A SHORT PROOF OF THE DEFORMATION PROPERTY OF BRIDGELAND STABILITY CONDITIONS

AREND BAYER

ABSTRACT. The key result in the theory of Bridgeland stability conditions is the property that they form a complex manifold. This comes from the fact that given any small deformation of the central charge, there is a unique way to correspondingly deform the stability condition.

We give a short direct proof of a strong version of this deformation property.

CONTENTS

1. INTRODUCTION

Stability conditions on triangulated categories, introduced in [Bri07], have been hugely influential, due to their connections to physics [BS15, GMN13], to mirror symmetry [Bri09a] and to representation theory [ABM15], and due to their applications within algebraic geometry, for example to Donaldson-Thomas invariants [Tod14], to the derived category itself [Huy11, BB13], or to the birational geometry of moduli spaces [ABCH13, BM14, Bay16, MS16].

Their distinguishing property, crucial for all applications, is a strong deformation property: by the main result of [Bri07], there is a complex manifold of stability conditions, with map to a vector space that is a locally an isomorphism. We give a short proof of an effective version of this result.

Result. We refer to Section 2 for the complete definitions; here we briefly review notation and the support property. Let $\mathcal{D}$ be a triangulated category, and let $v\colon K(\mathcal{D}) \to \Lambda$ be a homomorphism from its K-group to a finitely generated free abelian group Λ. A pre-stability condition on $\mathcal{D}$ with respect to v is a pair $\sigma = (Z, \mathcal{P})$, where $\mathcal{P}$ is a *slicing* (see Definition 2.1) and $Z\colon \Lambda \to \mathbb{C}$ is a compatible (see Definition 2.2) group homomorphism.

Key words and phrases. Bridgeland stability conditions, Derived category, Wall-crossing.

Definition 1.1 ([Bri07], [KS08])**.** Let $Q\colon \Lambda_{\mathbb{R}} \to \mathbb{R}$ be a quadratic form. We say that a pre-stability condition $(Z, \mathcal{P})$ satisfies the support property with respect to Q if

(a) the kernel $\operatorname{Ker} Z \subset \Lambda_{\mathbb{R}}$ of the central charge is negative definite with respect to Q, and
(b) for any semistable object E, i.e. $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$, we have $Q(v(E)) \geq 0$.

If such Q exists, we call σ a stability condition. Let $\operatorname{Stab}_\Lambda(\mathcal{D})$ denote the topological space (see Section 2) of stability conditions. It comes with a canonical map $\mathcal{Z}\colon \operatorname{Stab}_\Lambda(\mathcal{D}) \to \operatorname{Hom}(\Lambda, \mathbb{C})$ given by $\mathcal{Z}(Z, \mathcal{P}) = Z$. We will prove:

Theorem 1.2. *Let Q be a quadratic form on $\Lambda \otimes \mathbb{R}$, and assume that the stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to Q. Then:*

(a) There is an open neighboorhood $\sigma \in U_\sigma \subset \operatorname{Stab}_\Lambda(\mathcal{D})$ such that the restriction $\mathcal{Z}\colon U_\sigma \to \operatorname{Hom}(\Lambda, \mathbb{C})$ is a covering of the set of Z' such that Q is negative definite on $\operatorname{Ker} Z'$.
(b) All $\sigma \in U_\sigma$ satisfy the support property with respect to Q.

In other words, $\operatorname{Stab}_\Lambda(\mathcal{D})$ is a manifold, and any path $Z_t \in \operatorname{Hom}(\Lambda, \mathbb{C})$ for $t \in [0,1]$ with $Z_0 = Z$ and $\operatorname{Ker} Z_t$ negative definite for all $t \in [0,1]$ lifts uniquely to a continuous path $\sigma_t = (Z_t, \mathcal{P}_t)$ in the space of stability conditions starting at $\sigma_0 = \sigma$.

Part (a) is an effective variant of [Bri07, Theorem 1.2] (which says that there is *some* neighbourhood of Z_0 in which paths can be lifted uniquely). The entire result first appeared as [BMS16, Proposition A.5] with an indirect proof based on reduction to Bridgeland's previous result.

Remarks. The support property can be a deep and interesting property in itself: a quadratic Bogomolov-Gieseker type inequality for Chern classes of semi-stable objects which, by Theorem 1.2, is preserved under wall-crossing.

Theorem 1.2 was crucial in [BMS16] in order to describe an entire component of the space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds. It also greatly simplifies the construction of stability conditions on surfaces (or of *tilt-stability* on higher-dimensional varieties [BMT14]). In this case, the quadratic form Q is the classical Bogomolov-Gieseker inequality, and Theorem 1.2 gives an open subset of stability conditions that otherwise has to be glued together from many small pieces (see e.g. [BM11, Section 4]).

Theorem 1.2 of [Bri07] also allows for components of the space of stability conditions modelled on a linear subspace $L \subset \operatorname{Hom}(\Lambda, \mathbb{C})$. When L is defined over $\mathbb{Q}$, we can recover that statement by replacing Λ with $\Lambda / \operatorname{Ker} L$. (See [MP14] for examples where this is not satisfied; however, to achieve well-behaved wall-crossing one always has to assume that L is defined over $\mathbb{Q}$.)

Proof idea. Our proof is based on two ideas. First, we reduce to the case where the imaginary part of Z is constant; then we only have to deform stability in a fixed abelian category. Secondly, we use the elementary convex geometry of the *Harder-Narasimhan polygon*, see Section 3.

This avoids the need for *quasi-abelian categories*, of ϵ or of $\frac{1}{8}$. It also avoids some of the more technical arguments of [Bri07, Section 7]. We still need a few arguments similar to ones in [Bri07]; we have reproduced most of them, except for the proofs of Proposition 2.6 and Lemma 2.9.

Application. Assume that $\mathcal{D}$ is a 2-Calabi-Yau category, i.e. for all $E, F \in \mathcal{D}$ we have a bi-functorial isomorphism $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, E[2])^\vee$. Let Λ be the *numerical K-group* of $\mathcal{D}$, and assume that Λ is finitely generated. Then there is a surjection $v\colon K(\mathcal{D}) \to \Lambda$, and Λ admits a non-degenerate bilinear form $(\_, \_)$, called Mukai-pairing, with

$$\chi(E, F) = -\big(v(E), v(F)\big).$$

Let $\mathcal{P}_0(\mathcal{D}) \subset \mathrm{Hom}(\Lambda, \mathbb{C})$ be the set of central charges Z such that $\mathrm{Ker}\, Z$ is negative definite with respect to the Mukai pairing, and such that $\mathrm{Ker}\, Z$ contains no root $\delta \in \Lambda, (\delta, \delta) = -2$.

Corollary 1.3. *The restriction $\mathcal{Z}^{-1}(\mathcal{P}_0(\mathcal{D})) \xrightarrow{\mathcal{Z}} \mathcal{P}_0(\mathcal{D})$ is a covering map.*

The proof, given in Section 7, is fairly similar to the case of K3 surfaces [Bri08, Proposition 8.3]. The point of including it here is to show that in terms of the support property via quadratic forms, and equipped with Theorem 1.2, the proof becomes natural and short. This result was also proved previously for preprojective algebras of quivers in [Tho08, Bri09b, Ike14]. In each of these cases, there is in fact a connected component of $\mathrm{Stab}(\mathcal{D})$ that is a covering of a connected of $\mathcal{P}_0(X)$; such deeper statements rely crucially on *non-emptiness* of moduli spaces of stable objects.

Acknowledgements. I would like to thank Emanuele Macrì and Paolo Stellari; as indicated above, Theorem 1.2 first appeared with a different proof in our joint work [BMS16]. I presented a clumsier version of the arguments in this article at my lectures at the Hausdorff school on derived categories in Bonn, April 2016; I am grateful to the organisers for the opportunity, and the participants for their feedback. My work was supported by the ERC starting grant WallXBirGeom 337039.

2. Review: definitions and basic properties

Throughout, $\mathcal{D}$ will be a triangulated category, equipped with a group homomorphism

$$v\colon K(\mathcal{D}) \to \Lambda$$

from its K-group to an abelian group $\Lambda \cong \mathbb{Z}^m$.

Definitions. We first recall the main definitions from [Bri07].

Definition 2.1. A *slicing* $\mathcal{P}$ on $\mathcal{D}$ is a collection of full subcategories $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ with

(a) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$;
(b) for $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i), i = 1, 2$, we have $\mathrm{Hom}(E_1, E_2) = 0$; and
(c) for any $E \in \mathcal{D}$ there is a sequence of maps $0 = E_0 \xrightarrow{i_1} E_1 \to \dots \xrightarrow{i_m} E_m$ and of real numbers $\phi_1 > \phi_2 > \dots > \phi_m$ such that the cone of i_j is in $\mathcal{P}(\phi_j)$ for $j = 1, \dots, m$.

The objects of $\mathcal{P}(\phi)$ are called *semistable of phase* ϕ; its simple objects are called *stable*. The sequence of maps in (c) is called the HN filtration of E.

Definition 2.2. A pre-stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ where $\mathcal{P}$ is a slicing, and $Z\colon \Lambda \to \mathbb{C}$ is a group homomorphism, that satisfy the following condition: for all $0 \neq E \in \mathcal{P}(\phi)$, we have $Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$.

We will abuse notation and write $Z(E)$ instead of $Z(v(E))$.

Basic properties. Let $\mathrm{GL}_2^+(\mathbb{R})$ denote the group of real 2×2-matrices with positive determinant, and let $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ be its universal cover. Since $\mathrm{GL}_2^+(\mathbb{R})$ acts on S^1, its universal cover acts on the universal cover $\mathbb{R} \to S^1$ given explicitly by $\phi \mapsto e^{i\pi\phi}$. For $\tilde{g} \in \widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ we will write g for the corresponding element of $\mathrm{GL}_2^+(\mathbb{R})$, and $\tilde{g}.\phi$ for the given action on $\mathbb{R}$.

Proposition 2.3. *There is a natural action of $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ on the set of pre-stability conditions given by $\tilde{g}.(Z, \mathcal{P}) = (Z', \mathcal{P}')$ with*

$$Z' = g \circ Z \quad \text{and} \quad \mathcal{P}'(\tilde{g}.\phi) = \mathcal{P}(\phi).$$

The *heart of a bounded t-structure* is a full subcategory $\mathcal{A} \subset \mathcal{D}$ such that

$$\mathcal{S}(\phi) := \begin{cases} \mathcal{A}[\phi] & \text{if } \phi \in \mathbb{Z} \\ \emptyset & \text{if } \phi \notin \mathbb{Z} \end{cases}$$

is a slicing (see [Bri07, Lemma 3.2]). It is automatically an abelian subcategory; and stability conditions on $\mathcal{D}$ can be constructed from slope-stability in $\mathcal{A}$.

Definition 2.4. A stability function Z on an abelian category $\mathcal{A}$ is a morphism $Z \colon K(\mathcal{A}) \to \mathbb{C}$ of abelian groups such that for all $0 \neq E \in \mathcal{A}$, the complex number $Z(E)$ is in the semi-closed upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} : \Im Z > 0, \quad \text{or } \Im Z = 0 \text{ and } \Re Z < 0\}.$$

For $0 \neq E \in \mathcal{A}$ we define its phase by $\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1]$. An object $E \in \mathcal{A}$ is called Z-semistable if for all subobjects $A \hookrightarrow E$, we have $\phi(A) \leq \phi(E)$.

Definition 2.5. We say that a stability function Z on an abelian category $\mathcal{A}$ satisfies the *HN property* if every object $E \in \mathcal{A}$ admits a Harder-Narasimhan (HN) filtration: a sequence $0 = E_0 \hookrightarrow E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow E_m = E$ such that E_i/E_{i-1} is Z-semistable for $i = 1, \ldots, m$, with

$$\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots > \phi(E_m/E_{m-1}).$$

Proposition 2.6 ([Bri07, Proposition 5.3])**.** *To give a pre-stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A}$ of a bounded t-structure, and a stability function Z on $\mathcal{A}$ with the HN property.*

Here we tacitly assume that the stability function Z on $\mathcal{A}$ also factors via $K(\mathcal{A}) = K(\mathcal{D}) \xrightarrow{v} \Lambda$. Given $(Z, \mathcal{A})$, the slicing is determined by setting $\mathcal{P}(\phi)$ to be the Z-semistable objects $E \in \mathcal{A}$ of phase ϕ for $\phi \in (0, 1]$. Conversely, given $(Z, \mathcal{P})$, the heart $\mathcal{A}$ is the smallest extension-closed subcategory of $\mathcal{D}$ containing $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$.

Definition 2.7. A stability condition σ is a pre-stability condition that satisfies the support property in the sense of Definition 1.1 with respect to some quadratic form Q on $\Lambda \otimes \mathbb{R}$.

Topology and local injectivity. There is a generalised metric, and thus a topology, on the set of slicings $\mathrm{Slice}(\mathcal{D})$ given as follows. Given two slicings $\mathcal{P}, \mathcal{Q}$, we write $\phi^{\pm}(E)$ and $\psi^{\pm}(E)$ for the largest and smallest phase in the associated HN filtration of an object E for $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then we define the distance of $\mathcal{P}$ and $\mathcal{Q}$ by

$$d(\mathcal{P}, \mathcal{Q}) := \sup \left\{ \left|\phi^+(E) - \psi^+(E)\right|, \left|\phi^-(E) - \psi^-(E)\right| \,:\, E \in \mathcal{D} \right\} \in [0, +\infty].$$

We recall that this distance can be computed by considering $\mathcal{P}$-semistable objects alone:

Lemma 2.8 ([Bri07, Lemma 6.1])**.** *We have* $d(\mathcal{P}, \mathcal{Q}) = d'(\mathcal{P}, \mathcal{Q})$*, where the latter is defined by*

$$d'(\mathcal{P}, \mathcal{Q}) := \sup \left\{ \psi^+(E) - \phi, \phi - \psi^-(E) : \phi \in \mathbb{R}, 0 \neq E \in \mathcal{P}(\phi) \right\}.$$

Proof. The inequality $d(\mathcal{P}, \mathcal{Q}) \geq d'(\mathcal{P}, \mathcal{Q})$ is immediate. For the converse, consider $E \in \mathcal{D}$, and let A_i be one of its HN factors with respect to $\mathcal{P}$. Then $\psi^+(A_i) \leq \phi(A_i) + d'(\mathcal{P}, \mathcal{Q}) \leq \phi^+(E) + d'(\mathcal{P}, \mathcal{Q})$. Hence E admits no maps from $\mathcal{Q}$-stable objects of phase bigger than $\phi^+(E) + d'(\mathcal{P}, \mathcal{Q})$, and so $\psi^+(E) \leq \phi^+(E) + d'(\mathcal{P}, \mathcal{Q})$. The analogous inequality for $\psi^-(E)$ follows similarly; combined, they imply the claim. □

The topology on $\operatorname{Stab}_\Lambda(\mathcal{D})$ (and similarly on the set of pre-stability conditions) is the finest topology such that both maps

$$\operatorname{Stab}_\Lambda(\mathcal{D}) \to \operatorname{Slice}(\mathcal{D}), \quad (Z, \mathcal{P}) \mapsto \mathcal{P}$$
$$\operatorname{Stab}_\Lambda(\mathcal{D}) \to \operatorname{Hom}(\Lambda, \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z$$

are continuous.

Lemma 2.9 ([Bri07, Lemma 6.4])**.** *If* $\sigma = (Z, \mathcal{P})$ *and* $\tau = (Z, \mathcal{Q})$ *are two pre-stability conditions with the same central charge* Z *and* $d(\mathcal{P}, \mathcal{Q}) < 1$*, then* $\sigma = \tau$*.*

Corollary 2.10. *The map* $\operatorname{Stab}_\Lambda(\mathcal{D}) \to \operatorname{Hom}(\Lambda, \mathbb{C}), (Z, \mathcal{P}) \mapsto Z$ *is locally injective.*

3. HARDER-NARASIMHAN FILTRATIONS VIA THE HARDER-NARASIMHAN POLYGON

Throughout this section, let $\mathcal{A}$ be an abelian category with a stability function Z.

Definition 3.1. The *Harder-Narasimhan polygon* $\mathrm{HN}^Z(E)$ of an object $E \in \mathcal{A}$ is the convex hull of the central charges $Z(A)$ of all subobjects $A \subset E$ of E.

(The trivial subobjects $A = 0$ or $A = E$ are included in the definition.) The idea to consider this convex set in the context of slope-stability goes back at least 40 years [Sha76].

Definition 3.2. We say that the Harder-Narasimhan polygon $\mathrm{HN}^Z(E)$ of an object $E \in \mathcal{A}$ is *polyhedral on the left* if the set has finitely many extremal points $0 = z_0, z_1, \ldots, z_m = Z(E)$ such that $\mathrm{HN}^Z(E)$ lies to the right of the path $z_0 z_1 z_2 \ldots z_m$; see fig. 1.

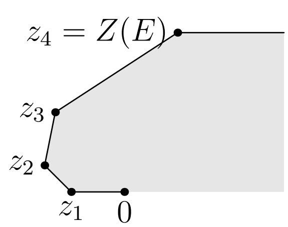

FIGURE 1. Polyhedral on the left

In other words, the intersection of $\mathrm{HN}^Z(E)$ with the closed half-plane to the left of the line through 0 and $Z(E)$ is the polygon with vertices $z_0, z_1, \ldots, z_m$. Our proof of Theorem 1.2 is based on the following well-known statement; we provide a proof for completeness:

Proposition 3.3. *The object E has a Harder-Narasimhan filtration with respect to Z if and only if its Harder-Narasimhan polygon $\mathrm{HN}^Z(E)$ is polyhedral on the left.*

Assume that $\mathrm{HN}^Z(E)$ is polyhedral on the left. For each $i = 1, \dots, m$, choose a subobject $E_i \subset E$ such that $Z(E_i) = z_i$. (This exists as z_i is extremal.)

Lemma 3.4. *This is a filtration, i.e. $E_{i-1} \subset E_i$ for $i = 1, \dots, m$.*

Proof. Let $A := E_{i-1} \cap E_i \subset E$ be the intersection of two subsequent objects, and $B := E_{i-1} + E_i \subset E$ be their span inside E. Then there is a short exact sequence

$$A \hookrightarrow E_{i-1} \oplus E_i \twoheadrightarrow B.$$

Hence the midpoint of $Z(A)$ and $Z(B)$ is also the midpoint of z_{i-1} and z_i, see also figure 2.

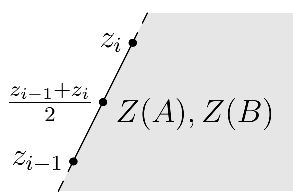

FIGURE 2. Lemma 3.4

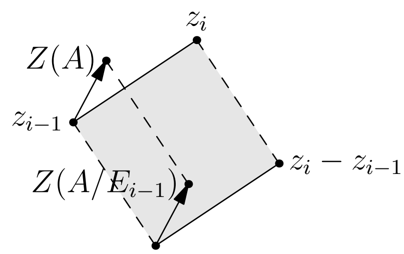

FIGURE 3. Lemma 3.5

On the other hand, $Z(A), Z(B)$ lie in $\mathrm{HN}^Z(E)$; by convexity and the choice of z_{i-1}, z_i, they both have to lie either in the open half-plane to the right of the line $(z_{i-1}z_i)$, or on the line segment $\overline{z_{i-1}z_i}$. The former would be a contradiction to the previous paragraph, and so $Z(A) \in \overline{z_{i-1}z_i}$.

Since $A \subset E_{i-1}$, this implies $Z(A) = z_{i-1}$ and $A \cong E_{i-1}$; therefore, $E_{i-1} \subset E_i$. □

Lemma 3.5. *The filtration quotient E_i/E_{i-1} is semistable.*

Proof. Otherwise, there is an object A with $E_{i-1} \subset A \subset E_i$ such that A/E_{i-1} has bigger phase than E_i/E_{i-1}, see fig. 3. It follows that $Z(A)$ lies to the left of the line segment $\overline{z_{i-1}z_i}$. Since $A \subset E$ and hence $Z(A) \in \mathrm{HN}^Z(E)$, this is a contradiction. □

Proof of Propostion 3.3. The phase of E_i/E_{i-1} is determined by the argument of $z_i - z_{i-1}$; by convexity this shows $\phi(E_1/E_0) > \dots > \phi(E_m/E_{m-1})$, and so the E_i form a HN filtration.

Conversely, assume that we are given a HN filtration $0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_m$ and a subobject $A \hookrightarrow E$. We have to show that $Z(A)$ lies to the right of the path $z_0 z_1 \dots z_m$ with vertices $z_i := Z(E_i)$. By induction on m, we may assume that $Z(A \cap E_{m-1})$ lies to the right of the path $z_0 z_1 \dots z_{m-1}$. On the other hand, $A/\left(A \cap E_{m-1}\right)$ is a subobject of E_m/E_{m-1}, which is semistable; thus the central charge of $Z(A/\left(A \cap E_{m-1}\right)$ lies to the right of the line segment from 0 to $z_m - z_{m-1}$. Therefore, $Z(A) = Z(A \cap E_{m-1}) + Z(A/\left(A \cap E_{m-1}\right)$ lies to the right of the path $z_0 z_1 \dots z_m$ as claimed. □

Corollary 3.6. *Given $E \in \mathcal{A}$, assume that there are only finitely many classes $v(A)$ of subobjects $A \subset E$ with $\Re Z(A) < \max\{0, \Re Z(E)\}$. Then E admits a HN filtration.*

4. PROOF OF THE DEFORMATION PROPERTY

Throughout Section 4 and 5, we will make the following assumption:

Assumption 4.1. *The quadratic form Q has signature* $(2, \operatorname{rk}\Lambda - 2)$.

Lemma 4.2. *Up to the action of $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$, we may assume that we are in the following situation. There is a norm $\|\cdot\|$ on $\operatorname{Ker} Z$ such that if $p\colon \Lambda_\mathbb{R} \to \operatorname{Ker} Z$ denotes the orthogonal projection with respect to Q, then*

$$Q(v) = |Z(v)|^2 - \|p(v)\|^2 .$$

Proof. Let K the kernel of Z, and let $K^\perp$ denote its orthogonal complement. Then Q is negative definite on K; let $\|\cdot\|$ be the norm associated to $-Q$. As $Z|_{K^\perp}$ is injective, Assumption 4.1 can only hold if Q is positive definite on $K^\perp$, and if we have an isomorphism of real vector spaces

$$Z|_{K^\perp}\colon K^\perp \to \mathbb{C}.$$

Using the $\mathrm{GL}_2^+(\mathbb{R})$-action, we may assume this to be an isometry. Then the the claim follows. □

Remark 4.3. In different context, namely for the Mukai quadratic form instead of Q, the analogous normalisation is used extensively in [Bri08].

Consider the subset in $\operatorname{Hom}(\Lambda, \mathbb{C})$ of central charges whose kernel is negative definite with respect to Q; let $\mathcal{P}_Z(Q)$ be its connected component containing Z.

Lemma 4.4. *Assume we are in the situation of Lemma 4.2. Up to the action of $\mathrm{GL}_2^+(\mathbb{R})$, each $Z' \in \mathcal{P}_Z(Q)$ is of the form*

$$Z' = Z + u \circ p$$

where $u\colon \operatorname{Ker} Z \to \mathbb{C}$ is a linear map with operator norm satisfying $\|u\| < 1$.

Proof. As in the previous Lemma, let $K^\perp$ be the orthogonal complement of $\operatorname{Ker} Z$. The restriction of Z' to $K^\perp$ is an isomorphism for any $Z' \in \mathcal{P}_Z(Q)$. Hence for any path $Z(t)$ in $\mathcal{P}_Z(Q)$ starting at Z there is a corresponding path $\gamma(t) \in \mathrm{GL}_2^+(\mathbb{R})$ such that $\gamma(t) \circ Z(t)$ is constant. So we may assume that Z' and Z agree when restricted to $K^\perp$. Let u be the restriction of Z' to $\operatorname{Ker} Z$, and the claim follows. □

Lemma 4.5. *In order to prove Theorem 1.2, it is enough to show the following: given any stability conditions $\sigma_0 = (Z_0, \mathcal{P}_0)$, and any path of central charges of the form $t \mapsto Z_t = Z + t \cdot u \circ p$ for $t \in [0, 1]$, where $u\colon \operatorname{Ker} Z \to \mathbb{R}$ is a linear map to the* real numbers *with $\|u\| < 1$, there exists a continuous lift $t \mapsto \sigma_t$ to the space of stability conditions; moreover, all σ_t satisfy the support property with respect to the same quadratic form Q.*

Proof. Due to Corollary 2.10, it is enough to prove the *existence* of a lift for any given path, and moreover we can freely replace any path in $\mathcal{P}_Z(Q)$ by a homotopic one. Observe that due the $\widetilde{\mathrm{GL}_2^+}(\mathbb{R})$-action such a result would equally hold when u is purely imaginary. Now write $Z_1 = Z + u \circ p$ and $u = \Re u + i\Im u$. Since $\|\Re u\| \le \|u\|$, we first obtain a path from $\sigma_0 = \sigma$ to a stability condition $\sigma_1 = (Z_1, \mathcal{P}_1)$ with $Z_1 = Z + \Re u \circ p$. By part (b) of Theorem 1.2, we can apply the result again starting at σ_1 to construct the desired stability condition with central charge $Z + u \circ p = Z_1 + i\Im u \circ p$. □

Our next key observation is that when u is real, we may (and in fact, have to) leave the heart $\mathcal{A} := \mathcal{P}(0, 1]$ unchanged. Hence we will apply Proposition 2.6 and prove that $(\mathcal{A}, Z_t)$ produces a stability condition for all $t \in [0, 1]$. Clearly we just need to prove the case $t = 1$.

Lemma 4.6. *Let Z, u be as in Lemma 4.5. Then $Z_1 = Z + u \circ p$ is a stability function on $\mathcal{A}$.*

Proof. Consider $E \in \mathcal{A}$; if $\Im Z(E) = \Im Z_1(E) > 0$, there is nothing to prove. Otherwise, E is semistable with $Z(E) \in \mathbb{R}_{<0}$ and thus $\|p(E)\| \leq -Z(E)$. From $\|u\| < 1$ we conclude

$$Z_1(E) = Z(E) + u \circ p(E) \leq Z(E) + \|u\| \, \|p(E)\| < Z(E) - Z(E) = 0.$$

□

Next, we want to prove that $(\mathcal{A}, Z_1)$ satisfies the HN property. We will use Proposition 3.3 and Corollary 3.6.

Let us define the *mass* $m^Z(E)$ of E with respect to Z as the length of the boundary of $\mathrm{HN}^Z(E)$ on the left between 0 and $Z(E)$.

Lemma 4.7. *For all $E \in \mathcal{A}$ we have $\|p(E)\| \leq m^Z(E)$.*

Proof. If E is semistable, then $0 \leq Q(E) = |Z(E)|^2 - \|p(E)\|^2 = \left(m^Z(E)\right)^2 - \|p(E)\|^2$, which is exactly the claim. Otherwise, consider the HN filtration $E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_m = E$. Combined with the triangle inequality, this gives

$$\|p(E)\| \leq \sum_i \|p(E_i/E_{i-1})\| \leq \sum_i |Z(E_i/E_{i-1})| = \sum_i |Z(E_i) - Z(E_{i-1})| = m^Z(E).$$

□

The following Lemma needs no proof:

Lemma 4.8. *If $A \subset E$, then $\mathrm{HN}^Z(A) \subset \mathrm{HN}^Z(E)$.*

Lemma 4.9. *Given any subobject $A \subset E$, we have*

$$m^Z(A) - \Re Z(A) \leq m^Z(E) - \Re Z(E).$$

Proof. This follows from the previous Lemma, convexity and a picture, see fig. 4. Indeed, choose $x > \Re Z(A), \Re Z(E)$; let $a = x + i\Im Z(A)$ and $e = x + i\Im Z(E)$. Let γ_A be the path that follows by boundary of $\mathrm{HN}^Z(A)$ from 0 to $Z(A)$, and then continues horizontally to a; similarly γ_E follows the boundary of $\mathrm{HN}^Z(E)$ and then continues to e. Their lengths are given as

$$|\gamma_A| = m^Z(A) + x - \Re Z(A), \quad |\gamma_E| = m^Z(E) + x - \Re Z(E).$$

On the other hand, convexity and Lemma 4.8 imply $|\gamma_A| \leq |\gamma_E|$; for example, if γ_I denotes the intermediate path that follows the boundary of $\mathrm{HN}^Z(E)$ up to height $\Im Z(A)$ and then goes horizontally to a, we clearly have $|\gamma_A| \leq |\gamma_I| \leq |\gamma_E|$. □

Lemma 4.10. *Given $C \in \mathbb{R}$, there are only finitely classes $v(A)$ of subobjects $A \subset E$ with $\Re\left(Z + u \circ p\right)(A) < C$.*

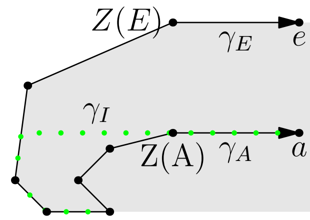

FIGURE 4. Proof of Lemma 4.9

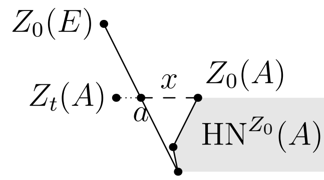

FIGURE 5. Proof of Lemma 4.11

Proof. Given any such A, we use Lemmas 4.7 and 4.9 to obtain

$$C > (\Re Z + u \circ p)(A) \geq \Re Z(A) - \|u\| \, \|p(A)\| > (1 - \|u\|)\Re Z(A) - \|u\| \left(m^Z(A) - \Re Z(A)\right)$$
$$\geq (1 - \|u\|)\Re Z(A) - \|u\| \left(m^Z(E) - \Re Z(E)\right).$$

Since $\|u\| < 1$, this bounds $\Re Z(A)$ from above. On the other hand, $Z(A) \in \mathrm{HN}^Z(E)$, and thus $Z(A)$ is constrained to lie in a compact region of $\mathbb{C}$. Using Lemmas 4.9 and 4.7 again, this gives an upper bound first for $m^Z(A)$ and consequently for $\|p(A)\|$. Hence $v(A)$ is contained in a compact region of $\Lambda \otimes \mathbb{R}$ depending only on E and C, and the claim follows. □

Therefore, Corollary 3.6 implies the existence of HN filtrations for Z_1 on $\mathcal{A}$.

Continuity. So far, we have constructed a pre-stability condition $\sigma_t = (\mathcal{A}, Z_t)$ for each $t \in [0, 1]$.

Lemma 4.11. *The map $t \mapsto \sigma_t = (\mathcal{A}, Z_t)$ defines a continuous path in the space of pre-stability conditions.*

Proof. Let $\mathcal{P}_t$ denote the associated slicing. It is enough to prove that for t sufficiently small, the distance $d(\mathcal{P}_0, \mathcal{P}_t)$ becomes arbitrarily small. We will apply Lemma 2.8. Thus consider a $\mathcal{P}_0$-semistable object $E \in \mathcal{D}$; up to shift, we may assume $E \in \mathcal{A}$. Let $A \hookrightarrow E$ be the leading HN filtration factor of E with respect to Z_t. Write $Z_0(A) = a + x$ where $a \in \mathbb{C}$ has the same phase as $Z_0(E)$ and $x \geq 0$, see fig. 5. By convexity, $m^{Z_0}(A) \leq |a| + x$. Therefore

$$\Re Z_t(A) \geq \Re Z_0(A) - t\,\|u\|\,\|p(A)\| \geq \Re Z_0(A) - t m^{Z_0}(A) \geq \Re Z_0(A) - t\,(|a| + x) \geq \Re a - t\,|a|.$$

Note that $\pi \cdot (\psi^+(E) - \phi(E))$ is the argument of $\frac{1}{a} Z_t(A)$; hence

$$\psi^+(E) - \phi(E) \leq \frac{1}{\pi} \sin^{-1} \frac{|Z_t(A) - a|}{|a|} < \frac{1}{\pi} \sin^{-1}(t).$$

Combined with an analogous argument for $\psi^-(E)$ we obtain $d'(\mathcal{P}_0, \mathcal{P}_t) \leq \frac{1}{\pi} t$ as claimed. □

5. PRESERVATION OF THE QUADRATIC INEQUALITY

It remains to show that the pre-stability condition $(Z_1, \mathcal{A})$ satisfies the support property with respect to Q, i.e. that that $Q(v(E)) \geq 0$ for all $E \in \mathcal{A}$ that are Z_1-stable. The basic reason is that the quadratic inequality is preserved by wall-crossing:

Lemma 5.1. *Let $\sigma = (Z, \mathcal{P})$ be pre-stability condition. Assume that Q is a non-degenerate quadratic form on $\Lambda_{\mathbb{R}}$ of signature $(2, \operatorname{rk}\Lambda - 2)$ such that Q is negative definite on $\operatorname{Ker} Z$. If E is strictly σ-semistable and admits a Jordan-Hölder filtration with factors $E_1, \dots, E_m$, and if $Q(v(E_i)) \geq 0$ for $i = 1, \dots, m$, then $Q(v(E)) = 0$.*

Proof. We apply Lemma 4.2; then $Q(v) \geq 0$ is equivalent to $|Z(v)| \geq \|p(v)\|$. We obtain

$$|Z(E)| = \sum_i |Z(E_i)| \geq \sum_i \|p(v(E_i))\| \geq \left\| \sum_i p(v(E_i)) \right\| = \|p(v(E))\|$$

where the first equality holds since the central charges of all E_i are aligned, the first inequality holds by assumption, and the second inequality is the triangle inequality. □

The proof strategy is thus clear: if $E \in \mathcal{A}$ is Z_1-stable with $Q(v(E)) < 0$, then it must be Z_0-unstable; wall-crossing gives a $t \in [0, 1)$ such that E is strictly Z_t-semistable; by the Lemma, one of its Jordan-Hölder factors will also violate the inequality, and we proceed by induction. To make this argument work, we have to show that we can find such a wall, and that this process terminates.

Lemma 5.2. *Given two objects $A, E \in \mathcal{A}$, denote their phases with respect to Z_t by $\phi^t(A), \phi^t(E)$, respectively. If the set of $t \in [0, 1]$ with $\phi^t(A) \geq \phi^t(E)$ is non-empty, then it is a closed subinterval of $[0, 1]$ containing one of its endpoints.*

Proof. The condition is equivalent to $\frac{-\Re Z_t(A)}{\Im Z_t(A)} \geq \frac{-\Re Z_t(E)}{\Im Z_t(E)}$, which is a linear inequality in t. □

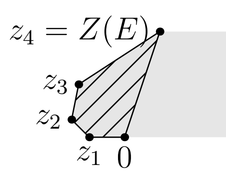

FIGURE 6. The truncated HN polygon

Consider the polygon whose vertices are the extremal points of $\mathrm{HN}^{Z_0}(E)$ on the left; we will call this the *truncated HN polygon of E*, see fig. 6. Note that if $A \subset E$ is a subobject with $\phi_0(A) \geq \phi_0(E)$, then $Z_0(A)$ is contained in the truncated HN polygon of E; by Lemmas 4.9 and 4.7 there are only finitely many classes $v(A)$ of such subobjects.

Lemma 5.3. *Every Z_1-semistable object $E \in \mathcal{A}$ satisfies $Q(E) \geq 0$.*

Proof. Otherwise, E must be Z_0-unstable. By Lemma 5.2 and the following observation, there are only finitely many classes $v(A)$ of subobjects $A \hookrightarrow E$ that destabilise E with respect Z_t for any $t \in [0, 1]$. Hence there is a wall $t_1 \in (0, 1]$ such that E is strictly semistable with respect to Z_{t_1}, and moreover E admits a Jordan-Hölder filtration with respect to Z_{t_1}. By Lemma 5.1, there are subobjects $G_1 \hookrightarrow F_1 \hookrightarrow E$ of the same phase, such that $Q(v(F_1/G_1)) < 0$.

Applying the same logic to F_1/G_1, we obtain $t_2 \in (0, t_1)$ and subobjects $G_1 \subset G_2 \subset F_2 \subset F_1 \subset E$ such that $F_2/G_1, G_2/G_1$ and F_1/G_1 all have the same phase with respect to t_2, and such that $Q(v(F_2/G_2)) < 0$. Continuing by induction, we obtain a sequence $t_1 > t_2 > t_3 > \dots$ of real number and chains $G_1 \subset G_2 \subset G_3 \subset \dots \subset E$ and $E \supset F_1 \supset F_2 \supset F_3 \supset \dots$ of subobjects of E.

Lemma 5.2 gives $\phi^{t_2}(F_1) \geq \phi^{t_2}(E)$ and $\phi^{t_2}(G_1) \geq \phi^{t_2}(E)$. Since the central charge of $Z_{t_2}(F_2)$ lies on the line segment connecting $Z_{t_2}(F_1)$ and $Z_{t_2}(G_1)$, we also have $\phi^{t_2}(F_2) \geq \phi^{t_2}(E)$ (and therefore $\phi^t(F_2) \geq \phi^t(E)$ for all $t \in [0, t_2]$; similarly for G_2. Continuing this argument by induction, we see that $Z_0(F_i)$ and $Z_0(G_i)$ are all contained in the truncated HN polygon of E. Thus this process terminates. □

By Lemma 4.5, this concludes the proof of Theorem 1.2 whenever Assumption 4.1 holds.

6. Reductions

Finally, we will show that we can always reduce the general situation to the case where Assumption 4.1 holds. By abuse of language, we call a quadratic form degenerate or non-degenerate if the associated symmetric bilinear form is degenerate or non-degenerate, respectively.

Lemma 6.1. *Assume that the quadratic form Q on $\Lambda_\mathbb{R}$ is degenerate. Then there exists an injective map $\Lambda_\mathbb{R} \hookrightarrow \overline{\Lambda}$ of real vector spaces and a non-degenerate quadratic form $\overline{Q}$ on $\overline{\Lambda}$, extending Q, such that any central charge $Z\colon \Lambda_\mathbb{R} \to \mathbb{C}$ whose kernel is negative definite with respect to Q extends to a central charge $\overline{Z}\colon \overline{\Lambda} \to \mathbb{C}$ whose kernel is negative definite with respect to $\overline{Q}$.*

Proof. Let $N \hookrightarrow \Lambda_\mathbb{R}$ be the null space of Q; we will only treat the case $\dim_\mathbb{R} N = 1$ (otherwise, we can iterate the construction that follows). Choose a splitting $\Lambda_\mathbb{R} \cong N \oplus C$; then for $n \in N, c \in C$, we have $Q(n \oplus c) = Q(c)$. Let $\overline{\Lambda_\mathbb{R}} := N \oplus N^\vee \oplus C$, let q be the canonical quadratic form on the hyperbolic plane $N \oplus N^\vee$, and set $\overline{Q} := q \oplus Q|_C$.

Given Z as above, the restriction $Z|_N$ is injective, and we may assume that Z maps N to the real line. Let $n \in N$ be such that $Z(n) = 1$, and let $n^\vee \in N^\vee$ be the dual vector with $(n, n^\vee) = 1$. We claim that for $\alpha \gg 0$, the extension of Z defined by $Z'(n^\vee) = \alpha$ has the desired property.

Let $K := \operatorname{Ker} Z$; then the kernel of Z' is contained in $N \oplus N^\vee \oplus K$, and given by vectors of the form $a \cdot n - \frac{a}{\alpha} \cdot n^\vee + k$ for $k \in K, a \in \mathbb{R}$. For such vectors, we have

$$Q\left(a \cdot n - \frac{a}{\alpha} \cdot n^\vee + k\right) = -\frac{2a^2}{\alpha} - \frac{2a}{\alpha}(n^\vee, k) + Q(k).$$

This is a quadratic function in a with negative constant term; its discriminant is negative if

$$\alpha > \max\left\{\frac{(n^\vee, k)^2}{-Q(k)} : \ k \in K, k \neq 0\right\}$$

(which is finite since $-Q(\cdot)$ is a positive definite form on K). □

Replacing Λ by $\Lambda \oplus \mathbb{Z}$ and v by

$$K(\mathcal{D}) \xrightarrow{v} \Lambda \hookrightarrow \Lambda \oplus \mathbb{Z}$$

we can therefore restrict to the case where Q is non-degenerate: given a path Z_t of central charges in $\operatorname{Hom}(\Lambda_\mathbb{R}, \mathbb{C})$ that are negative definite with respect to Q, we can choose extensions $\overline{Z}_t$ as in the Lemma that form a continuous path in $\operatorname{Hom}(\overline{\Lambda}, \mathbb{C})$. If we can lift the latter path to a path of stability conditions $\overline{\sigma}_t = (\overline{Z}_t, \mathcal{P}_t)$ that satisfy the support property with respect to $\overline{Q}$, then $\sigma_t := (Z_t, \mathcal{P}_t)$

is a path of stability conditions satisfying the support property with respect to Q. The reduction to the case where Q has signature $(2, \operatorname{rk}\Lambda - 2)$ works similarly:

Lemma 6.2. *Assume that Q is non-degenerate and of signature $(p, \operatorname{rk}\Lambda - p)$ for $p \in \{0, 1\}$. Let $\overline{\Lambda} := \Lambda_{\mathbb{R}} \oplus \mathbb{R}$, and let $\overline{Q}$ be the extension given by $\overline{Q}(v, \alpha) = Q(v) + \alpha^2$ for $v \in \Lambda_{\mathbb{R}}$ and $\alpha \in \mathbb{R}$. Then any central charge Z on $\Lambda_{\mathbb{R}}$ whose kernel is negative definite with respect to Q extends to a central charge $\overline{Z}$ on $\overline{\Lambda}$ whose kernel is negative definite with respect to $\overline{Q}$.*

Proof. We claim that there exists $z \in \mathbb{C}$ such that for all $v \in \Lambda_{\mathbb{R}}$ with $Z(v) = z$, we have $Q(v) < -1$. Indeed, let $K \subset \Lambda_{\mathbb{R}}$ be the kernel of Z, and let $K^{\perp}$ be its orthogonal complement. Then clearly we may assume $v \in K^{\perp}$. Since the restriction of Z to $K^{\perp}$ is injective, and since $K^{\perp}$ either has rank one, or has signature $(1, -1)$ with respect to Q, the claim is evident.

Using the claim, we can set $\overline{Z}(v, \alpha) := Z(v) + \alpha z$. □

This concludes the proof of Theorem 1.2.

7. Application

Proof of Corollary 1.3. Using the same arguments as in the previous section, we may assume that the Mukai pairing on Λ has signature $(2, \operatorname{rk}\Lambda - 2)$.

By Serre duality, any σ-stable object $E \in \mathcal{D}$ satisfies $\operatorname{Hom}(E, E[i]) = 0$ for $i < 0$ or $i > 3$ and $\operatorname{Hom}(E, E) = \mathbb{C} = \operatorname{Hom}(E, E[2])$; therefore, $(v(E), v(E)) \geq -2$. Moreover, Serre duality induces a non-degenerate symplectic form on $\operatorname{Ext}^1(E, E)$, and it has even dimension; thus $(v(E), v(E)) = -2$ or $(v(E), v(E)) \geq 0$.

Let $\sigma = (Z, \mathcal{P})$ be a stability condition with $Z \in \mathcal{P}_0(\mathcal{D})$. By the same argument as in Lemma 4.2 we may assume

$$(v, v) = |Z(v)|^2 - \|p(v)\|^2,$$

where $p \colon \Lambda_{\mathbb{R}} \to \operatorname{Ker} Z$ is the orthogonal projection onto the kernel of Z, and where $\|\cdot\|$ denotes the norm on $\operatorname{Ker} Z$ induced by the negative of the Mukai pairing. We claim that

$$C := \inf \{|Z(\delta)| \,:\, \delta \in \Lambda, (\delta, \delta) = -2\} > 0. \tag{1}$$

Indeed, if $|Z(\delta)| \leq 1$, then $\|p(\delta)\| \leq \sqrt{3}$; as $|Z(\cdot)| + \|p(\cdot)\|$ is a norm on $\Lambda_{\mathbb{R}}$, there are only finitely many integral classes satisfying both inequalities. Since $Z(\delta) \neq 0$ by assumption, the claim follows.

Now set

$$Q(v) := (v, v) + \frac{2}{C^2} |Z(v)|^2.$$

Clearly Q is negative definite on $\operatorname{Ker} Z$. Moreover, if E is σ-stable, then either $(v(E), v(E)) \geq 0$ or $v(E) = \delta$ is a root in Λ; in both cases, $Q(v(E)) \geq 0$ is evident from the construction.

Therefore, σ satisfies the support property with respect to Q. Theorem 1.2 gives an open neighbourhood $\mathcal{P}_Z(Q) \subset \operatorname{Hom}(\Lambda, \mathbb{C})$ of Z and an open neighbourhood U_σ of σ such that $U_\sigma \xrightarrow{\mathcal{Z}} \mathcal{P}_Z(Q)$ is a covering.

By compactness, any path Z_t in $\mathcal{P}_0(\mathcal{D})$ is contained in a finite number of such sets $\mathcal{P}_{Z_{t_i}}(Q_i)$, where Q_i the quadratic form associated to Z_{t_i}. Thus Z_t lifts uniquely to a path in $\mathcal{Z}^{-1}(\mathcal{P}_0(\mathcal{D}))$. □

REFERENCES

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga. The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability. *Adv. Math.*, 235:580–626, 2013, arXiv:1203.0316.

[ABM15] Rina Anno, Roman Bezrukavnikov, and Ivan Mirković. Stability conditions for Slodowy slices and real variations of stability. *Mosc. Math. J.*, 15(2):187–203, 403, 2015.

[Bay16] Arend Bayer. Wall-crossing implies Brill-Noether. Applications of stability conditions on surfaces, 2016, arXiv:1604.08261.

[BB13] Arend Bayer and Tom Bridgeland. Derived automorphism groups of $K3$ surfaces of Picard rank 1, 2013, arXiv:1310.8266. To appear in Duke Math. J.

[BM11] Arend Bayer and Emanuele Macrì. The space of stability conditions on the local projective plane. *Duke Math. J.*, 160(2):263–322, 2011, arXiv:0912.0043.

[BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014, arXiv:1301.6968.

[BMS16] Arend Bayer, Emanuele Macrì, and Paolo Stellari. The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds, 2016, arXiv:1410.1585.

[BMT14] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014, arXiv:1103.5010.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007, arXiv:math/0212237.

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008, arXiv:math/0307164.

[Bri09a] Tom Bridgeland. Spaces of stability conditions. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 1–21. Amer. Math. Soc., Providence, RI, 2009, arXiv:math.AG/0611510.

[Bri09b] Tom Bridgeland. Stability conditions and Kleinian singularities. *Int. Math. Res. Not. IMRN*, 21(21):4142–4157, 2009, arXiv:math/0508257.

[BS15] Tom Bridgeland and Ivan Smith. Quadratic differentials as stability conditions. *Publ. Math. Inst. Hautes Études Sci.*, 121:155–278, 2015.

[GMN13] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. Wall-crossing, Hitchin systems, and the WKB approximation. *Adv. Math.*, 234:239–403, 2013, arXiv:0907.3987.

[Huy11] Daniel Huybrechts. Introduction to stability conditions, 2011, arXiv:1111.1745.

[Ike14] Akishi Ikeda. Stability conditions for preprojective algebras and root systems of Kac-Moody Lie algebras, 2014, arXiv:1402.1392.

[KS08] Maxim Kontsevich and Yan Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, 2008, arXiv:0811.2435.

[MP14] Sven Meinhardt and Holger Partsch. Quotient categories, stability conditions, and birational geometry. *Geom. Dedicata*, 173:365–392, 2014.

[MS16] Emanuele Macrì and Benjamin Schmidt. Introduction to Bridgeland stability, 2016. Lecture notes for the CIMPA-ICTP-Mexico research school 2016. In preparation.

[Sha76] Stephen S. Shatz. Degeneration and specialization in algebraic families of vector bundles. *Bull. Amer. Math. Soc.*, 82(4):560–562, 1976.

[Tho08] R. P. Thomas. Stability conditions and the braid group. In *Superstring theory*, volume 1 of *Adv. Lect. Math. (ALM)*, pages 209–233. Int. Press, Somerville, MA, 2008, arXiv:math/0212214.

[Tod14] Yukinobu Toda. Derived category of coherent sheaves and counting invariants, 2014, arXiv:1404.3814.

SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE, UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UNITED KINGDOM

E-mail address: arend.bayer@ed.ac.uk

URL: http://www.maths.ed.ac.uk/~abayer/