Contents

1 Introduction

1.1 What is Mirror Symmetry?

It is now roughly 10 years ago when physicists first discovered that a special duality in conformal field theory might have interesting mathematical applications. Under the name of "mirror symmetry", this phaenomenom quickly became famous and rised interest among mathematicians and physicists. Since then, many purely mathematical statements arised from the rather vague conjecture, and a lot of effort has been put into setting up the right language to formulate them. However, it is still far from being well-understood.

The prediction came up in the study of a certain string theory. To each such string theory there is an associated manifold. Now the duality mentioned above implies that there are two possible choices for this manifold. Since there are a lot of properties of this string theory that can be deduced from geometrical properties of the manifold, these two manifolds will necessarily have corresponding structures on them. However, mathematically very surprising was the observation that the relevant properties of the manifolds were purely complex invariants on the one side and symplectic invariants on the other; mirror symmetry thus gave hints for relations between completely differently formulated mathematical structures that mathematicians might never have suspected.

This rather vague statement has been translated into very accurate predictions. To get a grasp of what can happen, it might be useful to have a quick look at the probably most famous single statement produced by mirror symmetry; it is dealing with numbers of rational curves on a Calabi-Yau hypersurface in $\mathbb{P}^4$ and involves a surprising identity:

For the symplectic side, we consider the generating function of rational curves

$$F(r) \sim \sum_{d \geq 1} N_d e^{dt}; \tag{1}$$

up to minor corrections, N_d is the number of holomorphic embeddings $\mathbb{P}^1(\mathbb{C}) \hookrightarrow \widehat{M}$ of degree d, the manifold $\widehat{M}$ itself is any hypersurface in $\mathbb{P}^4$ given by a polynomial in degree 5. Here the dimension 3 simplifies the formulation, since in this case the moduli space of rational curves of a certain degree is expected to be 0-dimensional, i. e. the rational curves are isolated; under general circumstances, this "number of curves" is computed by the theory of Gromov-Witten-invariants (see [15]). On the complex side we consider periods $\psi_i = \int_{\gamma_i} \omega$ (where ω is a Kähler form) of a certain class of Calabi-Yau manifolds parametrized by z. Now these functions are related by the simple equation

$$F\left(\frac{\psi_1}{\psi_0}\right) = \frac{5}{2}\frac{\psi_1\psi_2 - \psi_0\psi_3}{\psi_0^2}. \tag{2}$$

The proof of this statement (see [5]) is despite its physical origin purely mathematical.

It is not clear in general how to constuct the symplectic mirror manifold of a given complex manifold. However, in the case of 2-dimensional tori this mirror map is the simplest one possible: On the complex side, we have elliptic curves, which are as usual parametrized by a τ in the upper half-plane $\mathbb{H}$. The possible symplectic structures on $\mathbb{R}^2/\mathbb{Z}^2$ are characterized by $\rho = \int \omega$.

- The mirror map simply exchanges τ and ρ.

This essay will try to examine this relation. It will be concerned with a rather concrete side of mirror symmetry in the first part and a far more abstract one in the second. Section 2 is a detailed treatment of a generating function as in (1) for the case of the elliptic curve; the function considered there is extended to take into account maps from curves of higher genus. All other sections are more or less dealing with a categorical formulation of mirror symmetry proposed by Kontsevich [14].

1.2 Summary

In section 2, I give a proof of "theorem 2" (theorem 2.4) in Dijkgraaf's article on mirror symmetry on elliptic curves [7] (which is sketched in this paper) in full detail.

Section 3 is devoted to the statement of the homological mirror conjecture. After defining the two categories involved in sections 3.1 (sources: [25, 11]) and 3.2 ([23, 9, 14]) and proving a duality that arises in both of them in 3.3 (statement in [14]; [12, 25] used for the proof), I explain the construction of Polishchuk and Zaslow to proof the desired equivalence in the case of elliptic curves (section 3.4; [23, 25]).

Section 4 starts with a short report on Fourier-Mukai transforms (main source: [4]); it then compares the autoequivalences of the two categories that have been proved to be equivalent in section 3.

Section 5 gives rudiments of a construction somehow analogous to Fourier-Mukai-transforms for Fukaya categories that might not be found in the literature, which is however mentioned in [14].

The source for appendix A is [6].

2 Enumerative aspects (following R. Dijkgraaf)

This chapter will examine a generating function of numbers of curves similar to the one defined above (1). We won't be able to proof a statement as strong as equation (2), but proof a functional equation originating in its role in quantum field theory.

Let E be an elliptic curve. First of all, we need an appropriate definition of the "number of curves of genus g and degree d" $N_{g,d}$. The first idea would be to take the Euler characteristic of the Hurwitz space $H_{g,d} := \mathcal{M}_g(E, d)$, i. e. the moduli space of maps from genus g degree d-curves to E, but via the translation the elliptic curve acts freely on these spaces which hence have Euler characteristic zero. Instead, one uses the following definition:

By the Riemann-Hurwitz formula, a simple branched map (i. e. all branch points have ramification index 2) from a curve C_g of genus g to E has exactly

$2g-2$ branch points, hence we get a map $H_{g,d} \to E_{2g-2}$ to the configuration space of $2g-2$ points in E. This map has a finite fiber $X_{g,d}$. For any $\xi\colon C_g \to E$ in a fiber of an arbitrary $P \in E_{2g-2}$, we define the group $\operatorname{Aut}\xi$ to be the subgroup of the group of automorphisms of C_g commuting with ξ.

2.1 Definition *We define the "number of genus g degree d curves" in E to be*

$$N_{g,d} := \frac{1}{(2g-2)!} \sum_{\xi \in X_{g,d}} \frac{1}{\operatorname{Aut}\xi},$$

and get the generating functions

$$F_g(q) := \sum_{d=1}^{\infty} N_{g,d} q^d$$

(in fact there should be a slight modification in the case $g = 1$ about which I won't care here; apparently this corresponds to the fact that in this case the maps aren't stable).

Here q should be seen as the symplectic parameter of E (it is related to the parameter ρ of section 3.4.2 via $q = e^{2\pi i\rho}$).

As a rather tautological reformulation of the meaning of the numbers $N_{g,d}$, we have the following

2.2 Lemma *Let $E' := E \setminus \{P_1, \dots, P_{2g-2}\}$ be the $2g-2$-punctured curve E, and let S_d denote the symmetric group. Then let B be the set of isomorphism classes bundles over E' with fiber $\{1, \dots, d\}$ and group S_d. Now we have the subset B'' of B consisting of all bundles that have connected total space and monodromies around the points P_i in the conjugacy class of $(12) \in S_d$. Then*

$$N_{g,d} = \frac{1}{(2g-2)!} \sum_{\zeta \in B''} \frac{1}{|\operatorname{Aut}\zeta|},$$

where $\operatorname{Aut}\zeta$ is the usual automorphism group of a bundle (i. e. those automorphisms that have the identity as base map).

Proof. First oberve that if you are given any map $\xi\colon C_g \to E$ in the fibre $X_{g,d}$, then ξ restricted to the preimage of E' is a connected d-fold covering, which is nothing else than a bundle of the prescribed type. The branch points being simple exactly corresponds to the condition on the monodromies.

On the other hand, given such a bundle $\zeta\colon C \to E'$, we can first extend this to a map $\xi\colon C_g \to E$ by adding fibres of order $d-1$ at the branch points: around a point $P = P_i$ for some i, there is a neighbourhood $U \subset E$ of P such that $\zeta^{-1}(U \setminus \{P\})$ looks like $V \cup \bigcup_{i=1}^{d-2} U \setminus \{P\}$, where $\xi|_V\colon V \to U \setminus \{P\}$ is isomorphic to the mapping $z \mapsto z^2 : \Delta \setminus \{0\} \to \Delta \setminus \{0\}$; we insert P at each of the $d-2$ copies of $U \setminus \{P\}$ and another point to at the place of zero in the homeomorphism $V \approx \Delta \setminus \{0\}$. We then pull back the complex structure from E to the new total space C_g in the obvious way. The map ξ is then clearly a holomorphic simple branched map, and according to the Riemmann-Hurwitz formula the so-obtained Riemann surface C_g must be of genus g.

It is now clear that the automorphism groups of ζ and ξ as defined above correspond. □

It should be noted that this lemma, trivial as it is, translates an invariant that seems to involve the complex structure of the manifold into a purely symplectic one. This is a general feature of Gromov-Witten invariants.

The functions $F_g(q)$ have now the interesting property that they are in the space of quasi-modular forms $\mathbb{Q}[E_2, E_4, E_6]$. With the correspondence $q = e^{2\pi i\tau}$ where τ is the moduli parameter for the complex side (this is the mirror map for elliptic curves), this means that they are reasonably well-behaved functions on $\mathcal{M}_{1,1}$, the moduli space of elliptic curves: modular forms are sections of a power of the natural given line bundle on $\mathcal{M}_{1,1}$ that has as fibres the space of global sections of the cotangent bundle on the corresponding elliptic curve. This can be considered as a weak analogy to stronger mirror identity statements in higher dimensions, where one can get an explicit relation with corresponding functions on the mirror moduli space.

Now we combine all the functions F_g in one big function Z of two variables:

$$Z(q, \lambda) := e^{\sum_{g=1}^{\infty} \lambda^{2g-2} F_g(q)}$$

That this is useful can be seen by the following

2.3 Lemma *If we define the numbers $\hat{N}_{g,d}$ analogously to the interpretation of the numbers $N_{g,d}$ in lemma 2.2 by:*

$$\hat{N}_{g,d} = \frac{1}{(2g-2)!} \sum_{\zeta \in B'} \frac{1}{|\mathrm{Aut}\,\zeta|}$$

We just replaced B'' by the larger subset B' of B consisting of all bundles that have monodromies around the points P_i in the conjugacy class of $(12) \in S_d$ (we thus omit the condition of connectedness), then $Z(q, \lambda)$ has the expansion

$$Z(q, \lambda) = \sum_{g,d=1}^{\infty} \hat{N}_{g,d} q^d \lambda^{2g-2} \tag{3}$$

Proof.

$$Z(q, \lambda) = e^{\sum_{g=1}^{\infty} \lambda^{2g-2} F_g(q)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{g,d=1}^{\infty} \lambda^{2g-2} N_{g,d} q^d \right)^n$$

$$= \sum_{g,d=1}^{\infty} \lambda^{2g-2} q^d \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(g_i, d_i)_{(i=1..k)}} \prod_{i=1}^{k} N_{g_i, d_i} \tag{4}$$

Here the last sum goes over all k-tuples of pairs of integers (g_i, d_i) with the property that $\sum_{i=1}^k (2g_i - 2) = 2g - 2$ and $\sum_{i=1}^k d_i = d$. A close inspection of this sum shows that it exactly computes $\hat{N}_{g,d}$:

Let k be the number of connected components of such a bundle $\zeta \in B'$, and let ζ_i, $i = 1..k$, be the bundles that we get from the connected components of ζ. Let ζ_i have a fibre of order d_i and a total space of genus g_i. Since the Euler characteristic is additive on disjoint union, we in fact get $\sum_{i=1}^k (2g_i - 2) = 2g - 2$, while $\sum_{i=1}^k d_i = d$ is obvious. Now number the isomorphism classes of those connected components with j, $j = 1..l$, and let k_j be the number of components in each class; hence $\sum_j k_j = k$. If we use n-nomial coefficients, then there are $\binom{k}{k_1\, k_2 \ldots k_l}$ ways to number these connected components; thus the contribution of this family of bundles $\{\zeta_1, \ldots, \zeta_l\}$ to the sum in (4) will be

$$\frac{1}{k!} \binom{k}{k_1\, k_2\, \ldots\, k_l} \prod_{i=1}^{k} \frac{1}{(2g_i - 2)!\, |\mathrm{Aut}\, \zeta_i|} = \frac{1}{k_1! \cdot \ldots \cdot k_l!} \prod_{i=1}^{k} \frac{1}{(2g_i - 2)!\, |\mathrm{Aut}\, \zeta_i|} \tag{5}$$

(The factors $\frac{1}{(2g_i-2)!|\mathrm{Aut}\,\zeta_i|}$ come from the definition of N_{g_i,d_i}.)

On the other hand we want to see the contribution of this family $(\zeta_i)_i$ to the expansion (3), i. e. to the number $N_{g,d}$.

There are $\binom{2g-2}{2g_1-2 \ldots 2g_l-2}$ ways to distribute the $2g-2$ branch points among the connected components, only that $\prod_j' k_j!$ of these possible ways lead to an isomorphic bundle ζ; the prime in this product here indicates to omit a possible class of trivial bundles with one fibre (these are the only ones that have no branch points). Now the automorphism group of ζ is the direct product of the automorphism groups of the ζ_i and the group of permutations of those trivial bundles, if any. Hence, the contribution of this familiy to $\hat{N}_{g,d}$ will be

$$\frac{1}{(2g-2)!} \binom{2g-2}{2g_1 - 2\, \ldots\, 2g_l - 2} \cdot \prod_j{}' \frac{1}{k_j!} \cdot \frac{1}{|\mathrm{Aut}\, \zeta|}$$

$$= \frac{1}{k_1! \cdot \ldots \cdot k_l!} \prod_{i=1}^{k} \frac{1}{(2g_i - 2)!\, |\mathrm{Aut}\, \zeta_i|}$$

□

But this is not the only reason for combining this function into the partition function $Z(q, \lambda)$. This function has (up to the above-mentioned (definition 2.1) minor modifications for $g = 1$) a quantum field theoretic interpretation as a path-integral, and this is where the following equation was first observed ([8]):

2.4 Theorem

$$Z(q, \lambda) = \int_{\partial\Delta} \frac{dz}{2\pi i z} \prod_{n \in \mathbb{Z}_{\geq 0} + 1/2} \left(1 + z q^n e^{\lambda \frac{n^2}{2}}\right) \left(1 + z^{-1} q^n e^{-\lambda \frac{n^2}{2}}\right)$$

However, there is as well a purely mathematical proof that will be outlined in the following. This might be considered as a toy model of the claim in the introduction that properties of the quantum field theory can be deduced from properties of the associated manifold.

For a slight convenience, I will introduce the notion of a "pointed G-bundle" over a pointed space. The technical advantage will be that in our case they do not have any automorphisms, so that we won't need to keep track of the factor $\frac{1}{\operatorname{Aut} b}$ any more. (I do not know whether this notion already exists, but it seems only natural for pointed spaces.)

2.5 Definition *Let G be a group that acts faithfully from the left on a topological space F. A* pointed G-bundle with fiber F *over a pointed space (X, x_0) is a G-bundle $p\colon H \to X$ with fiber F and a specified G-homeomorphism $F \cong p^{-1}(x_0)$. Morphisms between pointed G-bundles are G-bundle homomorphisms that have a map of pointed spaces as base map and respect the given G-homeomorphisms of the fibres at the basepoints.*

The difference to the usual notion becomes most apparent in the case of F and G being finite and discrete: While a bundle corresponds to a conjugacy class of maps $\sigma\colon \pi_1(X, x_0) \to G$, a pointed bundle corresponds to a single one of these maps.

As can be seen by its faithful action on $p^{-1}(x_0)$, the automorphism group of an (ordinary) G-bundle is the centralizer of $\sigma(\pi_1(X, x_0))$ in G, i. e. the stabilizer of σ under the action of G on the set of all these maps by conjugation. This makes the following observation obvious:

2.6 Lemma *Let $(E', 0)$ be the pointed, $2g-2$-punctured elliptic curve E and $\tilde{N}_{g,d}$ be the number of pointed S_d-bundles on E' with fiber $\{1, \ldots, d\}$ and monodromies around the $2g-2$ holes in the conjugacy class of $(12) \in S_d$. Then*

$$\hat{N}_{g,d} = \frac{1}{(2g-2)!d!} \tilde{N}_{g,d}.$$

Proof. Let σ be a map $\sigma\colon \pi_1(E') \to S_d$; if its stabilizer in G under the action of conjugation has order s (so $s = |\operatorname{Aut} \zeta|$ for the corresponding bundle ζ), then the conjugacy class of σ has $\frac{d!}{s}$ elements. Thus the contribution of this conjugay class of maps to both sides of the above equation is $\frac{1}{(2g-2)!s}$. □

The task is now to count this number of pointed bundles. Due to the general one-to-one correspondence between principal G-bundles and G-bundles with fibre F on which G acts faithfully, we can restrict our attention to principal bundles. (This correspondence is immediately clear from the viewpoint of transition functions; the harder part in the proof of this statement is rather the correspondence between bundles and transition functions.) We will count them under a more general setting:

2.7 Definition *Let G be any finite group, R be the set of its irreducible representations, and C the set of its conjugacy classes. Let Σ be the oriented surface of genus h and N holes $P_1, \ldots, P_N$. If $c_1, \ldots, c_N \in C$ are conjugacy classes of*

G, we define $Z_h(c_1, \dots, c_N)$ to be the number of pointed principal G-bundles on Σ with monodromies around P_i in the conjugacy class c_i, divided by the order of G:

$$Z_h(c_1, \dots, c_N) := \frac{\#\textit{pointed principal } G\textit{-bundles with } \dots}{|G|}$$

(Note that with pointed bundles, there is still the 1-to-1 correspondence of principal G-bundles and G-bundles with a fiber F on which G acts faithfully.)

These numbers $Z_h(c_1, \dots, c_N)$ can be computed by the following

2.8 Lemma *For an irreducible representation $r \in R$ of G and a conjugacy class $c \in C$, we have the character $\chi_r(c)$ of r on c; further we define the Frobenius number*

$$f_r(c) := \frac{|c|\,\chi_r(c)}{\dim r}.$$

Then the number of pointed principal G-bundles on a surface of genus h with the above boundary conditions is given by

$$Z_h(c_1, \dots, c_N) = \sum_{r \in R} \left(\frac{|G|}{\dim r} \right)^{2h-2} \prod_{i=1}^{N} f_r(c_i)$$

Proof. The proof works by induction on h: for a punctured sphere, the lemma can be proved directly, and surfaces of higher genus are obtained by glueing holes together.

- The case $h = 0$:

  Let $\mathcal{H}$ be the center of the group algebra $\mathbb{C}[G]$ of G; the proof of this case more or less turns out to become an exercise in understanding $\mathcal{H}$.

  It is immediate that as a vector space, $\mathcal{H}$ is the subspace of $\mathbb{C}[G]$ generated by the elements $z_c = \sum_{g \in c} g$ corresponding to conjugacy classes c.

  Since z_c commutes with all elements of G, it induces a G-automorphism on every G-module. On an irreducible representation r, this map must be a homothety; by looking at the trace we see that this is multiplication with $f_r(c)$.

  We can define a linear form $\langle \cdot \rangle$ on $\mathcal{H}$ by setting $\langle z_c \rangle := \frac{1}{|G|}\delta_{c,e}$, where e is the neutral element in G. Now let $\mathcal{R} = \mathbb{C}[G]$ be the regular representation of G. Since $\mathrm{Tr}_{\mathcal{R}}(g) = |G|\,\delta_{g,e}$ for $g \in G$, we get

  $$\mathrm{Tr}_{\mathcal{R}}(\cdot) = |G|^2 \langle \cdot \rangle \tag{6}$$

  if we restrict $\mathrm{Tr}_{\mathcal{R}}$ to $\mathcal{H}$.

  Using the fact that the fundamental group $\pi_1(\Sigma)$ of an N-punctured sphere is generated by the N loops γ_i around the holes with the single relation $\prod_i \gamma_i = 1$, we can interpret the number $Z_h(c_1, \dots, c_N)$ in terms of our class algebra $\mathcal{H}$: Giving a morphism from $\pi_1(\Sigma)$ to G means chosing N elements σ_i of G with $\prod \sigma_i = e$. So a pointed G-bundles with the monodromies around P_i in c_i is given by an N-tuple $(\sigma_i)_i \in G^N$ with their product being e and σ_i being in the conjugacy class c_i. But the number of these N-tuples together with the normalizing factor $\frac{1}{|G|}$ of definition 2.7 can be easily expressed with our linear form:

$$Z_h(c_1, \dots, c_N) = \langle z_{c_1} \cdot \ldots \cdot z_{c_N} \rangle$$

Now we can use our earlier remarks on $\mathcal{H}$: First, using (6), we obtain

$$Z_h(c_1, \dots, c_N) = \frac{1}{|G|^2} \operatorname{Tr}_{\mathcal{R}} \langle z_{c_1} \cdot \ldots \cdot z_{c_N} \rangle$$

But $\mathcal{R} \cong \bigoplus_{r \in R} r^{\dim r}$; since we know the action of z_c on r, we can simplify this to $\operatorname{Tr}_{\mathcal{R}} \langle z_{c_1} \cdot \ldots \cdot z_{c_N} \rangle = \sum_{r \in R} \dim r^2 \prod_i f_r(c_i)$. So our claim follows:

$$Z_h(c_1, \dots, c_N) = \sum_{r \in R} \left(\frac{\dim r}{|G|} \right)^2 \prod_i f_r(c_i)$$

- Now let us consider the glueing:

  Let Σ be the oriented surface of genus $h - 1$ with $N + 2$ holes $\{P_1, \dots, P_{N+2}\}$, and let Σ' be the surface of genus h with N holes that is obtained by glueing annular neighbourhoods U_{N+1} and U_{N+2} of P_{N+1} and P_{N+2} in the usual way with a homeomorphism $\beta\colon U_{N+1} \to U_{N+2}$; let $\alpha\colon \Sigma \to \Sigma'$ denote the collapsing map.

  By pulling it back, we get a pointed bundle $\zeta = \alpha^* \zeta'$ on Σ from every pointed bundle ζ' on Σ'. Let $\gamma_{N+1}, \gamma_{N+2} \in \pi_1(\Sigma)$ be the loops around P_{N+1} and P_{N+2} in canonical orientation; by drawing a picture one can convince oneself that $\alpha_*(\gamma_{N+1})$ is a conjugate of the inverse of $\alpha_*(\gamma_{N+2})$ in $\pi_1 \Sigma'$. This shows that the conjugacy classes of the monodromies of ζ around P_{N+1} and P_{N+2} are inverses of each other.

  On the other hand, if we are given a pointed bundle ζ on Σ with monodromies around P_{N+1} and P_{N+2} in the conjugacy class c and c^{-1} respectively, then the bundles on U_{N+1} and U_{N+2} obtained by restricting ζ will be isomorphic, and the isomorphism can be obtained with the glueing map $\beta\colon U_{N+1} \to U_{N+2}$ as the map between the ground spaces. With any such isomorphism, we can glue ζ to get a pointed bundle on ζ'. So we need to know exactly how many of these isomorphisms exist, i. e. we need to know the order of the automorphism group of $\zeta|_{U_{N+1}}$. By the faithful action on any specific fiber $\zeta^{-1}(u_0) \cong G$, this automorphism group is isomorphic to a subgroup of G; more precisely, this subgroup is the centralizer of the monodromy of $\zeta|_{U_{N+1}}$ with respect to our fiber $\zeta^{-1}(u_0) \cong G$. Since G acts transitively by conjugation on c, the stabilizer of an element of c (which is its centralizer of course) has order $\frac{|G|}{|c|}$.

Now a simple computation yields our induction:

$$Z_h(c_1,\dots,c_N) = \sum_{c\in C} Z_{h-1}(c_1,\dots,c_N,c,c^{-1})\frac{|G|}{|c|}$$

$$= \sum_{r\in R}\left(\frac{|G|}{\dim r}\right)^{2h-4}\prod_{i=1}^{N} f_r(c_i)$$

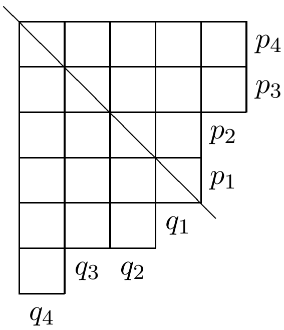
$$\sum_{c\in C}\frac{|G|}{|c|}\frac{|c|^2\,\chi_r(c)\chi_r(c^{-1})}{\dim r^2} = \sum_{r\in R}\frac{|G|^{2h-3}}{\dim r^{2h-2}}\prod_{i=1}^{N} f_r(c_i)\sum_{g\in G}\chi_r(g)\chi_r(g^{-1})$$

$$= \sum_{r\in R}\left(\frac{|G|}{\dim r}\right)^{2h-2}\prod_{i=1}^{N} f_r(c_i)$$

□

By specialising this result to our case, we will be able to prove theorem 2.4; however, first we need some facts about the representations of the symmetric group S_d:

Every irreducible representation r of S_d is given by a partition of d, or equivalently by a Young diagram. By slicing a Young diagram along the diagonal, we get subsets $P = \{p_1 < \dots < p_l\}$ and $Q = \{q_1 < \dots < q_l\}$ of the half-integers, i. e. $p_i, q_i \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$.



From these sets we define the numbers w_r^k to be

$$w_r^k := \sum_i p_i^k - \sum_i (-q_i)^k.$$

These numbers have a surprisingly good interpretation related to the representation r; for $k=0$ and $k=1$ this is obvious, for $k=2$ see [10]:

$$\begin{aligned} w_r^0 &= 0 \\ w_r^1 &= d \\ w_r^2 &= 2f_r(c) \end{aligned} \qquad (7)$$

Here c is the conjugacy class of $(1\,2)$, the single transposition, in S_d.

Proof. (of theorem 2.4)

By using lemma 2.8 and lemma 2.6 we see that

$$\hat{N}_{g,d} = \frac{1}{(2g-2)!} \sum_{r \in R_d} f_r(c)^{2g-2}.$$

With lemma 2.3 this yields

$$Z(q,\lambda) = \sum_{g,d} \hat{N}_{g,d} q^d \lambda^{2g-2} = \sum_{d,b=0}^{\infty} \frac{q^d \lambda^b}{b!} \sum_{r \in R_d} f_r(c)^b = \sum_{d=0}^{\infty} q^d \sum_{r \in R_d} e^{\lambda f_r(c)}$$

(The summands for odd b are zero; the nicest way to see this might be to go all our construction backwards, since a non-zero term would claim the existence of a map to E from a Riemann surface with odd Euler characteristic.)

With (7) and $w_r := \ln q \cdot w_r^1 + \frac{\lambda}{2} w_r^2$ this becomes

$$Z(q,\lambda) = \sum_{d=0}^{\infty} \sum_{r \in R_d} e^{w_r}$$

If we define the polynomial $t(p) := \ln z \cdot p^0 + \ln q \cdot p^1 + \frac{\lambda}{2} p^2$, we get $w_r = \sum_i t(p_i) - \sum_i t(-q_i)$; we can now replace the sum over all representations of all symmetric groups in the last equation by a sum over all pairs P, Q of subsets of the positive half-integers satisfying $|P| = |Q|$:

$$Z(q,\lambda) = \sum_{\substack{P,Q \subset \mathbb{Z}_{\geq 0}+1/2 \\ |P|=|Q|}} e^{w_r} = \sum_{\substack{P,Q \subset \mathbb{Z}_{\geq 0}+1/2 \\ |P|=|Q|}} e^{\sum_i t(p_i) - \sum_i t(-q_i)} \qquad (8)$$

Now consider the following generating function:

$$\prod_{p \in \mathbb{Z}_{\geq 0}+1/2} \left(1 + e^{t(p)}\right)\left(1 + e^{t(-p)}\right)$$

If we look at the constant term of its Laurent expansion in z, we only take into account those summands with equally as many $e^{t(p)}$ and $e^{t(-p)}$ factors, we thus get exactly the sum of equation (8). By computing the constant Laurent coefficient with the usual integral, we proof our theorem. □

3 Homological Mirror Symmetry

In his talk [14] at the ICM 1994, Kontsevich proposed a completely new mirror symmetry statement, claiming the equivalence of two categories associated to a complex variety and its symplectic mirror. He believed that what he called homological mirror conjecture would put mirror symmetry on more algebraic grounds, and that the enumerative statements of mirror symmetry could be deduced from his conjecture:

3.1 Conjecture *The derived category of coherent sheaves on a complex manifold M is equivalent to the Fukaya category associated to its symplectic mirror manifold $\widehat{M}$.*

However so far, nearly five years later, there seems to be no single example where enumerative or other predictions have been proved from the assumption of this conjecture. The statement itself has been proved in the simplest case, the elliptic curve, by Polishchuk and Zaslow [23], and some slightly modified statement for some elliptic K3 surfaces by Bartocci, Bruzzo and Sanguinetti [2]. However, both methods don't seem to be accessible to an obvious generalization; while P. and Z. used a very explicit description of both categories involved that can't be given similarly for higher-dimensional cases, the Italians made a very strong use of the 2-dimensionality of their case.

First, I will restate the definitions of the two categories involved.

3.1 The derived category of coherent sheaves

3.1.1 The construction

For any abelian category $\mathcal{A}$, we can form the derived category $\mathbf{D}^b(\mathcal{A})$ by taking the category $\mathbf{K}^b(\mathcal{A})$ with cochain complexes as objects and chain homotopy equivalence classes of chain maps as morphisms, and then inverting all quasi-isomorphisms to become isomorphisms in $\mathbf{D}^b(\mathcal{A})$. In the following, I will give the definitions and constructions, but I will omit all necessary proofs and especially all set-theoretic considerations; they can be found in [25].

Let $\mathcal{A}$ be any abelian category. We define

3.2 Definition the category $\mathbf{Ch}(\mathcal{A})$ of complexes in $\mathcal{A}$ *as follows: an object $C^\bullet$ of $\mathbf{Ch}(\mathcal{A})$ is a sequence $C^n, n \in \mathbb{Z}$ of objects together with differentials $\partial^n \in \mathrm{Hom}_{\mathcal{A}}(C^n, C^{n+1})$, such that the composition of two differentials $\partial^{n+1}\partial^n$ is zero. A Morphism f between two complexes $C^\bullet$ and $D^\bullet$ is a collection of morphisms $f^n\colon C^n \to D^n$ that commute with the differentials.*

In $\mathbf{Ch}(\mathcal{A})$, we have the full subcategories $\mathbf{Ch}^+(\mathcal{A})$, $\mathbf{Ch}^-(\mathcal{A})$ or $\mathbf{Ch}^b(\mathcal{A})$ of complexes whose non-zero objects are, respectively, bounded above, bounded below or bounded in degree.

We will only need the derived category constructed from $\mathbf{Ch}^b(\mathcal{A})$ in the following, so I will restrict myself to this case.

On the sets $\mathrm{Hom}_{\mathbf{Ch}^b(\mathcal{A})}(X, Y)$, we have the equivalence relation of chain homotopy; we can now define a new category $\mathbf{K}^b(\mathcal{A})$ that has the same objects as $\mathbf{Ch}^b(\mathcal{A})$ and chain homotopy equivalence classes of $\mathrm{Hom}_{\mathbf{Ch}^b(\mathcal{A})}(X, Y)$ as morphisms $\mathrm{Hom}_{\mathbf{K}^b(\mathcal{A})}(X, Y)$. It has to be checked that this is again a category.

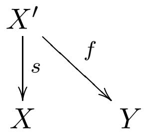
Among the morphisms in $\mathbf{K}^b(\mathcal{A})$, we have the subset S of quasi-isomorphisms, i. e. those isomorphisms that induce an isomorphism on cohomology; we want to *localize* the category at this set S:

3.3 Definition *Let $\mathcal{C}$ be a category, and let S be a class of morphisms in $\mathcal{C}$. A* localization with respect to S *is a category $S^{-1}\mathcal{C}$ with a functor $i\colon \mathcal{C} \to S^{-1}\mathcal{C}$ such that*

- *for every $s \in S$, $i(s)$ is an isomorphism in $S^{-1}\mathcal{C}$, and*
- *every other functor $F\colon \mathcal{C} \to \mathcal{B}$ that maps S to isomorphisms factors through i.*

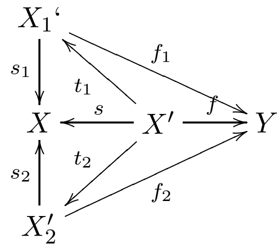
In case S satisfies some properties, the localized category can be constructed in the following way which looks quite similar to the construction of localizing a ring at a multiplicative subset (it is a generalization in fact):

3.4 Definition *The fractional category $S^{-1}\mathcal{C}$ is the category that has the same objects as $\mathcal{C}$, while the morphisms in $\operatorname{Hom}_{S^{-1}\mathcal{C}}(X, Y)$ are given by "hats" (corresponding to fractions in the case of a ring):*



Here s has to be in S, and f is any morphism in $\mathcal{C}$.

Two of such morphisms (s_1, f_1) and (s_2, f_2) are regarded as equal if there is a commutative diagram



with $s, t_1, t_2 \in S$.

The conditions imposed on S are rather worked backwards from what one needs to prove that this construction produces a category again. It can be checked that the quasi-isomorphisms fulfill these conditions.

3.5 Definition *The (bounded) derived category $\mathbf{D}^b(\mathcal{A})$ is the category obtained from $\mathbf{K}^b(\mathcal{A})$ by localizing at the collection S of quasi-isomorphisms.*

Now suppose X is a compact complex manifold.

3.6 Definition *A coherent sheaf $\mathcal{G}$ on X is a sheaf that is everywhere locally the cokernel of a map $\mathcal{O}_X^m \to \mathcal{O}_X^n$ where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.*

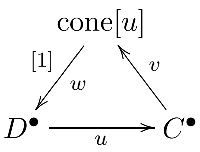
This sheaves form the abelian category $\operatorname{Coh} X$. From this we form the bounded derived category of coherent sheaves $\mathbf{D}^b(\operatorname{Coh} X)$.

3.1.2 Triangulated categories

The derived category has the additional structure of being a *triangulated category*. I won't give a full account of the definition of the notion of an exact triangle here, and I won't give the axioms of a triangulated category until the end of this section; instead, I will give a sketch of the construction and state the properties I need.

In the category of complexes $\mathbf{Ch}^b(\mathcal{A})$, we have the construction of a *mapping cone* cone(u) for every morphism $u\colon C^\bullet \to D^\bullet$. This is a complex $E^\bullet$ which has the same objects as the direct sum $C^\bullet \oplus D^\bullet[1]$, only the differentials have a certain twist involving u. Only in the case of $u = 0$, the cone is exactly the direct sum.

With the obvious maps $C^\bullet \xrightarrow{v} \mathrm{cone}(u) \xrightarrow{w} D^\bullet[1]$, this construction gives a triangle of maps:



Now the observation that for every such triangle we have at the same time $C^\bullet \simeq \mathrm{cone}(w)$ and $D^\bullet \simeq \mathrm{cone}(v)$ shows that in $\mathbf{K}^b(\mathcal{A})$, it makes sense to define the notion of an exact triangle: we will call a triangle of maps in $\mathbf{K}^b(\mathcal{A})$ exact if it is isomorphic (which means homotopic in $\mathbf{Ch}^b(\mathcal{A})$ to a triangle constructed with the mapping cone; a triangle in $\mathbf{D}^b(\mathcal{A})$ will be called exact if it is quasi-isomorphic to an exact triangle in $\mathbf{K}^b(\mathcal{A})$.

The importance of the notion of exact triangles is that in the derived category, they play the role of short exact sequences in abelian categories: in abelian categories, we have the notion of a *cohomological functor* (or δ-functor); its characterizing property is that it gives rise to a long exact cohomology sequence for every short exact sequence in the category. Now a cohomological functor from a triangulated category is a functor F that, for every exact triangle $A \xrightarrow{u} B \xrightarrow{v} C \underset{[1]}{\xrightarrow{w}} A$, yields a long exact sequence

$$\cdots \to F(A) \stackrel{F(u)}{\to} F(B) \stackrel{F(v)}{\to} F(C) \stackrel{F(w)}{\to} F(A[1]) \stackrel{F(u)}{\to} F(B[1]) \to \cdots .$$

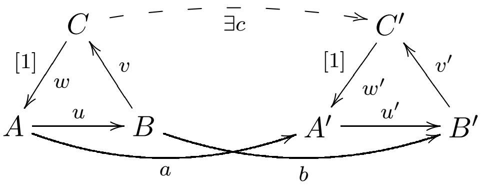
This means that we have replaced the natural transformations δ^n for δ-functors by a simple functorial map.

The derived category $\mathbf{D}^b(\mathcal{A})$ has the additional useful property that for every short exact sequence of complexes $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ in $\mathbf{Ch}^b(\mathcal{A})$, we can find a map $C \xrightarrow{w} A$ such that (u, v, w) becomes an exact triangle; this seems to be an important technical advantage of $\mathbf{D}^b(\mathcal{A})$ compared to $\mathbf{K}^b(\mathcal{A})$.

A set of axioms for triangulated categories that is now commonly used has been given by Verdier in [24]; they do not seem to be completely satisfactorily (as we will see later in section 3.4.3), but are in most cases technically sufficient:

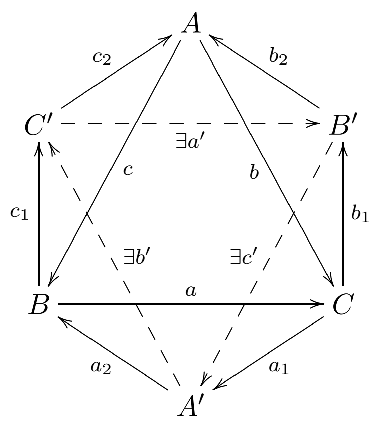
3.7 Definition *An additive category $\mathcal{A}$ with an autoequivalence T as shift functor and a class of distinguished (or "exact") triangles is called a triangulated category if it satisfies the folowing axioms:*

1. (a) *Every morphism can be embedded in a distinguished triangle (that is, we can construct a cone).*
   (b) *If a triangle is isomorphic to a distinguished triangle, then it is itself distinguished.*
   (c) *The triangle* $A \xrightarrow{\text{id}} A \to 0 \to T(A)$ *is distinguished.*
2. *If* (u, v, w) *is a distinguished triangle, then* $(v, w, -T(u))$ *is distinguished as well.*
3. *If we have a commutative diagram*



with exact triangles (u, v, w) *and* (u', v', w') *and given maps* a *and* b*, we can always complete it with a map* $c\colon C \to C'$ *such that still everything commutes.*

4. (Octahedral axiom) *Given objects* A, B, C, A', B', C' *and maps* $a, b, c, a_1, a_2, b_1, b_2, c_1, c_2$ *as in the diagram such that* (a, a_1, a_2), (b, b_1, b_2) *and* (c, c_1, c_2) *are exact triangles,*



then we can complete this diagram with maps a', b' *and* c' *such that*

- (a', c', b') *is an exact triangle,*
- *the four faces that are not exact triangles commute and*
- *the two possible squares involving* B *and* B' *commute, that is* $a'c_1 = b_1a$ *and* $a_2c' = cb_2$.

Note that the three maps a', b', c' are uniquely determined if they exist.

3.1.3 Derived functors

If we have a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of finite cohomological dimension (where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories), and if A has enough injectives (that is, every object can be embedded into an injective one), then we get a derived functor $\mathbf{R}F\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}^b(\mathcal{B})$ (the circumstances under which the derived functor can be constructed are actually a bit more general than that). This derived functor is a functor of triangulated categories and the natural extension of F to the derived categories.

On complexes with all objects being injective, it can be given simply by applying F to every object of the complex; other complexes are always quasi-isomorphic to an injective one. The cohomology of the complex $F(A)$ for $A \in \mathbf{D}^b(\mathcal{A})$ is computed by the hyper-derived functors $(\mathbb{R}^q F)(A)$.

At some points we will have to use spectral sequences. However, since they are more a tool for computing than essential for getting an understanding, and since an introduction to spectral sequences would necessarily be very longwided and technical, this will be omitted.

3.2 The Fukaya Category

In general, it seems not yet completely clear how to define the Fukaya category to get the desired equivalence. I will follow the definition of Polishchuk and Zaslow [23].

3.2.1 A_∞-categories

The Fukaya Category itself is not a true category (but there will be natural way to make one out of it); instead, it is an so-called A_∞-category:

3.8 Definition *An A_∞-category $\mathcal{C}$ consists of the following data:*

- *Objects* $\operatorname{Ob}\mathcal{C}$
- *For each pair X, Y of objects, we have a $\mathbb{Z}$-graded group of homorphisms* $\operatorname{Hom}(X, Y)$
- *Instead of a composition, we have for each positive integer k a linear map*

  $$m_k\colon \operatorname{Hom}(X_1, X_2) \otimes \cdots \otimes \operatorname{Hom}(X_k, X_{k+1}) \to \operatorname{Hom}(X_1, X_{k+1})$$

  of degree $2-k$, satisfying the following consistency conditions:

  $$\sum_{r=1}^{n} \sum_{s=1}^{n-r+1} (-1)^{\epsilon} m_{n-r+1}(a_1 \otimes \cdots \otimes a_{s-1} \otimes m_r(a_s \otimes \cdots \otimes a_{s+r-1}) \otimes \cdots \otimes a_n) = 0$$

  The sign factor ϵ depends on r, on s and on the degrees of the a_i:
  $\epsilon = (r+1)s + r\left(n + \sum_{i=1}^{s-1} \deg a_i\right)$.

The first conditions says that m_1 is a degree one map on $\mathrm{Hom}(X, Y)$ with $m_1^2 = 0$, i. e. all the homomorphism groups are complexes with coboundary operator $m_1 =: d$. The second condition says that m_2 is a map of complexes:

$$d(m_2(a_1 \otimes a_2)) = m_2(da_1 \otimes a_2) + (-1)^{\deg a_1} m_2(a_1 \otimes da_2) \tag{9}$$

So m_2 induces a product on cohomology. The third condition

$$m_2(m_2(a_1 \otimes a_2) \otimes a_3) - m_2(a_1 \otimes m_2(a_2 \otimes a_3))$$

$$= d(m_3(a_1 \otimes a_2 \otimes a_3))$$

$$-m_3\Big(da_1 \otimes a_2 \otimes a_3 + (-1)^{\deg a_1} a_1 \otimes da_2 \otimes a_3 + (-1)^{\deg a_1 + \deg a_2} a_1 \otimes a_2 \otimes da_3\Big)$$

tells us that m_3 is a homotopy between the two possible ways to compose three morphisms with m_2; in other words, the product induced by m_2 on the level of cohomology is associative. Hence, we can always get a proper category $H^0(\mathcal{C})$ by taking the zero cohomology of the homomorphism groups.

3.2.2 Fukaya category: The objects

Let $\widehat{M}$ be a n-dimensional Calabi-Yau Kähler manifold with (possibly complexified) Kähler form ω and nowhere vanishing holomorphic n-form Ω (we do not, however, require the condition of $h^{1,0}$ to be zero); more generally, we can also take a symplectic manifold that has an almost-complex structure compatible with the symplectic form. On M, we have the bundle $\mathcal{L}$ of Lagrangian planes, which has as fibre at $x \in M$ the Lagrangian Grassmanian consisting of all n-dimensional subspaces of the tangent space $T_x\widehat{M}$ on which ω restricts to zero. The fibres have fundamental group $\mathbb{Z}$. By glueing the universal covers of the fibres, we get a $\mathbb{Z}$-covering $\widetilde{\mathcal{L}}$ of $\mathcal{L}$.

3.9 Definition *A simple object in the Fukaya category $\mathcal{F}$ consists of*

- *a minimal (or special) Langrangian submanifold U of $\widehat{M}$, i. e. a closed real-)n-dimensional submanifold for which the Kähler form restricts to zero, i. e. $\iota^*(\Omega) = 0$, and for which $\mathrm{Im}(z \cdot \iota^*\Omega) = 0$ for some $z \in \mathbb{C}$;*
- *a lift $\tilde{\gamma} \colon U \to \widetilde{\mathcal{L}}|_U$ of the naturally given map $\gamma \colon U \to \mathcal{L}|_U$, and*
- *a local system $\mathcal{E}$ on U: we want $\mathcal{E}$ to be a complex vector bundle with a flat connection, all of whose monodromies have eigenvalues of modulus 1.*

An object in $\mathcal{F}$ is a direct sum of finitely many simple objects.

For our context it is sufficient to regard a connection of a bundles as a map Θ that associates to each piecewise smooth path $\gamma \colon [a, b] \to U$ an isomorphism of the fibres of the bundle at the endpoints: $\Theta(\gamma) \in \mathrm{Hom}(\mathcal{E}|_{\gamma(a)}, \mathcal{E}|_{\gamma(b)}$. This map Θ has to be compatible with composition:

$$\Theta(\gamma_1 \circ \gamma_2) = \Theta(\gamma_1) \circ \Theta(\gamma_2)$$

The flatness of the connection then says that Θ depends only on the homotopy class (relative the endpoints) of the path.

The existence of the lift $\tilde{\gamma}$ is not guaranteed; it is instead a further condition on U.

Kontsevich first suggested $\mathcal{E}$ to be an $U(n)$-bundle, but Polishchuk and Zaslow needed this slight enlargement to prove the desired equivalence. For higher dimensions, there might be further enlargements needed as Kontsevich already believed in 1994 and as the result of [2] indicates.

3.2.3 Fukaya category: The morphisms

As for the morphisms, we first need to refer to the Maslov index that we need to define the required $\mathbb{Z}$-grading on the morphisms. A definition is given in appendix A; it is a natural extension of the map $\pi_1(\mathrm{Lag}\,\widehat{M}) \cong \mathbb{Z}$ (where $\mathrm{Lag}\,\widehat{M}$ is the Lagrangian Grassmannian) to open paths.

3.10 Definition *Let $(\mathcal{E}_1, U_1)$ and $(\mathcal{E}_2, U_2)$ be objects of $\mathcal{F}$ so that U_1 and U_2 intersect transversely, i.e. in a finite number of points $x_1, \ldots, x_k$. Then we define the group of morphisms between the two objects as the vector space*

$$\mathrm{Hom}_{\mathcal{F}}((\mathcal{E}_1, U_1), (\mathcal{E}_2, U_2)) := \bigoplus_i \mathrm{Hom}(\mathcal{E}_1|_{x_i}, \mathcal{E}_2|_{x_i}).$$

The grading is given on each point x_i by the Maslov index μ_{x_i} of the path in $\widetilde{\mathcal{L}}|_{x_i}$ given by $\tilde{\gamma}_1(x_1)$ and $\tilde{\gamma}_2(x_2)$.

In case the two submanifolds don't intersect transversely, Fukaya didn't define the space of morphisms, and it does not seem quite clear what would be the appropriate definition. Of course in the case of elliptic curves, one can work backwards and define them in the way that one gets the desired equivalence; for the not very enlightning result see appendix B.

Finally, we need to define the composition maps m_k. So assume that we are given $a_i \in \mathrm{Hom}_{\mathcal{F}}((\mathcal{E}_i, U_i), (\mathcal{E}_{i+1}, U_{i+1}))$ for $i = 1..k$; by linearity, we may assume that each a_i is only non-zero at one point $x_i \in U_i \cup U_{i+1}$.

Let Δ be the closed unit ball inside $\mathbb{C}$. Let x_{k+1} be a point of $U_{k+1} \cap U_1$; we want to compute the summand of $m_k(a_1 \otimes \cdots \otimes a_k)$ in $\mathrm{Hom}_{\mathcal{F}}((\mathcal{E}_1, U_1), (\mathcal{E}_{k+1}, U_{k+1}))$. Consider all pseudo-holomorphic maps $\phi\colon \Delta \to \widehat{M}$ such that, for some $z_1, \ldots, z_{k+1} \in \partial\Delta$, we have

- $\phi(z_i) = x_i$ for all $i = 1..k+1$,
- $\phi([z_i, z_{i+1}]) \subset U_{i+1}$ for $i = 1..k$ and $\phi([z_{k+1}, z_1]) \subset U_1$.

We call two such maps ϕ, ϕ' projective equivalent iff $\phi = \phi' \circ \varphi$ for some automorphism φ of Δ.

3.11 Definition *If we decompose the composition m_k as*

$$m_k(a_1 \otimes \cdots \otimes a_k) = \sum_{x_{k+1} \in U_{k+1} \cap U_1} m_k(a_1 \otimes \cdots \otimes a_k)_{x_{k+1}},$$

then the components are computed by

$$m_k(a_1 \otimes \cdots \otimes a_k)_{x_{k+1}} :=$$

$$\sum_{\phi} e^{2\pi i \int \phi^* \omega} \cdot \Theta_{k+1}(\phi|_{[z_k, z_{k+1}]}) \circ a_k \circ \Theta_k(\phi|_{[z_{k-1}, z_k]}) \circ a_{k-1} \circ \cdots$$

$$\cdots \circ a_1 \circ \Theta_1(\phi|_{[z_{k+1}, z_1]}) \qquad (10)$$

where Θ_i *stands for the given connection of the bundle* $\mathcal{E}_i$*.*

The sum goes over all pseudo-holomorphic discs with the properties described above up to projective equivalence.

The long expression in Θs and a_is is simply the most obvious way to go stepwise from $\mathcal{E}_1|_{x_{k+1}}$ to $\mathcal{E}_{k+1}|_{x_{k+1}}$ along the path $\phi|_{\partial\Delta}$.

As it is, this definition is a bit vague. First, one seems to need conditions on the homotopy class of $\phi|_{\partial\Delta}$ to get a discrete set of such maps; further, it is a non-verified conjecture by Kontsevich that this sum converges for appropriate ω. This is discussed in [9].

The A_∞-structure of this category doesn't seem to be quite obvious. Apparently, the only reference for it is Fukaya's article [9]. However, in this paper he doesn't really provide his proof, he rather reports the existence of it. Also, he considered only $\mathbb{C}$-bundles as local systems; it seems to be commonly believed [14, 23] that this enlargement of the category does not affect the proof.

3.3 A duality in both categories

The derived category of coherent sheaves certainly contains a good deal of information about the complex structure of a manifold (in large classes of varieties, the single variety can even be reconstructed from its associated derived category); on the other hand, there is a wealth of information about pseudo-holomorphic mappings that is used in the composition maps. Thus, the desired equivalence will certainly do a big deal in relating complex and symplectic structures. However, a concrete relation to correspondences between more traditional complex and symplectic invariants remains unclear.

In his talk [14], Kontsevich lists several observations that made him proposing this conjecture. I don't understand all of them, but would like to explain one point in detail, since it will also be relevant for the treatment of the elliptic curve.

There is a natural $\mathbb{Z}$-action on both categories; this is the shifting of complexes in $\mathbf{D}^b(\mathrm{Coh}(X))$, corresponding to moving the lift $\tilde{\gamma}$ of γ in the universal covering $\widetilde{\mathcal{L}}$. Now if n is the dimension of X and $\widehat{X}$, then the shift induces a duality:

3.12 Remark *If* C *is either the*

- *the derived category of coherent sheaves on a* n*-dimensional Calabi-Yau manifold* X*, or*

- *the zero-cohomology part $H^0(\mathcal{F}(\widehat{X}))$ of the Fukaya category of an (complex-)n-dimensional Kähler manifold $\widehat{X}$,*

then $(\mathrm{Hom}_C(A,B))^* \cong \mathrm{Hom}_C(B, A[n])$*, where* $[n]$ *denotes the above-mentioned shift in the respective category.*

Proof. In the Fukaya category, this is the asymmetry of the Maslov index (see appendix): $\mu_x(U_1, U_2) + \mu_x(U_2, U_1) = n$

On the complex side, this is the natural formulation of Serre duality for the derived category. I won't give a complete proof of Serre duality, but show how to translate the usual statement into the language of the derived category.

In our case (that is, with trivial canonical sheaf ω_X), Serre duality states that for every coherent sheaf $\mathcal{F}$, we have a natural isomorphism (that is, more exactly, an isomorphism of functors) $H^i(\mathcal{F}) \cong H^{n-i}(\mathcal{F}^*)^*$.

Since every coherent sheaf has a finite resolution by vector bundles, every bounded complex is quasi-isomorphic to one consisting of vector bundles. So we may assume that the two given complexes $C^\bullet$ and $D^\bullet$ in $\mathbf{D}^b(\mathrm{Coh}\,X)$ are of that form. We want to compute $\mathrm{Hom}_{\mathbf{D}^b}(C^\bullet, D^\bullet)$ and $\mathrm{Hom}_{\mathbf{D}^b}(D^\bullet, C^\bullet[n])$. By general theorems (see e. g. Weibel [25]), we know that if $\mathbf{R}\mathrm{Hom}\colon (\mathbf{D}^b(\mathrm{Coh}\,X))^o \times \mathbf{D}^b(\mathrm{Coh}\,X) \to \mathbf{D}^b(\underline{\mathrm{Ab}})$ denotes the right derived functor of Hom and $\mathbb{R}^i\,\mathrm{Hom}$ the hyper-derived functor, then we have

$$\mathrm{Hom}_{\mathbf{D}^b}(C^\bullet, D^\bullet) = H^0\,\mathbf{R}\mathrm{Hom}(C^\bullet, D^\bullet) = \mathbb{R}^0\,\mathrm{Hom}(C^\bullet, D^\bullet).$$

Hence we need to compute the hyper-derived functor. The functor Hom is the composition $\Gamma \circ \underline{\mathrm{Hom}}$, where $\underline{\mathrm{Hom}}$ denotes the local Hom-functor and $\Gamma = H^0$ the global section functor. Since all objects are are vector bundles, $\underline{\mathrm{Hom}}$ is exact on them; thus we have

$$(\mathbb{R}^0\,\mathrm{Hom})(C^\bullet, D^\bullet) = (\mathbb{R}^0\Gamma)\,\underline{\mathrm{Hom}}^\bullet(C^\bullet, D^\bullet), \tag{11}$$

where $\underline{\mathrm{Hom}}^\bullet$ denotes the total complex of the double complex induced by $\underline{\mathrm{Hom}}$.

In the other case to be considered, we get:

$$\mathrm{Hom}_{\mathbf{D}^b}(D^\bullet, C^\bullet[n]) = \mathbb{R}^n\,\mathrm{Hom}(D^\bullet, C^\bullet) = (\mathbb{R}^n\Gamma)\,\underline{\mathrm{Hom}}(D^\bullet, C^\bullet)$$

For every complex $C^\bullet$, we can form the dual complex $C^{\bullet *}$ with $(C^*)^n = (C^{-n})^*$ and the differentials being the corresponding dual maps. With this notion it is clear that $\underline{\mathrm{Hom}}(D^\bullet, C^\bullet)$ is the dual complex of $\underline{\mathrm{Hom}}(C^\bullet, D^\bullet)$. So if we can prove that for a complex $C^\bullet$ and its dual $C^{\bullet *}$ we have $(\mathbb{R}^0\Gamma)(C^\bullet) \cong (\mathbb{R}^n\Gamma)(C^{\bullet *})^*$, then we are done.

This can be verified by comparing the two spectral sequences

$$E_2^{pq} = H^p(R^q\Gamma)(C^\bullet) \Longrightarrow (\mathbb{R}^{p+q}\Gamma)C^\bullet$$

and

$$E_2^{pq} = H^p(R^q\Gamma)(C^{\bullet *}) \Longrightarrow (\mathbb{R}^{p+q}\Gamma)C^{\bullet *}.$$

By our isomorphism of functors above, we know $R^q\Gamma(C^\bullet) \cong R^{n-q}\Gamma(C^{\bullet *})^*$, so on objects these two spectral sequences are dual to each other; by the functoriality of our isomorphism, the differentials should be dual as well. Hence the converging terms have to be dual as well, and by inspection one verifies that the corresponding terms are $\mathbb{R}^0\Gamma(C^\bullet)$ and $\mathbb{R}^n\Gamma(C^{\bullet *})$. □

3.4 The elliptic curve

The case of the elliptic curve is a lot easier than all other higher dimensional cases for several reasons. On the one hand, as we'll see, the derived category $\mathbf{D}^b(\mathrm{Coh}(E))$ has the simplifying property that every object is isomorphic to a direct sums of objects that are non-zero only in one degree; further, the vector bundles on it have been satisfactorily classified already in the 1950s [1]. Even more important might be the fact that the special Lagrangian submanifolds of the symplectic torus $\widehat{E}$ are simply the straight lines with rational slopes, so we don't need any analysis.

The proof of Polishchuk and Zaslow works by mere computation: The objects in $\mathbf{D}^b(\mathrm{Coh}(E))$ can be described quite explicitly, and the homomorphisms can be described by θ-functions. So they simply explicitly write down the functor yielding the desired equivalence; the main task is then proof that this definition respects the composition.

3.4.1 The derived catory on the elliptic curve

The above remark about the derived category of coherent sheaves follows from the following Lemma, since in $\mathrm{Coh}(E)$, the functor Hom has cohomological dimension 1. This can be seen from the following argument:

The Hom-functor is the composition $\Gamma \circ \underline{\mathrm{Hom}}$, where Γ is standing for the global section functor and $\underline{\mathrm{Hom}}$ for the local Hom-functor; so Ext can be computed by the Grothendieck spectral sequence of the derived functor of a composition:

$$E^2_{pq} = (R^p\Gamma)(\underline{\mathrm{Ext}}^q)(A, B) \Longrightarrow \mathrm{Ext}^{p+q}(A, B)$$

Now both $\underline{\mathrm{Hom}}$ and Γ are of cohomological dimension 1, so an Ext^2-term could only come from a term of type $\mathbb{R}^1\Gamma(\underline{\mathrm{Ext}}^1)$; but the sheafs obtained by $\underline{\mathrm{Ext}}^1$ can easily seen to be skyscraper sheaves, which have vanishing cohomology.

3.13 Lemma *Let $\mathcal{A}$ be an abelian category for which* Ext^2 *(and every higher* Ext*) vanishes. Then every object $C^\bullet$ in $\mathbf{D}^b(\mathcal{A})$ is quasi-isomorphic to the complex $H^\bullet(C^\bullet)$ of its cohomology (with zero differentials).*

Proof. For convenience, we assume that 0 is the highest degree in which C has non-zero cohomology. Now define the truncated complex $\tau_{\leq 0}C$ in the following way: We have $\tau_{\leq 0}C^i = C^i$ for $i < 0$ and $\tau_{\leq 0}C^i = 0$ for $i > 0$; in degree zero we take $\tau_{\leq 0}C^0 = \ker \partial^0$. Then the obvious map $C^\bullet \to \tau_{\leq o}C^\bullet$ induces an isomophism in cohomology; so since we are in the derived category, we can assume that $C^\bullet$ is zero for positive degrees.

Now consider the following short exact sequence of complexes in $\mathbf{Ch}^b(\mathcal{A})$:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{im}\partial^{-1} & \longrightarrow & C^0 & \longrightarrow & H^0(C^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^{-1} & \longrightarrow & C^{-1} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

In $\mathbf{D}^b(\mathcal{A})$, this leads to the following exact triangle, where we have denoted the first complex by $\tau_{<0}C^\bullet$:

$$
\begin{array}{ccc}
 & H^0(C^\bullet) & \\
{\scriptstyle [1]}\swarrow {\scriptstyle u} & & \nwarrow \\
\tau_{<0}C^\bullet & \longrightarrow & C^\bullet
\end{array}
\tag{12}
$$

I claim that $\operatorname{Hom}_{\mathbf{D}^b(\operatorname{Coh}(E))}(H^0(C^\bullet), (\tau_{<0}C^\bullet)[1])$ is zero; from this we could easily deduce our claim: the map u in (12) must be zero; since we have an exact triangle, we must have $C^\bullet \cong \operatorname{cone}(u)$; however, the cone over the zero map is exactly the direct sum, i. e. $C^\bullet \cong H^0(C^\bullet) \oplus \tau_{<0}C^\bullet$. From this we can proceed by induction.

To proof our claim, let F denote the functor $\operatorname{Hom}(H^0(C^\bullet), \_\_)$ in $\mathcal{A}$. Note that $\operatorname{Hom}_{\mathbf{D}^b}(C, D) = H^0\,\mathbf{R}\operatorname{Hom}(C, D)$, where $\mathbf{R}\operatorname{Hom}$ denotes the derived functor of Hom (see Theorem 10.7.4 of [25]). So we need to prove $H^0\mathbf{R}F(D) = 0$. However, if $\mathbb{R}F$ denotes the hyper-derived functor of F, we know that $H^0\mathbf{R}F(D) = \mathbb{R}^0F(D)$; this hyper-derived functor can be computed by a hypercohomology spectral sequence

$$E_2^{pq} = (R^pF)(H^qD) \Longrightarrow \mathbb{R}^{p+q}F(D).$$

But since we have vanishing *Ext* from degree 2 on, we know $R^pF = 0$ for $p \geq 2$. With $H^q(D) = 0$ for $q > -2$, we conclude $\mathbb{R}^0F(D) = 0$. □

A further property that is only valid for the one-dimensional case is the following

3.14 Lemma *Every sheaf $\mathcal{F}$ in* $\operatorname{Coh}(E)$ *is a direct sum of vector bundles with skyscraper sheaves.*

Proof. Let $\mathcal{F}_{tor}$ be the torsion part of $\mathcal{F}$, and let $\mathcal{G}$ be the cokernel of its inclusion, giving a short exact sequence

$$0 \to \mathcal{F}_{tor} \to \mathcal{F} \to \mathcal{G} \to 0.$$

So $\mathcal{F}$ is an extension of $\mathcal{F}_{tor}$ with $\mathcal{G}$. Since $\mathcal{G}$ is torsion free, it must be a vector bundle. But since the Ext^1-group of a skyscraper sheaf with a vector bundle $\mathrm{Ext}^1(\mathcal{F}_{tor}, \mathcal{G})$ vanishes, this extension must be trivial, i. e. it is a direct sum. □

Obviously, the last two lemmas would be true for every curve (that is, for every one-dimensional compact complex manifold).

The vector bundles on an elliptic curve have been classified by Atiyah [1], and admit the following very explicit description.

We have a $\mathbb{Z}$-covering of our elliptic curve $E = E_q$ by $\mathbb{C}^*$; more concretely $E_q = \mathbb{C}^*/_{u\sim qu}$. The moduli parameter q is of course related to the more traditional parameter τ (where $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$) by the equation $q = e^{2\pi i\tau}$.

From the trivial vector bundle $\mathbb{C}^* \times V$ on $\mathbb{C}^*$ with any vector space V, we can form a bundle on E_q by giving an automorphic factor:

3.15 Definition *Let V be a vector space and $A\colon \mathbb{C}^* \to \mathrm{GL}(V)$ a holomorphic function. We define the bundle $F_q(V, A)$ on E_q to be the quotient*

$$F_q(V, A) := \mathbb{C}^* \times V/(u, v) \sim (qu, A(u)\langle v\rangle).$$

If $V = \mathbb{C}$ is one-dimensional and $A = \varphi$ is a holomorphic function, then we will also write $L_q(\varphi)$ for the so-obtained line bundle.

From now on, we will fix the automorphic function $\varphi_0 := \frac{1}{\sqrt{qu}}$. The distinguished role that the line bundle $L := L_q(\varphi_0)$ will play in the constuction is rather arbitrary; it is just for convenience since it has the classical theta function as a global section.

Further, from the obvious projection map $\pi_r\colon E_{q^r} \to E_q$, we get the functors of push-forward π_{r*} and pull-back π_r^*. Nearly by definition, these functors are adjoint of each other: If F_1 is a bundle on E_{q^r} and F_2 one on E_q, we have

$$\mathrm{Hom}(F_1, \pi_{r*}F_2) = \mathrm{Hom}(\pi_r^* F_1, F_2) \text{ and } \mathrm{Hom}(F_1, \pi_r^* F_2) = \mathrm{Hom}(\pi_{r*}F_1, F_2).$$

(While the first adjointness is true in general, the second one is only true because π_r is a covering map—an étale map in algebraic terms.)

The starting point for the description of the desired functor is the following classification of vector bundles on E_q; P. and Z. say that it follows easily from Atiyahs classification (I didn't check this):

3.16 Proposition *All indecomposable bundles on E_q are given by*

$$\pi_{r*}(L_{q^r}(\phi) \otimes F_{q^r}(\mathbb{C}^k, \exp N)),$$

where $\phi = t_x^*\varphi_0 \cdot \varphi^{n-1}$ with $n \in \mathbb{Z}$ and t_x denoting the translation by $x \in \mathbb{C}^*$, and N is a (constant) nilpotent matrix.

For the desription of the homomorphisms between two such objects, the following proposition will be very helpful:

3.17 Proposition *Let $\phi := t_x^*\varphi_0 \cdot \varphi_0^{n-1}$ with $n > 0$, and N be a nilpotent endomorphism $N \in \mathrm{End}(V)$. Then we have an isomorphism:*

$$\nu_{\phi,N}\colon H^0(L(\phi)) \otimes V \to H^0(L(\phi) \otimes F(V, \exp N))$$

Proof. Let f be a section of $L(\phi)$, i. e. a function on $\mathbb{C}^*$ having ϕ as automorphic factor, and let $v \in V$. Then $\nu_{\phi,N}(f \otimes v) := e^{\frac{DN}{n}} f \cdot v$, where $D = -u\frac{d}{du}$. It is an easy computation to check that the so-obtained V-valued function has the correct automorphic factor to be a section of this vector bundle. Conversely, from a section $v(u)$ of the right vector bundle, we get an element $e^{\frac{-DN}{n}} v(u)$ of $H^0(L(\phi) \otimes V) \cong H^0(L(\phi)) \otimes V$. □

3.4.2 Description of the functor

I will try to intinsically motivate the construction of P. and Z. [23] as good as possible, showing which choices they had to make.

So we are given an elliptic curve E_q and its mirror $\widehat{E}_\rho$. In terms of the complexified Kähler form ω that we used in the definition of the Fukaya category, ρ is given by $\rho = \int_{\widehat{E}} \omega$. If we look at $\widehat{E}$ as $\widehat{E} \cong \mathbb{R}^2/\mathbb{Z}^2$, then we can specify ω as $\omega = \rho dx \wedge dy$; its imaginary part is the usual Kähler form. Further we have a global 1-form $dz = dx + idy$.

The minimal one-dimensional submanifolds (the Lagrangian condition is trivial of course) are then clearly the straight lines. This leads to m_1 being zero, i. e. the complex structure on $\mathrm{Hom}(U_1, U_2)$ is trivial.

The "bundle of Lagrangian planes" over the torus is simply $\mathbb{P}^1 \times \widehat{E}$, and its $\mathbb{Z}$-covering is $\mathbb{R} \times \widehat{E}$. Giving the lift $\tilde{\gamma}$ of the definition above thus means simply giving an angle $\pi\alpha$ to the zero line $(x, 0)$ as a number $\alpha \in \mathbb{R}$ and *not* just in $\mathbb{R}/\mathbb{Z}$. The maslov index for two "lifts" $\alpha_1, \alpha_2 \in \mathbb{R}$ is then given by $\mu(\alpha_1, \alpha_2) = -\lfloor \alpha_2 - \alpha_1 \rfloor$.

So we want to construct a functor $\Phi \colon \mathbf{D}^b(\mathrm{Coh}\, E_q) \to H^0(\mathcal{F}(\widehat{E}_\rho))$; as mentioned in the introduction, the mirror map for tori is the simplest one possible, saying $\rho = \tau$, where $e^{2\pi\tau} = q$.

First, let us consider complexes $C^\bullet$ in $D^b(\mathrm{Coh}(E_q))$ that are concentrated in degree zero having a line bundle as C^0. All line bundles on E_q are given by $L_i(x) := t_x^* L \otimes L^{i-1}$ where L is the "canonical" line bundle on E_q defined in the previous section.

It is only fair to assume that line bundles are mapped to one-dimensional local systems.

We start by mapping the trivial line bundle $\mathcal{O}_E$ to the submanifold $\mathcal{L}_0 = \{(t, 0)\}$ with a trivial C-bundle on it, and specify its angle by 0. For $n \in \mathbb{Z} > 0$, we know that $\mathrm{Hom}(\mathcal{O}_E, L_n(x))$ is n-dimensional. Thus, there must be n intersection points between $\mathcal{L}_0$ and the submanifold that we want as an image for $L_n(x)$, and their relative Maslov index must be zero. So we are forced to map $L_n(x)$ to a line of slope n with angle $\pi\alpha \in [0, \pi/2)$.

We now make the choice to map $L = L_1(0)$ to the line $\{(t, t)\}$ with trivial local system on it; this is really arbitrary, but it suggests to map $L^n = L_n(0)$ to $\{(t, nt)\}$, again with trivial $\mathbb{C}$-bundle on it. We still have to decide how to use the two real parameters β, γ among the line bundles $L_n(\beta\tau + \gamma)$ of Chern class n; we have to use it for a translation of the submanifold and for introducing a non-trivial connection on it. Here again, the choice is rather arbitrary (and must be, since β, γ are themselves dependent on the $\tau \in \mathbb{H}$ that we chose). P.

and Z. decided to let β move the submanifold, getting $\{t, nt-\beta\}$, and to let γ introduce a monodromy of factor $e^{-2\pi i\gamma}$ as connection.

To proceed with our map on the objects classified by Proposition 3.16, we have to decide what tensoring with $F(V, \exp N)$ should correspond to on the Fukaya objects; one cannot resist the temptation to tensor with the local system that has fiber V and a connection with mondromy $\exp N$, and this does it indeed.

Let us summarize what we have defined so far:

$$\Phi(t^*_{\beta\tau+\gamma} L_q \otimes L_q^{n-1} \otimes F(V, \exp N) := (V, \{t, nt-\beta\})$$

The connection in V is given by the monodromy $e^{-2\pi i\gamma} \cdot \exp N$.

It is a bit more tricky to define the map on morphisms; again let us look first at the case of two line bundles $L_q(\phi_1)$ and $L_q(\phi_2)$ with $\phi_i = t^*_{\beta_i\tau+\gamma}\varphi_0 \cdot \varphi_0^{n_i-1}$. The homomorphism space between them is

$$\mathrm{Hom}(L_q(\phi_1), L_q(\phi_2)) = H^0(L_q(\phi_2\phi_1^{-1})) = H^0(L_q(t_{\beta_{12}\tau+\gamma_{12}}\phi^{n_2-n_1})). \qquad (13)$$

Here we set $\beta_{12} = \frac{\beta_2-\beta_1}{n_2-n_1}$ and $\gamma_{12} = \frac{\gamma_2-\gamma_1}{n_2-n_1}$.

For $n_2 < n_1$, this space is zero, as is the corresponding Hom-space in the Fukaya category. If $n_2 = n_1$, then either the two line bundles are isomorphic; in this case the morphisms are just multiplication by a scalar multiple, otherwise there are no morphisms – in both cases this translates one-to-one to the corresponding objects in the Fukaya category. If however $n_2 > n_1$, then we need theta functions to describe the global sections of this line bundle: If we define

$$\theta[c', c''](\tau, z) := \sum_{m\in\mathbb{Z}} e^{2\pi i[\frac{\tau}{2}(m+c')^2+(m+c')(z+c'')]}$$

then the functions

$$f_k := t^*_{\beta_{12}\tau+\gamma_{12}}\theta\left[\frac{k}{n_2-n_1}, 0\right]((n_2-n_1)\tau, (n_2-n_1)z) \qquad (14)$$

for $k \in \mathbb{Z}/(n_2-n_1)\mathbb{Z}$ form a basis for this vector space.

In the Fukaya category, the space of morphisms is equally a k-dimensional vector space; there is the canonical basis given by the intersection points of the given straight lines, which are easily determined to be

$$e_k = \left(\frac{k+\beta_2-\beta_1}{n_2-n_1}, \frac{n_1k+n_1\beta_2-n_2\beta_1}{n_2-n_1}\right)$$

for $k \in \mathbb{Z}$.

The functor Φ now simply maps f_k to e_k; however, and this is the first definition that is not obvious at all, we need to introduce a constant factor, precisely we let

$$f_k \overset{\Phi}{\mapsto} e^{-\pi i\tau\frac{(\beta_2-\beta_1)^2}{n_2-n_1}} \cdot e^{-2\pi i\frac{(\beta_2-\beta_1)(\gamma_2-\gamma_1)}{n_2-n_1}} \cdot e_k. \qquad (15)$$

The only reason for this factor I can give is that it turns out to work: One needs to check that Φ respects composition. Between the line bundles, this composition is just a multiplication of theta functions; by the addition formula, this product can be decomposed in terms of our canonical basis. The coefficients of this decomposition have to be correctly computed by our constant factors and the composition law in the Fukaya category. I don't want to copy the computation in full detail, but just point out why the composition can as well be computed by theta functions:

Given three lines Λ_i on our torus, and given three points of pairwise intersections $x_i \in \Lambda_i \cap \Lambda_{i+1}$, we need to evaluate the sum given in (10). A map $\phi\colon \Delta \to \mathbb{R}^2/\mathbb{Z}^2$ lifts to a map to the universal cover $\mathbb{R}^2$; here Δ must map holomorphically to a triangle. If we choose a fixed preimage X_1 of x_1 in the universal cover, then this triangle is determined by the choice of the second corner X_2 in the preimage of x_1. Now this set is labelled by $n \in \mathbb{Z}$, and it is very obvious that the area of this triangle with respect to ω can be expressed by a quadratic form $(an+b)^2$. Thus we get a sum of the form $\sum_n e^{-(an+b)^2}$, which can easily be expressed as a value of a theta function. If we add holonomy via a non-trivial connection in the local systems, then we just get another factor of form e^{cn+d}, so this argument is still valid.

Next, we have to consider homomorphisms between indecomposable vector bundles given as $E_i := L_q(\phi_i) \otimes F_q(V_i, \exp N_i)$. The homomorphisms between these are well described by proposition 3.17; with the same notation as above for equation (13) we get (in the case of $n_2 > n_1$)

$$\mathrm{Hom}(E_1, E_2) \cong H^0(L_q(t^*_{\beta_{12}\tau+\gamma_{12}}\varphi_0^{n_2-n_1})) \otimes V_1^* \otimes V_2.$$

So any homomorphisms is given by a linear combination of elements of the form $f_k \otimes T$, where the f_k are defined in 14 and $T \in V_1^* \otimes V_2$.

This desription has of course an equivalent on the Fukaya side: if the submanifolds corresponding to $L_q(\varphi_i)$ are Λ_i, then the vector space $\mathrm{Hom}(\Phi(E_1), \Phi(E_2))$ is given by

$$\mathrm{Hom}(\Phi(E_1), \Phi(E_2)) = \bigoplus_{e_k \in \Lambda_1 \cap \Lambda_2} e_k \otimes (V_2^* \otimes V_1).$$

The functor Φ then sends $f_k \otimes T$ to $\Phi(f_k) \otimes \Phi(T)$; here $\Phi(f_k)$ is the scalar multiple of e_k as defined in (15), and the map on $V_2^* \otimes V_1$ is given by

$$\Phi(T) = e^{N_2^*} \otimes e^{-N_1} \cdot T$$

So far we have ignored any bundles that are given as the pushforward π_{r*} of a vector bundle according to proposition 3.16.

Now this map $\pi_r\colon E_{q^r} \to E_q$ has an equivalent in the map $\pi_r\colon (\mathbb{R}^2/\mathbb{Z}^2)_{r\rho} \to (\mathbb{R}^2/\mathbb{Z}^2)_\rho$ sending (x, y) to (rx, y) (where $r\rho$ is the complexified Kähler form on the left and ρ the one on the right); we get functors π_{r*} and π_r^* between the two Fukaya categories on it. This can be used to define Φ on bundles given as the pushforward π_{r*} of a bundle on E_{q^r} in Prop. 3.16, by making the diagram

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{Coh}(E_{q^r})) & \xrightarrow{\Phi} & H^0(\mathcal{F}(\widehat{E}_{r\rho})) \\ {\scriptstyle \pi_{r*}}\downarrow & & \downarrow{\scriptstyle \pi_{r*}} \\ \mathbf{D}^b(\mathrm{Coh}(E_q)) & \xrightarrow{\Phi} & H^0(\mathcal{F}(\widehat{E}_{\rho})) \end{array}$$

commutative; to define Φ on morphisms between vector bundles given as push-forwards π_{r_1*} and π_{r_2*} of something, one has to do the same thing for the fibre product $E_{q^{r_1}} \times_{E_q} E_{q^{r_2}}$, which is in general a disjoint union of several elliptic curves. Finally, to prove that this construction still gives a functor (i. e. that it still respects composition), one first proves that Φ respects the pull-back π_r^*; then the claim follows by considering the triple fibre product of $E_{q^{r_1}} \times E_{q^{r_2}} \times E_{q^{r_3}}$ over E_q and by using the adjointness of π_r^* and π_{r*} in both categories.

It remains to define Φ on skyscraper sheaves. Since the space of morphisms from any n-dimensional vector bundle to a skyscraper sheaf with r-dimensional fiber has dimension nr, it makes sense to map a skyscraper sheaf to a vertical line.

More precisely, we can describe every skyscraper sheaf as the pushforward of a skyscraper sheaf on $\mathbb{C}^*$ supported at one point $z_0 \in \mathbb{C}^*$. It is then given by a vector space V (the fibre of the sheaf at z_0) and a nilpotent endomorphism $N \in \mathrm{End}(V)$ that describes the action of the local parameter $z - z_0$ on V. Assume that z_0 is mapped to $\beta\tau + \gamma \in E$ where $\beta, \gamma \in \mathbb{R}/\mathbb{Z}$. Then we map this sheaf to the line $\{(-\beta, t)\}$ with the local system V on it; the connection is given by a monodromy of $e^{2\pi i\gamma}$. The Hom-spaces $\mathrm{Hom}(L(\phi) \otimes F(W, M), \mathcal{C})$ and the corresponding spaces in the Fukaya category are both canonically isomorphic to $W^* \otimes V$. The functor Φ is once again just a multiplication by a scalar; this scalar is easily computed by the requirement to respect the composition map $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes \mathrm{Hom}(\mathcal{G}, \mathcal{C}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{C})$ for vector bundles $\mathcal{F}$ and $\mathcal{G}$ of the form we have been considering.

Again, we extend Φ to morphisms from a vector bundle given as the push-forward π_{r*} of another vector bundle to our skyscraper sheaf by requiring commutativity with the functor π_{r*}.

So we have constructed our functor on coherent sheaves. Since the shift of complexes corresponds to a shift of the lift $\tilde{\gamma}$, and because of lemma 3.13, we thus now how to define the functor on all objects. But via the shift of the translation functor and Serre duality, we can map every map of complexes concentrated in one degree to a map in degree zero, and so our functor is well-defined on the whole category $\mathbf{D}^b(\mathrm{Coh}\, E)$.

By the construction it is immediately clear that Φ is fully faithful; it is an easy check that every object in $H^0(\mathcal{F}(E))$ is in the image of Φ. This completes the proof of the desired equivalence of categories.

3.4.3 Consequences

This equivalence of two so differently constructed category immediately suggests several questions.

- First, one wonders whether the derived category of coherent sheaves can naturally be constructed as the zero cohomology H^0 of an A_∞-category. This is correct and a construction has been done by Polishchuk in [22], using composition of Ext-groups; Polishchuk further conjectures that there is an equivalence of A_∞-categories between the Fukaya category and his category.

- There is a natural action of $SL_2(\mathbb{Z})$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$; this action preserves the complexified Kähler form ω. Therefore, $SL_2(\mathbb{Z})$ acts—up to shifts of the lifting of $\tilde{\gamma}$—as auto-equivalences on the Fukaya category. On wonders what the corresponding group of auto-equivalences of the derived category is; this will be discussed in the next section.

- We noted that the derived category has the additional structure of being a triangulated category; therefore, the Fukaya category must have the same property.

One might ask under which circumstances an A_∞-category $\mathcal{A}$ produces a triangulated category by applying H^0. Now the functor Hom in $\mathcal{A}$ is naturally a functor $\mathrm{Hom}^\bullet\colon \mathcal{A}^o \times \mathcal{A} \to \mathbf{K}(\underline{\mathrm{Ab}})$ (note that this is only true because the associativity in an A_∞-category holds homotopically by the homotopy m_3). It would be natural to require a compatibility of the triangulated structure on $\mathcal{F}(\widehat{M})$ with this functor; this would mean that the shift functor T has the property $\mathrm{Hom}^\bullet(A, T(B)) = \mathrm{Hom}^\bullet(A, B)[1]$, and that a triangle $A \to B \to C \to TA$ would be called exact iff for all $X \in \mathrm{Ob}\,\mathcal{A}$ the resulting triangle $\mathrm{Hom}^\bullet(X, A) \to \mathrm{Hom}^\bullet(X, B) \to \mathrm{Hom}^\bullet(X, C) \to \mathrm{Hom}^\bullet(X, A)[1]$ is exact in $\mathbf{K}(\underline{\mathrm{Ab}})$.

On the other hand, we hope that under certain circumstances construct the triangulated structure in the following way:

A morphism $a \in \mathrm{Hom}^\bullet(A, B)$ of degree zero induces a map $A_X\colon \mathrm{Hom}^\bullet(X, A) \to \mathrm{Hom}^\bullet(X, B)$ via $A_X(\mu) := m_2(\mu \otimes a)$. If a is closed with respect to $d = m_1$, then this map is a map of complexes due to the compatibility of m_2 (9). The same equation shows that if a is exact, then the induced map is homotopic to zero.

Hence the map $(a, X) \mapsto A_X$ is well defined as a map $\mathrm{Mor}(H^0(\mathcal{A})) \times \mathcal{A}^o \to \mathrm{Mor}(\mathbf{K}(\underline{\mathrm{Ab}}))$. If we now apply the functor cone in $\mathbf{K}(\underline{\mathrm{Ab}})$ (where the cone can really be given by functorial construction) to this morphism, we get a functor $\Delta_a\colon \mathcal{A}^0 \to \mathbf{K}(\underline{\mathrm{Ab}})$.

Further, we can define the functor $\Xi\colon \mathcal{A} \times \mathcal{A}^o$ by $\Xi(X, Y) = \mathrm{Hom}^\bullet(X, Y)[1]$.

It would be nice if one had the following

3.18 Lemma (without proof)
If the functor Δ_a is representable for all a, and if the functor Ξ is representable via an autoequivalence $T\colon \mathcal{A} \to \mathcal{A}$, then $H^0(\mathcal{A})$ is triangulated according to the following definition:

A triangle $A \to B \to C \to T(A)$ is exact if for all $X \in \mathcal{A}$, the following *induced triangle is exact in* $\mathbf{K}(\underline{\mathrm{Ab}})$*:*

$$\begin{array}{ccc} & \mathrm{Hom}^\bullet(X,C) & \\ {}^{[1]}\swarrow & & \nwarrow \\ \mathrm{Hom}^\bullet(X,A) & \longrightarrow & \mathrm{Hom}^\bullet(X,B)\end{array}$$

One would like to prove this in the following way: We can embed $H^0(\mathcal{A})$ as a full subcategory in the category of functors $H^0(\mathcal{A})^0 \to \mathbf{K}(\underline{\mathrm{Ab}})$. Now a category of functors to a triangulated category should naturally inherit the triangulation structure, and a full subcategory of a triangulated category that is closed under the construction of cones and under the translation functor is again triangulated. So we would need to prove:

3.19 Lemma (in general wrong!)
If $\mathcal{A}$ is any category and if $\mathcal{B}$ is a triangulated category, then the category of functors $\mathrm{Func}(\mathcal{A},\mathcal{B})$ *from $\mathcal{A}$ to $\mathcal{B}$ is naturally triangulated.*

This seems to be false, however. The problem lies in the non-functoriality of Verdier's axioms; the construction of the cone cannot necessarily be given by a functor, and a cartesian square can possibly be embedded into a morphism of triangles in several different ways. This is discussed in Gelfand/Manin ([11]), pp. 244-245.

It seems to be possible to replace the axioms of Verdier by the requirement of the existence of a functorial cone (we have such a functorial construction in the main examples of triangulated categories) that would need to fulfill some further conditions. If with such a set of axioms, one could prove lemma 3.19, this might already be a worthwile improvement; however, it might also be necessary to restrict the notion of a morphism of triangles to morphisms that induced by the cone-functor.

There is a new set of axioms for triangulated categories by A. Neeman that could help to solve this problem, but I couldn't get hold of his article [20]. In our special case, one could possibly still prove lemma 3.19; see [13], lemma 2.2 for a similar statement for DG categories (DG or differential graded categories are A_∞-categories with $m_3 = 0$).

Finally, to convince ourselves that we at least need a functorial cone to prove lemma 3.19, we just have to look at the simplest possible example of a natural transformation: If we take the catgory $\mathcal{A} = \mathrm{Mor}\,\mathcal{B}$ of morphisms in $\mathcal{B}$, then we have the two functors from $\mathcal{A}$ to $\mathcal{B}$ that send each morphism to its domain and its target; between them, we have the natural transformation that sends a morphism to itself. An embedding of this morphism of functors into an exact triangle is exactly a functorial construction of the mapping cone.

4 Fourier-Mukai transforms

4.1 A report

In 1982, Mukai discovered that an equivalence between the derived category of coherent sheaves on an abelian variety X and its dual variety $\hat{X}$ can be given with the use of the Poincaré bundle $\mathcal{P}$ on $X \times \hat{X}$ [17]. More precisely, let p_X and $p_{\hat{X}}$ be the projections of $X \times \hat{X}$ to its factors; then the Fourier-Mukai transform is given by the right derived functor $\mathbf{R}(\pi_{\hat{X}*}(\mathcal{P} \otimes \pi_x^*(\_\_))$; up to a shift of the complex, its inverse is given by the corresponding functor from $\mathbf{D}(\mathrm{Coh}(\hat{X})) \to \mathbf{D}(\mathrm{Coh}(\hat{\hat{X}}))$ together with the identification $X = \hat{\hat{X}}$.

This construction has been generalized to a variety of cases, see for example [18, 3]. The recent paper by T. Bridgeland [4] seems to summarize the state of the art in his theorem 1.1. First we need some definitions:

4.1 Definition *A vector bundle $\mathcal{F}$ over a complex manifold is called* simple *if the only automorphisms are the multiplications by a scalar.*

Now let $\mathcal{P}$ be a vector bundle on $X \times Y$, where X and Y are compact complex manifolds. Let $\mathcal{P}_y$ denote the bundle on X obtained by restricting $\mathcal{P}$ to $\{y\} \times X$. Then $\mathcal{P}$ is called strongly simple over Y *if for each point $y \in Y$, the bundle $\mathcal{P}_y$ is simple, and if for any two points $y_1, y_2 \in Y$, we have $\mathrm{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$ for all integers i.*

The theorem I mentioned is:

4.2 Theorem *Let $\mathcal{P}$ be a vector bundle on $X \times Y$, where X and Y are smooth projective manifolds; define a functor $F\colon \mathbf{D}(\mathrm{Coh}\, X)) \to \mathbf{D}(\mathrm{Coh}\, Y)$ as*

$$F(\_\_) = \mathbf{R}(\pi_{Y*}(\mathcal{P} \otimes \pi_X^*(\_\_))).$$

Then F is fully faithful if and only if $\mathcal{P}$ is strongly simple over Y. In this case, it is an equivalence of categories if, and only if, we have also $\mathcal{P}_y = \mathcal{P}_y \otimes \omega_X$ for all $y \in Y$. Further, in this case the inverse of the equivalence is, up to shifts of the complexes, given by $\mathcal{P}^*$ in the analogous way.

The last condition is of course very convenient for us since when we are dealing with Calabi-Yau manifolds.

4.2 Auto-equivalences of $\mathbf{D}^b(\mathrm{Coh}\, E)$

As an example, we want to look at the traditional Fourier-Mukai transform for the elliptic curve: let $\mathcal{P}$ be the Poincaré line bundle on $E \times E$. If m is the group law $m\colon E \times E \to E$ and p_1 and p_2 the projections to the first and second factor, respectively, then it is given by $\mathcal{P} = m^*\mathcal{O}(-x_0) \otimes p_1^*\mathcal{O}(x_0) \otimes p_2^*\mathcal{O}(x_0)$, where x_0 is any point in E; this is verified by the see-saw principle (see [19]).

It is clear that $\mathcal{P}$ is strongly simple over both factors: any line bundle is simple, and two non-isomorphic line bundles of degree zero on the elliptic curve have vanishing Ext^i for all i. Since further ω_E is trivial, $\mathcal{P}$ must induce an equivalence of categories.

It is now just a matter of checking all the cases (and due to the nature of the definition of the functor Φ there is no other, elegant way of doing this) to see that this equivalence corresponds to the action of the matrix $M_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the torus $\mathbb{R}^2/\mathbb{Z}^2$.

If we take an arbitrary element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{SL}_2\,\mathbb{Z}$ instead of M_1, we still get an action on the torus $\mathbb{R}^2/\mathbb{Z}^2$ that preserves our complexified form ω and hence an autoequivalence of the Fukaya category. With our equivalence Φ, we get an associated autoequivalence N of the derived category of coherent sheaves on E; one wonders how we can describe it.

- In a way, we have already done this: the matrix $M_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ acts as tensoring with L_q, and since M_1 and M_2 generate $\mathrm{SL}_2\,\mathbb{Z}$, this is enough to characterize the group action. (Note however that the action of $\mathrm{SL}_2\,\mathbb{Z}$ on the Fukaya category is only well-defined up to shift with an integer; more exactly, we should hence speak of the action of a central extension of $\mathbb{Z}$ with $\mathrm{SL}_2\,\mathbb{Z}$.)

- Since N is an equivalence of categories, a theorem by D. O. Orlov discussed later (5.8) applies to our situation, which tells us that N can be represented by an object $\mathcal{F}$ in $\mathbf{D}^b(\mathrm{Coh}\,E \times E)$ as a Fourier-Mukai functor; so we want to describe $\mathcal{F}$.

  As multiplication with $-\,\mathrm{Id}$ is easily exhibited as the multiplication with -1 on the elliptic curve, we may assume from now on that $b > 0$.

  First note that if we write the Chern class of a stable vector bundle on E as $\binom{r}{d}$, then M exactly represents the group homomorphism of Chern classes. Now T. Bridgeland has determined bundles on $E \times E$ that give such a map on Chern classes in his paper from 1996 [3]; we will follow him in this paragraph. Since $M \cdot \binom{0}{1} = \binom{b}{d}$, skyscraper sheafs are mapped to bundles of rank b and degree d. From proposition 3.16, it is clear that we can identify E with the moduli space H of stable bundles of Chern class $\binom{b}{d}$. Under this identification, what we need is what is called a *tautological bundle* on $H \times E$ (the definition of which exactly says that the restricion of it to $\{h\} \times E$ is the bundle represented by h); the existence of such a bundle is assured in general. From theorem 4.2 it is immediately clear that such a tautological bundle induces an equivalence of categories; hence the map of Chern classes is given by an invertible matrix $M' = \begin{pmatrix} a' & b \\ c' & d \end{pmatrix}$; an easy argument shows that this matrix must even be in $\mathrm{SL}_2\,\mathbb{Z}$.

  Now by tensoring with $\pi_E^*(L_q^n)$, we can change a' and c' by $a' \mapsto a' + nb$ and $c' \mapsto c' + nd$, from which we can obtain every matrix in $\mathrm{SL}_2\,\mathbb{Z}$ with the given second column.

  Up to a translation on E, this should give the correct functor for our situation.

- We would like to describe $\mathcal{F}$ yet more explicitly. This is indeed possible:

  The skyscraper sheaf with fibre $\mathbb{C}$ at $0 \in E$ is mapped to $\pi_{b*}L_{q^b}^d$ with all notations as in section 3.4.1: $\pi_b \colon E_{q^b} \to E_q$ is the standard b-fold covering, and L_{q^b} is the "standard" line bundle of degree 1 on E_{q^b} given by the automorphic function ϕ_0 as defined above. This corresponds to the simple equation $M \cdot \binom{0}{1} = \binom{b}{d}$. A little more precise tracing of all maps involved shows that the similar skyscraper sheaf concentrated at $x \in E$ is mapped to $\pi_{b*}\left(t_{-x}^* L_{q^b} \otimes L_{q^b}^{d-1}\right)$.

Since we know that the inverse functor associated to $\mathcal{F}$ is induced by $\mathcal{F}^*$, and since $M^{-1} \cdot \binom{0}{1} = \binom{-b}{a}$, we deduce that for all x, the restriction $\mathcal{F}|_{E\times\{x\}}$ in the other way must be a bundle of form $\pi_{b*}\left(t_{-x}^* L_{q^b} \otimes L_{q^b}^{a-1}\right)$.

By looking at how these bundles can be constructed from the trivial bundle $\mathbb{C}^b$ on $\mathbb{C}^*$ by automorphic factors, one gets the idea for the following construction of $\mathcal{F}$:

We view $E_q \times E_q$ as the quotient $(\mathbb{C}^* \times \mathbb{C}^*)/(q,1)^{\mathbb{Z}} \cdot (1,q)^{\mathbb{Z}}$. We take the trivial bundle with fibre $\mathbb{C}^b$ over $\mathbb{C} \times C$. By defining automorphic factors $A_1 \colon \mathbb{C} \times \mathbb{C} \to \mathrm{GL}_b(\mathbb{C})$, we can form a bundle $F_b(A_1, A_2)$ on $E \times E$:

$$F_b(A_1, A_2) := \mathbb{C}^b \times (\mathbb{C}^* \times \mathbb{C}^*)/\sim$$

where the equivalence relation $\sim$ is generated by

$$\begin{cases} (v, u_1, u_1) \sim (A_1(u_1,u_2)(v), qu_1, u_2) \\ (v, u_1, u_1) \sim (A_2(u_1,u_2)(v), u_1, qu_2) \end{cases}$$

Necessary for this definition to make sense is the condition

$$A_2(qu_1, u_2) \circ A_1(u_1, u_2) = A_1(u_1, qu_2) \circ A_2(u_1, u_2).$$

We choose the automorphic factors to be of the form:

$$A_1 := \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \psi_1 & & & 0 \end{pmatrix} \qquad A_2 := \begin{pmatrix} 0 & 0 & \cdots & \psi_2 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

Here $\psi_1 := u_2 u_1^{-a} q^{-\frac{ba}{2}}$ and $\psi_2 := u_1 u_2^{-d} q^{-\frac{bd}{2}}$. With this definitions we have $\mathcal{F} = F_b(A_1, A_2)$.

Again, it is rather boring to check for all cases that the Fourier-Mukai-functor associated to $\mathcal{F}$ corresponds under Φ to the autoequivalence of the Fukaya category induced by M. What has to be done is the following:

Suppose we are given a bundle $\mathcal{G}$ on E_q as

$$\mathcal{G} = \pi_{r*}\left(t_x^* L_{q^r} \otimes L_{q^r}^{n-1} \otimes F(V, \exp N)\right)$$

according to the classification of proposition 3.16. We can easily write $\mathcal{H} := \mathcal{F} \otimes \pi_1^*(\mathcal{G})$ as a bundle of form $F(A_1', A_2')$ with some automorphic factors A_1' and A_2'. The global sections of $\mathbb{H}|_{E\times\{\tau\}}$ can be given by theta functions (see proposition 3.17 for how to take care of the factor $F(V, \exp N)$). They give us local trivializations of $\pi_{2*}(\mathbb{H})$, and so it becomes an exercise in θ-functions to see that they can be described by the automorphic factor on $\mathbb{C}^*$ corresponding to the bundle on E_q that we want.

5 More about Fukaya categories

5.1 Transversal Fukaya category

We have seen that the category of coherent sheaves on $X \times Y$ can naturally be regarded as a category of functors $\mathbf{D}(\mathrm{Coh}(X)) \to \mathbf{D}(\mathrm{Coh}(Y))$. This construction has an analog on the Fukaya side, that I will sketch in this section.

First we need a slight modification of the objects in our Fukaya category:

5.1 Definition *From now on, an object in the Fukaya category on $\widehat{M}$ will be an triple $(\mathcal{E}, U, \iota)$, where $\mathcal{E}$ is a local system on U as before, where U is a n-dimensional manifold and $\iota\colon U \to \widehat{M}$ is an immersion. We still require $\iota^*(\omega) = 0$ and $\mathrm{Im}(z \cdot \iota^*(\Omega) = 0$ and all other conditions (and the map $\tilde{\gamma}$) mentioned in 3.9.*

Of course the only new thing is that we allow self-intersection.

Now let $\widehat{M}, \widehat{N}$ be $2n$-dimensional symplectic manifolds with symplectic forms ω, ψ, an almost-complex structure compatible with the symplectic form and nowhere-zero holomorphic n-forms Ω, Ψ. Then $\widehat{M} \times \widehat{N}$ can be regarded as a compact Calabi-Yau Kähler manifold with Kähler form $\omega - \psi$ and holomorphic $2n$-form $\Omega \wedge \Psi$ (we won't distinguish between the form ω on $\widehat{M}$ and its pull-back on $\widehat{M} \times \widehat{N}$). Thus we can consider the Fukaya category $\mathcal{F}(\widehat{M} \times \widehat{N})$. For our purposes we have to consider a slightly reduced category that I would like to call *transversal Fukaya category*:

5.2 Definition *The transversal Fukaya category $\overline{\mathcal{F}}$ on $\widehat{M} \times \widehat{N}$ is a full subcategory of the Fukaya category $\mathcal{F}$. It consists of those objects whose underlying immersed compact manifold $\iota : U \to \widehat{M} \times \widehat{N}$ has the property that $\pi_{\widehat{M}} \circ \iota$ and $\pi_{\widehat{N}}$ are local diffeomorphisms; in other words, the tangent spaces $V = \iota_* T_u U$ intersect the subspaces $V_M = T_{\iota(u)}\widehat{M}$ and $V_N = T_{\iota(u)}\widehat{N}$ of $T_{\iota(u)}(\widehat{M} \times \widehat{N})$ transversely, i. e. only in $\{0\}$. Further we require that the pull-back of Ω on U is a constant multiple of the pull-back of Ψ.*

Observe that the first condition (it is needed to assure transversality later on) implies that $\pi_{\widehat{M}} \circ \iota$ and $\pi_{\widehat{N}} \circ \iota$ are surjective and covering maps. The problem of these conditions is that I cannot tell how restricting they are.

5.2 Fukaya functors

This category provides us with functors. We will first define their action on objects:

5.3 Definition *For every object* $(\mathcal{E}, U, \iota) \in \overline{\mathcal{F}}$ *we associate a functor* $F_U : \mathcal{F}(\widehat{M}) \to \mathcal{F}(\widehat{N})$ *in the following way:*

Let $(\mathcal{E}_M, U_M, \iota_M)$ *be an object of* $\mathcal{F}(\widehat{M})$*; let* $\iota_M : U_M \to \widehat{M}$ *be the immersion. Then the condition imposed on* U *in 5.2 assures that* $\iota(U)$ *and* $\pi_{\widehat{M}}^{-1}\iota_M(U_M)$ *intersect locally transversely. More precisely, for every* $u_M \in U_M$ *the set* $\iota^{-1}\pi_{hM}^{-1}\iota_M(u_M)$ *is discrete in* U*; and for every* $u \in \iota^{-1}\pi_{\widehat{M}}^{-1}\iota_M(u_M)$ *we have neighbourhoods* $V \ni u$ *and* $V_M \ni u_M$ *such that* $\iota(V)$ *and* $\pi_{\widehat{M}}^{-1}\iota_M(V_M)$ *intersect transversely. The intersection is thus an immersed* n*-dimensional submanifold* $\iota'_N : U_N \to \widehat{M} \times \widehat{N}$*.*

From this, we get (again due to our condition in 5.2) an immersed manifold in $\widehat{N}$ *via* $\iota_N = \pi_{\widehat{N}} \circ \iota'_N$*.*

On U_N*, we get a bundle* $\mathcal{E}_N$ *together with a connection simply by the tensor product* $({\iota'_N}^* \pi_{\widehat{N}}^* \mathcal{E}_M) \otimes \mathcal{E}|_{U_N}$*.*

In a way, the manifold U just plays the role of a multi-valued symplectic diffeomorphism. So far I haven't worked out yet how to construct the map $\tilde{\gamma}_N$ needed for the Maslov index from the corresponding maps $\tilde{\gamma}$ and $\tilde{\gamma}_M$.

5.4 Lemma *The object obtained as* $F_U(\mathcal{E}_M, U_M, \iota_M)$ *in 5.3 has all necessary properties to be an object in the Fukaya category* $\mathcal{F}(\widehat{N})$*.*

Proof. In 5.3 we have already proven that ι_N is an immersion of the compact manifold U_N. It is also clear that the monodromies have eigenvalues 1, since they are tensor products of monodromies of $\mathcal{E}$ and $\mathcal{E}_M$.

Since $\iota^*(\omega - \psi) = 0$, we certainly have ${\iota'_N}^*(\omega - \psi) = 0$. Further, since ω restricts to zero on $\iota_{M*}U_M$, it will also restrict to zero on $\pi_{\widehat{M}}^* \iota_{M*}(U_M)$, hence we must have ${\iota'_N}^*(\omega) = 0$. So it follows ${\iota'_N}^*(\psi) = {\iota_N}^*(\psi) = 0$.

Finally, our condition on the compatibility of Ω and Ψ obviously ensures that for an appropriate phase of the pullback ${\iota_N}^*(\Psi)$ its imaginary part vanishes. □

The next step is to define the functor on morphisms:

5.5 Definition *Take two objects* $(\mathcal{E}_{M,1}, U_{M,1})$ *and* $(\mathcal{E}_{M,2}, U_{M,2})$ *in* $\mathcal{F}(\widehat{M})$ *that intersect transversely and a point* x_M *in their intersection. Let* $\{x_{N,1}, \dots, x_{N,d}\}$ *be the points on* $F_U(U_{M,1}) \cap F_U(U_{M,2})$ *corresponding to* x_M*. Now take an element of* $\mathrm{Hom}_{\mathcal{F}(\widehat{M})}((\mathcal{E}_{M,1}, U_{M,1}), (\mathcal{E}_{M,2}, U_{M,2}))$ *concentrated at* x_M*, i.e.* $a_M \in \mathrm{Hom}(\mathcal{E}_{M,1}|_{x_M}, \mathcal{E}_{M,2}|_{x_M})$*. Then we will map* a_M *to*

$$F_U(a_M) := \sum_{x_{N,i}} (a_M \otimes \mathrm{id}_{\mathcal{E}})$$

where the i*th summand is regarded as an element in the homomorphism space located at* $x_{N,i}$*.*

(Remember that the $F_U(U_{M,1})$ has as a local system at $x_{N,i}$ the vector space $\mathcal{E}_{M,1}|_{x_M} \otimes \mathcal{E}|_{x_{N,i}}$, similarly for $F_U(U_{M,2})$, so this definition makes sense.)

To rectify this defintion one would need to prove the following Lemma:

5.6 Lemma *The functor* F_U *commutes with the composition maps* m_k *in* $\mathcal{F}(\widehat{M})$ *and* $\mathcal{F}(\widehat{N})$*.*

However, I cannot give a complete proof of this; what seems to be missing is a way to deal with deformations of the almost-complex structure on $\widehat{N}$:

Proof. Suppose we are given special Lagrangian submanifolds $U_1, \dots, U_{k+1}$ in $\widehat{M}$ together with points of intersection $x_1, \dots, x_{k+1}$ and elements $a_1, \dots, a_k$ in the corresponding local Hom-space as in definition 3.11. Consider one pseudo-holomorphic map $\phi\colon \Delta \to \widehat{M}$ with the required boundary conditions. The idea is now that, since $\pi_{\widehat{M}} \circ \iota$ is a covering map, the map ϕ lifts to a map $\widehat{\phi}\colon \Delta \to U$; in fact for every point x_{1i} in the fibre $\iota^{-1}\pi^{-1}_{\widehat{M}}(x_1)$ there is exactly one such lift $\widehat{\phi}_i$.

Let $z_1, \dots, z_{k+1}$ be the points in $\partial\mathbb{D}$ that are mapped to $x_1, \dots, x_{k+1}$ under ϕ, repectively. If we then denote by x_{ij} the point $x_{ji} := \phi_i(z_j)$, then the (finite) set $\{x_{j1}, x_{j2}, \dots\}$ is exactly the fibre of x_j with respect to the covering map $\pi_{\widehat{M}} \circ \iota$.

Now with $\pi_{\widehat{N}} \circ \iota \circ \widehat{\phi}_i$, we get a corresponding map $\phi_i\colon \Delta \to \widehat{N}$. The fact that U is Lagrangian means that ϕ_i will have the same volume as ϕ:

$$\int_{\mathbb{D}} \phi_i^* \psi = \int_{\mathbb{D}} \widehat{\phi}_i^* \psi = \int_{\mathbb{D}} \widehat{\phi}_i^* \omega = \int_{\mathbb{D}} \phi^* \omega.$$

(The middle equal sign is due to U being Lagrangian.) This is already very encouraging for our aim to prove the desired compatibility. It remains to consider the effect of the connection:

Let ξ the summand in $\mathrm{Hom}(\mathcal{E}_1|_{x_{k+1}}, \mathcal{E}_{k+1}|_{x_{k+1}})$ that we get by the map ϕ and the morphisms $a_1, \dots, a_k$ in the summation of equation (10). Let a_{ji} be the summand of $F_U(a_i)$ at the point x_{ji}. Again by the summation term corresponding to ϕ_i and $a_{1i}, \dots, a_{ki}$ in (10), we get a morphism concentrated at $x_{(k+1)i}$ that we will denote by ξ_i.

The claim is now $F_U(\xi) = \sum_i \xi_i$. We denote the connection on U by Θ, the connection on U_i by Θ_i and the one on $F_U(U_i)$ by Θ'_i. By plugging in the definition (10) we get:

$$\begin{aligned}
\xi_i &= e^{2\pi i \int \phi_i^* \psi} \cdot \Theta'_{k+1}(\phi_i|_{[z_k, z_{k+1}]}) \circ a_{ki} \circ \cdots \circ a_{1i} \circ \Theta'_1(\phi_i|_{[z_{k+1}, z_1]}) \\
&= e^{2\pi i \int \phi^* \omega} \cdot \Big(\Theta_{k+1}(\phi|_{[z_k, z_{k+1}]}) \otimes \Theta(\widehat{\phi}|_{[z_k, z_{k+1}]})\Big) \circ \Big(a_k \otimes \mathrm{id}(\mathcal{E}|_{x_{ki}})\Big) \circ \cdots \\
&\qquad \cdots \circ \Big(a_1 \otimes \mathrm{id}(\mathcal{E}|_{x_{1i}})\Big) \circ \Big(\Theta_1(\phi|_{[z_{k+1}, z_1]}) \otimes \Theta(\widehat{\phi}|_{[z_{k+1}, z_1]})\Big) \\
&= e^{2\pi i \int \phi^* \omega} \cdot \Big(\Theta_{k+1}(\phi|_{[z_k, z_{k+1}]}) \circ a_k \circ \cdots \circ a_1 \circ \Theta_1(\phi|_{[z_{k+1}, z_1]})\Big) \\
&\qquad \otimes \Big(\Theta(\widehat{\phi}|_{[z_k, z_{k+1}]}) \circ \mathrm{id}(\mathcal{E}|_{x_{ki}}) \circ \cdots \circ \mathrm{id}(\mathcal{E}|_{x_{1i}}) \circ \Theta(\widehat{\phi}|_{[z_{k+1}, z_1]})\Big) \\
&= \xi \otimes \Theta(\widehat{\phi}|_{\partial\mathbb{D}}) \\
&= \xi \otimes \mathrm{id}(\mathcal{E}|_{x_{(k+1)i}})
\end{aligned}$$

(The last equality is due to the fact that $\widehat{\phi}|_{\partial\mathbb{D}}$ is contractible in U since it extends to $\mathbb{D}$.)

This proves our claim.

The compatibility of F_U with m_k now follows simply from the fact that we can get every map $\tilde{\phi}\colon \Delta \to \widehat{N}$ as a pushforward constructed as above of a map $\phi\colon \Delta \to \widehat{M}$; one gets ϕ simply by applying our construction backwards. □

The above proof has just one big flaw: The maps ϕ_i that we constructed will in general not be holomorphic. The easiest way to solve this problem would be to put a further restriction on the objects in $\overline{\mathcal{F}}(\widehat{M} \times \widehat{N})$; we could require the induced local diffeomorphisms from $\widehat{M}$ to $\widehat{N}$ to be pseudo-holomorphic. This, however, seems to be very restricting.

There might be another way to save the above "prove": We can (at least locally) get an almost-complex structure J'_N on $\widehat{N}$ by pushing the corresponding structure on $\widehat{M}$ forward via U. This almost-complex structure will be compatible with ψ. The space of all almost-complex structures that are compatible with ψ is contractible (see e. g. [16]), so one would hope that one can find a deformation argument to show that the existence of a pseudo-holomorphic map $\phi\colon \Delta \to \widehat{N}$ in a given homotopy type (where we allow homotopies that fix the image of the $z_j \in \partial\mathbb{D}$ and vary $\phi|_{[z_j,z_{j+1}]}$ only within U_j) does not depend on the choice of the almost-complex structure. Since the considered connections are flat, since ω is closed and since it restricts to zero on all U_j, the expression in equation (10) does only depend on this homotopy type, and so this argument would really resolve our difficulties.

This argument would also prove that the composition maps m_k do not depend on the almost-complex structure, which would mean that the Fukaya category is a (almost) purely symplectic invariant.

With an assumed triangulated structure on $\mathcal{F}(\widehat{M})$ and $\mathcal{F}(\widehat{N})$, one would like this functors to be exact (i. e. map exact triangles to exact triangles); but as I haven't found a general definition of exact triangles yet, I can't verify this. Further, we would like the morphisms in $\overline{\mathcal{F}}(\widehat{M} \times \widehat{N})$ to correspond to natural transformations between the associated functors; this, however, seems to be slightly more difficult to define than the functors.

A final

5.7 Remark *Given an object $(\mathcal{E}, U)$ in $\overline{\mathcal{F}}(\widehat{M} \times \widehat{N})$, consider the object $(\mathcal{E}^*, U)$ that has the same underlying manifold but the dual vector bundle (with dual connection) on it. Then the two functors $F_{(\mathcal{E},U)}\colon \mathcal{F}(\widehat{M}) \to \mathcal{F}(\widehat{N})$ and $F_{(\mathcal{E}^*,U)}\colon \mathcal{F}(\widehat{N}) \to \mathcal{F}(\widehat{M})$ induced by these two objects are adjoints of each other.*

Proof. This is only a very elegant formulation of the simple isomorphism

$$\mathrm{Hom}(\mathcal{E}_1|_{x_N} \otimes \mathcal{E}|_x, \mathcal{E}_2|_{x_M}) \cong \mathrm{Hom}(\mathcal{E}_1|_{x_N}, \mathcal{E}_2|_{x_M} \otimes \mathcal{E}^*|_x.$$

Here x_N is any point in N that is mapped to a point x_M in M via the point x in U. □

5.3 Conclusion

This construction might be interesting for the following reason:

Suppose the homological mirror conjecture is verified for two pairs $M, \widehat{M}$ and $N, \widehat{N}$. Now as constructed the (transversal) Fukaya category $\overline{\mathcal{F}}(\widehat{M}\times\widehat{N})$ provides us with functors $\mathcal{F}(\widehat{M}) \to \mathcal{F}(\widehat{N})$ and hence with functors $\mathbf{D}^b(\mathrm{Coh}\, M) \to$

$\mathbf{D}^b(\mathrm{Coh}\, N)$. We know that such functors can be constructed from the category $\mathbf{D}^b(\mathrm{Coh}\, M \times N)$ as Fourier-Mukai-transforms; yet more, under certain circumstances every functor between these two categories satisfying a few conditions can be constructed this way:

5.8 Theorem *Let N, M be smooth projective varieties, and let F be an exact functor from $\mathbf{D}^b(\mathrm{Coh}\, M)$ to $\mathbf{D}^b(\mathrm{Coh}\, N)$.*

If F is fully faithful and has a left and right adjoint, then F is a Fourier-Mukai-transform given by an object $E \in \mathbf{D}^b(\mathrm{Coh}(M \times N))$.

This is a result of D. O. Orlov, see [21]; he even believes that this is true for every exact functor.

So this identification might help to construct a functor $\overline{\mathcal{F}}(\widehat{M} \times \widehat{N}) \to \mathbf{D}^b(\mathrm{Coh}\, M \times N)$; doing this for the case where M, N are elliptic curves might give hints for the construction in the case of higher-dimensional tori. But maybe it is too much to expect the desired equivalence of categories to be compatible with the relation between Fourier-Mukai-transforms and the construction of this section.

Appendix

A The Maslov index

There are a whole bunch of definitions that are called "Maslov index"; the article by Cappell, Lee, Miller [6] summarizes many definitions they found in the literature and shows how they relate to each other. Following their treatment, I will outline the definition that seems to be the appropriate for the Fukaya category.

We are given a $2n$-dimensional symplectic vector space V equipped with a complex structure, i. e. an endomorphism J with $J^2 = -1$ that is compatible with the symplectic form ω. We want to define a function on open paths in the space $\operatorname{Lag} V$ of Lagrangian planes in V.

So let γ be a path in $\operatorname{Lag} V$, where the endpoints intersect only in $\{0\}$. (The Maslov index can be defined for non-trivial intersection as well, but we would loose the asymmetry formula (16).) Fix the plane $L_1 := \gamma(1)$.

We can choose an ϵ with $0 < \epsilon < \pi$ such that for all $0 < |\theta| < \epsilon$, the plane $e^{\theta J} L_1$ has trivial intersection with L_1 and $\gamma(0)$. Now choose an $0 < \theta < \epsilon$ and let $\bar{L} := e^{\theta J} L_1$ Let $M \subset \operatorname{Lag} V$ be the set of planes that have non-trivial intersection with $\bar{L}$. It can be shown that M is a codimension one subvariety of $\operatorname{Lag} V$. Further, M is naturally stratified by the dimension of the intersection with $\bar{L}$; let M_0 be the top stratum of planes with one-dimensional intersection with $\bar{L}$. It has been proved that both the singularities of M and $M \setminus M_0$ have codimension 3 in $\operatorname{Lag} V$.

We want to define a transverse orientation on M_0: for $L \in M_0$, we declare that the path $t \mapsto e^{tJ}$ crosses M_0 in positive direction. Of course it needs a further check to verify that this defines a consistent orientation (for all proofs, resp. references to proofs, see [6]).

Now we deform the path γ homotopically to a path $\hat{\gamma}$ that intersects M only transversely, outside the singularities and only in M_0. Then we define the Maslov index $\mu(\gamma)$ as the negative of the topological intersection number of $\hat{\gamma}$ with M; because of the codimension 3 property, all possible choices of $\hat{\gamma}$ can be deformed one into another avoiding the singularities and $M \setminus M_0$, so that this number is indeed well-defined.

From the computations in [6], section 9 (our definition relates to theirs as $\mu(\gamma) = -\overline{\mathcal{M}}(x, y)$), it follows then that for transversal intersection of $\gamma(0)$ and $\gamma(1)$, we have for a path γ and its reverse path γ^{-1}:

$$\mu(\gamma) + \mu(\gamma^{-1}) = n \tag{16}$$

B Morphisms for non-transversal intersection in the Fukaya category

Due to our functor Φ, we can describe how the space of homomorphisms between to objects $(\mathcal{E}_{1/2}, U_{1/2}, \alpha_{1/2})$ in the Fukaya category $\mathcal{F}(\widehat{E})$ over a torus has to be defined to get the desired equivalence:

So suppose $U_1 = U_2 =: U$ (which is the only case of non-transversal intersection on our torus). The bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ give a bundle $\mathcal{E}_1 \otimes \mathcal{E}_2^* =: \mathcal{E}$ on U with fiber V. Let $A \in \mathrm{End}(V)$ be the monodromy of the connection. Let V_1 be the largest subspace of V such that $A - \mathrm{Id}_V$ is nilpotent on V_1 (i. e. the characteristic subspace of 1 with respect to A).

Then we define

$$\mathrm{Hom}((\mathcal{E}_1, U_1), (\mathcal{E}_2, U_2)) := V_1 \oplus V_1^*$$

where V_1 is concentrated in degree $\alpha_2 - \alpha_1$ and V_1^* in degree $\alpha_2 - \alpha_1 + 1$.

However, it is still unclear how to define the composition m_2 in case of non-transversal intersection.

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