,\geq}(w,\overline{u},u)\|_{Y_1^s} + \|\mathbf{G}_{A,\geq}(w,u,\overline{u})\|_{Y_1^s}\\ &\qquad + \|\mathbf{G}_{B,\geq}(w,\overline{w},u)\|_{Y_1^s} + \|\mathbf{G}_{B,>}(w,w,\overline{u})\|_{Y_1^s}\Big)\\ &\leq \|w\|_{Z_0^s} + C_1C_2T^\theta\|w\|_{Z_0^s}\|u\|_{Y_0^{\frac{1}{2}}}\big(\|u\|_{Y_0^{\frac{1}{2}}} + \|w\|_{Z_0^{\frac{1}{2}}}\big)\\ &\leq A_2 + C_1C_2T^\theta A_2A_3(A_2+A_3),\end{aligned}$$

for some $\theta > 0$ and constants $C_1, C_2 > 0$. By choosing $A_3 = 2A_2$ and $C_1C_2T^\theta\frac{3}{4}A_3^2 \leq \frac{1}{2}$, we would obtain that $\|\Gamma_w(u)\|_{Y_0^s} \leq A_3$. Note that we can rewrite the condition on T to see that it only depends on A_2, i.e., we can choose T satisfying $3C_1C_2T^\theta A_2^2 \leq \frac{1}{2}$. Analogously, we can establish the difference estimate as follows

$$\begin{aligned}
\|\Gamma_w(u)-\Gamma_w(v)\|_{Y_0^s} &\le C_1T^\theta\Big(\big\|\mathbf{G}_{A,\ge}(w,\overline{u},u)-\mathbf{G}_{A,\ge}(w,\overline{v},v)\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{A,>}(w,u,\overline{u})-\mathbf{G}_{A,>}(w,v,\overline{v})\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{B,\ge}(w,\overline{w},u)-\mathbf{G}_{B,\ge}(w,\overline{w},v)\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{B,>}(w,w,\overline{u})-\mathbf{G}_{B,>}(w,w,\overline{v})\big\|_{Y_1^s}\Big)\\
&\le C_1T^\theta\Big(\big\|\mathbf{G}_{A,\ge}(w,\overline{u}-\overline{v},u)\big\|_{Y_1^s}+\big\|\mathbf{G}_{A,\ge}(w,\overline{v},u-v)\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{A,>}(w,u-v,\overline{u})\big\|_{Y_1^s}+\big\|\mathbf{G}_{A,>}(w,v,\overline{u}-\overline{v})\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{B,\ge}(w,\overline{w},u-v)\big\|_{Y_1^s}+\big\|\mathbf{G}_{B,>}(w,w,\overline{u}-\overline{v})\big\|_{Y_1^s}\Big)\\
&\le C_1C_2T^\theta\|w\|_{Z_0^s}\big(2\|u\|_{Y_0^{\frac12}}+2\|v\|_{Y_0^{\frac12}}+2\|w\|_{Z_0^{\frac12}}\big)\|u-v\|_{Y_0^{\frac12}}\\
&\le 2C_1C_2T^\theta A_2(2A_3+A_2)\|u-v\|_{Y_0^s},
\end{aligned}$$

for some $\theta>0$. Choosing T such that $10C_1C_2T^\theta A_2^2\le\frac12$, shows that Γ_w is a contraction on the A_3-ball of Y_0^s.

It only remains to show the map $w\mapsto u=u[w]$ is Lipschitz from the A_2-ball in Z_0^s to the A_3-ball in Y_0^s. Let w_1,w_2 belong to the A_2-ball and u_1,u_2 the corresponding fixed points of $\Gamma_{w_1},\Gamma_{w_2}$. Proceeding as before, we obtain

$$\begin{aligned}
\|u_1-u_2\|_{Y_0^s} &\le \|w_1-w_2\|_{Y_0^s}+C_1T^\theta\Big(\big\|\mathbf{G}_{A,\ge}(w_1,\overline{u}_1,u_1)-\mathbf{G}_{A,\ge}(w_2,\overline{u}_2,u_2)\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{A,>}(w_1,u_1,\overline{u}_1)-\mathbf{G}_{A,\ge}(w_2,u_2,\overline{u}_2)\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{B,\ge}(w_1,\overline{w}_1,u_1)-\mathbf{G}_{B,\ge}(w_2,\overline{w}_2,u_2)\big\|_{Y_1^s}\\
&\qquad+\big\|\mathbf{G}_{B,>}(w_1,w_1,\overline{u}_1)-\mathbf{G}_{B,>}(w_2,w_2,\overline{u}_2)\big\|_{Y_1^s}\Big)\\
&\le \|w_1-w_2\|_{Z_0^s}+2C_1C_2T^\theta\|w_1-w_2\|_{Z_0^s}\|u_1\|_{Y_0^s}\big(\|u_1\|_{Y_0^s}+\|w_1\|_{Y_0^s}+\|w_2\|_{Z_0^s}\big)\\
&\qquad+2C_1C_2T^\theta\|u_1-u_2\|_{Y_0^s}\|w_2\|_{Z_0^s}\big(\|u_1\|_{Y_0^s}+\|u_2\|_{Y_0^s}+\|w_2\|_{Z_0^s}\big)\\
&\le \big(1+2C_1C_2T^\theta A_3(2A_2+A_3)\big)\|w_1-w_2\|_{Z_0^s}+2C_1C_2T^\theta A_2(2A_3+A_2)\|u_1-u_2\|_{Y_0^s},
\end{aligned}$$

for some $\theta>0$. Choosing T such that $10C_1C_2T^\theta A_2^2\le\frac12$, we have that

$$\big\|u_1[w_1]-u_2[w_2]\big\|_{Y_0^s}\lesssim\|w_1-w_2\|_{Z_0^s},$$

showing that the map is locally Lipschitz, as intended. □

A similar proof holds for Proposition 3.3.2, by combining the estimates in Section 3.5 with the estimate in Lemma 2.1.1 for the linear solution and Lemma 3.1.4 to gain a small power of T.

It only remains to show that we can extend the solutions of mKdV2 (3.1) with initial data $u_0\in\mathcal{F}L^{s,p}(\mathbb{T})$ for $s\ge\frac12$ and $4\le p<\infty$ globally in time. As in Section 2.4, we will apply the a priori bounds by Oh-Wang [91] when $\frac12\le s<1-\frac1p$ and a persistence of regularity argument for higher regularity. As before, we can extend these a priori bounds to non-smooth solutions.

Corollary 3.6.1. *Let $4\le p<\infty$ and $\frac12\le s<1-\frac1p$. There exists $C=C(p)>0$ such that for any $u_0\in\mathcal{F}L^{s,p}(\mathbb{T})$ we have*

$$\|u\|_{L_T^\infty\mathcal{F}L^{s,p}}\lesssim(1+\|u_0\|_{\mathcal{F}L^{s,p}})^{\frac p2-1}\|u_0\|_{\mathcal{F}L^{s,p}},$$

where $u\in C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ is the corresponding solution of the complex-valued mKdV2 equation (3.1).

In the following, we include a sketch of the proof of global well-posedness in the high integrability case.

Proof of Theorem 1.1.6 when $4 \le p < \infty$. Let $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, $s \ge \frac{1}{2}$ and $4 \le p < \infty$, and consider the corresponding solution u of mKdV2 (3.1) obtained by Theorem 1.1.3. For $\frac{1}{2} \le 1-\frac{1}{p}$, we can globalize solutions by iterating the local well-posedness argument, since Corollary 3.6.1 gives a lower bound for the time of existence at each iteration.

Now consider the case when $s \ge 1 - \frac{1}{p}$. Since $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T}) \subset \mathcal{F}L^{\frac{1}{2},p}(\mathbb{T})$, there exists a unique global-in-time solution $u \in C\big(\mathbb{R}; \mathcal{F}L^{\frac{1}{2},p}(\mathbb{T})\big)$. Proceeding as in Section 2.4, using the a priori bound in Corollary 3.6.1 when running the contraction mapping arguments for u and w on $I = [t_0, t_0 + T]$, imposes a local time of existence $T \sim (1 + \|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}})^{-\alpha}$, for some $\alpha > 0$, and the following bounds

$$\|\tilde{\varphi}_I u\|_{Y_0^{\frac{1}{2}}} \le 2\|\tilde{\varphi}_I w\|_{Z_0^{\frac{1}{2}}} \le 2C\|u(t_0)\|_{\mathcal{F}L^{\frac{1}{2},p}} \le 2\tilde{C}(1 + \|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}})^{\frac{p}{2}-1}\|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}}, \tag{3.41}$$

for some constants $C, \tilde{C} > 0$, where $\tilde{\varphi}_I(t) = \varphi_T(\cdot - t_0)$. Note that the above estimate is uniform in t_0. Now we want to establish an estimate similar to that in (3.41) at higher regularity. Using the nonlinear estimates in Sections 3.4 and 3.5, we have that

$$\begin{aligned}
\|\tilde{\varphi}_I u\|_{Y_0^s} &\le \|w\|_{\tilde{\varphi}_I Z_0^s} + C_1 C_2 T^\theta \|\tilde{\varphi}_I w\|_{Z_0^s} \|\tilde{\varphi}_I u\|_{Y_0^{1/2}} \big(\|\tilde{\varphi}_I u\|_{Y_0^{1/2}} + \|\tilde{\varphi}_I w\|_{Z_0^{1/2}}\big),\\
\|\tilde{\varphi}_I w\|_{Z_0^s} &\le C_3 \|u(t_0)\|_{\mathcal{F}L^{s,p}} + C_1 C_4 T^\theta \big(\|\tilde{\varphi}_I w\|_{Z_0^s} + \|\tilde{\varphi}_I u\|_{Y_0^s}\big)\\
&\qquad \times \Big\{\big(\|\tilde{\varphi}_I w\|_{Z_0^{1/2}} + \|\tilde{\varphi}_I u\|_{Y_0^{1/2}}\big)^2 + \big(\|\tilde{\varphi}_I w\|_{Z_0^{1/2}} + \|\tilde{\varphi}_I u\|_{Y_0^{1/2}}\big)^4\\
&\qquad + \big(\|\tilde{\varphi}_I w\|_{Z_0^{1/2}} + \|\tilde{\varphi}_I u\|_{Y_0^{1/2}}\big)^6\Big\},
\end{aligned}$$

for some $\theta > 0$. Using (3.41), we have

$$\begin{aligned}
\|\tilde{\varphi}_I u\|_{Y_0^s} &\le \|\tilde{\varphi}_I w\|_{Z_0^s} + C_5 T^\theta \|u(t_0)\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \|\tilde{\varphi}_I w\|_{Z_0^s},\\
\|\tilde{\varphi}_I w\|_{Z_0^s} &\le C_3 \|u(t_0)\|_{\mathcal{F}L^{s,p}} + C_6 T^\theta \|u(t_0)\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \big(1 + \|u(t_0)\|_{\mathcal{F}L^{\frac{1}{2},p}}\big)^4 \big(\|\tilde{\varphi}_I w\|_{Z_0^s} + \|\tilde{\varphi}_I u\|_{Y_0^s}\big),
\end{aligned}$$

for $C_5, C_6 > 0$. Using (3.41), we have

$$\begin{aligned}
C_5 T^\theta \|u(t_0)\|^2_{\mathcal{F}L^{\frac{1}{2},p}} &\le C_7 T^\theta (1 + \|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}})^{p-2} \|u_0\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \le 1,\\
C_6 T^\theta \|u(t_0)\|^2_{\mathcal{F}L^{\frac{1}{2},p}} \big(1 + \|u(t_0)\|_{\mathcal{F}L^{\frac{1}{2},p}}\big)^4 &\le C_8 T^\theta (1 + \|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}})^{3p-6} \|u(t_0)\|^6_{\mathcal{F}L^{\frac{1}{2},p}} \le \frac{1}{2},
\end{aligned}$$

where the last inequalities hold by possibly refining the choice of α when choosing the local time of existence T. Substituting these inequalities, we obtain

$$\|\tilde{\varphi}_I u\|_{Y_0^s} \le 2\|\tilde{\varphi}_I w\|_{Z_0^s} \quad \text{and} \quad \|\tilde{\varphi}_I w\|_{Z_0^s} \le 2C_3 \|u(t_0)\|_{\mathcal{F}L^{s,p}},$$

from which we conclude that

$$\sup_{t \in I} \|u(t)\|_{\mathcal{F}L^{s,p}} \le \|\tilde{\varphi}_I u\|_{Y_0^s} \le 4C_3 \|u(t_0)\|_{\mathcal{F}L^{s,p}}.$$

Since the above estimate holds for any $I = [t_0, t_0 + T]$ uniformly in t_0, we can iterate the above argument to conclude that

$$\sup_{t \in [-T_*, T_*]} \|u(t)\|_{\mathcal{F}L^{s,p}} \le (4C_3)^{(1+\|u_0\|_{\mathcal{F}L^{\frac{1}{2},p}})^\alpha T^*} \|u_0\|_{\mathcal{F}L^{s,p}}$$

for any $T^* > 0$. This shows the global well-posedness of mKdV2 (3.1) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \ge \frac{1}{2}$ and $4 \le p < \infty$.

□

Chapter 4

The generalized Korteweg-de Vries equations

In this chapter, we study the Cauchy problem for the periodic generalized Korteweg-de Vries equation (gKdV):

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm\partial_x(u^k), \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{T}, \tag{4.1}$$

for $k \geq 4$. Due to the Hamiltonian structure of gKdV (4.1), we are interested in studying the corresponding Gibbs measure

$$d\mu = Z^{-1}e^{-H(u)}\,du = Z^{-1}e^{\mp\frac{1}{k+1}\int_{\mathbb{T}}u^{k+1}\,dx}e^{-\frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2\,dx}du, \tag{4.2}$$

where the Hamiltonian H is given by

$$H(u) = \frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2\,dx \pm \frac{1}{k+1}\int_{\mathbb{T}}u^{k+1}\,dx.$$

Our goal is to show almost sure global well-posedness of gKdV (4.1) and the invariance of the Gibbs measure μ under the gKdV dynamics, i.e.,

$$\mu\big(\Psi(-t)A\big) = \mu(A)$$

for all $t \in \mathbb{R}$ and $A \subset L^2(\mathbb{T})$ measurable, where Ψ denotes the data-to-solution map of (4.1). To this end, we will apply Bourgain's invariant measure argument to complete the program initiated in [11] on the invariance of the Gibbs measures for the gKdV equations. There, Bourgain focused on the mKdV equation ($k = 3$ in (4.1)) and exploited the invariance of the Gibbs measure associated with the truncated dynamics to globalize solutions of the original equation. The main difficulty resides in showing local well-posedness in the support of the Gibbs measure μ. In the absence of suitable conservation laws, the invariance of the Gibbs measure is used as a substitute in the globalization argument.

We start by studying the gauged gKdV equation ($\mathcal{G}$-gKdV):

$$\partial_t u + \partial_x^3 u = \pm\partial_x\big(u^k - k\mathbf{P}_0(u^{k-1})u\big), \tag{4.3}$$

where $\mathbf{P}_0$ denotes the mean $\mathbf{P}_0(f) = \fint_{\mathbb{T}} f(x)\,dx$. The equations (4.1) and (4.3) are related through the following gauge transform

$$\mathcal{G}[u](t,x) = u\Big(t, x \mp k\int_0^t \mathbf{P}_0\big(u^{k-1}(t')\big)\,dt'\Big). \tag{4.4}$$

In fact, u is a solution of gKdV (4.1) if and only if $\mathcal{G}[u]$ is a solution of $\mathcal{G}$-gKdV (4.3). The effect of the gauge transform (4.4) is to remove certain resonant frequency interactions from the nonlinearity, allowing us to establish the main nonlinear estimates to obtain local well-posedness through the Fourier-restriction norm method. As in Chapters 2 and 3, we will use the Fourier restriction spaces adapted to the Fourier-Lebesgue setting:

$$Z_p^{s,\frac{1}{2}} = X_{p,2}^{s,\frac{1}{2}} \cap X_{p,1}^{s,0} \hookrightarrow C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big).$$

In Section 4.1, we introduce preliminary estimates on the phase function, needed to guide our case separation for the nonlinear estimates, which are established in Section 4.2. We start by decomposing our nonlinearity into resonant and non-resonant contributions, based on our analysis of the phase function. In Section 4.2.1, we prove multilinear Strichartz estimates adapted to the Fourier-Lebesgue setting, which are essential for the proof of the nonlinear estimates. We then prove estimates for the resonant and non-resonant contributions (Sections 4.2.2 and 4.2.3, respectively) from which the local well-posedness of $\mathcal{G}$-gKdV (4.3) follows. From this result and by inverting the gauge transform, we obtain local well-posedness of gKdV (4.1) for $k \geq 4$ in the Fourier-Lebesgue spaces which include the support of the Gibbs measure μ, obtaining Theorem 1.2.2. We omit this proof as it follows the strategy in Section 2.3 for the mKdV equation.

Section 4.3 is dedicated to Bourgain's invariant measure argument. We first establish almost sure global well-posedness and invariance of the Gibbs measure for $\mathcal{G}$-gKdV (4.3) (Theorem 1.2.4), and then extend this result to the original dynamics (4.1) proving Theorem 1.2.5. Lastly, in Section 4.4, we establish some results on the gauge transform and the solution map of gKdV (4.1) which are missing from the literature.

4.1 The phase function

Recall the phase function $\phi_k(n, n_1, \ldots, n_k) = n^3 - n_1^3 - \ldots - n_k^3$, which we will denote by ϕ for simplicity. When $k = 2$ (KdV) or $k = 3$ (mKdV), the phase function restricted to $n = n_1 + \ldots + n_k$ satisfies the following factorizations

$$(n_1 + n_2)^3 - n_1^3 - n_2^3 = 3(n_1 + n_2)n_1 n_2, \tag{4.5}$$

$$(n_1 + n_2 + n_3)^3 - n_1^3 - n_2^3 - n_3^3 = 3(n_1 + n_2)(n_1 + n_3)(n_2 + n_3). \tag{4.6}$$

Unfortunately, such a factorization no longer holds for $k \geq 4$. We recall the well-known upper bound for ϕ.

Lemma 4.1.1 ([29, Lemma 4.1])**.** *If $|n_1| \geq \ldots \geq |n_k|$ and $n_1 + \ldots + n_k = 0$, then*

$$|n_1^3 + \ldots + n_k^3| \lesssim |n_1 n_2 n_3|.$$

In order to exploit the multilinear dispersion to establish the main nonlinear estimates for the local well-posedness, we want to gain a better understanding of the phase function, and the above upper bound is insufficient. In particular, we want to identify suitable non-resonant regions where ϕ is large enough to control the derivative loss in the nonlinearity, while imposing strong restrictions to the frequencies in the remaining resonant regions. The following lemma gives us additional information on the phase function ϕ.

Lemma 4.1.2. *Let $k \geq 4$, $n = n_1 + \ldots + n_k$ and $|n_1| \geq \ldots \geq |n_k| > 0$.*

A. If $|n| \sim |n_1| \gg |n_2|$, $n \neq n_1$, then one of the following holds

- *A.1. $|n_1|^2 |n - n_1| \lesssim |\phi|$;*
- *A.2. $|n_1|^2 |n - n_1| \lesssim |n_2 n_3 n_4|$.*

B. If $|n_1| \sim |n_2| \gg |n_3|$, $n \neq n_1$, $n \neq n_2$, $n_1 + n_2 \neq 0$, then one of the following holds

- *B.1. $|n_1|^2 |n_1 + n_2| \lesssim |\phi|$;*
- *B.2. $|n_1 + n_2| \ll |n_4|$.*

Proof. We start by proving A. Assume that $|n_1|^2|n-n_1| \gg \max(|\phi|, |n_2n_3n_4|)$. Using (4.5), we can rewrite ϕ as follows

$$\phi = 3nn_1(n-n_1) + (n-n_1)^3 - n_2^3 - \ldots - n_k^3.$$

Since $|nn_1(n-n_1)| \sim |n_1|^2|n-n_1|$ and using Lemma 4.1.1, we have

$$|n_1|^2|n-n_1| \sim |(n-n_1)^3 - n_2^3 - \ldots - n_k^3| \lesssim |n_2n_3| \max(|n-n_1|, |n_4|).$$

From the above estimate, we must have $|n_1|^2|n-n_1| \lesssim |n_2n_3n_4|$ which contradicts our initial assumption. To prove B, assume that $|n_1|^2|n_1+n_2| \gg |\phi|$ and $|n_1+n_2| \gtrsim |n_4|$. Using (4.6), we can rewrite ϕ as follows

$$\phi = 3(n-n_1)(n-n_2)(n_1+n_2) + (n_3+\ldots+n_k)^3 - n_3^3 - \ldots - n_k^3.$$

Since $|(n-n_1)(n-n_2)(n_1+n_2)| \sim |n_1|^2|n_1+n_2|$, using Lemma 4.1.1, we have

$$|n_1|^2|n_1+n_2| \sim |(n_3+\ldots+n_k)^3 - n_3^3 - \ldots - n_k^3| \lesssim |n_3|^2|n_4|.$$

From the above estimate, since $|n_1+n_2| \gtrsim |n_4|$, we must have $|n_1| \lesssim |n_3|$ which contradicts out assumptions on the frequencies. □

4.2 Nonlinear estimates

In this section, we state and prove the nonlinear estimates needed to show the local well-posedness of $\mathcal{G}$-gKdV (4.3) in Theorem 1.1.3. We will establish a nonlinear estimate for the more general multilinear operator

$$\mathcal{N}(u_0,\ldots,u_m) = \mathbf{P}(u_1\ldots u_m)\partial_x u_0 - \sum_{j=1}^m \mathbf{P}_0(u_j\partial_x u_0)\prod_{\substack{i=1\\ i\neq j}}^m u_i, \tag{4.7}$$

where $m = k-1 \geq 3$ and $\mathbf{P} = \mathrm{Id} - \mathbf{P}_0$. Note that $\pm k\mathcal{N}(u,\ldots,u)$ coincides with the nonlinearity of $\mathcal{G}$-gKdV (4.3) and that the quantities subtracted on the right-hand side of (4.7) effectively remove certain resonant frequency interactions. In fact, the spatial Fourier transform of $\mathcal{N}(u_0,\ldots,u_m)$ at n, omitting time dependence, is given by

$$\sum_{\substack{n=n_0+\ldots+n_m\\ nn_0\cdots n_m\neq 0}} \Big(1 - \mathbb{1}_{\{n=n_0\}} - \sum_{j=1}^m \mathbb{1}_{\{n_0+n_j=0\}}\Big) in_0\widehat{u}_0(n_0)\cdots\widehat{u}_m(n_m).$$

The main difficulty in estimating $\mathcal{N}(u_0,\ldots,u_m)$ lies in controlling the derivative. To that end, we want to exploit the multilinear dispersion through the phase function ϕ and use Lemma 4.1.2 to guide our case separation in the nonlinearity. Due to the restrictions in Lemma 4.1.2, consider the following resonant regions in frequency space:

$$A_j(n) = \begin{cases} \{(n_0,\ldots,n_m)\in\mathbb{Z}_*^{m+1} : \ n=n_j\}, & j=0,\ldots,m,\\ \{(n_0,\ldots,n_m)\in\mathbb{Z}_*^{m+1} : \ n_0+n_\ell=0\}, & -j=\ell=1,\ldots,m, \end{cases}$$

where $\mathbb{Z}_* = \mathbb{Z}\setminus\{0\}$. Note that the terms on the right-hand side of (4.7) are localized to A_0 and A_{-j}, $j=1,\ldots,m$, respectively. We are further interested in removing the resonances associated with the sets A_j, $j=1,\ldots,m$, when defining the non-resonant contribution $\mathcal{N}_0$. We then decompose the nonlinearity as $\mathcal{N} = \mathcal{N}_0 + \mathcal{R}$, where the non-resonant and resonant contributions are respectively defined as

$$\mathcal{F}_x\big(\mathcal{N}_0(u_0,\ldots,u_m)\big)(n) = \sum_{\substack{n=n_0+\ldots+n_m\\ nn_0\cdots n_m\neq 0}} \mathbb{1}_{\bigcap_{j=-m}^m A_j^c} in_0\widehat{u}_0(n_0)\cdots\widehat{u}_m(n_m), \tag{4.8}$$

$$\mathcal{F}_x\big(\mathcal{R}(u_0,\dots,u_m)\big)(n) = \sum_{\substack{n=n_0+\ldots+n_m\\ nn_0\cdots n_m\neq 0}} \Big[\sum_{J\in\mathcal{C}}(-1)^{|J|+1}\mathbb{1}_{\bigcap_{j\in J}A_j}\Big] in_0\widehat{u}_0(n_0)\cdots\widehat{u}_m(n_m), \tag{4.9}$$

where $J\in\mathcal{C}$ if $J=\{j\}$, $j=1,\dots,m$, or $J\subset\{-m,\dots,m\}$ and $|J|\geq 2$.

The following proposition states the main nonlinear estimates, from which the local well-posedness of $\mathcal{G}$-gKdV (4.3) follows.

Proposition 4.2.1. *For $2<p<\infty$ there exists $\frac{1}{2}<s_*(p)<1-\frac{1}{p}$ such that for any $s>s_*(p)$ the following estimates hold*

$$\|\mathcal{N}_0(u_0,\dots,u_m)\|_{Z_p^{s,-\frac12}(T)} \lesssim T^\delta\prod_{j=0}^m\|u_j\|_{Z_p^{s,\frac12}(T)},$$
$$\|\mathcal{R}(u_0,\dots,u_m)\|_{Z_p^{s,-\frac12}(T)} \lesssim T^\delta\prod_{j=0}^m\|u_j\|_{Z_p^{s,\frac12}(T)},$$

for some $0<\delta<1$ and any $0<T\leq 1$.

Remark 4.2.2. It will suffice to show the above estimates for $v_0,\dots,v_m$ extensions of $u_0,\dots,u_m$ on $[-T,T]$. Consequently, in the remaining of this section we will show the estimates in $Z_p^{s,-\frac12}$ and $Z_p^{s,\frac12}$, instead of the time localized versions $Z_p^{s,-\frac12}(T)$ and $Z_p^{s,\frac12}(T)$. Moreover, we will establish stronger estimates which allow us to gain a small power of T by applying Lemma 2.1.2.

4.2.1 Bilinear and trilinear Strichartz estimates

In order to show Proposition 4.2.1, we first establish bilinear and trilinear Strichartz estimates adapted to the Fourier-Lebesgue setting. Recall that $\mathbf{P}_0(f)=\fint_{\mathbb{T}} f\,dx$ denotes the mean and $\mathbf{P}=\mathrm{Id}-\mathbf{P}_0$ the projection onto mean zero functions. The following lemma generalizes the periodic L^4-Strichartz of Bourgain in [10] to the Fourier-Lebesgue setting.

Lemma 4.2.3. *The following estimate holds for any $2\leq p\leq\infty$ and $b>\max\big(\frac13,\frac{3p-2}{8p}\big)$*

$$\|\mathbf{P}\big(\mathbf{P}u_1\cdot\mathbf{P}u_2\big)\|_{X^{0,0}_{p,2}} \lesssim \|u_1\|_{X^{0,b}_{p,2}}\|u_2\|_{X^{0,b}_{2,2}}. \tag{4.10}$$

Proof. The proof is adapted from the standard bilinear argument for $p=2$ by Nikolay Tzvetkov (see [102, Proposition 2.13], for instance). Let $M_1,M_2\geq 1$ denote dyadic numbers, $\mathbf{P}_{M_j}$ the projection onto space-time frequencies $\{\langle\tau-n^3\rangle\sim M_j\}$ and $u_{M_1}=\mathbf{P}_{M_1}u_1$, $v_{M_2}=\mathbf{P}_{M_2}u_2$. Since

$$\big\|\mathbf{P}\big(\mathbf{P}u_1\cdot\mathbf{P}u_2\big)\big\|_{X^{0,0}_{p,2}} \leq \sum_{M_1,M_2}\big\|\mathbf{P}\big(\mathbf{P}u_{M_1}\cdot\mathbf{P}v_{M_2}\big)\big\|_{X^{0,0}_{p,2}},$$

it suffices to show that

$$\big\|\mathbf{P}\big(\mathbf{P}u_{M_1}\cdot\mathbf{P}v_{M_2}\big)\big\|_{X^{0,0}_{p,2}} \lesssim M_1^bM_2^b\|u_{M_1}\|_{X^{0,0}_{p,2}}\|v_{M_2}\|_{X^{0,0}_{2,2}}. \tag{4.11}$$

for any $b>\max\big(\frac13,\frac{3p-2}{8p}\big)$. We assume $M_1\leq M_2$, while the same proof applies to the other case. Using Hölder's inequality, we get

$$\big\|\mathbf{P}\big(\mathbf{P}u_{M_1}\cdot\mathbf{P}v_{M_2}\big)\big\|_{X^{0,0}_{p,2}} \lesssim \Big\|M_1^{\frac1q}\big|A(\tau,n)\big|^{\frac1q}\Big[|\widehat{u}_{M_1}|^{q'}*_{\tau,n}|\widehat{v}_{M_2}|^{q'}\Big]^{\frac{1}{q'}}\Big\|_{\ell^p_nL^2_\tau}, \tag{4.12}$$

for $q>1$, where

$$A(\tau,n)=\big\{n_1\in\mathbb{Z}_*:\ 0\neq 3nn_1(n-n_1)=-\tau+n^3+\mathcal{O}(M_2)\big\}.$$

Since $n_1(n-n_1) = -\big(n_1 - \frac{n}{2}\big)^2 + \frac{n^2}{4}$, we can rewrite the set above as

$$A(\tau,n) = \bigg\{ n_1 \in \mathbb{Z}_* : \ \Big(n_1 - \frac{n}{2}\Big)^2 = \frac{1}{3n}\Big(\tau - \frac{n^3}{4}\Big) + \mathcal{O}\Big(\frac{M_2}{3|n|}\Big) \bigg\},$$

then we conclude that there are at most $\mathcal{O}\big(1 + (\frac{M_2}{|n|})^{\frac{1}{2}}\big)$ elements in $A(\tau,n)$. We first consider the case when $|n| > M_2$ and thus $|A(\tau,n)| \lesssim 1$. From (4.12) with $q = 2$ and Young's inequality, we have

$$\begin{aligned} \text{RHS of (4.12)} &\lesssim M_1^{\frac{1}{2}} \big\| |\widehat{u}_{M_1}|^2 * |\widehat{v}_{M_2}|^2 \big\|^{\frac{1}{2}}_{\ell_n^{\frac{p}{2}} L_\tau^1} \\ &\le M_1^{\frac{1}{2}} \big\| |\widehat{u}_{M_1}|^2 \big\|^{\frac{1}{2}}_{\ell_n^{\frac{p}{2}} L_\tau^1} \big\| |\widehat{v}_{M_2}|^2 \big\|^{\frac{1}{2}}_{\ell_n^1 L_\tau^1} \\ &= M_1^{\frac{1}{2}} \big\| \widehat{u}_{M_1} \big\|_{\ell_n^p L_\tau^2} \big\| \widehat{v}_{M_2} \big\|_{\ell_n^2 L_\tau^2}, \end{aligned}$$

which implies (4.11), since $M_1 \le M_2$ and $\frac{1}{4} \le \max\big(\frac{1}{3}, \frac{3p-2}{8p}\big)$. For the case when $|n| \le M_2$ and $|A(\tau,n)| \lesssim \big(\frac{M_2}{|n|}\big)^{\frac{1}{2}}$, we set

$$\Big(\frac{1}{q}, \frac{1}{r}\Big) = \begin{cases} \big(\frac{1}{3} + \varepsilon, \frac{1}{p} - \frac{1}{6} + \varepsilon\big) & \text{for } 2 \le p \le 6, \\ \big(\frac{1}{2} - \frac{1}{p}, 0\big) & \text{for } 6 < p \le \infty, \end{cases}$$

with sufficiently small $\varepsilon > 0$. Note that

$$\frac{1}{p} - \frac{1}{r} < \frac{1}{2q}, \quad \frac{q'}{r} + 1 = \frac{q'}{p} + \frac{q'}{2}, \quad 1 < q' \le 2 \le p \le r \le \infty.$$

Applying first Hölder's inequality in n, then following the above computation, and using Hölder's inequality in τ, we have

$$\begin{aligned} \text{RHS of (4.12)} &\lesssim M_1^{\frac{1}{q}} M_2^{\frac{1}{2q}} \Big\| |n|^{-\frac{1}{2q}} \Big[|\widehat{u}_{M_1}|^{q'} * |\widehat{v}_{M_2}|^{q'} \Big]^{\frac{1}{q'}} \Big\|_{\ell_n^p L_\tau^2} \\ &\lesssim M_1^{\frac{1}{q}} M_2^{\frac{1}{2q}} \Big\| \Big[|\widehat{u}_{M_1}|^{q'} * |\widehat{v}_{M_2}|^{q'} \Big]^{\frac{1}{q'}} \Big\|_{\ell_n^r L_\tau^2} \\ &\le M_1^{\frac{1}{q}} M_2^{\frac{1}{2q}} \big\| \widehat{u}_{M_1} \big\|_{\ell_n^p L_\tau^{q'}} \big\| \widehat{v}_{M_2} \big\|_{\ell_n^2 L_\tau^2} \\ &\lesssim M_1^{\frac{1}{2}} M_2^{\frac{1}{2q}} \big\| \widehat{u}_{M_1} \big\|_{\ell_n^p L_\tau^2} \big\| \widehat{v}_{M_2} \big\|_{\ell_n^2 L_\tau^2}. \end{aligned}$$

Since $M_1 \le M_2$ and $\frac{1}{2} + \frac{1}{2q} \le 2\max\big(\frac{1}{3}, \frac{3p-2}{8p}\big) + \frac{\varepsilon}{2}$, we obtain (4.11), from which the estimate follows. $\square$

We can then establish the following estimate.

Lemma 4.2.4. *The following estimate holds for any $2 \le p \le \infty$*

$$\big\| \mathbf{P}\big(\mathbf{P}u_1 \cdot \mathbf{P}u_2\big) \big\|_{X_{2,2}^{0,-\frac{1}{2}+}} \lesssim \|u_1\|_{X_{p,2}^{0,\frac{1}{2}-}} \|u_2\|_{X_{p',2}^{0,0}}. \tag{4.13}$$

Proof. Using duality, Hölder's inequality, and (4.10) we obtain

$$\begin{aligned} \big\| \mathbf{P}\big(\mathbf{P}u_1 \cdot \mathbf{P}u_2\big) \big\|_{X_{2,2}^{0,-\frac{1}{2}+}} &\lesssim \sup_{\|u_3\|_{X_{2,2}^{0,\frac{1}{2}-}} \le 1} \bigg| \int_{0=\tau_1+\tau_2+\tau_3} \sum_{\substack{0=n_1+n_2+n_3 \\ n_1 n_2 n_3 \neq 0}} \widehat{u}_1(\tau_1,n_1)\widehat{u}_2(\tau_2,n_2)\widehat{u}_3(\tau_3,n_3) \bigg| \\ &\le \sup_{\|u_3\|_{X_{2,2}^{0,\frac{1}{2}-}} \le 1} \|u_2\|_{X_{p',2}^{0,0}} \big\| \mathbf{P}\big(\mathbf{P}u_1 \cdot \mathbf{P}u_3\big) \big\|_{X_{p,2}^{0,0}} \\ &\lesssim \sup_{\|u_3\|_{X_{2,2}^{0,\frac{1}{2}-}} \le 1} \|u_2\|_{X_{p',2}^{0,0}} \|u_1\|_{X_{p,2}^{0,\frac{1}{2}-}} \|u_3\|_{X_{2,2}^{0,\frac{1}{2}-}} \end{aligned}$$

$$\lesssim \|u_1\|_{X^{0,\frac{1}{2}-}_{p,2}} \|u_2\|_{X^{0,0}_{p',2}},$$

as intended. ☐

Lemma 4.2.5. *The following estimate holds for any $1 \leq p, q \leq \infty$*

$$\|\mathbf{P}_0(u_1 u_2) u_3\|_{X^{0,0}_{p,2}} \lesssim \|u_1\|_{X^{0,\frac{1}{3}+}_{q,2}} \|u_2\|_{X^{0,\frac{1}{3}+}_{q',2}} \|u_3\|_{X^{0,\frac{1}{3}+}_{p,2}}. \tag{4.14}$$

Proof. By Young's and Hölder's inequalities, it follows that

$$\begin{aligned}
\|\mathbf{P}_0(u_1 u_2) u_3\|_{X^{0,0}_{p,2}} &\lesssim \Big\| \sum_{n_1} \int_{\tau=\tau_1+\tau_2+\tau_3} \widehat{u}_1(\tau_1, n_1)\widehat{u}_2(\tau_2, -n_1)\widehat{u}_3(\tau_3, n) \Big\|_{\ell^p_n L^2_\tau} \\
&\lesssim \|u_1\|_{X^{0,0}_{q,\frac{6}{5}}} \|u_2\|_{X^{0,0}_{q',\frac{6}{5}}} \|u_3\|_{X^{0,0}_{p,\frac{6}{5}}} \\
&\lesssim \|u_1\|_{X^{0,\frac{1}{3}+}_{q,2}} \|u_2\|_{X^{0,\frac{1}{3}+}_{q',2}} \|u_3\|_{X^{0,\frac{1}{3}+}_{p,2}}
\end{aligned}$$

for any $1 \leq q \leq \infty$. ☐

The following trilinear estimate can be seen as a multilinear analogue of Bourgain's L^6-Strichartz in [10] adapted to the Fourier-Lebesgue spaces.

Lemma 4.2.6. *Let $2 \leq p \leq \infty$, n_j denote the spatial frequency corresponding to $\widehat{u}_j$, $j = 1, 2, 3$, and assume that $(n_1 + n_2)(n_1 + n_3)(n_2 + n_3) \neq 0$. We have the following estimate*

$$\|u_1 u_2 u_3\|_{X^{0,0}_{p,2}} \lesssim \|u_1\|_{X^{0+,\frac{1}{2}}_{p,2}} \|u_2\|_{X^{0+,\frac{1}{2}}_{2,2}} \|u_3\|_{X^{0+,\frac{1}{2}}_{2,2}}. \tag{4.15}$$

Proof. Let $\phi = 3(n_1 + n_2)(n_1 + n_3)(n_2 + n_3)$. Note that by using Cauchy-Schwarz inequality and Lemma 2.1.4, we have

$$\begin{aligned}
\|u_1 u_2 u_3\|_{X^{0,0}_{p,2}} &= \Big\| \sum_{n=n_1+n_2+n_3} \int_{\tau=\tau_1+\tau_2+\tau_3} \prod_{j=1}^{3} \widehat{u}_j(\tau_j, n_j) \Big\|_{\ell^p_n L^2_\tau} \\
&\lesssim \Big\| \sum_{n=n_1+n_2+n_3} \frac{1}{\langle \tau - n^3 + \phi \rangle^{\frac{1}{2}(1-\varepsilon)}} \Big\| \prod_{j=1}^{3} \langle \tau_j - n_j^3 \rangle^{\frac{1}{2}} \widehat{u}_j(n_j, \tau_j) \Big\|_{L^2_{\tau_2} L^2_{\tau_3}} \Big\|_{\ell^p_n L^2_\tau},
\end{aligned}$$

for any $\varepsilon > 0$. Since $\langle x + y \rangle \lesssim \langle x \rangle \langle y \rangle$ for any x, y, we have the following

$$\frac{1}{\langle n_1 \rangle^{2\theta} \langle n_2 \rangle^{2\theta} \langle n_3 \rangle^{2\theta}} = \frac{1}{\langle n - n_2 - n_3 \rangle^{2\theta} \langle n_2 \rangle^{2\theta} \langle n_3 \rangle^{2\theta}} \lesssim \frac{1}{\langle n_2 + n_3 \rangle^{\theta} \langle n - n_2 \rangle^{\theta} \langle n - n_3 \rangle^{\theta}}.$$

for $\theta > 0$. Letting $\theta = 2\varepsilon$ and using Cauchy-Schwarz inequality, we obtain

$$\|u_1 u_2 u_3\|_{X^{0,0}_{p,2}} \lesssim \sup_{\tau, n} \big(\mathrm{I}(\tau, n)\big)^{\frac{1}{2}} \Big\| \prod_{j=1}^{3} \langle n_j \rangle^{4\varepsilon} \langle \tau_j - n_j^3 \rangle^{\frac{1}{2}} \widehat{u}_j(n_j, \tau_j) \Big\|_{\ell^p_n L^2_\tau \ell^2_{n_2} \ell^2_{n_3} L^2_{\tau_2} L^2_{\tau_3}},$$

where

$$\mathrm{I}(\tau, n) = \sum_{n_2, n_3} \frac{1}{\langle n_2 + n_3 \rangle^{4\varepsilon} \langle n - n_2 \rangle^{4\varepsilon} \langle n - n_3 \rangle^{4\varepsilon} \langle \tau - n^3 + \phi \rangle^{1-\varepsilon}}.$$

The estimate follows from Minkowski's inequality and showing a uniform bound on $\mathrm{I}(\tau, n)$. Let

$a = \tau - n^3$ fixed, then we can rewrite $\mathrm{I}(\tau, n)$ and estimate it as follows

$$
\begin{aligned}
\mathrm{I}(\tau, n) &\lesssim \sum_{n_2,n_3} \frac{1}{\langle n_2+n_3\rangle^{4\varepsilon}\langle n-n_2\rangle^{4\varepsilon}\langle n-n_3\rangle^{4\varepsilon}\langle a+\phi\rangle^{1-\varepsilon}} \\
&= \sum_{\substack{\xi_1,\xi_2\neq 0\\ 2n-\xi_1-\xi_2\neq 0}} \frac{1}{\langle 2n-\xi_1-\xi_2\rangle^{4\varepsilon}\langle \xi_1\rangle^{4\varepsilon}\langle \xi_2\rangle^{4\varepsilon}\langle a+3(2n-\xi_1-\xi_2)\xi_1\xi_2\rangle^{1-\varepsilon}} \\
&\lesssim \sum_{r\neq 0} \sum_{\substack{(\xi_1,\xi_2)\\ r=(2n-\xi_1-\xi_2)\xi_1\xi_2}} \frac{1}{\langle r\rangle^{4\varepsilon}\langle a+3r\rangle^{1-\varepsilon}} \\
&\lesssim \sum_{r\neq 0} \frac{1}{\langle r\rangle^{4\varepsilon}\langle a+3r\rangle^{1-\varepsilon}} \left|\left\{(\xi_1,\xi_2)\in\mathbb{Z}^2_* : \ r=(2n-\xi_1-\xi_2)\xi_1\xi_2\right\}\right| \\
&\lesssim \sum_{r\neq 0} \frac{1}{\langle r\rangle^{4\varepsilon}\langle a+3r\rangle^{1-\varepsilon}} |r|^{\varepsilon'},
\end{aligned}
$$

from the standard divisor counting estimate (Lemma 3.1.6), for any $\varepsilon' > 0$. Choosing $\varepsilon' \leq 2\varepsilon$ gives

$$
\mathrm{I}(\tau, n) \lesssim \sum_{r\neq 0} \frac{1}{\langle r\rangle^{2\varepsilon}\langle a+3r\rangle^{1-\varepsilon}}.
$$

Let $a \in \mathbb{R}$. Then we can write $a = 3m + b$, where $m \in \mathbb{Z}$ and $b \in [0, 3)$. Then, it follows that

$$
\begin{aligned}
&\sum_{r\neq 0} \frac{1}{\langle r\rangle^{2\varepsilon}\langle a+3r\rangle^{1-\varepsilon}} \\
&= \sum_{r\neq 0} \frac{1}{\langle r\rangle^{2\varepsilon}\langle 3m+3r+b\rangle^{1-\varepsilon}} \\
&\leq \sum_{|m+r|\geq 2} \frac{1}{\langle r\rangle^{2\varepsilon}\langle 3m+3r+b\rangle^{1-\varepsilon}} \\
&\qquad + \frac{1}{\langle m-1\rangle^{2\varepsilon}\langle 3+b\rangle^{1-\varepsilon}} + \frac{1}{\langle m+1\rangle^{2\varepsilon}\langle -3+b\rangle^{1-\varepsilon}} + \frac{1}{\langle m\rangle^{2\varepsilon}\langle b\rangle^{1-\varepsilon}} \\
&\leq 3 + \sum_{r} \frac{1}{\langle r\rangle^{2\varepsilon}\langle m+r\rangle^{1-\varepsilon}},
\end{aligned}
$$

since $2|m+r| \geq 4 \geq |b| \implies |3(m+r)+b| \geq 3|(m+r)| - |b| \geq |m+r|$. Consequently, from Lemma 2.1.5, we get $\mathrm{I}(\tau, n) \lesssim 1$ uniformly, from which the estimate follows.

□

Lastly, we include a lemma which will not be required to establish the nonlinear estimates in Proposition 4.2.1 but which we have established while considering this problem.

Lemma 4.2.7. *The following estimates hold for $2 \leq p < \infty$ and any $0 < \varepsilon'(\varepsilon), \varepsilon \ll 1$*

$$
\begin{aligned}
\|u_1u_2u_3\|_{X^{0,-\frac12+\varepsilon'}_{p,2}} &\lesssim \min\left(\|u_1\|_{X^{0,\frac12}_{p,2}}\|u_2\|_{X^{0,\frac12}_{2,2}}, \|u_1\|_{X^{0,\frac12}_{2,2}}\|u_2\|_{X^{0,\frac12}_{p,2}}\right)\|u_3\|_{X^{0,\varepsilon}_{p',2}}, \\
\|u_1u_2u_3\|_{X^{0,-\frac12+\varepsilon'}_{p,2}} &\lesssim \|u_1\|_{X^{0,\frac12}_{2,2}}\|u_2\|_{X^{0,\frac12}_{2,2}}\|u_3\|_{X^{0,\varepsilon}_{2,2}},
\end{aligned}
$$

where $|\sigma_3| \gtrsim |\sigma|, |\sigma_1|, |\sigma_2|$ and $(n_1+n_2)(n_1+n_3)(n_2+n_3) \neq 0$, for σ_j and n_j the modulation and spatial frequency associated with u_j, respectively, $j = 1, 2, 3$.

Proof. Let $f_j(\tau, n) = \langle \tau - n^3\rangle^{\frac12}|\widehat{u}_j(\tau, n)|$, $j = 1, 2$, $f_3(\tau, n) = \langle \tau - n^3\rangle^{\varepsilon}|\widehat{u}_3(\tau, n)|$ and $g \in \ell^{p'}_n L^2_\tau$. From the assumption on the modulations, we have $0 \neq |\phi| = |3(n_1+n_2)(n_1+n_3)(n_2+n_3)| =$

$|\sigma - \sigma_1 - \sigma_2 - \sigma_3| \lesssim |\sigma_3|$. Using duality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\|u_1u_2u_3\|_{X^{0,-\frac{1}{2}+\varepsilon'}_{p,2}} \\
&\quad\lesssim \sum_{n,n=n_1+n_2+n_3} \int_{\tau,\tau=\tau_1+\tau_2+\tau_3} \frac{1}{\langle\sigma\rangle^{\frac{1}{2}-\varepsilon'}\langle\sigma_1\rangle^{\frac{1}{2}}\langle\sigma_2\rangle^{\frac{1}{2}}\langle\phi\rangle^{\varepsilon}} g(\tau,n)\prod_{j=1}^{3} f_j(\tau_j,n_j) \\
&\quad\lesssim \sum_{n,n=n_1+n_2+n_3} \int_{\tau_3} \frac{f_3(\tau_3,n_3)}{\langle\phi\rangle^{\varepsilon}\langle\tau_3-n_3^3+\phi\rangle^{\frac{1}{2}-\varepsilon'}} \bigg(\int_{\tau_1,\tau_2} |f_1f_2g|^2\bigg)^{\frac{1}{2}} \\
&\quad\lesssim \sum_{n_3}\int_{\tau_3} f_3(\tau_3,n_3)\bigg(\sum_{n_1,n_2}\frac{1}{\langle\phi\rangle^{2\varepsilon}\langle\tau_3-n_3^3+\phi\rangle^{1-2\varepsilon'}}\bigg)^{\frac{1}{2}}\bigg(\sum_{n_1,n_2}\int_{\tau_1,\tau_2}|f_1f_2g|^2\bigg)^{\frac{1}{2}}
\end{aligned}$$

from Lemma 2.1.4. Choosing $2\varepsilon' = \varepsilon$ and with $a = \tau_3 - n_3^3$, we have

$$\sum_{n_1,n_2}\frac{1}{\langle a-\phi\rangle^{1-\varepsilon}\langle\phi\rangle^{2\varepsilon}} \lesssim \sum_{r\neq 0}\sum_{\substack{n_1,n_2\\(n_1+n_2)(n-n_1)(n-n_2)=r}}\frac{1}{\langle a-3r\rangle^{1-\varepsilon}\langle r\rangle^{2\varepsilon}} \lesssim \sum_{r\neq 0}\frac{|r|^{\varepsilon''}}{\langle a-3r\rangle^{1-\varepsilon}\langle r\rangle^{2\varepsilon}},$$

by applying the standard divisor counting estimate (Lemma 3.1.6), for any $0 < \varepsilon'' \ll 1$, which we will choose to be $\varepsilon'' < \varepsilon$. Since we can write $a = 3m + b$ for some $m \in \mathbb{Z}$ and $b \in [0,3)$ and $\langle a-3r\rangle = \langle b+3(m-r)\rangle \sim \langle m-r\rangle$, the sum in r converges by applying Lemma 2.1.5. Using the above estimate and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|u_1u_2u_3\|_{X^{0,-\frac{1}{2}+\varepsilon'}_{p,2}} &\lesssim \sum_{n_3}\int_{\tau_3} f_3\bigg(\sum_{n_1,n_2}\int_{\tau_1,\tau_2}|f_1f_2g|^2\bigg)^{\frac{1}{2}} \\
&\lesssim \sum_{n_3}\|f_3(n_3)\|_{L^2_\tau}\bigg(\sum_{n_1,n_2}\|f_1(n_1)\|^2_{L^2_\tau}\|f_2(n_2)\|^2_{L^2_\tau}\|g(n_1+n_2+n_3)\|^2_{L^2_\tau}\bigg)^{\frac{1}{2}} \\
&\lesssim \|u_1\|_{X^{0,\frac{1}{2}}_{p,2}}\|u_2\|_{X^{0,\frac{1}{2}}_{2,2}}\|u_3\|_{X^{0,\varepsilon}_{p',2}},
\end{aligned}$$

from Hölder's and Minkowski's inequalities. □

4.2.2 Resonant contributions

We start by considering the terms in $\mathcal{R}$ (4.9) where J satisfies $\{0,j\} \subset J$, $\{-j,0\} \subset J$ or $\{-j,j\} \subset J$, for some $j = 1,\dots,m$. The intended estimate essentially follows from the stronger estimate in Lemma 4.2.8.

Lemma 4.2.8. *Let $2 \le p < \infty$ and $s > 1 - \frac{1}{p} - \frac{p-2}{mp}$. Then the following estimate holds*

$$\bigg\|\int_{\tau=\tau_0+\ldots+\tau_m}\sum_{n=n_2+\ldots+n_m}\langle n\rangle^{s+1}|\widehat{u}_0(\tau_0,n)\widehat{u}_1(\tau_1,n)|\prod_{j=2}^{m}|\widehat{u}_j(\tau_j,n_j)|\bigg\|_{\ell^p_nL^2_\tau} \lesssim \prod_{j=0}^{m}\|u_j\|_{X^{s,0}_{p,2}\cap X^{s,0}_{p,1}}. \quad (4.16)$$

Proof. Assume that $|n_2| \ge \ldots \ge |n_m|$, without loss of generality. Then $|n| \lesssim |n_2|$. Using Young's and Hölder's inequalities, we have

$$\text{LHS of (4.16)} \lesssim \|u_0\|_{X^{s,0}_{\infty,2}}\|u_1\|_{X^{s,0}_{\infty,1}}\bigg\|\langle n\rangle^{1-s}\sum_{n=n_2+\ldots+n_m}\prod_{j=2}^{m}\|\widehat{u}_j(n_j)\|_{L^1_\tau}\bigg\|_{\ell^p_n}$$

$$\lesssim \sup_n \Big(\sum_{n=n_2+\ldots+n_m} \Big| \frac{1}{\langle n_2\rangle^{2s-1}\langle n_3\rangle^s\cdots\langle n_m\rangle^s}\Big|^{p'}\Big)^{\frac{1}{p'}} \|u_0\|_{X^{s,0}_{p,2}} \prod_{j=1}^m \|u_j\|_{X^{s,0}_{p,1}}.$$

The estimate follows from Lemma 2.1.5 for $s > \max(\frac{1}{2}, 1-\frac{1}{p}-\frac{p-2}{mp}) = 1-\frac{1}{p}-\frac{p-2}{mp}$. □

The following lemma establishes an estimate for $\mathcal{R}$ in (4.9) when $J \subset \{-m, \ldots, -1\}$.

Lemma 4.2.9. *For $2 < p < \infty$ and $s > 1 - \frac{1}{p} - \min\big(\frac{1}{p}, \frac{p-2}{mp}, \frac{1}{2(m-3)\mathbb{1}_{m>4}}\big)$, we have*

$$\Big\| \int_{\tau=\tau_0+\ldots+\tau_m} \sum_{n=n_2+\ldots+n_m} \langle n\rangle^s |n_2|\, |\widehat{u}_0(\tau_0,-n_2)\widehat{u}_1(\tau_1,n_2)\widehat{u}_2(\tau_2,n_2)| \prod_{j=3}^m |\widehat{u}_j(\tau_j,n_j)| \Big\|_{\ell^p_n L^2_\tau}$$
$$\lesssim \prod_{j=0}^m \|u_j\|_{X^{s,0}_{p,2}\cap X^{s,0}_{p,1}}. \quad (4.17)$$

Proof. Assume that $|n_3| \geq \ldots \geq |n_m|$, without loss of generality. We will consider two cases: $|n_2| \gtrsim |n_3|$ and $|n_2| \ll |n_3|$. If $|n_2| \gtrsim |n_3|$, using Young's and Hölder's inequalities gives

$$\text{LHS of (4.17)} \lesssim \sup_n \Big(\sum_{n=n_2+\ldots+n_m} \Big| \frac{1}{\langle n_2\rangle^{2s-1}\langle n_3\rangle^s\cdots\langle n_m\rangle^s}\Big|^{p'}\Big)^{\frac{1}{p'}} \|u_0\|_{X^{s,0}_{p,2}} \prod_{j=1}^m \|u_j\|_{X^{s,0}_{p,1}}.$$

The estimate follows from Lemma 2.1.5 if

$$s > 1 - \frac{1}{p} - \frac{p-2}{mp}. \quad (4.18)$$

If $|n_2| \ll |n_3|$, then $|n| \lesssim |n_3|$. Let $v(t,x) = \mathcal{F}_x^{-1}\big(\langle\cdot\rangle\widehat{u}_0(t,-\cdot)\widehat{u}_1(t,\cdot)\widehat{u}_2(t,\cdot)\big)$. If $m=3$, from Young's inequality, we have

$$\text{LHS (4.17)} \lesssim \|v\cdot D^s u_3\|_{X^{0,0}_{p,2}} \lesssim \|v\|_{X^{0,0}_{1,1}} \|u_3\|_{X^{s,0}_{p,2}}.$$

Note that using Young's inequality in time and Hölder's in space, we have

$$\|v\|_{X^{0,0}_{1,1}} \lesssim \prod_{j=0}^2 \|\langle n\rangle^{\frac{1}{3}}\widehat{u}_j(\tau,n)\|_{\ell^3_n L^1_\tau} \lesssim \prod_{j=0}^2 \|u_j\|_{X^{s,0}_{p,1}}, \quad (4.19)$$

for $s \geq \frac{1}{3}$, $2 \leq p \leq 3$ or $s > \frac{2}{3} - \frac{1}{p}$, $p > 3$, which are less restrictive than (4.18). If $m \geq 4$ and $n_3 + n_4 \neq 0$, we use Young's inequality to obtain

$$\text{LHS of (4.17)} \lesssim \big\|v\, \mathbf{P}(D^s u_3\cdot u_4)u_5\cdots u_m\big\|_{X^{0,0}_{p,2}} \lesssim \|v\|_{X^{0,0}_{1,1}} \|\mathbf{P}\big(D^s u_3\cdot u_4\big)\|_{X^{0,0}_{p,2}} \prod_{j=5}^m \|u_j\|_{X^{0,0}_{1,1}}.$$

The first factor is estimated as in (4.19). For the remaining factors we use the bilinear estimate (4.10) and Hölder's inequality, and the fact that $|n_4| \geq \ldots \geq |n_m|$,

$$\|\mathbf{P}\big(D^s u_3\cdot u_4\big)\|_{X^{0,0}_{p,2}} \prod_{j=5}^m \|u_j\|_{X^{0,0}_{1,1}} \lesssim \|u_3\|_{X^{s,\frac{1}{2}-}_{p,2}} \|u_4\|_{X^{0,\frac{1}{2}-}_{2,2}} \prod_{j=5}^m \|u_j\|_{X^{0,0}_{1,1}}$$
$$\lesssim \|u_3\|_{X^{s,\frac{1}{2}}_{p,2}} \|u_4\|_{X^{\frac{1}{2}-\frac{1}{p}+,\frac{1}{2}}_{p,2}} \prod_{j=5}^m \|u_j\|_{X^{1-\frac{1}{p}+,0}_{p,1}}.$$

The estimate follows if $s > \frac{1}{2} - \frac{1}{p}$ and, for $m > 4$, if $s > 1 - \frac{1}{p} - \frac{1}{2(m-3)}$. If $n_3 + n_4 = 0$, applying

Young's inequality in τ and Hölder's inequality in n, we have the following

$$\text{LHS of (4.17)} \lesssim \bigg\| \sum_{\substack{n=n_2+n_5+\\ \ldots+n_m}} \|\widehat{v}(n_2)\|_{L^1_\tau} \prod_{j=5}^m \|\widehat{u}_j(n_j)\|_{L^1_\tau} \Big(\sum_{n_3} \langle n_3\rangle^s \|\widehat{u}_3(n_3)\|_{L^2_\tau} \|\widehat{u}_4(-n_3)\|_{L^1_\tau}\Big) \bigg\|_{\ell^p_n}$$
$$\lesssim \Big(\sup_n J(n)\Big)^{\frac{1}{p'}} \|v\|_{X^{3s-1,0}_{p,1}} \|u_3\|_{X^{\frac{s}{2},0}_{2,2}} \|u_4\|_{X^{\frac{s}{2},0}_{2,1}} \prod_{j=5}^m \|u_j\|_{X^{s,0}_{p,1}}.$$

From Hölder's and Young's inequalities, we obtain $\|v\|_{X^{3s-1,0}_{p,1}} \lesssim \|u_0\|_{X^{s,0}_{p,1}} \|u_1\|_{X^{s,0}_{p,1}} \|u_2\|_{X^{s,0}_{p,1}}$ and $\|u_j\|_{X^{\frac{s}{2},0}_{2,q}} \lesssim \|u_j\|_{X^{\frac{s}{2}+\frac{1}{2}-\frac{1}{p}+,0}_{p,q}} \lesssim \|u_j\|_{X^{s,0}_{p,q}}$ for $s > 1-\frac{2}{p}$, $j=3,4$. Lastly, from Lemma 2.1.5, if $s > \frac{1}{3}$ and, additionally if $s > 1 - \frac{1}{p} - \frac{2p-3}{(m-1)p}$ when $m > 4$, we have

$$J(n) = \sum_{n=n_2+n_5+\ldots+n_m} \bigg| \frac{1}{\langle n_2\rangle^{3s-1} \langle n_5\rangle^s \cdots \langle n_m\rangle^s} \bigg|^{p'} \lesssim 1.$$

The estimate follows. □

Lastly, we consider $\mathcal{R}$ restricted to $J = J_+ \cup (-J_-)$, where $J_+, J_- \subset \{1,\ldots,m\}$ are *disjoint* sets and $|J_+| \geq 1$. The following lemma estimates the case when $J_+ = \{1,\ldots,m\}$.

Lemma 4.2.10. *The following estimate holds for any $1 \leq p < \infty$ and $s \geq \frac{1}{m}$*

$$\bigg\| \int_{\tau=\tau_0+\ldots+\tau_m} \langle n\rangle^s |(m-1)n|\, |\widehat{u}_0(\tau_0, -(m-1)n)| \prod_{j=1}^m |\widehat{u}_j(\tau_j, n)| \bigg\|_{\ell^p_n L^2_\tau} \lesssim \prod_{j=0}^m \|u_j\|_{X^{s,0}_{p,2} \cap X^{s,0}_{p,1}}.$$

Proof. Using Young's inequality in time and taking a supremum in n, we can estimate the intended quantity by placing u_0 in $X^{s,0}_{p,2}$ and the remaining terms in $X^{\frac{1}{m},0}_{\infty,1}$. The estimate follows for $s \geq \frac{1}{m}$. □

For the remaining cases, we fix J_+ and gather the contributions from $J_- \subset \{1,\ldots,m\} \setminus J_+$. Appealing to symmetry, let $J_+ = \{1,\ldots,\ell\}$ for some $1 \leq \ell \leq m-1$. Then, the net contribution can be rewritten as follows

$$\mathcal{R}_\ell(u_0,\ldots,u_m) := \int_{\tau=\tau_0+\ldots+\tau_m} \sum_{n=n_0+\ldots+n_m} \bigg[\sum_{J_-\subset\{\ell+1,\ldots,m\}} (-1)^{\ell+|J_-|+1} \mathbb{1}_{\big(\bigcap_{i=1}^{\ell} A_i\big) \cap \big(\bigcap_{j\in J_-} A_{-j}\big)} \bigg]$$
$$\times\, i n_0 \widehat{u}_0(\tau_0, n_0) \cdots \widehat{u}_m(\tau_m, n_m)$$
$$= (-1)^{\ell+1} \int_{\tau=\tau_0+\ldots+\tau_m} \sum_{n=n_0+\ldots+n_m} \mathbb{1}_{\big(\bigcap_{i=1}^{\ell} A_i\big) \cap \big(\bigcap_{j=\ell+1}^{m} A^c_{-j}\big)} i n_0 \widehat{u}_0(\tau_0, n_0) \cdots \widehat{u}_m(\tau_m, n_m),$$

which is estimated in the following lemma.

Lemma 4.2.11. *Let $1 \leq \ell \leq m-1$, $2 < p < \infty$, and $s > 1 - \frac{1}{p} - \min\big(\frac{p-2}{4p}, \frac{p-2}{mp}, \frac{1}{2m}\big)$. Then, the following holds*

$$\Big\| \langle n\rangle^s \mathcal{R}_\ell(u_0,\ldots,u_m) \Big\|_{\ell^p_n L^2_\tau} \lesssim \prod_{j=0}^m \|u_j\|_{Z^{s,\frac{1}{2}}_p}. \tag{4.20}$$

Proof. Fix $1 \leq \ell \leq m-1$ and assume without loss of generality that $|n_{\ell+1}| \geq \ldots \geq |n_m|$. If $\ell = 1$, then $0 = n_0 + n_2 + \ldots + n_m$, $n_0 + n_j \neq 0$ for $j = 2,\ldots,m$ and $|n_0| \lesssim |n_2|$. Using Young's inequality, we obtain

$$\text{LHS of (4.20)} \lesssim \bigg\| \int_{\tau=\tau_0+\ldots+\tau_m} \langle n\rangle^s \widehat{u}_1(\tau_1,n) \sum_{\substack{0=n_0+n_2+\ldots+n_m\\ n_0+n_2\neq 0}} n_0\widehat{u}_0(\tau_0,n_0)\prod_{j=2}^m \widehat{u}(\tau_j,n_j)\bigg\|_{\ell^p_n L^2_\tau}$$
$$\lesssim \|u_1\|_{X^{s,0}_{p,1}} \Big\| \mathbf{P}\big(D^s u_0\cdot D^{1-s}u_2\big)\cdot u_3\cdots u_m\Big\|_{X^{0,0}_{\infty,2}}$$
$$\lesssim \|u_1\|_{X^{s,0}_{p,1}} \big\| \mathbf{P}\big(D^s u_0\cdot D^{1-s}u_2\big)\big\|_{X^{0,0}_{p,2}} \prod_{j=3}^m \|u_j\|_{X^{0,0}_{q,1}}$$

where $(m-2)=\frac{1}{p}+\frac{m-2}{q}$. Using the bilinear estimate (4.10) and Hölder's inequality, we have

$$\big\|\mathbf{P}\big(D^s u_0\cdot D^{1-s}u_2\big)\big\|_{X^{0,0}_{p,2}} \prod_{j=3}^m \|u_j\|_{X^{0,0}_{q,1}} \lesssim \|u_0\|_{X^{s,\frac12}_{p,2}} \|u_2\|_{X^{1-s,\frac12}_{2,2}} \prod_{j=3}^m \|u_j\|_{X^{0,0}_{q,1}}$$
$$\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}} \|u_2\|_{X^{\frac32-s-\frac1p+,\frac12}_{p,2}} \prod_{j=3}^m \|u_j\|_{X^{\frac1q-\frac1p+,0}_{p,1}}$$

and we must impose

$$s > \max\Big(\frac12, 1-\frac1p-\frac{p-2}{4p}, 1-\frac1p-\frac{1}{2m}\Big) = 1-\frac1p-\min\Big(\frac{p-2}{4p},\frac{1}{2m}\Big). \tag{4.21}$$

Now, let $1<\ell<m$. Then, $(1-\ell)n = n_0+n_{\ell+1}+\ldots+n_m$ and $n_0+n_j\neq 0$ for $j=\ell+1,\ldots,m$. Assuming that $|n|\sim|n_0|\gg|n_{\ell+1}|$ and using Hölder's and Young's inequalities, we have

$$\text{LHS of (4.20)} \lesssim \bigg\| \int_{\tau=\tau_0+\ldots+\tau_m} \langle n\rangle^s \prod_{i=1}^{\ell}\widehat{u}_i(\tau_i,n) \sum_{\substack{(1-\ell)n=n_0\\ +n_{\ell+1}+\ldots+n_m}} |n_0|\widehat{u}_0(\tau_0,n_0)\prod_{j=\ell+1}^m \widehat{u}_j(\tau_j,n_j)\bigg\|_{\ell^p_n L^2_\tau}$$
$$\lesssim \prod_{i=1}^{\ell}\|u_i\|_{X^{s,0}_{\infty,1}} \bigg\| \sum_{\substack{(1-\ell)n=n_0\\ +n_{\ell+1}+\ldots+n_m}} |n_0|^{1-(\ell-1)s}\|\widehat{u}_0(n_0)\|_{L^2_\tau}\prod_{j=\ell+1}^m \|\widehat{u}_j(n_j)\|_{L^1_\tau}\bigg\|_{\ell^p_n}$$
$$\lesssim \|u_0\|_{X^{s,0}_{p,2}} \Big(\prod_{i=1}^{\ell}\|u_i\|_{X^{s,0}_{p,1}}\Big)\Big(\prod_{j=\ell+1}^m \|u_j\|_{X^{-\alpha,0}_{1,1}}\Big),$$

where $\alpha=\frac{\ell s-1}{m-\ell}>0$, since $s>\frac{1}{\ell}$, for $\ell\geq 2$. The estimate follows from Young's inequality if $s>\max\big(\frac1\ell, 1-\frac1p-\frac{(\ell-1)p-\ell}{mp}\big)$, which is less restrictive than

$$s>\max\Big(\frac12, 1-\frac1p-\frac{p-2}{mp}\Big) = 1-\frac1p-\frac{p-2}{mp}.$$

If $|n_0|\lesssim|n_{\ell+1}|$, then we proceed as in the case when $\ell=1$,

$$\text{LHS of (4.20)} \lesssim \prod_{i=1}^{\ell}\|u_i\|_{X^{s,0}_{p,1}} \big\|\mathbf{P}\big(D^s u_0\cdot D^{1-s}u_{\ell+1}\big)\cdot u_{\ell+2}\cdots u_m\big\|_{X^{0,0}_{\infty,2}}$$
$$\lesssim \big\|\mathbf{P}\big(D^s u_0\cdot D^{1-s}u_{\ell+1}\big)\big\|_{X^{0,0}_{p,2}} \Big(\prod_{i=1}^{\ell}\|u_i\|_{X^{s,0}_{p,1}}\Big)\Big(\prod_{j=\ell+2}^m \|u_j\|_{X^{0,0}_{q,1}}\Big),$$

where $(m-\ell-1)=\frac1p+\frac{m-\ell-1}{q}$. Using the bilinear estimate (4.10) and Hölder's inequality, we obtain

$$\text{LHS of (4.20)} \lesssim \|u_0\|_{X^{s,\frac12}_{p,2}} \|u_{\ell+1}\|_{X^{1-s,\frac12}_{2,2}} \Big(\prod_{i=1}^{\ell} \|u_i\|_{X^{s,0}_{p,1}}\Big)\Big(\prod_{j=\ell+2}^{m} \|u_j\|_{X^{0,0}_{q,1}}\Big)$$

$$\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}} \|u_{\ell+1}\|_{X^{\frac32-s-\frac1p+,\frac12}_{p,2}} \Big(\prod_{i=1}^{\ell} \|u_i\|_{X^{s,0}_{p,1}}\Big)\Big(\prod_{j=\ell+2}^{m} \|u_j\|_{X^{\frac1q-\frac1p+,0}_{p,1}}\Big)$$

and the estimate follows if $s > 1 - \frac{1}{p} - \min(\frac{p-2}{4p}, \frac{1}{2(m-\ell+1)\mathbb{1}_{\ell+2\le m}})$, which is less restrictive than (4.21). □

4.2.3 Non-resonant contributions

In this section, we establish the estimate for the non-resonant contribution $\mathcal{N}_0$ in (4.8). Without loss of generality, we can assume that $|n_1| \ge \ldots \ge |n_m|$. We further split the non-resonant contribution as follows

$$\mathcal{N}_0 = \mathcal{N}_1 + \mathcal{N}_3 + \ldots + \mathcal{N}_m \qquad \text{if } m \text{ is odd,}$$
$$\mathcal{N}_0 = \mathcal{N}_1 + \mathcal{N}_3 + \ldots + \mathcal{N}_{m-1} \qquad \text{if } m \text{ is even,}$$

where $\mathcal{N}_\alpha$, for odd $1 \le \alpha \le m-1$, corresponds to $\mathcal{N}_0$ further restricted to the region

$$\Lambda_\alpha(n) = \big\{(n_0,\ldots,n_m) \in \mathbb{Z}_*^{m+1} : |n_1| \ge \ldots \ge |n_m|,\; n_j + n_{j+1} = 0,\ 1 \le j \le \alpha-1 \text{ odd},\; n_\alpha + n_{\alpha+1} \neq 0\big\}$$

and $\Lambda_m(n) = \big\{(n_0,\ldots,n_m) \in \mathbb{Z}_*^{m+1} :\ |n_1| \ge \ldots \ge |n_m|,\ n_1+n_2 = \ldots = n_{m-2}+n_{m-1} = 0\big\}$, for odd m. We will start by estimating the most difficult contribution $\mathcal{N}_1$. Guided by Lemma 4.1.2, we will consider the following case separation:

- **Case 1**: $|n| \sim |n_0| \gg |n_1|$
  - **Case 1.1:** $|n_0|^2|n - n_0| \lesssim |\phi|$
  - **Case 1.2:** $|n_0|^2|n - n_0| \lesssim |n_1 n_2 n_3|$
- **Case 2**: $|n_0| \sim |n_1| \gg |n_2|$
  - **Case 2.1:** $|n_0|^2|n_0 + n_1| \lesssim |\phi|$
  - **Case 2.2:** $|n_0 + n_1| \ll |n_3|$
- **Case 3**: $|n_0| \sim |n_1| \sim |n_2| \gg |n_3|$
  - **Case 3.1:** $|(n - n_1)(n - n_2)(n_1 + n_2)| \lesssim |\phi|$
  - **Case 3.2:** $|(n - n_1)(n - n_2)(n_1 + n_2)| \lesssim |n_0|^2|n_3|$
- **Case 4**: $|n| \sim |n_1| \gg |n_2|$, $|n_1| \gg |n_0| \gg |n_3|$
  - **Case 4.1:** $|n_1|^2|n - n_1| \lesssim |\phi|$
  - **Case 4.2:** $|n_1|^2|n - n_1| \lesssim |n_0 n_2 n_3|$
- **Case 5**: $|n_1| \sim |n_2| \gg |n_0| \gg |n_3|$
  - **Case 5.1:** $|n_1|^2|n_1 + n_2| \lesssim |\phi|$
  - **Case 5.2:** $|n_1 + n_2| \ll |n_3|$
- **Case 6**: $|n_0| \lesssim |n_3|$

We see that this covers all the cases. First, the frequency region $|n_0| \gg |n_3|$ divides into **Cases 1,2,3/4,5** according to $|n_0| \sim / \ll \max(|n_0|, |n_1|)$, respectively. Then, Lemma 4.1.2 divides each of **Cases 1,2,4,5** into the two subcases mentioned above, while the division of **Case 3** is based on the fact that $\phi = 3(n-n_1)(n-n_2)(n_1+n_2) + (n_0+n_3+\ldots+n_m)^3 - (n_0^3+n_3^3+\ldots+n_m^3) = \mathcal{O}(|n_0|^2|n_3|)$. We also observe that in **Case 3** we have

$$\max(|n-n_1|, |n-n_2|, |n_1+n_2|) \gtrsim |n_0|.$$

<u>**Cases 1.1–5.1:**</u>

Let $\sigma = \tau - n^3$ and $\sigma_j = \tau_j - n_j^3$, $j = 0, \ldots, m$, denote the modulations. Then, we have the following upper bound for the resonance relation

$$|\phi| = |\sigma - \sigma_0 - \ldots - \sigma_m| \lesssim \max(|\sigma|, |\sigma_0|, \ldots, |\sigma_m|) = \sigma_{\max},$$

which we can use to gain a power of ϕ. First, let $\sigma_{\max} = |\sigma|$. Using Cauchy-Schwarz inequality and Lemma 2.1.4, we have

$$\begin{aligned}\|\mathcal{N}_1(u_0,\ldots,u_m)\|_{X^{s,-1}_{p,1}} &\lesssim \bigg\| \sum_{n=n_0+\ldots+n_m} \frac{\langle n\rangle^s|n_0|}{\langle\phi\rangle^{\frac12}} \bigg(\int_{\tau,\tau=\tau_0+\ldots+\tau_m} \frac{1}{\langle\sigma\rangle\langle\sigma_0\rangle^{1-}\cdots\langle\sigma_m\rangle^{1-}}\bigg)^{\frac12} \\ &\qquad\times \prod_{j=0}^m \|\langle\sigma_j\rangle^{\frac12-}\widehat{u}_j(n_j)\|_{L^2_\tau}\bigg\|_{\ell^p_n} \\ &\lesssim \bigg\| \sum_{n=n_0+\ldots+n_m} \frac{\langle n\rangle^s|n_0|}{\langle\phi\rangle^{\frac12}} \prod_{j=0}^m \|\langle\sigma_j\rangle^{\frac12-}\widehat{u}_j(n_j)\|_{L^2_\tau}\bigg\|_{\ell^p_n}. \qquad (4.22)\end{aligned}$$

For the $X^{s,-\frac12}_{p,2}$-norm, the same approach holds. When $\sigma_{\max} = |\sigma_j|$ for some $j = 0, \ldots, m$, since $X^{s,-\frac12+}_{p,2} \subset Z^{s,-\frac12}_p$, it suffices to estimate the $X^{s,-\frac12+}_{p,2}$-norm of $\mathcal{N}_1$. Using duality and Hölder's inequality, we can estimate the $X^{s,-\frac12+}_{p,2}$-norm by (4.22) with $\langle\sigma_j\rangle^{\frac12}$ instead of $\langle\sigma_j\rangle^{\frac12-}$. Consequently, it suffices to estimate (4.22). By using Hölder's inequality, we obtain

$$(4.22) \lesssim \sup_n \Big(\mathrm{I}_\phi(n)\Big)^{\frac{1}{p'}} \max_{j=0,\ldots,k} \bigg(\|u_j\|_{X^{s,\frac12}_{p,2}} \prod_{\substack{i=0\\ i\neq j}}^m \|u_i\|_{X^{s,\frac12-}_{p,2}}\bigg),$$

where

$$\mathrm{I}_\phi(n) = \sum_{n=n_0+\ldots+n_m} \bigg| \frac{\langle n\rangle^s|n_0|}{\langle\phi\rangle^{\frac12}\langle n_0\rangle^s\langle n_1\rangle^s\cdots\langle n_m\rangle^s}\bigg|^{p'},$$

and it suffices to bound $\mathrm{I}_\phi(n)$ uniformly in n. To this end, we must consider the lower bound for ϕ. In **Case 1.1**, we have

$$\mathrm{I}_\phi(n) \lesssim \sum_{n_1,\ldots,n_m} \frac{1}{\langle n_1+\ldots+n_m\rangle^{\frac{p'}{2}}\langle n_1\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \lesssim 1$$

by applying Lemma 2.1.5, given that $s > 1 - \frac1p - \frac{1}{2m}$. In **Case 2.1**, we have $|n_0| \sim |n_1| \gg |n_2|$. If $|n| \lesssim |n_0+n_1|$, then

$$\mathrm{I}_\phi(n) \lesssim \sum_{n_1,\ldots,n_m} \frac{1}{\langle n_1\rangle^{(s+\frac12)p'}\langle n_2\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \lesssim \bigg(\sum_n \frac{1}{\langle n\rangle^{(s+\frac{1}{2m})p'}}\bigg)^m \lesssim 1,$$

under the following assumption

$$s > \max\Big(\frac12, 1 - \frac1p - \frac{1}{2m}\Big). \qquad (4.23)$$

If $|n| \gg |n_0+n_1|$, then $|n| \sim |n_2+\ldots+n_m| \lesssim |n_2|$ and we have

$$\begin{aligned}
\mathrm{I}_\phi(n) &\lesssim \sum_{n=n_0+\ldots+n_m} \frac{1}{\langle n_0+n_1\rangle^{\frac{p'}{2}}\langle n_0\rangle^{2sp'}\langle n_3\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \\
&\lesssim \Big(\sum_{n_0,n_1} \frac{1}{\langle n_0+n_1\rangle^{(s+\frac{1}{2m})p'}\langle n_0\rangle^{(s+\frac{1}{2m})p'}}\Big)\Big(\sum_{n}\frac{1}{\langle n\rangle^{(s+\frac{1}{2m})p'}}\Big)^{m-2} \lesssim 1,
\end{aligned}$$

if (4.23) holds. In **Case 3.1**, $|n_0| \sim |n_1| \sim |n_2| \gg |n_3|$ and at least one of the factors $|(n-n_1)(n-n_2)(n_1+n_2)|$ must be comparable to $|n_0|$. If $|n_0||n-n_1||n-n_2| \lesssim |\phi|$, we use Lemma 2.1.5 to obtain

$$\begin{aligned}
\mathrm{I}_\phi(n) &\lesssim \sum_{n_0,n_2,\ldots,n_m} \frac{1}{\langle n_0+n_2+\ldots+n_m\rangle^{\frac{p'}{2}}\langle n-n_2\rangle^{\frac{p'}{2}}\langle n_0\rangle^{(2s-\frac12)p'}\langle n_3\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \\
&\lesssim \sum_{n_0,n_3,\ldots,n_m} \frac{1}{\langle n+n_0+n_3+\ldots+n_m\rangle^{p'-1}\langle n_0\rangle^{\beta}\langle n_3\rangle^{\beta}\cdots\langle n_m\rangle^{\beta}} \lesssim 1,
\end{aligned}$$

for $\beta = \frac{1}{m-1}(ms-\frac12)p'$ and the estimate follows from Lemma 2.1.5 if (4.23) holds. If $|n_0||n_1+n_2||n-n_1| \lesssim |\phi|$, then

$$\begin{aligned}
\mathrm{I}_\phi(n) &\lesssim \sum_{n_1,\ldots,n_m} \frac{1}{\langle n_1+n_2\rangle^{\frac{p'}{2}}\langle n-n_1\rangle^{\frac{p'}{2}}\langle n_2\rangle^{(2s-\frac12)p'}\langle n_3\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \\
&\lesssim \sum_{n_2,\ldots,n_m} \frac{1}{\langle n+n_2\rangle^{p'-1}\langle n_2\rangle^{(2s-\frac12)p'}\langle n_3\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \lesssim 1,
\end{aligned}$$

proceeding as in the previous cases by splitting the power of $\langle n_2\rangle$ between the other frequencies. By exchanging the roles of n_1 and n_2, we obtain the estimate when $|n_0||n_1+n_2||n-n_2| \lesssim |\phi|$. In **Case 4.1**, since $|n_1| \gg |n_0|$ and $|\phi| \gtrsim |n_1|^2|n-n_1|$, we have

$$\mathrm{I}_\phi(n) \lesssim \sum_{n_0,n_2,\ldots,n_m} \frac{1}{\langle n_0+n_2+\ldots+n_m\rangle^{\frac{p'}{2}}\langle n_0\rangle^{sp'}\langle n_2\rangle^{sp'}\cdots\langle n_m\rangle^{sp'}} \lesssim 1$$

and the estimate follows from that of Case 1.1, by exchanging the roles of n_0 and n_1. Similarly, the estimate in **Case 5.1** follows Case 2.1 by exchanging the roles of (n_0,n_1) with (n_1,n_2).

In **Cases 1.2–5.2** and **Case 6**, we can no longer use the largest modulation. However, note that it suffices to control the stronger norm $X^{s,-\frac12+}_{p,2}$.

Case 1.2:

Here we have $|n| \sim |n_0| \gg |n_1|$ and $|n_0|^2|n-n_0| \lesssim |n_1n_2n_3|$, thus we can control the multiplier as follows

$$\langle n\rangle^s|n_0| \lesssim \langle n_0\rangle^s \frac{|n_1n_2n_3|^{\frac12}}{|n_1+\ldots+n_m|^{\frac12}}.$$

Using Lemma 4.2.4, we have

$$\begin{aligned}
\|\mathcal{N}_1(u_0,\ldots,u_m)\|_{X^{s,-\frac12+}_{p,2}} &\lesssim \big\|(D^s u_0)\cdot D^{-\frac12}\big(\mathbf{P}(D^{\frac12}u_1\cdot D^{\frac12}u_2)\cdot D^{\frac12}u_3\cdot u_4\cdots u_m\big)\big\|_{X^{0,-\frac12+}_{p,2}} \\
&\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}}\big\|\mathbf{P}(D^{\frac12}u_1\cdot D^{\frac12}u_2)\cdot D^{\frac12}u_3\cdot u_4\cdots u_m\big\|_{X^{-\frac12,0}_{p',2}}.
\end{aligned}$$

Using Hölder's and Young's inequality (if $m \geq 4$), we get

$$\begin{aligned}
\|\mathcal{N}_1(u_0,\ldots,u_m)\|_{X^{s,-\frac12+}_{p,2}} &\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}}\big\|\mathbf{P}(D^{\frac12}u_1\cdot D^{\frac12}u_2)\cdot D^{\frac12}u_3\cdot u_4\cdots u_m\big\|_{X^{0,0}_{q,2}} \\
&\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}}\big\|\mathbf{P}(D^{\frac12}u_1\cdot D^{\frac12}u_2)\cdot D^{\frac12}u_3\big\|_{X^{0,0}_{q,2}}\prod_{j=4}^{m}\|u_j\|_{X^{0,0}_{1,1}},
\end{aligned}$$

where $q = p$ for $2 < p < 4$ and $q = \frac{2p}{p-2}-$ for $4 \leq p < \infty$. Applying the trilinear estimates (4.14) or (4.15), we obtain

$$\begin{aligned}\big\|D^{\frac12}u_2 \cdot \mathbf{P}_0(D^{\frac12}u_1 \cdot D^{\frac12}u_3)\big\|_{X^{0,0}_{q,2}} + \big\|D^{\frac12}u_1 \cdot \mathbf{P}_0(D^{\frac12}u_2 \cdot D^{\frac12}u_3)\big\|_{X^{0,0}_{q,2}} \\ + \big\|D^{\frac12}u_1 \cdot D^{\frac12}u_2 \cdot D^{\frac12}u_3 \mathbb{1}_{\phi'\neq 0}\big\|_{X^{0,0}_{q,2}} \\ \lesssim \|u_1\|_{X^{\frac12,\frac12}_{q,2}} \|u_2\|_{X^{\frac12,\frac12}_{q,2}} \|u_3\|_{X^{\frac12,\frac12}_{q',2}} + \|u_1\|_{X^{\frac12+,\frac12}_{q,2}} \|u_2\|_{X^{\frac12+,\frac12}_{2,2}} \|u_3\|_{X^{\frac12+,\frac12}_{2,2}}, \quad (4.24)\end{aligned}$$

where $\phi' = (n_1+n_2)(n_1+n_3)(n_2+n_3)$. Using the fact that $|n_1| \geq \ldots \geq |n_m|$, the estimate follows from Hölder's inequality if $2 < p < 4$ and $s > 1 - \frac1p - \frac{p-2}{2pm}$, or $4 \leq p < \infty$ and $s > 1 - \frac1p - \frac{1}{pm}$.

Case 2.2:

In this case we have $|n_0| \sim |n_1| \gg |n_2|$ and $|n_0 + n_1| \ll |n_3|$. Thus, $|n| \lesssim |n_2|$, $|n_0 + n_1 + n_3 + \ldots + n_m| \lesssim |n_3|$, and we can estimate the multiplier as

$$\langle n\rangle^s |n_0| \lesssim \langle n_2\rangle^s \langle n_0\rangle^{\frac12}\langle n_1\rangle^{\frac12} \lesssim \frac{\langle n_2\rangle^s \langle n_0\rangle^{\frac12}\langle n_1\rangle^{\frac12}\langle n_3\rangle^{\frac12}}{\langle n_0+n_1+n_3+\ldots+n_m\rangle^{\frac12}}.$$

Using (4.13) since $nn_2(n-n_2) \neq 0$ and applying Young's inequality, we have

$$\begin{aligned}\|\mathcal{N}_1(u_0,\ldots,u_m)\|_{X^{s,-\frac12+}_{p,2}} &\lesssim \big\|(D^s u_2)D^{-\frac12}\big(D^{\frac12}u_0 \cdot D^{\frac12}u_1 \cdot D^{\frac12}u_3 \cdot u_4 \cdots u_m\big)\big\|_{X^{0,-\frac12+}_{p,2}} \\ &\lesssim \|u_2\|_{X^{s,\frac12}_{p,2}} \big\|D^{\frac12}u_0 \cdot D^{\frac12}u_1 \cdot D^{\frac12}u_3 \cdot u_4 \cdots u_m\big\|_{X^{-\frac12,0}_{p',2}}.\end{aligned}$$

We apply Hölder's inequality and Young's inequality (if $m \geq 4$), to obtain

$$\|\mathcal{N}_1(u_0,\ldots,u_m)\|_{X^{s,-\frac12+}_{p,2}} \lesssim \|u_2\|_{X^{s,\frac12}_{p,2}} \big\|D^{\frac12}u_0 \cdot D^{\frac12}u_1 \cdot D^{\frac12}u_3\big\|_{X^{0,0}_{q,2}} \prod_{j=4}^m \|u_j\|_{X^{0,0}_{1,1}}$$

where $\frac1q > \frac12 - \frac1p$. Since $(n_0+n_1)(n_0+n_3)(n_1+n_3) \neq 0$, with the last factor nonzero because $|n_1| \gg |n_3|$, we only need to apply the trilinear estimate (4.15) and the estimate follows from Case 1.2 exchanging the roles of (u_0,u_1,u_2,u_3) by (u_2,u_0,u_1,u_3).

Case 3.2:

Since $|n_0| \sim |n_1| \sim |n_2| \gg |n_3|$ and $|(n-n_1)(n-n_2)(n_1+n_2)| \lesssim |n_0|^2|n_3|$, for $N_{\min} = \min(|n-n_1|, |n-n_2|, |n_1+n_2|)$, we get

$$N^2_{\min}|n_0| \lesssim |n_0|^2|n_3| \implies N_{\min} \lesssim |n_0 n_3|^{\frac12}.$$

If $N_{\min} = |n_1+n_2|$, then $|n_1 + \ldots + n_m| \lesssim |n_1+n_2| + |n_3| \lesssim |n_0n_3|^{\frac12}$ and we can estimate the multiplier as follows

$$\langle n\rangle^s|n_0| \lesssim \langle n_0\rangle^s |n_1n_2|^{\frac12}\frac{|n_0n_3|^{\frac{\delta}{2}}}{|n_1+\ldots+n_m|^{\delta}} \sim \langle n_0\rangle^s \frac{|n_1|^{\frac12+\frac{\delta}{4}}|n_2|^{\frac12+\frac{\delta}{4}}|n_3|^{\frac{\delta}{2}}}{|n_1+\ldots+n_m|^{\delta}},$$

where $\delta = \frac32 - \frac3p+$ for $2 < p < 4$ and $\delta = 1 - \frac1p+$ for $4 \leq p < \infty$. Using (4.13) and Young's inequality, we obtain the following

$$\begin{aligned}\|\mathcal{N}_1(u_0,\ldots,u_m)\|_{X^{s,-\frac12+}_{p,2}} &\lesssim \Big\|D^s u_0 \cdot D^{-\delta}\big(D^{\frac12+\frac{\delta}{4}}u_1 \cdot D^{\frac12+\frac{\delta}{4}}u_2 \cdot D^{\frac{\delta}{2}}u_3 \cdot u_4 \cdots u_m\big)\Big\|_{X^{0,-\frac12+}_{p,2}} \\ &\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}} \big\|D^{\frac12+\frac{\delta}{4}}u_1 \cdot D^{\frac12+\frac{\delta}{4}}u_2 \cdot D^{\frac{\delta}{2}}u_3 \cdot u_4 \cdots u_m\big\|_{X^{-\delta,0}_{p',2}} \\ &\lesssim \|u_0\|_{X^{s,\frac12}_{p,2}} \big\|D^{\frac12+\frac{\delta}{4}}u_1 \cdot D^{\frac12+\frac{\delta}{4}}u_2 \cdot D^{\frac{\delta}{2}}u_3\big\|_{X^{0,0}_{q,2}} \prod_{j=4}^m \|u_j\|_{X^{0,0}_{1,1}},\end{aligned}$$

where $q = \frac{2p}{4-p}$ for $2 < p < 4$ and $q = \infty$ for $4 \leq p < \infty$, which satisfies $\frac{1}{p'} - \frac{1}{q} < \delta$ and $2 < q \leq \infty$. Since $n_1 + n_2 \neq 0$ and $|n_1| \sim |n_2| \gg |n_3|$, then $(n_1+n_2)(n_1+n_3)(n_2+n_3) \neq 0$ and we can apply the trilinear estimate (4.15) to obtain

$$\begin{aligned}
&\big\| D^{\frac{1}{2}+\frac{\delta}{4}} u_1 \cdot D^{\frac{1}{2}+\frac{\delta}{4}} u_2 \cdot D^{\frac{\delta}{2}} u_3 \big\|_{X^{0,0}_{q,2}} \\
&\quad \lesssim \min\big(\|u_1\|_{X^{\frac{1}{2}+\frac{\delta}{4}+,\frac{1}{2}}_{q,2}} \|u_2\|_{X^{\frac{1}{2}+\frac{\delta}{4}+,\frac{1}{2}}_{2,2}}, \|u_1\|_{X^{\frac{1}{2}+\frac{\delta}{4}+,\frac{1}{2}}_{2,2}} \|u_2\|_{X^{\frac{1}{2}+\frac{\delta}{4}+,\frac{1}{2}}_{q,2}} \big) \|u_3\|_{X^{\frac{\delta}{2}+,\frac{1}{2}}_{2,2}} \\
&\quad \lesssim \|u_1\|_{X^{\frac{1}{2}+\frac{\delta}{4}+,\frac{1}{2}}_{r,2}} \|u_2\|_{X^{\frac{1}{2}+\frac{\delta}{4}+,\frac{1}{2}}_{r,2}} \|u_3\|_{X^{\frac{\delta}{2}+,\frac{1}{2}}_{2,2}},
\end{aligned}$$

using multilinear interpolation for the last inequality, where $r = p$ for $2 < p < 4$ and $r = 4$ for $4 \leq p < \infty$. The estimate follows if $2 < p < 4$ and

$$s > 1 - \frac{1}{p} - \min\Big(\frac{p-2}{8p}, \frac{1}{3p}, \frac{1}{\mathbb{1}_{m\geq 4} mp} \Big) = 1 - \frac{1}{p} - \min\Big(\frac{p-2}{8p}, \frac{1}{mp} \Big),$$

or $4 \leq p < \infty$ and

$$s > 1 - \frac{1}{p} \min\Big(\frac{1}{4p}, \frac{1}{3p}, \frac{1}{\mathbb{1}_{m\geq 4} mp} \Big) = 1 - \frac{1}{p} - \min\Big(\frac{1}{4p}, \frac{1}{mp} \Big).$$

If $N_{\min} = |n - n_1| = |n_0 + n_2 + \ldots + n_m|$, then

$$\langle n \rangle^s |n_0| \lesssim \langle n_1 \rangle^s |n_0 n_2|^{\frac{1}{2}} \frac{|n_0 n_3|^{\frac{\delta}{2}}}{|n_0 + n_2 + \ldots + n_m|^{\delta}} \sim \langle n_1 \rangle^s \frac{|n_0|^{\frac{1}{2}+\frac{\delta}{4}} |n_2|^{\frac{1}{2}+\frac{\delta}{4}} |n_3|^{\frac{\delta}{2}}}{|n_0 + n_2 + \ldots + n_m|^{\delta}},$$

and the estimate follows from the previous argument, exchanging the roles of u_0 and u_1. Similarly, if $N_{\min} = |n - n_2| = |n_0 + n_1 + n_3 + \ldots + n_m|$, we can control the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n_2 \rangle^s |n_0 n_1|^{\frac{1}{2}} \frac{|n_0 n_3|^{\frac{\delta}{2}}}{|n_0 + n_1 + n_3 + \ldots + n_m|^{\delta}} \sim \langle n_2 \rangle^s \frac{|n_0|^{\frac{1}{2}+\frac{\delta}{4}} |n_1|^{\frac{1}{2}+\frac{\delta}{4}} |n_3|^{\frac{\delta}{2}}}{|n_0 + n_1 + n_3 + \ldots + n_m|^{\delta}},$$

and the estimate follows from the same arguments.

<u>**Case 4.2:**</u>
Since $|n| \sim |n_1| \gg |n_2|$, $|n_1| \gg |n_0| \gg |n_3|$ and $|n_1|^2 |n - n_1| \lesssim |n_0 n_2 n_3|$, we estimate the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n_1 \rangle^s \frac{|n_0 n_2 n_3|^{\frac{1}{2}}}{|n_0 + n_2 + \ldots + n_m|^{\frac{1}{2}}}.$$

The estimate follows the strategy in Case 1.2, exchanging the roles of u_0 and u_1, considering the cases $(n_0+n_2)(n_0+n_3)(n_2+n_3) \neq 0$, and $n_2 + n_3 = 0$, $|n_0| \gg |n_2|$, when applying the trilinear estimates in (4.24).

<u>**Case 5.2:**</u>
In this case we have $|n_1| \sim |n_2| \gg |n_0| \gg |n_3|$ and $|n_1 + n_2| \ll |n_3|$. Then, $|n| \sim |n_0|$, $|n_1 + \ldots + n_m| \lesssim |n_3|$ and we estimate the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n_0 \rangle^s \frac{|n_1 n_2 n_3|^{\frac{1}{2}}}{|n_1 + \ldots + n_m|^{\frac{1}{2}}}.$$

The estimate follows from the approach in Case 1.2, with $(n_1+n_2)(n_1+n_3)(n_2+n_3) \neq 0$.

<u>**Case 6:**</u>
Let $|n_0| \lesssim |n_3|$. Let us first consider the case when $n_2 + n_3 = 0$. Using Young's and Hölder's inequalities, we obtain the following

$$\|\mathcal{N}_1(u_0.\ldots, u_m)\|_{X^{s,-\frac{1}{2}+}_{p,2}} \lesssim \Big\| D^{s+\frac{1}{p}-\frac{1}{2}-} u_0 \cdot D^s u_1 \cdot \mathbf{P}_0 (D^{\frac{3}{2}-\frac{1}{p}-s+} u_2 \cdot u_3) \cdot u_4 \cdots u_m \Big\|_{X^{0,-\frac{1}{2}+}_{p,2}}$$

$$\lesssim \big\| D^{s+\frac{1}{p}-\frac{1}{2}-} u_0 \cdot D^s u_1 \big\|_{X^{0,0}_{p,2}} \|u_2\|_{X^{s,0}_{p,1}} \|u_3\|_{X^{\frac{5}{2}-\frac{3}{p}-2s+,0}_{p,1}} \prod_{j=4}^m \|u_j\|_{X^{0,0}_{1,1}}.$$

Using (4.10) and Hölder's inequality, we have

$$\big\| D^{s+\frac{1}{p}-\frac{1}{2}-} u_0 \cdot D^s u_1 \big\|_{X^{0,0}_{p,2}} \lesssim \|u_0\|_{X^{s+\frac{1}{p}-\frac{1}{2}-,\frac{1}{2}}_{2,2}} \|u_1\|_{X^{s,\frac{1}{2}}_{p,2}} \lesssim \|u_0\|_{X^{s,\frac{1}{2}}_{p,2}} \|u_1\|_{X^{s,\frac{1}{2}}_{p,2}}.$$

Then, the estimate follows from $|n_3| \geq \ldots \geq |n_m|$ given that $s > \max(\frac{1}{2} - \frac{1}{p}, 1 - \frac{1}{p} - \frac{1}{2m}) = 1 - \frac{1}{p} - \frac{1}{2m}$. If $n_2 + n_3 \neq 0$, note that $|n_0 + n_2 + \ldots + n_m| \lesssim |n_2|$, so we can estimate the multiplier as follows

$$\langle n \rangle^s |n_0| \lesssim \langle n_1 \rangle^s |n_0 n_3|^{\frac{1}{2}} \lesssim \langle n_1 \rangle^s \frac{|n_0 n_2 n_3|^{\frac{1}{2}}}{|n_0 + n_2 + \ldots + n_m|^{\frac{1}{2}}}.$$

Then, we can proceed as in Case 1.2, exchanging the roles of (u_0, u_1, u_2, u_3) by (u_1, u_2, u_3, u_0), and using the fact that $(n_0 + n_2)(n_0 + n_3)(n_2 + n_3) \neq 0$.

This completes the estimate of $\mathcal{N}_1$.

Lastly, we want to estimate $\mathcal{N}_\alpha$ for odd $3 \leq \alpha \leq m$. Note that

$$\mathcal{N}_\alpha(u_0, \ldots, u_m) = \mathbf{P}_0(u_1 u_2) \cdots \mathbf{P}_0(u_{\alpha-2} u_{\alpha-1}) \mathcal{N}'_\alpha(u_0, u_\alpha, \ldots, u_m),$$

where

$$\mathcal{F}_x\big(\mathcal{N}'_\alpha(u_0, \ldots, u_m)\big)(t, n) = \sum_{\substack{n = n_0 + n_\alpha + \ldots + n_m \\ n n_0 \cdots n_m \neq 0 \\ n_\alpha + n_{\alpha+1} \neq 0}} i n_0 \widehat{u}_0(n_0) \widehat{u}_\alpha(n_\alpha) \cdots \widehat{u}_m(n_m).$$

For $\alpha = m$ or $\alpha = m - 1$, the phase function satisfies $|\phi| \sim |n n_0 n_m|$ and $|\phi| \sim |(n - n_0)(n - n_{m-1})(n - n_m)|$, respectively. Thus, we can proceed as in Cases 1.1–5.1, following the same strategy in time and using Cauchy-Schwarz inequality in space on the terms $\mathbf{P}_0(u_1 u_2), \ldots, \mathbf{P}_0(u_{\alpha-2} u_{\alpha-1})$. For $3 \leq \alpha \leq m - 2$, we follow a case separation analogous to $\alpha = 1$ by replacing (n_1, n_2, n_3) with $(n_\alpha, n_{\alpha+1}, n_{\alpha+2})$. In Cases 1.1–5.1 we follow the strategy mentioned above. To illustrate the strategy in the remaining cases, consider Case 1.2. Following the strategy for $\mathcal{N}_1$, we have

$$\begin{aligned}
&\|\mathcal{N}_\alpha(u_0, \ldots, u_m)\|_{X^{s,-\frac{1}{2}+}_{p,2}} \\
&\lesssim \bigg\| (D^s u_0) \cdot D^{-\frac{1}{2}} \bigg(\Big(\prod_{\substack{j=1 \\ \text{odd}}}^{\alpha-2} \mathbf{P}_0(u_j u_{j+1}) \Big) \cdot \mathbf{P}(D^{\frac{1}{2}} u_\alpha \cdot D^{\frac{1}{2}} u_{\alpha+1}) \cdot D^{\frac{1}{2}} u_{\alpha+2} \cdot \prod_{i=\alpha+3}^m u_i \bigg) \bigg\|_{X^{0,-\frac{1}{2}+}_{p,2}} \\
&\lesssim \|u_0\|_{X^{s,\frac{1}{2}}_{p,2}} \bigg\| \Big(\prod_{\substack{j=1 \\ \text{odd}}}^{\alpha-2} \mathbf{P}_0(u_j u_{j+1}) \Big) \cdot \mathbf{P}(D^{\frac{1}{2}} u_\alpha \cdot D^{\frac{1}{2}} u_{\alpha+1}) \cdot D^{\frac{1}{2}} u_{\alpha+2} \cdot \prod_{i=\alpha+3}^m u_i \bigg\|_{X^{-\frac{1}{2},0}_{p',2}} \\
&\lesssim \|u_0\|_{X^{s,\frac{1}{2}}_{p,2}} \Big(\prod_{j=1}^{\alpha-1} \|u_j\|_{X^{\frac{1}{2}-\frac{1}{p}+,0}_{p,1}} \Big) \bigg\| \mathbf{P}(D^{\frac{1}{2}} u_\alpha \cdot D^{\frac{1}{2}} u_{\alpha+1}) \cdot D^{\frac{1}{2}} u_{\alpha+2} \cdot \prod_{i=\alpha+3}^m u_i \bigg\|_{X^{-\frac{1}{2},0}_{p',2}},
\end{aligned}$$

using Young's and Cauchy-Schwarz inequalities in the last step. The last term can be estimated following the same approach as for $\mathcal{N}_1$.

This completes the estimate for the non-resonant contribution $\mathcal{N}_0$ in Proposition 4.2.1.

4.3 Almost sure global well-posedness and invariance of the Gibbs measure

In this section, we start by extending the solutions of $\mathcal{G}$-gKdV (4.3) in Theorem 1.2.2 globally-in-time and show invariance of the Gibbs measure under the dynamics of $\mathcal{G}$-gKdV (4.1). We closely follow the argument in [80]. In addition, we establish the invariance of the Gibbs measure under the dynamics of the original gKdV equation (4.1).

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{g_n\}_{n\in\mathbb{Z}_*}$, $\mathbb{Z}_* = \mathbb{Z}\setminus\{0\}$, a sequence of complex-valued standard Gaussian random variables with $g_{-n} = \overline{g_n}$. We can define the Gaussian measure ρ as the induced probability measure under the map

$$\omega \mapsto u^\omega(x) = \sum_{n\in\mathbb{Z}_*} \frac{g_n(\omega)}{|n|} e^{inx} \in \bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T}) \text{ a.s.},$$

or equivalently, as $\rho = \mathbb{P}\circ u^{-1}$ the push-forward of the map above, with the following density

$$d\rho = Z^{-1} e^{-\frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2} du,$$

with a normalizing constant Z. Further details on the construction of Gaussian measures in Banach spaces can be found in [41, 68], for example. Before discussing the construction of the Gibbs measure μ, we recall the following tail estimate for ρ. This lemma follows from the fact that $\mathcal{F}L^{s,p}(\mathbb{T})$ is an abstract Wiener space for $(s-1)p < -1$ (see [5, 80]) and from Fernique's theorem [36]. We also give a direct proof using Chebyshev's inequality.

Lemma 4.3.1. *Let (s,p) satisfy $(s-1)p < -1$ and $K > 0$. Then, the following estimate holds*

$$\rho\big(\|u\|_{\mathcal{F}L^{s,p}} > K\big) \le Ce^{-cK^2},$$

for some constants $C, c > 0$ depending only on s and p.

Proof. Using Chebyshev's inequality and Minkowski's integral inequality, we have

$$\begin{aligned}\rho\big(\|u\|_{\mathcal{F}L^{s,p}} > K\big) &= \mathbb{P}\big(\|u^\omega\|_{\mathcal{F}L^{s,p}} > K\big)\\ &\le K^{-q}\mathbb{E}\big[\|u^\omega\|^q_{\mathcal{F}L^{s,p}}\big]\\ &\le K^{-q}\big\|\langle n\rangle^{s-1}\|g_n(\omega)\|_{L^q(\Omega)}\big\|^q_{\ell^p_n},\end{aligned}$$

for $q \ge p$. Since $\{g_n\}_{n\in\mathbb{Z}_*}$ are standard Gaussian random variables, we know that

$$\mathbb{E}\big[|g_n|^{2m}\big] \lesssim C^m m! \le C^m m^m$$

for any $m \in \mathbb{N}$, which implies that

$$\rho\big(\|u\|_{\mathcal{F}L^{s,p}} > K\big) \le \Big(\frac{C}{K}\Big)^q \Big(\frac{q}{2}\Big)^{\frac{q}{2}} \big\|\langle n\rangle^{s-1}\big\|^q_{\ell^p_n} \le \Big(\frac{\tilde{C}q^{\frac{1}{2}}}{K}\Big)^q.$$

Since ρ is a probability measure, we may assume $K \gg \tilde{C}p^{\frac{1}{2}}$, so that we can choose $q \in 2\mathbb{N}$ satisfying $q \ge p$ and

$$e^{-2} \le \frac{\tilde{C}q^{\frac{1}{2}}}{K} \le e^{-1} \quad\Longrightarrow\quad \frac{e^{-4}K^2}{\tilde{C}^2} \le q \le \frac{e^{-2}K^2}{\tilde{C}^2}.$$

Then,

$$\rho\big(\|u\|_{\mathcal{F}L^{s,p}} > K\big) \le e^{-q} \le e^{-C_0K^2},$$

for a constant $C_0 = C_0(s,p) > 0$, from which the intended estimate follows. □

Now, we view the Gibbs measure μ in (4.2) as a weighted Gaussian measure

$$d\mu = Z^{-1} e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}} d\rho(u),$$

with a normalizing constant Z. In the defocusing case ('+' in (4.1)) and for odd $k \ge 3$, the measure μ is a well-defined probability measure in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $1 \le p \le \infty$ and $s < 1 - \frac{1}{p}$, and it is absolutely continuous with respect to ρ. This follows easily from Sobolev and Hölder's inequalities, since

$$\|u\|_{L^{k+1}(\mathbb{T})} \lesssim \|u\|_{H^{\frac{1}{2}-\frac{1}{k+1}+}(\mathbb{T})} \lesssim \|u\|_{\mathcal{F}L^{1-\frac{1}{p}-\frac{1}{k+1}+}(\mathbb{T})}.$$

The non-defocusing case is more challenging, i.e., when we have '$-$' in (4.1) or $k \geq 2$ is even. In this case, Lebowitz-Rose-Speer [71] and Bourgain [11] proposed the introduction of a mass cutoff and instead studied the following Gibbs measure

$$d\mu = Z^{-1}\mathbb{1}_{\{\|u\|_{L^2}\leq R\}} e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx} d\rho(u),$$

for some $R > 0$. This new measure is known to be normalizable under certain additional assumptions, as stated in the following theorem, which we restate from Chapter 1.

Theorem 4.3.2 ([71, 11, 89]). *Let $k \geq 2$, $R > 0$, and define $F(u)$ by*

$$F(u) = e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx}\mathbb{1}_{\{\|u\|_{L^2}<R\}}, \tag{4.25}$$

where '$\mp$' above corresponds to '$\pm$' in the equation (1.27). Then, for $1 \leq q < \infty$, we have that $F(u) \in L^q(d\rho)$ if one of the following assumptions hold:

(a) *$2 \leq k \leq 4$ and any finite $R > 0$;*

(b) *$k = 5$ and $0 < R < \|Q\|_{L^2(\mathbb{R})}$, where Q is the (unique) optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ with $\|Q\|^6_{L^6(\mathbb{R})} = 3\|Q'\|^2_{L^2(\mathbb{R})}$. If $R = \|Q\|_{L^2(\mathbb{R})}$, then we further impose $q = 1$.*

Remark 4.3.3. (i) Theorem 4.3.2 was first claimed in [71]. Unfortunately, there was a gap in the argument for (b) as remarked in [22]. In [11], Bourgain presented a more analytic proof of Theorem 4.3.2 (a) for any finite R and (b) for small enough R. The optimal threshold $R = \|Q\|_{L^2(\mathbb{R})}$ for $k = 5$ and $q = 1$ was only recently established by Oh-Sosoe-Tolomeo in [89].

(ii) The gKdV equations (4.1) and the nonlinear Schrödinger equation (NLS) share a Hamiltonian. Consequently, they have the same associated Gibbs measure. An analogue of Theorem 4.3.2 for NLS was also shown in [89]. In fact, the critical threshold $k = 5$ is related to the existence of finite time blow-up solutions of NLS on $\mathbb{T}$ due to Ogawa-Tsutsumi [84]. Although the quintic focusing gKdV equation (4.1) exhibits finite time blow-up solutions on the real line [72, 74, 73], such a result is not known on $\mathbb{T}$.

(iii) Note that the assumptions in Theorem 1.2.4 (b) follow from those in Theorem 4.3.2 needed to rigorously construct the Gibbs measure μ in the non-defocusing case. In fact, the measure is not normalizable in the non-defocusing case when $k > 5$ or when $k = 5$ and $R > \|Q\|_{L^2(\mathbb{R})}$ [71, 89].

(iv) It follows from Theorem 4.3.2 that, for $F_N(u) := F(\mathbf{P}_{\leq N}u)$ and any $1 \leq q < \infty$, the estimate

$$\|F_N\|_{L^q(d\rho)} \leq C < \infty, \tag{4.26}$$

holds uniformly in N.

For simplicity, we choose to take the '$-$' sign in the definition of μ, as it will not play a role in the results. Lastly, we state the following known result on the convergence of the truncated measures μ_N defined by

$$d\mu_N(u) = Z_N^{-1}F_N(u)d\rho(u),$$

with Z_N^{-1} a normalization constant.

Lemma 4.3.4. *For all $1 \leq q < \infty$, we have*

$$F_N(u) \to F(u) \quad \text{in } L^q(d\rho) \text{ as } N \to \infty.$$

Moreover, for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ and any measurable set $A \subset \mathcal{F}L^{s,p}(\mathbb{T})$, for $1 \leq p < \infty$ and $s < 1 - \frac{1}{p}$, the following holds

$$|\mu_N(A) - \mu(A)| < \varepsilon.$$

Proof. We first note that

$$\mathbb{1}_{\{\|\mathbf{P}_{\leq N} u\|_{L^2} \leq R\}} \to \mathbb{1}_{\{\|u\|_{L^2} \leq R\}} \quad \rho\text{-a.s.},$$

as $N \to \infty$. From the continuity of the exponential function, it suffices to show that $\int_{\mathbb{T}} (\mathbf{P}_{\leq N} u))^{k+1} dx \to \int_{\mathbb{T}} u^{k+1} dx$ ρ-a.s. Using Sobolev embedding and Hölder's inequality, the convergence follows from $\mathcal{F}L^{s,p}(\mathbb{T}) \hookrightarrow L^{k+1}(\mathbb{T})$ for $s \in (1 - \frac{1}{p} - \frac{1}{k+1}, 1 - \frac{1}{p})$. Since ρ is a probability measure, almost sure convergence implies convergence in measure

$$\lim_{N \to \infty} \rho\big(|F_N(u) - F(u)| \geq \varepsilon\big) = 0, \qquad \text{for every } \varepsilon > 0. \tag{4.27}$$

Fix $\varepsilon > 0$ and let $\Omega_{N,\varepsilon} = \{u \in \mathcal{F}L^{s,p}(\mathbb{T}) : \; |F_N(u) - F(u)| < \frac{\varepsilon}{2}\}$. Then, we have

$$\begin{aligned}
\|F_N - F\|_{L^q(d\rho)} &\leq \|(F_N - F)\mathbb{1}_{\Omega_{N,\varepsilon}}\|_{L^q(d\rho)} + \|(F_N - F)\mathbb{1}_{\Omega^c_{N,\varepsilon}}\|_{L^q(d\rho)} \\
&\leq \frac{\varepsilon}{2} \rho(\Omega_{N,\varepsilon})^{\frac{1}{q}} + \big(\|F_N\|_{L^{2q}(d\rho)} + \|F\|_{L^{2q}(d\rho)}\big) \rho(\Omega^c_{N,\varepsilon})^{\frac{1}{2q}} \\
&\leq \frac{\varepsilon}{2} + C \rho(\Omega^c_{N,\varepsilon})^{\frac{1}{2q}}
\end{aligned}$$

from Hölder's inequality and the uniform bound in (4.26) from Theorem 4.3.2 for any $1 \leq q < \infty$. From (4.27), we see that $\rho(\Omega^c_{N,\varepsilon})^{\frac{1}{2q}} < \frac{\varepsilon}{2C}$, for N large enough, proving the intended convergence of F_N in $L^q(d\rho)$. Lastly, let $A \in \mathcal{F}L^{s,p}(\mathbb{T})$ be any measurable set. Then, we have

$$|\mu_N(A) - \mu(A)| = \left| \int_A \left(\frac{F_N(u)}{\|F_N\|_{L^1(d\rho)}} - \frac{F(u)}{\|F\|_{L^1(d\rho)}} \right) d\rho(u) \right| \lesssim \|F_N - F\|_{L^1(d\rho)} \to 0$$

as $N \to \infty$. $\square$

Consider the following truncated gauged gKdV equation ($\mathcal{G}$-gKdV$_N$)

$$\begin{cases} \partial_t u_N + \partial_x^3 u_N = k \mathbf{P}_{\leq N} \Big(\partial_x (\mathbf{P}_{\leq N} u_N) \cdot \mathbf{P} (\mathbf{P}_{\leq N} u_N)^{k-1} \Big), \\ u_N|_{t=0} = u_0. \end{cases} \tag{4.28}$$

The local well-posedness of (4.28) follows from the proof of Theorem 1.2.2, with the same time of existence $\delta \sim (1 + \|u_0\|_{\mathcal{F}L^{s,p}})^{-\gamma}$ as the solution u of (4.3). Moreover, as we see below, (4.28) is globally well-posed. Note that we can decompose u_N into high and low frequencies $u_N = u_{\text{low}} + u_{\text{high}}$, which solve the following equations

$$\begin{aligned}
\partial_t u_{\text{high}} + \partial_x^3 u_{\text{high}} &= 0, \\
\partial_t u_{\text{low}} + \partial_x^3 u_{\text{low}} &= k \mathbf{P}_{\leq N} \Big(\partial_x u_{\text{low}} \cdot \mathbf{P} (u_{\text{low}})^{k-1} \Big),
\end{aligned}$$

allowing us to discuss the two decoupled flows Φ_{high} and Φ_{low}, respectively. The high frequency part evolves linearly, therefore $\Phi_{\text{high}}(t) = S(t)\mathbf{P}_{>N}$. We can view the low frequency part as a finite-dimensional system of nonlinear ODEs on the Fourier coefficients of u_N. In fact, for $0 < |n| \leq N$ and $c_n = \widehat{u_N}(n)$, we want to solve the following system for $c = \{c_n\}_{0<|n|\leq N} \in \mathbb{C}^{2N}$ with $c_{-n} = \overline{c}_n$,

$$\frac{d}{dt} c_n = i n^3 c_n + k \sum_{\substack{n = n_0 + \ldots + n_{k-1} \\ n \neq n_0}} i n_0 c_{n_0} \cdots c_{n_{k-1}} = \mathbf{N}_n(c). \tag{4.29}$$

Since $\mathbf{N} = \{\mathbf{N}_n\}_{0<|n|\leq N}$ is Lipschitz, we can conclude by the Cauchy-Lipschitz theorem that the system of ODEs is locally well-posed. Furthermore, we can extend these solutions globally-in-time since the L^2-norm of u_N is conserved:

$$\begin{aligned}\frac{d}{dt}M\big(u_N(t)\big) &= 2\int_{\mathbb{T}} u_N\Big(-\partial_x^3 u_N + k\mathbf{P}_{\le N}\big(\partial_x\mathbf{P}_{\le N}u_N\cdot\mathbf{P}(\mathbf{P}_{\le N}u_N)^{k-1}\big)\Big)\,dx\\ &= \int_{\mathbb{T}}\partial_x\Big(\partial_x u_N\Big)^2 dx + \frac{2k}{k+1}\int_{\mathbb{T}}\partial_x\big(\mathbf{P}_{\le N}u_N\big)^{k+1}dx\\ &\qquad\qquad - k\mathbf{P}_0\big(\mathbf{P}_{\le N}u_N\big)^{k-1}\int_{\mathbb{T}}\partial_x\big(\mathbf{P}_{\le N}u_N\big)^2 dx = 0.\end{aligned}\tag{4.30}$$

Thus, $M\big(u_N(t)\big) = M(u_0)$. In addition, the mass is also conserved for u_{low}, $M\big(u_{\text{low}}(t)\big) = M(\mathbf{P}_{\le N}u_0)$, and the solution of (4.29) exists globally-in-time, proving that u_N extends to a global solution of (4.28).

We now focus on proving invariance of the Gibbs measure associated with $\mathcal{G}$-KdV$_N$ (4.28). We first decompose the measure $\rho = \rho_N \otimes \rho_N^{\perp}$, where

$$\begin{aligned} d\rho_N &= Z_N^{-1}e^{-\frac{1}{2}\sum_{0<|n|\le N}|g_n|^2}\prod_{0<|n|\le N}dg_n,\\ d\rho_N^{\perp} &= \tilde{Z}_N^{-1}e^{-\frac{1}{2}\sum_{|n|>N}|g_n|^2}\prod_{|n|>N}dg_n,\end{aligned}$$

for normalization constants $Z_N, \tilde{Z}_N$. Note that ρ_N and $\rho_N^{\perp}$ are also probability measures in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s < 1 - \frac{1}{p}$. Let $\tilde{\mu}_N$ denote the finite-dimensional Gibbs measure with density

$$d\tilde{\mu}_N(u) = Z_N^{-1}F_N(u)d\rho_N(u).$$

Then, $\mu_N = \tilde{\mu}_N \otimes \rho_N^{\perp}$ is the Gibbs measure associated with $\mathcal{G}$-gKdV$_N$ (4.28).

Proposition 4.3.5. *The finite-dimensional Gibbs measure $\tilde{\mu}_N$ is invariant under the flow Φ_{low}. Moreover, the Gibbs measure μ_N is invariant under the flow of $\mathcal{G}$-gKdV$_N$ (4.28).*

Proof. We follow the strategy in [80]. We start by establishing the invariance of $\tilde{\mu}_N$ under the flow of Φ_{low}. The conservation of mass for u_{low} follows from the calculation in (4.30) by replacing u_N by $u_{\text{low}} = \mathbf{P}_{\le N}u_N$. An analogous straightforward computation establishes the conservation of the Hamiltonian for u_{low}. It remains to show the invariance of the Lebesgue measure on $\mathbb{C}^{2N}$ with respect to the system defined in (4.29). We can rewrite the system as

$$\frac{d}{dt}a_n = \operatorname{Re}\big(\mathbf{N}_n(\{a_n, b_n\})\big), \qquad \frac{d}{dt}b_n = \operatorname{Im}\big(\mathbf{N}_n(\{a_n, b_n\})\big),$$

where $c_n = a_n + ib_n$. Thus, the invariance of the Lebesgue measure follows from Liouville's theorem once we establish that the divergence of the vector field vanishes:

$$\sum_{1\le|n|\le N}\Big(\frac{\partial\operatorname{Re}(\mathbf{N}_n)}{\partial a_n} + \frac{\partial\operatorname{Im}(\mathbf{N}_n)}{\partial b_n}\Big) = 0.\tag{4.31}$$

For $1 \le |n| \le N$, we have

$$\begin{aligned}\frac{\partial\operatorname{Re}(\mathbf{N}_n)}{\partial a_n} &= \frac{\partial}{\partial a_n}\Bigg(-n^3 b_n + \frac{k}{2}\sum_{\substack{n=n_0+\ldots+n_{k-1}\\ n\ne n_0}}\big(in_0c_{n_0}\cdots c_{n_{k-1}} - in_0\overline{c_{n_0}}\cdots\overline{c_{n_{k-1}}}\big)\Bigg)\\ &= \frac{k}{2}\sum_{\substack{n=n_0+\ldots+n_{k-1}\\ n\ne n_0}}\sum_{j=1}^{k-1}\Bigg(in_0c_{n_0}\delta(n-n_j)\prod_{\substack{i=1\\ i\ne j}}^{k-1}c_{n_i} - in_0\overline{c_{n_0}}\delta(n-n_j)\prod_{\substack{i=1\\ i\ne j}}^{k-1}\overline{c_{n_i}}\Bigg)\\ &= \frac{k(k-1)}{2}\sum_{\substack{0=n_0+\ldots+n_{k-2}\\ n\ne n_0}}\big(in_0c_{n_0}\cdots c_{n_{k-2}} - in_0\overline{c_{n_0}}\cdots\overline{c_{n_{k-2}}}\big)\end{aligned}$$

$$= k(k-1)\,\mathrm{Re}\bigg(\sum_{\substack{0=n_0+\ldots+n_{k-2}\\ n\neq n_0}} in_0 c_{n_0}\cdots c_{n_{k-2}}\bigg).$$

Similarly, we have

$$\begin{aligned}\frac{\partial\,\mathrm{Im}(\mathbf{N}_n)}{\partial b_n} &= \frac{\partial}{\partial b_n}\bigg(n^3 a_n + \frac{k}{2i}\sum_{\substack{n=n_0+\ldots+n_{k-1}\\ n\neq n_0}} \big(in_0 c_{n_0}\cdots c_{n_k} + in_0\overline{c_{n_0}}\cdots\overline{c_{n_{k-1}}}\big)\bigg)\\ &= k(k-1)\,\mathrm{Re}\bigg(\sum_{\substack{0=n_0+\ldots+n_{k-2}\\ n\neq n_0}} in_0 c_{n_0}\cdots c_{n_{k-2}}\bigg).\end{aligned}$$

Since

$$\sum_{0=n_0+\ldots+n_{k-2}} in_0 c_{n_0}\cdots c_{n_{k-2}} = \int_{\mathbb{T}} \partial_x u_{\mathrm{low}}\cdot u_{\mathrm{low}}^{k-2}\,dx = 0,$$

we conclude (4.31). Lastly, the invariance of $\mu_N = \tilde{\mu}_N \otimes \rho_N^{\perp}$ under the flow $\Phi_N(t) = \big(\Phi_{\mathrm{low}}(t), \Phi_{\mathrm{high}}(t)\big)$ follows from that of $\tilde{\mu}_N$ under the flow of Φ_{low} and the invariance of Gaussian measures under rotation. $\square$

Let $2 < p < \infty$ and $s_* = s_*(p)$ given by Theorem 1.2.2 such that gKdV (4.1) and $\mathcal{G}$-gKdV (4.3) are locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s_* < s < 1 - \frac{1}{p}$. The following two lemmas can be shown through the method in [11] (see also [20, 105, 85, 80]). The proof of Lemma 4.3.6 requires the tail estimate in Lemma 4.3.1, Theorem 1.2.2, Proposition 4.3.5 and (4.26). Lemma 4.3.7 is purely deterministic and follows from the local theory for $\mathcal{G}$-gKdV (4.3).

Lemma 4.3.6. *Let $s_* < s < 1 - \frac{1}{p}$. Then, there exists $C_0 > 0$ (independent of s) and $C_s > 0$ such that: for all $N \in \mathbb{N}$, $T \geq 1$, $0 < \varepsilon \leq \frac{1}{2}$, $A \geq 1$, there exists $\Omega_N^s(T,\varepsilon,A) \subset \mathcal{F}L^{s,p}(\mathbb{T})$ such that:*

(a) $\mu_N\big(\mathcal{F}L^{s,p}(\mathbb{T}) \setminus \Omega_N^s(T,\varepsilon,A)\big) < \varepsilon$.

(b) *For $u_0 \in \Omega_N^s(T,\varepsilon,A)$, the solution u_N of (4.28) satisfies*

$$\big\|u_N(t)\big\|_{\mathcal{F}L^{s,p}} \leq AC_0C_s\big(\log\tfrac{T}{\varepsilon}\big)^{\frac{1}{2}}, \quad |t| \leq T.$$

(c) *For $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, if the solution u_N of (4.28) satisfies*

$$\big\|u_N(t)\big\|_{\mathcal{F}L^{s,p}} \leq AC_s\big(\log\tfrac{T}{\varepsilon}\big)^{\frac{1}{2}}, \quad |t| \leq T,$$

then $u_0 \in \Omega_N^s(T,\varepsilon,A)$.

Proof. From Theorem 1.2.2, we know that for $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ with $\|u_0\|_{\mathcal{F}L^{s,p}} \leq K$, the corresponding solution u_N of $\mathcal{G}$-gKdV$_N$ (4.28) satisfies

$$\big\|u_N(t)\big\|_{\mathcal{F}L^{s,p}} \leq C_0K,$$

for $|t| \leq \delta \sim K^{-\gamma}$, for some $\gamma > 0$, where $C_0 > 0$ does not depend on s. Note also that the constants can be taken uniformly in N. We want to establish a bound on $u_N(t)$ for all $|t| \leq T$. Let $[x]$ denote the integer part of a real number x and define

$$\Omega_N^s(T,\varepsilon,A) = \bigcap_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \Phi_N(j\delta)\Big(\Big\{\|u_0\|_{\mathcal{F}L^{s,p}} \leq K\Big\}\Big),$$

where $\Phi_N(t)$ denotes the solution map for (4.28) and $K = AC_s\big(\log\frac{T}{\varepsilon}\big)^{\frac{1}{2}}$ with a constant $C_s > 0$ to be chosen later.

We start by showing (a). Let $B_K = \{\|u_0\|_{\mathcal{F}L^{s,p}} \leq K\}$. From the uniqueness of solution of (4.28) in each time interval $[j\delta, (j+1)\delta]$, we see that the solution map is invertible and

$$\big[\Phi_N(j\delta)(B_K)\big]^c = \Phi_N(j\delta)(B_K^c).$$

Consequently,

$$\begin{aligned}\mu_N\big([\Omega_N^s(T,\varepsilon,A)]^c\big) &= \mu_N\Bigg(\bigcup_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \Phi_N(j\delta)(B_K^c)\Bigg)\\ &\leq \sum_{j=-\left[\frac{T}{\delta}\right]}^{\left[\frac{T}{\delta}\right]} \mu_N\big(\Phi_N(j\delta)(B_K^c)\big) = 2\Big[\frac{T}{\delta}\Big]\mu_N(B_K^c)\end{aligned}$$

from the invariance of μ_N under the flow of (4.28) in Proposition 4.3.5. From Cauchy-Schwarz inequality, Lemma 4.3.1, and (4.26), we have

$$\mu_N\big([\Omega_N^s(T,\varepsilon,A)]^c\big) \lesssim \frac{T}{\delta}\int_{B_K^c} F_N(u)\, d\rho(u) \lesssim \frac{T}{\delta}\|F_N\|_{L^2(d\rho)}\rho(B_K^c)^{\frac12} \lesssim \frac{T}{\delta}e^{-cK^2} \sim TK^\gamma e^{-cK^2}.$$

Since $\log\frac{T}{\varepsilon} \geq \log 2$ by the assumption, there exists $C_s > 0$ such that if $K \geq C_s\big(\log\frac{T}{\varepsilon}\big)^{\frac12}$, then $TK^\gamma e^{-cK^2} \leq Te^{-\frac{c}{2}K^2} \ll \varepsilon$. Hence, the above estimate, for $K = AC_s\big(\log\frac{T}{\varepsilon}\big)^{\frac12}$ with $A \geq 1$ and such a constant C_s, ensures that $\mu_N\big([\Omega_N^s(T,\varepsilon,A)]^c\big) < \varepsilon$, establishing (a). With the invertibility of the solution map, (b) is a consequence of the local bound mentioned at the beginning, and (c) immediately follows from the definition of $\Omega_N^s(T,\varepsilon,A)$. □

Lemma 4.3.7. *For any $s_* < s < \sigma < 1 - \frac1p$, $T \geq 1$, and $K \geq 1$, there exists $N_0 \in \mathbb{N}$ such that:*

(a) *Let $N \geq N_0$ and $u_N \in C\big(\mathbb{R}; \mathcal{F}L^{\sigma,p}(\mathbb{T})\big)$ be the solution of $\mathcal{G}$-gKdV$_N$ (4.28) with initial data $u_0 \in \mathcal{F}L^{\sigma,p}(\mathbb{T})$. Assume that $\|u_N(t)\|_{\mathcal{F}L^{\sigma,p}} \leq K$ for $|t| \leq T$. Then, there exists a unique solution $u \in C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big) \cap Z_p^{s,\frac12}(T)$ of $\mathcal{G}$-gKdV (4.3) with $u(0) = u_0$ satisfying*

$$\|u(t) - \mathbf{P}_{\leq N}u_N(t)\|_{\mathcal{F}L^{s,p}} \leq \big(\tfrac{N_0}{N}\big)^{\sigma - s}K, \qquad |t| \leq T.$$

In particular, $\|u(t)\|_{\mathcal{F}L^{s,p}} \leq 2K$ for $|t| \leq T$.

(b) *Let $u \in C\big([-T,T]; \mathcal{F}L^{\sigma,p}(\mathbb{T})\big) \cap Z_p^{\sigma,\frac12}(T)$ be a solution of $\mathcal{G}$-gKdV (4.3) with $u(0) = u_0$ satisfying $\|u(t)\|_{\mathcal{F}L^{\sigma,p}} \leq K$ for $|t| \leq T$. Then, for any $N \geq N_0$, the solution u_N of $\mathcal{G}$-gKdV$_N$ (4.28) with initial data u_0 satisfies*

$$\|u(t) - \mathbf{P}_{\leq N}u_N(t)\|_{\mathcal{F}L^{s,p}} \leq \big(\tfrac{N_0}{N}\big)^{\sigma - s}K, \qquad |t| \leq T.$$

In particular, $\|u_N(t)\|_{\mathcal{F}L^{s,p}} \leq 3K$ for $|t| \leq T$.

Proof. We only consider the positive time direction. We start by showing (a). Let $\mathcal{N}(u) := k\mathbf{P}(u^{k-1})\partial_x u$. By the local theory, with $\delta \sim (1+K)^{-\gamma}$ the solution u_N of (4.28) satisfies

$$\|u_N\|_{Z_p^{\sigma,\frac12}([j\delta,(j+1)\delta])} \leq C_2 K, \qquad 0 \leq j < [\tfrac{T}{\delta}] \tag{4.32}$$

for some $C_2 > 0$. Note that the solution of (4.28) in $C\big([-T,T]; \mathcal{F}L^{\sigma,p}(\mathbb{T})\big)$ coincides on each interval $[j\delta, (j+1)\delta]$ with the solution constructed by the iteration argument in $Z_p^{\sigma,\frac12}$, and also that

$$\mathbf{P}_{\leq N}u_N(t) = S(t - j\delta)\mathbf{P}_{\leq N}u_N(j\delta) + \int_{j\delta}^t S(t-t')\mathbf{P}_{\leq N}\mathcal{N}(\mathbf{P}_{\leq N}u_N(t'))\, dt', \quad t \in [j\delta, (j+1)\delta]$$

for any $0 \le j < [\frac{T}{\delta}]$. We want to construct a solution u of

$$u(t) = S(t - j\delta)u(j\delta) + \int_{j\delta}^{t} S(t - t')\mathcal{N}(u(t'))\, dt', \qquad t \in [j\delta, (j+1)\delta]$$

for each $j = 0, 1, \ldots, [\frac{T}{\delta}] - 1$. This amounts to constructing $w(t) := u(t) - \mathbf{P}_{\le N} u_N(t)$, which solves

$$\begin{aligned} w(t) = \Xi_j[w](t) &:= S(t - j\delta)w(j\delta) + \int_{j\delta}^{t} S(t - t')\mathbf{P}_{>N}\mathcal{N}(\mathbf{P}_{\le N} u_N)(t')\, dt' \\ &\quad + \int_{j\delta}^{t} S(t - t')\big\{\mathcal{N}(w + \mathbf{P}_{\le N} u_N) - \mathcal{N}(\mathbf{P}_{\le N} u_N)\big\}(t')\, dt'. \end{aligned} \tag{4.33}$$

By the nonlinear estimates in $Z_p^{s,\frac{1}{2}}$ and $Z_p^{\sigma,\frac{1}{2}}$, together with (4.32), we have

$$\begin{aligned} &\|\Xi_j[w]\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} \\ &\quad \le C_0 \|w(j\delta)\|_{\mathcal{F}L^{s,p}} + C_1 \delta^{\theta} \Big(\|w\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} + C_2 K \Big)^{k-1} \|w\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} \\ &\quad\quad + C_1 N^{-(\sigma - s)} \delta^{\theta} (C_2 K)^k, \\ &\|\Xi_j[w] - \Xi_j[\tilde{w}]\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} \\ &\quad \le C_1 \delta^{\theta} \Big(\|w\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} + \|\tilde{w}\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} + C_2 K \Big)^{k-1} \|w - \tilde{w}\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])}, \end{aligned}$$

for some $C_0 > 0$, $C_1 = C_1(s,p) > 0$, and $\theta = \theta(s,p) > 0$. Therefore, taking smaller $\delta \sim_{s,p} (1 + K)^{-\gamma}$ if necessary ($\gamma = \frac{k-1}{\theta}$), we can show that Ξ_j is a contraction on

$$\big\{ w \in Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta]) : \|w\|_{Z_p^{s,\frac{1}{2}}([j\delta,(j+1)\delta])} \le 2C_0 \|w(j\delta)\|_{\mathcal{F}L^{s,p}} + N^{-(\sigma - s)} K \big\}$$

as long as

$$\|w(j\delta)\|_{\mathcal{F}L^{s,p}} \le K.$$

Starting from $\|w(0)\|_{\mathcal{F}L^{s,p}} \le N^{-(\sigma-s)}K$, we obtain the solution w of (4.33) on $[j\delta, (j+1)\delta]$ with

$$\|w((j+1)\delta)\|_{\mathcal{F}L^{s,p}} \le \tilde{C}_0^{j+1} N^{-(\sigma-s)} K, \qquad j = 0, 1, \ldots, \Big[\frac{T}{\delta}\Big] - 1,$$

for some $\tilde{C}_0 > 0$. In particular, the solution can be extended up to $t = T$ if N satisfies

$$N^{\sigma - s} \ge e^{C_3 (1+K)^{\gamma} T} \ (\ge \tilde{C}_0^{[\frac{T}{\delta}]})$$

for some $C_3 = C_3(s,p) > 0$. Consequently, for N large enough, we obtain

$$\max_{0 \le t \le T} \|w(t)\|_{\mathcal{F}L^{s,p}} \le \max_{0 \le j < [\frac{T}{\delta}]} \tilde{C}_0^{j+1} N^{-(\sigma-s)} K \le e^{C_3(1+K)^{\gamma} T} N^{-(\sigma-s)} K.$$

The estimate follows by further imposing $N \ge N_0$ where $N_0 \sim \exp\big(\frac{CK^{\gamma}T}{\sigma - s}\big)$.

To establish (b), note that we can also write $w(t)$ as follows

$$\begin{aligned} w(t) = \tilde{\Xi}_j[w](t) &:= S(t - j\delta)w(j\delta) + \int_{j\delta}^{t} S(t - t')\mathbf{P}_{>N}\mathcal{N}(u)(t')\, dt' \\ &\quad + \int_{j\delta}^{t} S(t - t')\mathbf{P}_{\le N}\big\{\mathcal{N}(u) - \mathcal{N}(u - w)\big\}(t')\, dt'. \end{aligned}$$

The estimate then follows from the same arguments as for (a). □

Remark 4.3.8. We can choose $N_0 \sim \exp\big(\frac{CK^{\gamma}T}{\sigma - s}\big)$, for example.

Using Lemma 4.3.6 and Lemma 4.3.7, we establish almost a.s. global well-posedness of the $\mathcal{G}$-gKdV equation (4.3).

Proposition 4.3.9. *Let $s_* < s < 1 - \frac{1}{p}$, $T \geq 1$, and $0 < \varepsilon \leq \frac{1}{2}$. For any $A \geq 1$, there exists $N_1 = N_1(A) \in \mathbb{N}$ such that the set $\Sigma^s_{T,\varepsilon}(A) := \Omega^\sigma_{N_1}(T, \frac{\varepsilon}{2}, A)$, with $\sigma = \frac{1}{2}(s + 1 - \frac{1}{p})$, satisfies:*

(a) $\mu\big(\mathcal{F}L^{s,p}(\mathbb{T}) \setminus \Sigma^s_{T,\varepsilon}(A)\big) < \varepsilon$;

(b) *For $u_0 \in \Sigma^s_{T,\varepsilon}(A)$, there exists a unique corresponding solution $u \in C\big([-T,T]; \mathcal{F}L^{s,p}(\mathbb{T})\big) \cap Z_p^{s,\frac{1}{2}}(T)$ of $\mathcal{G}$-gKdV (4.3) on $[-T,T]$ such that*

$$\|u(t)\|_{\mathcal{F}L^{s,p}} \leq 2\sqrt{2} A C_0 C_\sigma \big(\log \tfrac{T}{\varepsilon}\big)^{\frac{1}{2}}, \quad |t| \leq T.$$

Proof. Lemma 4.3.6 (b) shows that for $u_0 \in \Sigma^s_{T,\varepsilon}(A)$ we have

$$\|\Phi_{N_1}(t)(u_0)\|_{\mathcal{F}L^{\sigma,p}} \leq A C_0 C_\sigma \big(\log \tfrac{2T}{\varepsilon}\big)^{\frac{1}{2}}, \quad |t| \leq T.$$

From Lemma 4.3.7 (a), there exists a unique solution u of $\mathcal{G}$-gKdV on $[-T,T]$ with $u(0) = u_0$ satisfying

$$\|u(t)\|_{\mathcal{F}L^{s,p}} \leq 2 A C_0 C_\sigma \big(\log \tfrac{2T}{\varepsilon}\big)^{\frac{1}{2}}, \qquad |t| \leq T,$$

provided N_1 is large enough. The intended estimate follows from $\log(2x) \leq 2\log x$ for $x \geq 2$. Note that from Lemma 4.3.4 there exists $N_2 \in \mathbb{N}$ such that

$$\int |F_N(u) - F(u)| d\rho(u) < \tfrac{\varepsilon}{2},$$

for $N \geq N_2$. By taking N_1 larger so that the previous bound holds, using Lemma 4.3.6 (a) and the fact that $\mathcal{F}L^{s,p}(\mathbb{T})$, $\mathcal{F}L^{\sigma,p}(\mathbb{T})$ have full μ-measure, we have

$$\mu\big(\mathcal{F}L^{s,p}(\mathbb{T}) \setminus \Sigma^s_{T,\varepsilon}(A)\big) \leq \mu_{N_1}\big(\mathcal{F}L^{\sigma,p}(\mathbb{T}) \setminus \Omega^\sigma_{N_1}(T, \tfrac{\varepsilon}{2}, A)\big) + \int |F_{N_1}(u) - F(u)| d\rho(u) < \varepsilon,$$

which completes the proof. □

We can now show Theorem 1.2.4.

Proof of Theorem 1.2.4. This proof follows the approaches in [105, 80]. We first establish almost sure global well-posedness of $\mathcal{G}$-gKdV. Define an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ by $s_1 = \frac{1}{2}(s_* + 1 - \frac{1}{p})$ and $s_{j+1} = \frac{1}{2}(s_j + 1 - \frac{1}{p})$, which converges to $1 - \frac{1}{p}$ as $j \to \infty$. Fix $0 < \varepsilon \leq 1$ and let $T_j = 2^j$, $\varepsilon_j = 2^{-j}\varepsilon$, $j \in \mathbb{N}$. For $\Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$ as defined in Proposition 4.3.9, with $s = s_j$ and $\sigma = s_{j+1}$, let

$$\Sigma_\varepsilon = \bigcap_{j=1}^\infty \Sigma^{s_j}_{T_j,\varepsilon_j} = \bigcap_{j=1}^\infty \Big(\bigcup_{k=1}^\infty \Sigma^{s_j}_{T_j,\varepsilon_j}(2^k) \Big).$$

Lastly, let $\Sigma = \bigcup_{n=1}^\infty \Sigma_{\frac{1}{n}}$.

First note that $\Sigma \subset \bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T})$. From Lemma 4.3.6 and Proposition 4.3.9, $\Sigma^{s_j}_{T_j,\varepsilon_j} \subset \mathcal{F}L^{s_{j+1},p}(\mathbb{T})$, which implies that

$$\Sigma_{\frac{1}{n}} = \bigcap_{j=1}^\infty \Sigma^{s_j}_{T_j,\varepsilon_j} \subset \bigcap_{j=1}^\infty \mathcal{F}L^{s_{j+1},p}(\mathbb{T}) \subset \bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T}).$$

The last inclusion follows from the fact that $\mathcal{F}L^{s,p}(\mathbb{T}) \subset \mathcal{F}L^{\sigma,p}(\mathbb{T})$ for $\sigma \leq s$, and for each fixed $s < 1 - \frac{1}{p}$, since $(s_j)_{j\in\mathbb{N}}$ is an increasing sequence converging to $1 - \frac{1}{p}$, there exists $j* \in \mathbb{N}$ such that $s \leq s_{j*} < 1 - \frac{1}{p}$. The conclusion for Σ follows from taking a union over $n \in \mathbb{N}$.

Let $u_0 \in \Sigma$. Then, for some $n, k \in \mathbb{N}$ and for any $j \in \mathbb{N}$, we have $u_0 \in \Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$ where $\varepsilon_j = 2^{-j}\frac{1}{n}$. Hence, by Proposition 4.3.9, there exists a solution $u \in C\big([-T_j, T_j]; \mathcal{F}L^{s_j,p}(\mathbb{T})\big) \cap Z^{s_j,\frac{1}{2}}_p(T_j)$ of $\mathcal{G}$-gKdV with $u(0) = u_0$. By uniqueness of local solutions in $Z^{s,\frac{1}{2}}_p(T)$, we obtain a unique global solution $u \in \bigcap_{s<1-\frac{1}{p}} C\big(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})\big)$. Moreover, since $\Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$ is closed in $\mathcal{F}L^{s_1,p}(\mathbb{T})$ and

$$\mu\big(\Sigma^c_{\frac{1}{n}}\big) \leq \sum_{j=1}^{\infty} \mu\big((\Sigma^{s_j}_{T_j,\varepsilon_j})^c\big) < \sum_{j=1}^{\infty} 2^{-j}\frac{1}{n} < \frac{1}{n}.$$

Therefore, Σ is μ-measurable and $\mu(\Sigma^c) \leq \liminf_{n\to\infty} \frac{1}{n} = 0$.

We now establish that $\Phi(t)\Sigma = \Sigma$ for any $t \in \mathbb{R}$, where $\Phi(t) : u_0 \mapsto u(t)$ denotes the solution map of $\mathcal{G}$-gKdV defined above. Fixing $\tau \in \mathbb{R}$, we focus on showing that $\Phi(\tau)\Sigma \subset \Sigma$. Note from this property and the reversibility of the flow, we have that $\Sigma = \Phi(-\tau)\big[\Phi(\tau)\Sigma\big] \subset \Phi(-\tau)\Sigma$, from which we can conclude the equality of the sets. From the definition of Σ, it suffices to show that $\Phi(\tau)\Sigma_\varepsilon \subset \Sigma_\varepsilon$, for each fixed $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$. In fact, we will establish that if $|\tau| \leq T_\ell$ for some $\ell \in \mathbb{N}$, then for every $i \in \mathbb{N}$,

$$\Phi(\tau)\Sigma^{s_j}_{T_j,\varepsilon_j} \subset \Sigma^{s_i}_{T_i,\varepsilon_i}, \qquad \text{for } j = \max(i+2, \ell+1),$$

from which the intended result follows, since this implies that

$$\Phi(\tau)\Sigma_\varepsilon \subset \bigcap_{j=1}^{\infty} \Phi(\tau)\Sigma^{s_j}_{T_j,\varepsilon_j} \subset \bigcap_{i=1}^{\infty} \Phi(\tau)\Sigma^{s_{j(i,\ell)}}_{T_{j(i,\ell)},\varepsilon_{j(i,\ell)}} \subset \bigcap_{i=1}^{\infty} \Sigma^{s_i}_{T_i,\varepsilon_i}.$$

Let $u_0 \in \Sigma^{s_j}_{T_j,\varepsilon_j}$. Then, there exists $k \in \mathbb{N}$ such that $u_0 \in \Sigma^{s_j}_{T_j,\varepsilon_j}(2^k)$. From Proposition 4.3.9, there exists a solution $u(t)$ of $\mathcal{G}$-gKdV for $|t| \leq T_j$ satisfying

$$\|u(t)\|_{\mathcal{F}L^{s_j,p}} \leq (2\sqrt{2})\, 2^k C_0 C_{s_{j+1}} \big(\log \tfrac{T_j}{\varepsilon_j}\big)^{\frac{1}{2}}, \qquad |t| \leq T_j.$$

Note that $u_\tau(t) = u(\tau+t)$ is a solution of $\mathcal{G}$-gKdV with $u_\tau(0) = u(\tau) = \Phi(\tau)u_0$, which belongs to $C\big([-T_{j-1}, T_{j-1}]; \mathcal{F}L^{s_j,p}(\mathbb{T})\big) \cap Z^{s_j,\frac{1}{2}}_p(T_{j-1})$, because $\ell \leq j-1$ and then $|t+\tau| \leq T_{j-1} + T_\ell \leq T_j$. Since the above estimate holds for $u_\tau(t)$ if $|t| \leq T_{j-1}$, from Lemma 4.3.7 (b), it follows that

$$\|\Phi_N(t)\Phi(\tau)u_0\|_{\mathcal{F}L^{s_{j-1},p}} \leq (6\sqrt{2})\, 2^k C_0 C_{s_{j+1}} \big(\log \tfrac{T_j}{\varepsilon_j}\big)^{\frac{1}{2}}, \qquad |t| \leq T_{j-1},$$

for any $N \geq N_0$. Since $i \leq j-2$ and $\frac{T_j}{\varepsilon_j} \leq \big(\frac{2T_i}{\varepsilon_i}\big)^{j/i}$ for $0 < \varepsilon \leq 1$, we get that

$$\|\Phi_N(t)\Phi(\tau)u_0\|_{\mathcal{F}L^{s_{i+1},p}} \leq (6\sqrt{2j/i})\, 2^k C_0 C_{s_{j+1}} \big(\log \tfrac{2T_i}{\varepsilon_i}\big)^{\frac{1}{2}}, \qquad |t| \leq T_{i+1}.$$

Consequently, by choosing $\tilde{k} \in \mathbb{N}$ such that $(6\sqrt{2j/i})\, 2^k C_0 C_{s_{j+1}} \leq 2^{\tilde{k}} C_{s_{i+1}}$ and $N_1(\tilde{k}) \geq N_0$, and applying Lemma 4.3.6 (c), we conclude that $\Phi(\tau)u_0 \in \Sigma^{s_i}_{T_i,\varepsilon_i}(\tilde{k})$. The group property of $\Phi(t)$ follows from uniqueness of local solutions in $Z^{s,\frac{1}{2}}_p(T)$.

Before showing the invariance of μ under the flow map $\Phi(t)$, we show that $\Phi(t)$ is μ-measurable for every $t \in \mathbb{R}$. It suffices to show the continuity of the map in the topology induced by $\mathcal{F}L^{s_1,p}(\mathbb{T})$. Fix $t \in \mathbb{R}$ and $u_0 \in \Sigma$. Consider a sequence $\{u_{0,k}\}_{k\in\mathbb{N}} \subset \Sigma$ converging to u_0 in $\mathcal{F}L^{s_1,p}(\mathbb{T})$. Let $j \in \mathbb{N}$ such that $|t| \leq T_j$. Then, $u_0 \in \Sigma^{s_j}_{T_j,\varepsilon_j}(A)$ for some $\varepsilon = \frac{1}{n}$ and some A. By Proposition 4.3.9, we have

$$\sup_{|t|\leq T_j} \|\Phi(t)u_0\|_{\mathcal{F}L^{s_j,p}} \leq 2\sqrt{2}AC_0C_{s_{j+1}} \log\big(\tfrac{T_j}{\varepsilon_j}\big)^{\frac{1}{2}} =: \Lambda.$$

Let T be the local time of existence for data of size 2Λ in $\mathcal{F}L^{s_1,p}(\mathbb{T})$. From the Lipschitz continuity of the solution map, we obtain

$$\|\Phi(t)u_0 - \Phi(t)u_{0,k}\|_{\mathcal{F}L^{s_1,p}} \le C^{\left[\frac{|t|}{T}\right]}\|u_0 - u_{0,k}\|_{\mathcal{F}L^{s_1,p}},$$

as long as the right-hand side is bounded by Λ, which holds for k large enough. Consequently, by taking $k \to \infty$, we conclude that $\Phi(t)u_{0,k} \to \Phi(t)u_0$ in $\mathcal{F}L^{s_1,p}(\mathbb{T})$.

It remains to show the invariance of the Gibbs measure μ under the flow $\Phi(t)$ of $\mathcal{G}$-gKdV (4.3). Having established the flow property of $\Phi(t)$, it suffices to show that for all $G \in L^1\big(\mathcal{F}L^{s_1,p}(\mathbb{T}), d\mu\big)$ and $t \in \mathbb{R}$, we have

$$\int_\Sigma G\big(\Phi(t)u\big)d\mu(u) = \int_\Sigma G(u)d\mu(u). \tag{4.34}$$

Moreover, it suffices to show (4.34) for G in a dense subset $\mathcal{H}$ of $L^1\big(\mathcal{F}L^{s_1,p}(\mathbb{T}), d\mu\big)$. In particular, we choose $\mathcal{H}$ as the set of continuous and bounded functions on $\mathcal{F}L^{s_1,p}(\mathbb{T})$. Fix $G \in \mathcal{H}$, $t \in \mathbb{R}$ and $\kappa > 0$. We have the following

$$\begin{aligned}
\left|\int_\Sigma G\big(\Phi(t)u\big)d\mu(u) - \int_\Sigma G(u)d\mu(u)\right| &\le \left|\int_\Sigma G\big(\Phi(t)u\big)d\mu(u) - \int_\Sigma G\big(\Phi(t)u\big)d\mu_N(u)\right| \\
&\quad + \left|\int_\Sigma G\big(\Phi(t)u\big)d\mu_N(t) - \int_\Sigma G\big(\Phi_N(t)u\big)d\mu_N(u)\right| \\
&\quad + \left|\int_\Sigma G\big(\Phi_N(t)u\big)d\mu_N(t) - \int_\Sigma G(u)d\mu_N(u)\right| \\
&\quad + \left|\int_\Sigma G(u)d\mu_N(u) - \int_\Sigma G(u)d\mu(u)\right| \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

From Lemma 4.3.4, we have

$$\int \tilde{G}(u)d\mu_N(u) - \int \tilde{G}(u)d\mu(u) = \int \tilde{G}(u)\Big(\frac{F_N(u)}{\|F_N\|_{L^1(d\rho)}} - \frac{F(u)}{\|F\|_{L^1(d\rho)}}\Big)d\rho(u) \to 0, \quad N \to \infty$$

for every bounded measurable function $\tilde{G}$ on $\mathcal{F}L^{s_1,p}(\mathbb{T})$. Consequently, since G is bounded and continuous and $\Phi(t)$ is measurable, there exists $N_0 \in \mathbb{N}$ such that $\mathrm{I} + \mathrm{IV} < \frac{\kappa}{2}$, for $N \ge N_0$. From Proposition 4.3.5, the measure μ_N is invariant under the flow $\Phi_N(t)$, thus $\mathrm{III} = 0$. It only remains to estimate II. For $0 < \varepsilon \le \frac{1}{2}$, consider the set $\Sigma(t,\varepsilon) = \Sigma^{s_2}_{1+|t|,\varepsilon}(1) \subset \mathcal{F}L^{s_3,p}(\mathbb{T})$. From Lemma 4.3.4, there exists $N_1 \in \mathbb{N}$ such that $\mu_N(\Sigma(t,\varepsilon)^c) < \mu(\Sigma(t,\varepsilon)^c) + \varepsilon$ for $N \ge N_1$. Since $\mu(\Sigma(t,\varepsilon)^c) < \varepsilon$ by Proposition 4.3.9, we see that

$$\left|\int_{\Sigma\setminus\Sigma(t,\varepsilon)} G\big(\Phi(t)u\big)d\mu_N(u) - \int_{\Sigma\setminus\Sigma(t,\varepsilon)} G\big(\Phi_N(t)u\big)d\mu_N(u)\right| \le 2\|G\|_{L^\infty}\big(\mu\big(\Sigma(t,\varepsilon)^c\big) + \varepsilon\big) < \frac{\kappa}{4},$$

for $N \ge N_1$ and by choosing $\varepsilon \le \frac{\kappa}{16\|G\|_{L^\infty}}$. In order to estimate the contribution restricted to $\Sigma(t,\varepsilon)$, we want to exploit the continuity of G. In particular, we want to use the fact that there exists $\gamma > 0$ such that if $\big\|\Phi(t)u_0 - \Phi_N(t)u_0\big\|_{\mathcal{F}L^{s_1,p}} < \gamma$, then

$$\big|G(\Phi(t)u_0) - G(\Phi_N(t)u_0)\big| < \frac{\kappa}{4}. \tag{4.35}$$

Consequently, we want to show that we can choose N large enough such that the difference of the flows is less than γ. For $u_0 \in \Sigma \cap \Sigma(t,\varepsilon)$, from Proposition 4.3.9 and uniqueness, we have

$$\|u_0\|_{\mathcal{F}L^{s_2,p}},\ \|\Phi(s)u_0\|_{\mathcal{F}L^{s_2,p}} \le 2\sqrt{2}C_0C_{s_3}\big(\log\tfrac{1+|t|}{\varepsilon}\big)^{\frac{1}{2}}, \qquad |s| \le 1 + |t|.$$

Then, from Lemma 4.3.7 (b) we have

$$\big\|\Phi(t)u_0 - \mathbf{P}_{\le N}\Phi_N(t)u_0\big\|_{\mathcal{F}L^{s_1,p}} \le C(t,\varepsilon)N^{-(s_2-s_1)}$$

for any N large enough. Thus, it follows that

$$\begin{aligned}\big\|\Phi(t)u_0 - \Phi_N(t)u_0\big\|_{\mathcal{F}L^{s_1,p}} &\le \big\|\Phi(t)u_0 - \mathbf{P}_{\le N}\Phi_N(t)u_0\big\|_{\mathcal{F}L^{s_1,p}} + \big\|\mathbf{P}_{>N}\Phi_N(t)u_0\big\|_{\mathcal{F}L^{s_1,p}}\\ &\le C(t,\varepsilon)N^{-(s_2-s_1)} + N^{-(s_2-s_1)}\|u_0\|_{\mathcal{F}L^{s_2,p}}\\ &\le C(t,\varepsilon)N^{-(s_2-s_1)} < \gamma,\end{aligned}$$

by choosing N large enough, say $N \ge N_2$, for some $N_2 \in \mathbb{N}$. Consequently, (4.35) holds and we can estimate the remaining piece of $\mathbb{I}$

$$\bigg|\int_{\Sigma\cap\Sigma(t,\varepsilon)} G\big(\Phi(t)u\big)d\mu_N(u) - \int_{\Sigma\cap\Sigma(t,\varepsilon)} G\big(\Phi_N(t)u\big)d\mu_N(u)\bigg| \le \int \frac{\kappa}{4}\, d\mu_N(u) = \frac{\kappa}{4}.$$

Combining all the estimates, we obtain

$$\bigg|\int_\Sigma G\big(\Phi(t)u\big)d\mu(u) - \int_\Sigma G\big(u\big)d\mu(u)\bigg| < \kappa.$$

Since κ is arbitrarily small, we obtain (4.34), as intended. ☐

Before we establish the invariance of the Gibbs measure μ under the dynamics of the original gKdV equation (4.1), we must consider its solution map $\Psi(t)$. We can define the map $\Psi(t_1,t_2)$ for $t_1,t_2 \in \mathbb{R}$ as

$$\Psi(t_1,t_2)u_0 = \big[\Phi(t_2-t_1)u_0\big]\bigg(x \pm k\int_{t_1}^{t_2} \mathbf{P}_0\big(\Phi(t'-t_1)u_0\big)^{k-1}\,dt'\bigg),$$

which is a solution of gKdV (4.1) at time t_2, with initial data u_0 at time t_1. Since $\Psi(t_1,t_2) = \Psi(0,t_2-t_1)$, we can denote the solution map of gKdV (4.1) at time t as $\Psi(t) := \Psi(0,t)$. The following lemma establishes that the solution map $\Psi(t)$ satisfies the group property.

Lemma 4.3.10. *For any $t,s\in\mathbb{R}$ we have that $\Psi(t+s) = \Psi(t)\Psi(s)$.*

Proof. Let $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$ and $t_1,t_2\in\mathbb{R}$. From the definition of Ψ, we have

$$\Psi(t_1+t_2)u_0 = \big[\Phi(t_1+t_2)u_0\big]\bigg(x \pm k\int_0^{t_1+t_2} \mathbf{P}_0\big(\Phi(t')u_0\big)^{k-1}\,dt'\bigg).$$

Using the group property of Φ and a change of variables, we obtain

$$\begin{aligned}\Psi(t_2)\Psi(t_1)u_0 &= \Psi(t_2)\bigg[\big[\Phi(t_1)u_0\big]\bigg(x \pm k\int_0^{t_1} \mathbf{P}_0\big(\Phi(t')u_0\big)^{k-1}\,dt'\bigg)\bigg]\\ &= \big[\Phi(t_1+t_2)u_0\big]\bigg(x \pm k\int_0^{t_1} \mathbf{P}_0\big(\Phi(t')u_0\big)^{k-1}\,dt' \pm k\int_0^{t_2} \mathbf{P}_0\big(\Phi(t_1+t')u_0\big)^{k-1}\,dt'\bigg)\\ &= \big[\Phi(t_1+t_2)u_0\big]\bigg(x \pm k\int_0^{t_1+t_2} \mathbf{P}_0\big(\Phi(t')u_0\big)^{k-1}\,dt'\bigg),\end{aligned}$$

which is equal to $\Psi(t_1+t_2)u_0$, establishing the group property of the map. ☐

We can now establish the intended invariance of the Gibbs measure under the original flow $\Psi(t)$.

Proof of Theorem 1.2.5. Let Σ be the subset of $\bigcap_{s<1-\frac{1}{p}} \mathcal{F}L^{s,p}(\mathbb{T})$ constructed in Theorem 1.2.4 and denote by $T(y)$, for $y\in\mathbb{T}$, the spatial translation operator $f(x)\mapsto f(x-y)$. Note that Σ is invariant under $T(y)$. Consequently, we can establish the global-in-time dynamics on Σ for the gKdV equation (4.1) with the solution map $\Psi(t)$ satisfying the flow property in Lemma 4.3.10.

It remains to prove the invariance of the Gibbs measure (4.34).[1] Let m denote the Haar measure on $\mathbb{T}$. Fix $A\subset\Sigma$ and $t\in\mathbb{R}$. Using the invariance of μ under $T(y)$, the fact that $T(y)$

[1]This argument was suggested by Terence Tao and Rowan Killip.

and $\Psi(t)$ commute, and Fubini's Theorem, we have

$$\begin{aligned}\mu\big(\Psi(-t)A\big) &= \frac{1}{2\pi}\int_{\mathbb{T}} \mu\big(T(-y)\Psi(-t)A\big)\,d\,\mathrm{m}(y)\\ &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\Sigma} \mathbb{1}_A\big(T(y)\Psi(t)u_0\big)\,d\mu(u_0)\,d\,\mathrm{m}(y)\\ &= \frac{1}{2\pi}\int_{\Sigma}\int_{\mathbb{T}} \mathbb{1}_A\Big[T\Big(y\mp k\int_0^t \mathbf{P}_0(\Phi(t')u_0)^{k-1}dt'\Big)\Phi(t)u_0\Big]\,d\,\mathrm{m}(y)\,d\mu(u_0).\end{aligned}$$

From the translation invariance of m, Fubini's Theorem, and the fact that $\Phi(t)$ commutes with $T(y)$, we have that

$$\begin{aligned}\mu\big(\Psi(-t)A\big) &= \frac{1}{2\pi}\int_{\Sigma}\int_{\mathbb{T}} \mathbb{1}_A\big(T(y)\Phi(t)u_0\big)\,d\,\mathrm{m}(y)\,d\mu(u_0)\\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \mu\big(T(-y)\Phi(-t)A\big)\,d\,\mathrm{m}(y).\end{aligned}$$

Since μ is invariant under $T(y)$ and under the flow map $\Phi(t)$ of (4.3) from Theorem 1.2.4, we get $\mu\big(\Psi(-t)A\big) = \mu\big(\Phi(-t)A\big) = \mu(A)$, as intended. □

4.4 Gauge transform

We start by establishing continuity of the (inverse) gauge transform.

Lemma 4.4.1. *The (inverse) gauge transform in (4.4) is a continuous map on $C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ given that $1\le p<\infty$ and $s>1-\frac{1}{p}-\frac{1}{k-1}$.*

Proof. Let u be any function in $C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$. Consider $\{u_m\}_{m\in\mathbb{N}} \subset C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ a sequence converging to u and fix $t\in[-T,T]$. Then,

$$\begin{aligned}&\big\|\mathcal{G}_{0,t}\big(u(t)\big)-\mathcal{G}_{0,t}\big(u_m(t)\big)\big\|_{\mathcal{F}L^{s,p}}\\ &\quad= \big\|\langle n\rangle^s\big(e^{ink\int_0^t \mathbf{P}_0(u^{k-1}(t'))dt'}\widehat{u}(t,n) - e^{ink\int_0^t \mathbf{P}_0(u_m^{k-1}(t'))dt'}\widehat{u}_m(t,n)\big)\big\|_{\ell^p_n}\\ &\quad\le 2\big\|\mathbb{1}_{|n|>N}\langle n\rangle^s\widehat{u}(t,n)\big\|_{\ell^p_n} + \|u(t)-u_m(t)\|_{\mathcal{F}L^{s,p}}\\ &\qquad + \|u(t)\|_{\mathcal{F}L^{s,p}}\big\|\mathbb{1}_{|n|\le N}\big(e^{ink\int_0^t \mathbf{P}_0(u^{k-1}(t'))dt'} - e^{ink\int_0^t \mathbf{P}_0(u_m^{k-1}(t'))dt'}\big)\big\|_{\ell^\infty_n}.\end{aligned}$$

The first two terms on the right-hand side of the estimate converge to zero as $N\to\infty$ and $m\to\infty$. Thus, it only remains to consider the last one. Using the mean-value theorem, we have

$$\big\|\mathbb{1}_{|n|\le N}\big(e^{ink\int_0^t \mathbf{P}_0(u^{k-1}(t'))dt'} - e^{ink\int_0^t \mathbf{P}_0(u_m^{k-1}(t'))dt'}\big)\big\|_{\ell^\infty_n} \le |t|N\|u^{k-1}-u_m^{k-1}\|_{C_{|t|}L^1}.$$

Since $\mathcal{F}L^{s,p}(\mathbb{T})\hookrightarrow L^{k-1}(\mathbb{T})$ for $s>1-\frac{1}{p}-\frac{1}{k-1}$, then the above quantity converges to zero for each fixed N, establishing the continuity of $\mathcal{G}_{0,t}$. An analogous proof works for $\mathcal{G}_{0,t}^{-1}$. □

Following the argument in [44], we establish the following result for the (inverse) gauge transform in (4.4).

Proposition 4.4.2. *Let $1\le p<\infty$ and $s>1-\frac{1}{p}-\frac{1}{k-1}$. Then, the (inverse) gauge transform in (4.4) is not uniformly continuous on arbitrarily small balls of $C\big([-T,T];\mathcal{F}L^{s,p}(\mathbb{T})\big)$ centered at the origin.*

Proof. Let $R>0$ and $N\in\mathbb{N}$. Define $\{u_{N,j}\}_{N\in\mathbb{N}}$ for $j=1,2$ as follows

$$\begin{aligned}u_{N,1}(t,x) &= RN^{-s}(e^{iNx}+e^{-iNx}) + N^{-\frac{1}{k-1}}(e^{iMx}+e^{-iMx}),\\ u_{N,2}(t,x) &= RN^{-s}(e^{iNx}+e^{-iNx}),\end{aligned}$$

with $M = 0$ for k even, and $M = 1$ for k odd. Note that

$$\|u_{N,1}\|_{C_T \mathcal{F}L^{s,p}} \lesssim R,$$

for N large enough, and $\|u_{N,2}\|_{C_T \mathcal{F}L^{s,p}} \sim R$. Moreover,

$$\|u_{N,1} - u_{N,2}\|_{C_T \mathcal{F}L^{s,p}} \sim N^{-\frac{1}{k-1}} \to 0,$$

as $N \to \infty$. Using mean value theorem, we obtain

$$\|\mathcal{G}_{0,t}(u_{N,1}) - \mathcal{G}_{0,t}(u_{N,2})\|_{C_T \mathcal{F}L^{s,p}} \geq TN \Big| \int_{\mathbb{T}} \big(u_{N,1}^{k-1}(x) - u_{N,2}^{k-1}(x)\big)\, dx \Big|.$$

Calculating $\int_{\mathbb{T}} (u_{N,1}^{k-1} - u_{N,2}^{k-1})\, dx$, we have

$$\sim \sum_{\substack{1 \leq j \leq k-1 \\ 0 \leq l \leq k-1-j \\ 0 \leq m \leq j}} \binom{k-1}{j} N^{-s(k-1-j) - \frac{j}{k-1}} \int_{\mathbb{T}} e^{iNx(k-j-2l-1) + iMx(j-2m)}.$$

Thus, the nonzero contributions correspond to the choices of indices satisfying $k - 1 - j = 2l$ and $M(j - 2m) = 0$, since $N \gg M$. Consequently, we see that the quantity is dominated by the contribution at $j = k - 1$, therefore

$$\|\mathcal{G}_{0,t}(u_{N,1}) - \mathcal{G}_{0,t}(u_{N,2})\|_{C_T \mathcal{F}L^{s,p}} \gtrsim 1,$$

which does not decay as $N \to \infty$. □

Appendix A

Appendix

A.1 Choice of η

We can, for example, choose η as follows. Consider another function ψ satisfying $\psi(t) = e^{it}\eta(t)$. Then, $\widehat{\psi}(\tau) = \widehat{\eta}(\tau - 1)$ and the conditions on η impose

$$\begin{aligned} \widehat{\psi}(0) &= 0, \\ \mathcal{H}\widehat{\psi}(0) &= -1. \end{aligned}$$

The first one means that ψ is a mean-zero function, i.e.,

$$\widehat{\psi}(0) = \frac{1}{2\pi} \int\limits_{\mathbb{R}} \psi(t)\ dt = 0.$$

To understand the second condition, note that

$$\mathcal{F}_\tau(\mathcal{H}\widehat{\psi}(\tau))(t) = i\,\mathrm{sgn}(t)\mathcal{F}_\tau(\widehat{\psi}(\tau))(t) = \frac{i}{2\pi}\,\mathrm{sgn}(t)\psi(-t).$$

Then, from the second condition we have

$$\begin{aligned} -1 = \mathcal{H}\widehat{\psi}(0) &= \mathcal{F}_\tau^{-1}\Big(\frac{i}{2\pi}\,\mathrm{sgn}(t)\psi(-t)\Big)(0) \\ &= \frac{1}{2\pi}\int\limits_{\mathbb{R}} i\,\mathrm{sgn}(t)\psi(-t)\ dt \\ &= \frac{i}{2\pi}\Big(\int_{-\infty}^{0}\psi(t)\ dt - \int_0^\infty \psi(t)\ dt\Big) \\ &= \frac{i}{2\pi}\Big(\int_{-\infty}^{0}\psi(t)\ dt + \underbrace{\int\limits_{\mathbb{R}}\psi(t)\ dt}_{=0} - \int_0^\infty \psi(t)\ dt\Big) \\ &= \frac{i}{\pi}\int_{-\infty}^{0}\psi(t)\ dt. \end{aligned}$$

Thus, we have

$$\int_0^\infty \psi(t)\ dt = -i\pi.$$

Rewriting these assumptions with respect to η, we get

$$\int\limits_{\mathbb{R}} e^{it}\eta(t)\ dt = 0,$$

$$\int_0^\infty e^{it}\eta(t)\ dt = -i\pi.$$

An example of a function ψ satisfying the conditions above is $\psi(t) = -\frac{4\pi i}{\sqrt{2\pi}} t e^{-t^2}$.

A.2 First step of the second iteration process for w

Here we present the equation for w after using second iteration once, following the strategy described in Section 3.3.

$$\begin{aligned}
w = {} & \varphi(t)S(t)u_0 + \varphi_T \cdot \mathcal{DR}(u,u,u) \\
& + \varphi_T\big(\mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq}\big)(w,\overline{w},w) + \varphi_T\big(\mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>}\big)(w,w,\overline{w}) \\
& + \varphi_T\big(\mathbf{B}^0_{A,\geq}(w,\overline{u},u) + \mathbf{B}^1_{A,\geq}(w,\overline{u},u) + \mathbf{B}^2_{A,\geq}(w,\overline{w},u) + \mathbf{B}^3_{A,\geq}(w,\overline{u},w)\big) \\
& + \varphi_T\big(\mathbf{B}^0_{A,>}(w,u,\overline{u}) + \mathbf{B}^1_{A,>}(w,u,\overline{u}) + \mathbf{B}^2_{A,>}(w,w,\overline{u}) + \mathbf{B}^3_{A,>}(w,u,\overline{w})\big) \\
& + \varphi_T\big(\mathbf{B}^0_{B,\geq}(w,\overline{w},u) + \mathbf{B}^1_{B,\geq}(w,\overline{w},u) + \mathbf{B}^2_{B,\geq}(w,\overline{w},u) + \mathbf{B}^3_{B,\geq}(w,\overline{w},w)\big) \\
& + \varphi_T\big(\mathbf{B}^0_{B,>}(w,w,\overline{u}) + \mathbf{B}^1_{B,>}(w,w,\overline{u}) + \mathbf{B}^2_{B,>}(w,w,\overline{u}) + \mathbf{B}^3_{B,>}(w,w,\overline{w})\big) \\
& + \varphi_T(\mathcal{DNR}_{A,\geq} + \mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],\overline{u},u\big) \\
& + \varphi_T(\mathcal{DNR}_{A,\geq} + \mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],\overline{u},u\big) \\
& + \varphi_T(\mathcal{DNR}_{A,\geq} + \mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u],\overline{u},u\big) \\
& + \varphi_T(\mathcal{DNR}_{A,\geq} + \mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}],\overline{u},u\big) \\
& + \varphi_T(\mathcal{DNR}_{A,>} + \mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],u,\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{A,>} + \mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],u,\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{A,>} + \mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u],u,\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{A,>} + \mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}],u,\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]},u\big) \\
& + \varphi_T(\mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]},u\big) \\
& + \varphi_T(\mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]},u\big) \\
& + \varphi_T(\mathcal{DNR}_{B,\geq} + \mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]},u\big) \\
& + \varphi_T(\mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u],\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{B,>} + \mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}],\overline{u}\big) \\
& + \varphi_T(\mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]\big) \\
& + \varphi_T(\mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]\big) \\
& + \varphi_T(\mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]\big) \\
& + \varphi_T(\mathcal{DNR}_{C,\geq} + \mathcal{DNR}_{D,\geq})\big(w,\overline{w},\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]\big) \\
& + \varphi_T(\mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,w,\overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]}\big) \\
& + \varphi_T(\mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,w,\overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]}\big) \\
& + \varphi_T(\mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,w,\overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]}\big) \\
& + \varphi_T(\mathcal{DNR}_{C,>} + \mathcal{DNR}_{D,>})\big(w,w,\overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]}\big) \\
& + \varphi_T\Big(\mathbf{B}^2_{A,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]},u\big) + \mathbf{B}^2_{A,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]},u\big)\Big) \\
& + \varphi_T\Big(\mathbf{B}^2_{A,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]},u\big) + \mathbf{B}^2_{A,\geq}\big(w,\overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]},u\big)\Big) \\
& + \varphi_T\Big(\mathbf{B}^2_{A,>}\big(w,\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u],\overline{u}\big) + \mathbf{B}^2_{A,>}\big(w,\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}],\overline{u}\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \varphi_T \Big(\mathbf{B}^2_{A,>}\big(w, \varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u], \overline{u}\big) + \mathbf{B}^2_{A,>}\big(w, \varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}], \overline{u}\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{A,\geq}\big(w, \overline{u}, \varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]\big) + \mathbf{B}^3_{A,\geq}\big(w, \overline{u}, \varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{A,\geq}\big(w, \overline{u}, \varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]\big) + \mathbf{B}^3_{A,\geq}\big(w, \overline{u}, \varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{A,>}\big(w, u, \overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]}\big) + \mathbf{B}^3_{A,>}\big(w, u, \overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]}\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{A,>}\big(w, u, \overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]}\big) + \mathbf{B}^3_{A,>}\big(w, u, \overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]}\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{B,\geq}\big(w, \overline{w}, \varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]\big) + \mathbf{B}^3_{B,\geq}\big(w, \overline{w}, \varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{B,\geq}\big(w, \overline{w}, \varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]\big) + \mathbf{B}^3_{B,\geq}\big(w, \overline{w}, \varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{B,>}\big(w, w, \overline{\varphi_T \cdot \mathbf{G}_{A,\geq}[w,\overline{u},u]}\big) + \mathbf{B}^3_{B,>}\big(w, w, \overline{\varphi_T \cdot \mathbf{G}_{A,>}[w,u,\overline{u}]}\big) \Big) \\
&+ \varphi_T \Big(\mathbf{B}^3_{B,>}\big(w, w, \overline{\varphi_T \cdot \mathbf{G}_{B,\geq}[w,\overline{w},u]}\big) + \mathbf{B}^3_{B,>}\big(w, w, \overline{\varphi_T \cdot \mathbf{G}_{B,>}[w,w,\overline{u}]}\big) \Big). \qquad \text{(A.1)}
\end{aligned}
$$

A.3 Second step of the second iteration process for w

We recall how to control the contributions in Section 3.5.2. In Proposition 3.5.7, we establish an estimate which can be applied to these terms given that one of the following conditions is satisfied:

1. There are no pairings in $(n_1, \dots, n_5)$ and the largest frequency corresponds to a function in Z_0^s;

2. There is one pairing $n_i + n_j = 0$ and the largest frequency in $\{|n_k| :\ 1 \leq k \leq 5,\ k \neq i, j\}$ corresponds to a function in Z_0^s;

3. There are two pairings and the remaining frequency corresponds to a functions in Z_0^s.

If the contributions do not satisfy any of the above conditions, then the largest frequency that is not in a pairing corresponds to a function u and we want to use the equation for u again. This leads to one quintic term that satisfies the assumptions above and four septic terms. In the following, we decompose the contributions in (3.29), indicating when it suffices to apply Proposition 3.5.7 and when we have to substitute a particular term by the equation for u. The resulting cubic term can be estimated by Proposition 3.5.7 and we apply Proposition 3.5.9 to the four septic terms.

Let $\mathbb{X}_B = \mathbb{X}_{B_1} \cup \mathbb{X}_{B_2}$ where

$$
\begin{aligned}
\mathbb{X}_{B_1}(n) &= \big\{(n_1, n_2, n_3) \in \mathbb{Z}^3 :\ n = n_1 + n_2 + n_3,\ |n_3| \ll |n| \lesssim |n_1| \sim |n_2|\big\}, \\
\mathbb{X}_{B_2}(n) &= \big\{(n_1, n_2, n_3) \in \mathbb{Z}^3 :\ n = n_1 + n_2 + n_3,\ |n_3| \ll |n_1| \ll |n| \sim |n_2|\big\}.
\end{aligned}
$$

- $\underline{\mathcal{DNR}_A\big(\varphi_T \cdot \mathbf{G}_A[w_1, u_2, u_3], u_4, u_5\big)}$: The frequencies satisfy the following

$$
\begin{aligned}
|n_5| \leq |n_4| \ll |n_0| \sim |n_1| \sim |n|, \\
|n_3| \leq |n_2| \ll |n_0|,
\end{aligned}
$$

with possible pairings $(2,4), (2,5), (3,4), (3,5)$. Since $|n| \sim |n_1|$ and n_1 is never part of pairing we can apply Proposition 3.5.7 directly, with $w_1 \in Z_0^s$.

- $\underline{\mathcal{DNR}_A\big(\varphi_T \cdot \mathbf{G}_B[w_1, w_2, u_3], u_4, u_5\big)}$: If $(n_1, n_2, n_3) \in \mathbb{X}_{B_1}(n_0)$, then

$$
\begin{aligned}
|n_4| \leq |n_5| \ll |n_0| \sim |n| \lesssim |n_1| \sim |n_2|, \\
|n_3| \ll |n_0|,
\end{aligned}
$$

so the only pairings are $(3,4),(3,5)$ and $|n| \lesssim |n_1| \sim |n_2|$, so we can apply Proposition 3.5.7 with $w_1 \in Z_0^s$. If $(n_1,n_2,n_3) \in \mathbb{X}_{B_2}(n_0)$, then

$$|n_4| \leq |n_5| \ll |n_0| \sim |n_2| \sim |n|,$$
$$|n_3| \ll |n_1| \ll |n_0|,$$

so the possible pairings are $(1,4),(1,5),(3,4),(3,5)$ and $|n| \sim |n_2|$, so we can apply Proposition 3.5.7 with $w_2 \in Z_0^s$.

• $\underline{\mathcal{DNR}_B\big(\varphi_T \cdot \mathbf{G}_A[w_1,u_2,u_3],u_4,u_5\big)}$: If $(n_0,n_4,n_5) \in \mathbb{X}_{B_1}(n)$, then

$$|n_5| \ll |n| \lesssim |n_0| \sim |n_1| \sim |n_4|,$$
$$|n_3| \leq |n_2| \ll |n_0|,$$

so the possible pairings are $(1,4),(2,5),(3,5)$. We proceed as follows:

- $(1,4)$ not a pairing: $|n| \sim |n_1|$, use Proposition 3.5.7 and with $w_1 \in Z_0^s$;
- $(1,4)$ only pairing: $|n| \lesssim \max(|n_2|,|n_3|,|n_5|) = |n_2|$, use equation on u_2;
- $(1,4),(2,5)$ pairings: $n = n_3$, use equation on u_3;
- $(1,4),(3,5)$ pairings: $n = n_2$, use equation on u_2.

Now consider $(n_0,n_4,n_5) \in \mathbb{X}_{B_2}(n)$, then

$$|n_5| \ll |n_0| \sim |n_1| \ll |n| \sim |n_4|,$$
$$|n_3| \leq |n_2| \ll |n_0|,$$

so the possible pairings are $(2,5),(3,5)$ and $|n| \sim |n_4|$. Since $u_4 \notin Z_0^s$ but n_4 is never in a pairing, it suffices to use the equation on u_4.

• $\underline{\mathcal{DNR}_B\big(\varphi_T \cdot \mathbf{G}_B[w_1,w_2,u_3],u_4,u_5\big)}$: If $(n_0,n_4,n_5) \in \mathbb{X}_{B_1}(n), (n_1,n_2,n_3) \in \mathbb{X}_{B_1}(n_0)$, we have

$$|n_5| \ll |n| \lesssim |n_0| \sim |n_4| \lesssim |n_1| \sim |n_2|,$$
$$|n_3| \ll |n_0|,$$

with possible pairings $(1,4),(2,4),(3,5)$. Since $|n| \lesssim |n_1| \sim |n_2|$, and n_1,n_2 cannot be in a pairing at the same time, we can use Proposition 3.5.7 and place w_1 or w_2 in Z_0^s.

If $(n_0,n_4,n_5) \in \mathbb{X}_{B_1}(n), (n_1,n_2,n_3) \in \mathbb{X}_{B_2}(n_0)$, we have

$$|n_5| \ll |n| \lesssim |n_0| \sim |n_2| \sim |n_4|,$$
$$|n_3| \ll |n_1| \ll |n_0|,$$

with possible pairings $(2,4),(1,5),(3,5)$. We proceed as follows:

- $(2,4)$ not a pairing: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- $(2,4)$ only pairing: $|n| \sim |n_1|$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- $(2,4),(1,5)$ pairings: $n = n_3$ and use the equation on u_3;
- $(2,4),(3,5)$ pairings: $n = n_1$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$.

If $(n_0,n_4,n_5) \in \mathbb{X}_{B_2}(n), (n_1,n_2,n_3) \in \mathbb{X}_{B_1}(n_0)$, then

$$|n_5| \ll |n_0| \ll |n| \sim |n_4|,$$
$$|n_3| \ll |n_0| \lesssim |n_1| \sim |n_2|,$$

so the possible pairings are $(1,4),(2,4),(3,5)$. We proceed as follows:

- $(1,4)$ and $(2,4)$ not a pairing: $|n| \sim |n_4|$ and use equation on u_4;

- $(1,4)$ is a pairing: $|n| \sim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- $(2,4)$ is a pairing: $|n| \sim |n_1|$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$.

If $(n_0, n_4, n_5) \in \mathbb{X}_{B_2}(n), (n_1, n_2, n_3) \in \mathbb{X}_{B_2}(n_0)$, then

$$|n_5| \ll |n_0| \sim |n_2| \ll |n| \sim |n_4|,$$
$$|n_3| \ll |n_1| \ll |n_0|,$$

with possible pairings $(1,5),(3,5)$. Since $|n| \sim |n_4|$ and n_4 is never in a pairing, we use the equation on u_4.

• $\underline{\mathcal{DNR}_C\big(\varphi_T \cdot \mathbf{G}_A[w_1, u_2, u_3], u_4, u_5\big)}$: The frequencies satisfy the following:

$$|n| \lesssim |n_5| \ll |n_0| \sim |n_1| \sim |n_4|,$$
$$|n_3| \leq |n_2| \ll |n_0|,$$

with possible pairings $(1,4),(2,5),(3,5)$. We proceed as follows:

- $(1,4)$ is not a pairing: $|n| \sim |n_1|$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- $(1,4)$ only pairing: $|n| \lesssim |n_5|$ and use equation on u_5;
- $(1,4),(2,5)$ pairings: $n = n_3$ and use equation on u_3;
- $(1,4),(3,5)$ pairings: $n = n_2$ and use equation on u_2.

• $\underline{\mathcal{DNR}_C\big(\varphi_T \cdot \mathbf{G}_B[w_1, w_2, u_3], u_4, u_5\big)}$: If $(n_1, n_2, n_3) \in \mathbb{X}_{B_1}(n_0)$:

$$|n| \lesssim |n_5| \ll |n_0| \sim |n_4| \lesssim |n_1| \sim |n_2|,$$
$$|n_3| \ll |n_0|,$$

with possible pairings $(1,4),(2,4),(3,5)$. Since $|n| \lesssim |n_1| \sim |n_2|$ and n_1, n_2 are never in a pairing at the same time, we can apply Proposition 3.5.7 and place the function in $\{w_1, w_2\}$ whose frequency is not in a pairing in Z_0^s.

If $(n_1, n_2, n_3) \in \mathbb{X}_{B_2}(n_0)$, then:

$$|n| \lesssim |n_5| \ll |n_0| \sim |n_2| \sim |n_4|,$$
$$|n_3| \ll |n_1| \ll |n_0|,$$

with possible pairings $(2,4),(1,5),(3,5)$. We proceed as follows:

- $(2,4)$ not a pairing: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- $(2,4)$ only pairing: $|n| \lesssim |n_5|$ and use the equation on u_5;
- $(2,4),(1,5)$ pairings: $n = n_3$ and use the equation on u_3;
- $(2,4),(3,5)$ pairings: $n = n_1$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$.

• $\underline{\mathcal{DNR}_D\big(\varphi_T \cdot \mathbf{G}_A[w_1, u_2, u_3], u_4, u_5\big)}$: The frequencies satisfy the following:

$$|n_3| \leq |n_2| \ll |n_1| \sim |n_0| \lesssim |n_5| \leq |n_4|,$$

with possible pairings $(1,4),(1,5)$ and $|n| \lesssim |n_4|$. Thus, we use the equation on u_4.

• $\underline{\mathcal{DNR}_D\big(\varphi_T \cdot \mathbf{G}_B[w_1, w_2, u_3], u_4, u_5\big)}$: If $(n_1, n_2, n)_3 \in \mathbb{X}_{B_1}(n_0)$:

$$|n_0| \lesssim |n_5| \leq |n_4|,$$
$$|n_3| \ll |n_0| \lesssim |n_1| \sim |n_2|,$$

with possible pairings $(1,4),(1,5),(2,4),(2,5)$. We proceed as follows:

- $(1,4)$ and $(2,4)$ are not pairings: $|n| \lesssim |n_4|$ and we use the equation on u_4;
- $(1,4)$ or $(2,4)$ only pairing: $|n| \lesssim |n_5|$ and we use the equation on u_5;
- Two pairings: use the equation on u_3.

If $(n_1, n_2, n_4) \in \mathbb{X}_{B_2}(n_0)$:

$$|n_3| \ll |n_1| \ll |n_0| \sim |n_2| \lesssim |n_5| \leq |n_4|,$$

with possible pairings $(2,4), (2,5)$. We proceed as follows:

- $(2,4)$ is a pairing: $|n| \lesssim |n_5|$ and use the equation on u_5;
- $(2,5)$ is a pairing: $|n| \lesssim |n_4|$ and use the equation on u_4.

- $\underline{\mathcal{DNR}_B\big(w_1, \varphi_T \cdot \mathbf{G}_A[w_2, u_3, u_4], u_5\big)}$: If $(n_1, n_0, n_5) \in \mathbb{X}_{B_1}(n)$:

$$|n_5| \ll |n| \lesssim |n_0| \sim |n_1| \sim |n_2|,$$
$$|n_4| \leq |n_3| \ll |n_0|,$$

with pairings $(1,2), (3,5), (4,5)$. We proceed as follows:

- $(1,2)$ not a pairing: $|n| \lesssim |n_1|$ and we use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- $(1,2)$ only pairing: $|n| \lesssim |n_3|$ and use the equation on u_3;
- $(1,2), (3,5)$ pairings: $n = n_4$ and use the equation on u_4;
- $(1,2), (4,5)$ pairings: $n = n_3$ and use the equation on u_3.

If $(n_1, n_0, n_5) \in \mathbb{X}_{B_2}(n)$:

$$|n_5| \ll |n_1| \ll |n_0| \sim |n_2| \sim |n|,$$
$$|n_4| \leq |n_3| \ll |n_0|,$$

with pairings $(1,3), (1,4), (3,5), (4,5)$. Since $|n| \sim |n_2|$ and n_2 is not in a pairing, we can use Proposition 3.5.7 with $w_2 \in Z_0^s$.

- $\underline{\mathcal{DNR}_B\big(w_1, \varphi_T \cdot \mathbf{G}_B[w_2, w_3, u_4], u_5\big)}$: If $(n_1, n_0, n_5), (n_2, n_3, n_4) \in \mathbb{X}_{B_1}(n)$:

$$|n_5| \ll |n| \lesssim |n_0| \sim |n_1| \lesssim |n_2| \sim |n_3|,$$
$$|n_4| \ll |n_0|,$$

with possible pairings $(1,2), (1,3), (4,5)$. Since $|n| \lesssim |n_1| \sim |n_2| \sim |n_3|$ and n_1, n_2, n_3 are not all in a pairing at the same time, we apply Proposition 3.5.7 with the term in $\{w_1, w_2, w_3\}$ which is not in a pairing placed in Z_0^s. If $(n_1, n_0, n_5) \in \mathbb{X}_{B_1}(n), (n_2, n_3, n_4) \in \mathbb{X}_{B_2}(n_0)$:

$$|n_5| \ll |n| \lesssim |n_0| \sim |n_1| \sim |n_3|,$$
$$|n_4| \ll |n_2| \ll |n_0|,$$

with possible pairings $(1,3), (2,4), (2,5)$. We proceed as follows:

- $(1,3)$ not a pairing: $|n| \lesssim |n_1|$ and we use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- $(1,3)$ is a pairing but not $(2,5)$: $|n| \lesssim |n_2|$ and we use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- If $(1,3), (2,5)$ pairings: $n = n_4$ and we use the equation on u_4.

If $(n_1, n_0, n_5) \in \mathbb{X}_{B_2}(n), (n_2, n_3, n_4) \in \mathbb{X}_{B_1}(n_0)$:

$$|n_5| \ll |n_1| \ll |n| \sim |n_0| \lesssim |n_2| \sim |n_3|,$$
$$|n_4| \ll |n|,$$

with possible pairings $(1,2),(1,3),(4,5)$. Since n_2, n_3 are not in a pairing at the same time and $|n| \sim |n_2| \sim |n_3|$, we can apply Proposition 3.5.7 with w_2 or w_3 in Z_0^s.

If $(n_1, n_0, n_5) \in \mathbb{X}_{B_2}(n), (n_2, n_3, n_4) \in \mathbb{X}_{B_2}(n_0)$:

$$|n_5| \ll |n_1| \ll |n_0| \sim |n_3| \sim |n|,$$
$$|n_4| \ll |n_2| \ll |n_0|,$$

with possible pairings $(1,2),(1,4),(2,5),(4,5)$. Since $|n| \sim |n_3|$ and n_3 is never in a pairing, we apply Proposition 3.5.7 with $w_3 \in Z_0^s$.

- $\underline{\mathcal{DNR}_C(w_1, \varphi_T \cdot \mathbf{G}_A[w_2, u_3, u_4], u_5)}$: We have the following assumptions:

$$|n| \lesssim |n_5| \ll |n_0| \sim |n_1| \sim |n_2|,$$
$$|n_4| \leq |n_3| \ll |n_0|,$$

with possible pairings $(1,2),(3,5),(4,5)$. We proceed as follows:

  - $(1,2)$ not a pairing: $|n| \lesssim |n_1|$ and we use Proposition 3.5.7 with $w_1 \in Z_0^s$;
  - $(1,2)$ only pairing: $|n| \lesssim |n_5|$ and we use the equation on u_5;
  - $(1,2),(3,5)$ pairings: $n = n_4$ and we use the equation on u_4;
  - $(1,2),(4,5)$ pairings: $n = n_3$ and we use the equation on u_3.

- $\underline{\mathcal{DNR}_C(w_1, \varphi_T \cdot \mathbf{G}_B[w_2, w_3, u_4], u_5)}$: If $(n_2, n_3, n_4) \in \mathbb{X}_{B_1}(n_0)$:

$$|n| \lesssim |n_5| \ll |n_0| \sim |n_1| \lesssim |n_2| \sim |n_3|,$$
$$|n_4| \ll |n_0|,$$

with possible pairings $(1,2),(1,3),(4,5)$. We proceed as follows:

  - $(1,2)$ and $(1,3)$ are not pairings: $|n| \lesssim |n_1|$ and we use Proposition 3.5.7 with $w_1 \in Z_0^s$;
  - $(1,2)$ is a pairing: $|n| \lesssim |n_3|$ and we use Proposition 3.5.7 with $w_3 \in Z_0^s$;
  - $(1,3)$ is a pairing: $|n| \lesssim |n_2|$ and we use Proposition 3.5.7 with $w_2 \in Z_0^s$.

If $(n_2, n_3, n_4) \in \mathbb{X}_{B_2}(n_0)$:

$$|n| \lesssim |n_5| \ll |n_0| \sim |n_1| \sim |n_3|,$$
$$|n_4| \ll |n_2| \ll |n_0|,$$

with possible pairings $(1,3),(2,5),(4,5)$. We proceed as follows:

  - $(1,3)$ not a pairing: $|n| \lesssim |n_1|$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$;
  - $(1,3)$ pairing but not $(2,5)$: $|n| \lesssim |n_2|$ and we can use Proposition 3.5.7 with $w_2 \in Z_0^s$;
  - $(1,3),(2,5)$ are pairings: $n = n_4$ and we use the equation on u_4.

- $\underline{\mathcal{DNR}_D(w_1, \varphi_T \cdot \mathbf{G}_A[w_2, u_3, u_4], u_5)}$: We have the following:

$$|n_1| \lesssim |n_5| \leq |n_0| \sim |n_2|,$$
$$|n_4| \leq |n_3| \ll |n_0|,$$

with possible pairings $(1,2),(1,3),(1,4),(2,5),(3,5),(4,5)$ We proceed as follows:

  - $(1,2)$ and $(2,5)$ not pairings: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
  - $(1,2)$ is a pairing but not $(2,5)$: $|n| \lesssim |n_1| = |n_5| = |n_0| = |n_2| \gg |n_3| \geq |n_4|$, so n_5 is not in a pairing and we use the equation on u_5;

- (2, 5) only pairing: $|n| \lesssim |n_1|$ use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- (2, 5), (1, 3) pairings: $n = n_4$ and use the equation on u_4;
- (2, 5), (1, 4) pairings: $n = n_3$ and use the equation on u_3.

- $\underline{\mathcal{DNR}_D\big(w_1, \varphi_T \cdot \mathbf{G}_B[w_2, w_3, u_4], u_5\big)}$: If $(n_2, n_3, n_4) \in \mathbb{X}_{B_1}(n_0)$:

$$|n_1| \lesssim |n_5| \leq |n_0| \lesssim |n_2| \sim |n_3|,$$
$$|n_4| \ll |n_0|,$$

with possible pairings (1, 2), (1, 3), (1, 4), (2, 5), (3, 5), (4, 5). We proceed as follows:

- (1, 2) and (2, 5) not pairings: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- (1, 3) and (3, 5) not pairings: $|n| \lesssim |n_3|$ and use Proposition 3.5.7 with $w_3 \in Z_0^s$;
- (1, 2), (3, 5) or (1, 3), (2, 5) pairings: $n = n_4$ and use the equation on u_4.

If $(n_2, n_3, n_4) \in \mathbb{X}_{B_2}(n_0)$:

$$|n_1| \lesssim |n_5| \leq |n_0| \sim |n_3|,$$
$$|n_4| \ll |n_2| \ll |n_0|,$$

with possible pairings (1, 2), (1, 3), (1, 4), (2, 5), (3, 5), (4, 5). We proceed as follows:

- (1, 3) and (3, 5) not pairings: $|n| \lesssim |n_3|$ and use Proposition 3.5.7 with $w_3 \in Z_0^s$;
- (1, 3) pairing but not (4, 5), (1, 4): $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- (1, 3), (4, 5) or (3, 5), (1, 4) pairings: $n = n_2$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- (3, 5) pairing but not (2, 5), (1, 2): $|n| \lesssim \max(|n_1|, |n_2|)$ and use Proposition 3.5.7 with w_1 or w_2 in Z_0^s;
- (1, 3), (2, 5) or (3, 5), (1, 2) pairings: $n = n_4$ and use the equation on u_4.

- $\underline{\mathcal{DNR}_D\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_A[w_3, u_4, u_5]\big)}$: We have the following:

$$|n_1| \lesssim |n_0| \sim |n_3| \leq |n_2|,$$
$$|n_5| \leq |n_4| \ll |n_0|$$

with possible pairings (1, 3), (1, 4), (1, 5), (2, 3). We proceed as follows:

- (2, 3) not a pairing: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- (2, 3) only pairing: $|n| \lesssim |n_1|$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- (2, 3), (1, 4) pairings: $n = n_5$ and use the equation on u_5;
- (1, 5), (2, 3) pairings: $n = n_4$ and use the equation on u_4.

- $\underline{\mathcal{DNR}_D\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5]\big)}$: If $(n_3, n_4, n_5) \in \mathbb{X}_{B_1}(n_0)$:

$$|n_1| \lesssim |n_0| \leq |n_2|,$$
$$|n_5| \ll |n_0| \lesssim |n_3| \sim |n_4|,$$

with possible pairings (1, 3), (1, 4), (1, 5), (2, 3), (2, 4). We proceed as follows:

- (2, 3) and (2, 4) not pairings: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- (2, 3) pairing but not (1, 4): $|n| \lesssim |n_4|$ and use Proposition 3.5.7 with $w_4 \in Z_0^s$;

- $(2,4)$ pairing but not $(1,3)$: $|n| \lesssim |n_3|$ and use Proposition 3.5.7 with $w_3 \in Z_0^s$;
- $(2,3),(1,4)$ or $(2,4),(1,3)$ pairings: $n = n_5$ and use the equation on u_5;

If $(n_3, n_4, n_5) \in \mathbb{X}_{B_2}(n_0)$:

$$|n_1| \lesssim |n_0| \sim |n_4| \le |n_2|,$$
$$|n_5| \ll |n_3| \ll |n_0|,$$

with possible pairings $(1,4),(1,3),(1,5),(2,4)$. We proceed as follows:

- $(2,4)$ not a pairing: $|n| \lesssim |n_2|$ and use Proposition 3.5.7 with $w_2 \in Z_0^s$;
- $(2,4)$ pairing but not $(1,3)$: $|n| \lesssim |n_1|$ and use Proposition 3.5.7 with $w_1 \in Z_0^s$;
- $(1,3),(2,4)$ pairings: $n = n_5$ and use the equation on u_5.

A.4 Remaining quintic terms

It remains to consider the terms in (3.31). Looking at the terms in (3.38), we want to determine the frequency regions where we can apply the standard quintic estimate (Proposition 3.5.7) or Propositions 3.5.10 or 3.5.11.

We start by considering the terms on the right-hand side of the inequalities in (3.38) with α_1, α_2. Note that for $*, \# \in \{A, B\}$, we have

$$\alpha_1(n, n_1, \ldots, n_5) = \frac{\max_{j=1,\ldots,5} \langle n_j \rangle^{9\theta} |n_1 n_2|}{\langle \phi(\overline{n}_{105}\rangle \langle \phi(\overline{n}_{234})\rangle} \lesssim \frac{\max_{j=1,\ldots,5} \langle n_j \rangle^{9\theta}}{\langle n_1 \rangle \max_{j=2,3,4} \langle n_j \rangle}, \qquad \overline{n}_{105} \in \mathbb{X}_A(n), \overline{n}_{234} \in \mathbb{X}_{\#}(n_0),$$

$$\alpha_2(n, n_1, \ldots, n_5) = \frac{\max_{j=1,\ldots,5} \langle n_j \rangle^{9\theta} |n_1 n_3|}{\langle \phi(\overline{n}_{120}\rangle \langle \phi(\overline{n}_{345})\rangle} \lesssim \frac{\max_{j=1,\ldots,5} \langle n_j \rangle^{9\theta}}{\max_{j=1,2} \langle n_j \rangle \max_{k=3,4,5} \langle n_k \rangle}, \quad \overline{n}_{120} \in \mathbb{X}_*(n), \overline{n}_{345} \in \mathbb{X}_{\#}(n_0),$$

due to the lower bounds on the phase functions. Consequently, we can apply Proposition 3.5.11.

Now, we consider the terms on the right-hand side of the inequalities in (3.38) with $\beta_1, \beta_2, \beta_3$.

- $\underline{\mathbf{B}_A^2(w_1, \varphi_T \cdot \mathbf{G}_A[w_2, u_3, u_4], u_5)}$: In this case, we have that

$$|n_4| \le |n_3| \ll |n_2| \sim |n_0| \ll |n_1| \sim |n|,$$

which implies that $\beta_1(n, n_1, \ldots, n_5) \lesssim \langle n_2 \rangle^{9\theta}$. Since n_1 is not in a pairing, we can apply Proposition 3.5.7 with $w_1 \in Z_0^s$.

- $\underline{\mathbf{B}_A^2(w_1, \varphi_T \cdot \mathbf{G}_B[w_2, w_3, u_4], u_5)}$: In this case, we have that

$$\beta_1(n, n_1, \ldots, n_5) \lesssim \frac{\max_{j=1,\ldots,5} \langle n_j \rangle^{9\theta} |n_2|}{|n_1|},$$

$|n| \sim |n_1| \gg |n_0| \ge |n_5|$ and $|n_3| \sim \max(|n_0|, |n_2|) \ge \min(|n_0|, |n_2|) \gg |n_4|$. We consider the following cases:

- $\underline{|n_1| \sim |n_2|}$: then we have $|n_4| \ll |n_0| \ll |n_1| \sim |n_2| \sim |n_3| \sim |n|$ and $|n_5| \le |n_0|$, so $\beta_1(n, n_1, \ldots, n_5) \lesssim \langle n_1 \rangle^{9\theta} \sim \langle n_2 \rangle^{9\theta} \sim \langle n_3 \rangle^{9\theta}$. Since n_1, n_2, n_3 cannot all be in pairings at the same time, we can apply Proposition 3.5.7 with $w_j \in Z_0^s$, for some $j \in \{1,2,3\}$.
- $\underline{|n_1| \gg |n_2|}$: since $\beta_1(n, n_1, \ldots, n_5) \lesssim \langle n_2 \rangle^{9\theta}$ and n_1 is never in a pairing, we use Proposition 3.5.7 with $w_1 \in Z_0^s$.
- $\underline{|n_1| \ll |n_2|}$: we have $|n_5| \le |n_0|$, $|n_4| \ll |n_0| \ll |n| \sim |n_1| \ll |n_2| \sim |n_3|$, $(2,3)$ is not a pairing, $\beta_1(n, n_1, \ldots, n_5) \lesssim \frac{|n_2|^{1+9\theta}}{|n_1|}$, and we can use Proposition 3.5.10 with $w_2 \in Z_0^s$ and $w_1, w_3 \in Z_0^{\frac{1}{2}}$.

- $\underline{\mathbf{B}_A^3(w_1, u_2, \varphi_T \cdot \mathbf{G}_A[w_3, u_4, u_5])}$: In this case, we have that

$$|n_5| \leq |n_4| \ll |n_3| \sim |n_0| \leq |n_2| \ll |n_1| \sim |n|,$$

which implies that $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_3 \rangle^{9\theta}$. Since n_1 is not in a pairing, we can apply Proposition 3.5.7 with $w_1 \in Z_0^s$.

- $\underline{\mathbf{B}_A^3(w_1, u_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5])}$: In this case, we have that

$$\beta_2(n, n_1, \dots, n_5) \lesssim \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_3|}{|n_1|},$$

$|n| \sim |n_1| \gg |n_2| \geq |n_0|$ and $|n_4| \sim \max(|n_0|, |n_3|) \geq \min(|n_0|, |n_3|) \gg |n_5|$. We consider the following cases:

  - $\underline{|n_1| \sim |n_3|}$: we must have $|n_1| \sim |n_3| \sim |n_4| \gg |n_0| \gg |n_5|$ and $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_1 \rangle^{9\theta} \sim \langle n_3 \rangle^{9\theta} \sim \langle n_4 \rangle^{9\theta}$. Since the possible pairings are $(1,3), (1,4), (2,5)$, there is always one function in u_1, u_3, u_4 which is not in a pairing and we can use Proposition 3.5.7 with $w_j \in Z_0^s$ for $j \in \{1, 3, 4\}$.

  - $\underline{|n_1| \gg |n_3|}$: then $|n_1| \sim |n| \gg |n_4| \gtrsim |n_3|$ and $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_3 \rangle^{9\theta}$. Since n_1 is never in a pairing, we can use Proposition 3.5.7 with $w_1 \in Z_0^s$.

  - $\underline{|n_1| \ll |n_3|}$: we have $|n_3| \sim |n_4| \gg |n_1| \sim |n| \gg |n_2| \geq |n_0| \gg |n_5|$, $(3,4)$ is not a pairing, $\beta_2(n, n_1, \dots, n_5) \lesssim \frac{|n_3|^{1+9\theta}}{|n_1|}$, and we can use Proposition 3.5.10 with $w_3 \in Z_0^s$ and $w_1, w_4 \in Z_0^{\frac{1}{2}}$.

- $\underline{\mathbf{B}_B^3(w_1, w_2, \varphi_T \cdot \mathbf{G}_A[w_3, w_4, u_5])}$: In this case, we have

$$|n_5| \leq |n_4| \ll |n_3| \sim |n_0| \ll \min(|n|, |n_1|) \leq \max(|n|, |n_1|) \sim |n_2|,$$

and $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_3 \rangle^{9\theta}$. Since n_2 is never in a pairing, we can apply Proposition 3.5.7 with $w_2 \in Z_0^s$.

- $\underline{\mathbf{B}_B^3(w_1, w_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5])}$: In this case, we have

$$\beta_2(n, n_1, \dots, n_5) \lesssim \frac{\max_{j=1,\dots,5} \langle n_j \rangle^{9\theta} |n_3|}{\max(|n_1|, |n_2|)},$$

$|n_2| \sim \max(|n|, |n_1|) \geq \min(|n|, |n_1|) \gg |n_0|$ and $|n_4| \sim \max(|n_0|, |n_3|) \geq \min(|n_0|, |n_3|) \gg |n_5|$.

  - $\underline{|n_4| \sim |n_3| \sim |n_0| \gg |n_5| \text{ or } |n_4| \sim |n_0| \gg |n_3| \gg |n_5|}$: then $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_3 \rangle^{9\theta}$ and since n_1 is not in a pairing, we can use Proposition 3.5.7 with $w_1 \in Z_0^s$.

  - $\underline{|n_1| \sim |n_2| \gtrsim |n| \gg |n_0| \text{ and } |n_3| \sim |n_4| \gg |n_0| \gg |n_5|}$:

    - $\underline{|n_2| \sim |n_3|}$: then $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_1 \rangle^{9\theta} \sim \langle n_2 \rangle^{9\theta} \sim \langle n_3 \rangle^{9\theta} \sim \langle n_4 \rangle^{9\theta}$, with possible pairings $(1,3), (1,4), (2,3), (2,4)$. We cannot have $|n| = |n_5|$ under these assumptions, thus we cannot have two pairings. Consequently, we can use Proposition 3.5.7 with $w_j \in Z_0^s$ for some $j \in \{1, 2, 3, 4\}$.
    - $\underline{|n_2| \gg |n_3|}$: then $\beta_2(n, n_1, \dots, n_5) \lesssim \langle n_1 \rangle^{9\theta} \sim \langle n_2 \rangle^{9\theta}$ and n_1, n_2 are not in any pairings, so apply Proposition 3.5.7 with $w_2 \in Z_0^s$.
    - $\underline{|n_2| \ll |n_3|}$: then $|n_3| \sim |n_4| \gg |n_2| \gtrsim |n|$, $(3,4)$ is not a pairing, and $\beta_2(n, n_1, \dots, n_5) \lesssim \frac{|n_3|^{1+9\theta}}{|n_2|}$. Thus, we can apply Proposition 3.5.11 with $w_3 \in Z_0^s$ and $w_2, w_4 \in Z_0^{\frac{1}{2}}$.

- $\underline{\mathcal{DNR}_C\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_A[w_3, u_4, u_5]\big)}$: In this case, we have that

$$\begin{aligned} |n| \lesssim |n_3| \sim |n_0| \ll |n_1| \sim |n_2|, \\ |n_5| \leq |n_4| \ll |n_3|, \end{aligned}$$

and $\beta_3(n, n_1, \ldots, n_5) \lesssim \frac{|n_1|}{|n_3|}$. Since $(1, 2)$ is not a pairing, we can apply Proposition 3.5.10 with $w_1 \in Z_0^s$ and $w_2, w_3 \in Z_0^{\frac{1}{2}}$.

- $\underline{\mathcal{DNR}_C\big(w_1, w_2, \varphi_T \cdot \mathbf{G}_B[w_3, w_4, u_5]\big)}$: In this case, we have that

$$\beta_3(n, n_1, \ldots, n_5) \lesssim \frac{\max\limits_{j=1,\ldots,5} \langle n_j \rangle^{9\theta} |n_1|}{\max(|n_3|, |n_4|)},$$

$|n_1| \sim |n_2| \gg |n_0| \gtrsim |n|$ and $|n_4| \sim \max(|n_0|, |n_3|) \geq \min(|n_0|, |n_3|) \gg |n_5|$.

  - $\underline{|n_0| \sim |n_4| \ll |n_1| \sim |n_2|}$: then $\beta_3(n, n_1, \ldots, n_5) \lesssim \frac{|n_1|}{|n_4|}$. Since $(1, 2)$ is not a pairing, we can use Proposition 3.5.10 with $w_1 \in Z_0^s$ and $w_2, w_4 \in Z_0^{\frac{1}{2}}$.

  - $\underline{|n_5| \ll |n_0| \ll |n_3| \sim |n_4|}$: if $|n_3| \gtrsim |n_1|$, then $\beta_3(n, n_1, \ldots, n_5) \lesssim 1$ and n_3 is not in a pairing, so we can apply Proposition 3.5.7 with $w_3 \in Z_0^s$. Otherwise $|n_3| \ll |n_1|$ which implies $|n| \lesssim |n_3| \ll |n_2| \sim |n_1|$. Since $\beta_3(n, n_1, \ldots, n_5) \lesssim \frac{|n_1|}{|n_3|}$ and $(1, 2)$ is not a pairing, we can apply Proposition 3.5.10 with $w_1 \in Z_0^s$ and $w_2, w_3 \in Z_0^{\frac{1}{2}}$.

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