WALL-CROSSING IMPLIES BRILL-NOETHER
APPLICATIONS OF STABILITY CONDITIONS ON SURFACES

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ABSTRACT. Over the last few years, wall-crossing for Bridgeland stability conditions has led to a large number of results in algebraic geometry, particular on birational geometry of moduli spaces.

We illustrate some of the methods behind these result by reproving Lazarsfeld's Brill-Noether theorem for curves on K3 surfaces via wall-crossing. We conclude with a survey of recent applications of stability conditions on surfaces.

The intended reader is an algebraic geometer with a limited working knowledge of derived categories. This article is based on the author's talk at the AMS Summer Institute on Algebraic Geometry in Utah, July 2015.

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1. INTRODUCTION

Merely following the logic of wall-crossing naturally leads one to reprove Lazarsfeld's Brill-Noether theorem for curves on K3 surfaces. I hope that explaining this proof will serve to illustrate the methods underlying many of the recent applications of wall-crossing for Bridgeland stability conditions on surfaces, in particular to the birational geometry of moduli spaces of sheaves.

To state our concrete goal, let (X, H) be a smooth polarised K3 surface.

Assumption (*): H^2 divides $H.D$ for all curve classes D on X.

Theorem 1.1 ([Laz86]). *Let (X, H) be a polarised K3 surface satisfying Assumption (*). Let C be any smooth curve in the linear system $|H|$. Then the Brill-Noether variety $W^r_d(C)$ has expected dimension $\rho(r, g, d)$; in particular, it is empty if and only if $\rho(r, g, d) < 0$.*

Here g is the genus of C, $\rho(r,d,g)$ is the Brill-Noether number, and $W^r_d(C)$ denotes the variety of globally generated degree d line bundles L on C with at least $r+1$ global sections; see Section 5 for more details. This, of course, is closely related to the space of morphisms $C \to \mathbb{P}^r$, and thus Lazarsfeld's theorem answers one of the most basic questions about the projective geometry of C. The corresponding statements for arbitrary generic curves was famously proved by degeneration in [GH80]; Lazarsfeld's proof instead uses vector bundles on the K3 surface and does not need degeneration.

We will prove a slightly more precise statement, allowing for singular curves and arbitrary pure torsion sheaves, see Theorem 5.2 and the discussion at the end of Section 5.

Background. Over the last few years, stability conditions and wall-crossing have produced many results in birational geometry completely unrelated to derived categories; we conclude this article with a survey of such results. While this development may have come as a surprise to many, myself included, it is, as often, quite a logical development in hindsight—as well as perhaps in the foresight of a few, more on that below.

There are many famous conjectures (due to Bondal, Orlov, Kawamata, Katzarkov, Kuznetsov, ...) predicting precise relations between the derived category of a variety and its birational geometry. But below the surface, wall-crossing is much closer connected to vector bundle techniques as introduced in the 1980s, and as used in Lazarsfeld's proof. I hope that this direct comparison will illuminate the additional insights coming from the derived category, stability conditions and wall-crossing.

Intended reader. I assume that the reader is an algebraic geometer with a passing familiarity of basic facts about the bounded derived category $\mathrm{D}^b(X) = \mathrm{D}^b(\operatorname{Coh} X)$ of coherent sheaves on smooth projective varieties X; for references, the reader may consult [Wei94, Chapter 10] or [Huy06, Chapters 1–2].

Omissions and apologies. This survey does not say anything on stability conditions on higher-dimensional varieties. It is also ignorant of applications of wall-crossing and stability conditions to Donaldson-Thomas theory (see [Tod14]), to the derived category itself (as e.g. in [BB13]), and of connections to mirror symmetry (see [Bri09]).

The survey would also like to apologise for not actually giving a definition of stability conditions (instead it only describes the construction of some stability conditions on a K3 surface). We refer the interested reader to the original articles [Bri07, Bri08], or to [Huy11, Bay10, MS16] for surveys.

Acknowledgements. Such a survey may be the right place to try to appropriately thank Aaron Bertram, who stubbornly convinced me and others of the power of wall-crossing for questions in birational geometry, and whose foresight influenced my approach to the topic to great extent. Of course, I am also very much indebted to Emanuele Macrì—this article is directly inspired by our joint work, and greatly benefitted from a number of additional conversations with him. I am also grateful for comments by Izzet Coskun, Gavril Farkas, Soheyla Feyzbakhsh, Davesh Maulik and Kota Yoshioka.

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Plan of the paper. Sections 2, 3 and 4 review properties of stability conditions on K3 surfaces and moduli space of stable objects; the key results are Proposition 2.3, Theorem 3.1 and Theorem 4.1. Section 5 recalls the basics about Brill-Noether for curves in K3 surfaces and the associated moduli space of torsion sheaves. The proof of Theorem 1.1 is contained in Sections 6 and 7. Section 8 reinterprets the results as a statement of the birational geometry of the moduli space of torsion sheaves. Section 9 systematically reviews results on birational geometry of moduli space obtained via wall-crossing, as well as other applications of stability conditions on surfaces.

2. The heart of the matter

The one key derived category technique that we will need is the construction of a certain abelian subcategory $\operatorname{Coh}^\beta X \subset \mathrm{D}^b(X)$ of two-term complexes, see Proposition 2.3. More technically, we will construct a *bounded t-structure* which has $\operatorname{Coh}^\beta X$ as its *heart*. In addition to H, it depends on a choice of real number β.

We recall the slope of a coherent sheaf E, for later convenience shifted by $\beta \in \mathbb{R}$:

$$\mu_\beta(E) := \begin{cases} \frac{H.c_1(E)}{\operatorname{rk}(E)} - \beta & \text{if } \operatorname{rk}(E) > 0 \\ +\infty & \text{otherwise} \end{cases}$$

The following slight modification (which I learned from Yukinobu Toda) of the definition of slope-stability implicitly accounts correctly for torsion sheaves:

Definition 2.1. We say that $E \in \operatorname{Coh} X$ is slope-(semi)stable if for all subsheaves $A \subset E$, we have $\mu_\beta(A) < (\leq)\ \mu_\beta(E/A)$.

Every sheaf E has a (unique and functorial) Harder-Narasimhan (HN) filtration: a sequence $0 = E_0 \hookrightarrow E_1 \hookrightarrow \dots \hookrightarrow E_m = E$ of coherent sheaves where E_i/E_{i-1} is slope-semistable for $1 \leq i \leq m$, and with

$$\mu_\beta^+(E) := \mu_\beta\left(E_1/E_0\right) > \mu_\beta\left(E_2/E_1\right) > \dots > \mu_\beta^-(E) := \mu_\beta\left(E_m/E_{m-1}\right).$$

Moreover, if E, F are slope-semistable with $\mu_\beta(E) > \mu_\beta(F)$, then $\operatorname{Hom}(E,F) = 0$.

We use the existence of HN filtrations to break the abelian category of coherent sheaves into two pieces $T^\beta, F^\beta \subset \operatorname{Coh} X$:

$$\begin{aligned} T^\beta &= \left\{\mathcal{T} :\ \mu_\beta^-(E) > 0\right\} = \left\{\mathcal{T} :\ \text{all HN-factors of } \mathcal{T} \text{ satisfy } \mu_\beta(\_) > 0\right\} \\ &= \left\{\mathcal{T} :\ \text{all quotients } \mathcal{T} \twoheadrightarrow \mathcal{E} \text{ satisfy } \mu_\beta(\mathcal{E}) > 0\right\} \\ &= \left\langle \mathcal{T} \colon \mathcal{T} \text{ is slope-semistable with } \mu_\beta(\mathcal{T}) > 0 \right\rangle \\ F^\beta &= \left\{\mathcal{T} :\ \mu_\beta^+(E) \leq 0\right\} = \left\{\mathcal{T} :\ \text{all HN-factors of } \mathcal{T} \text{ satisfy } \mu_\beta(\_) \leq 0\right\} \\ &= \left\{\mathcal{T} :\ \text{all subobjects } \mathcal{A} \hookrightarrow \mathcal{T} \text{ satisfy } \mu_\beta(\mathcal{A}) \leq 0\right\} \\ &= \left\langle \mathcal{T} \colon \mathcal{T} \text{ is slope-semistable with } \mu_\beta(\mathcal{T}) \leq 0 \right\rangle \end{aligned}$$

Here $\langle\cdot\rangle$ denotes the extension-closure, i.e., the smallest subcategory of $\operatorname{Coh} X$ containing the given objects and closed under extensions. The equivalence of the above formulations follows from the existence of Harder-Narasimhan filtrations, and the Hom-vanishing between stable objects mentioned above.

The formal properties of this pair of subcategories can be summarised as follows:

Proposition 2.2. *The pair (T^β, F^β) is a torsion pair, i.e.:*

(a) For $T \in T^\beta, F \in F^\beta$, we have $\operatorname{Hom}(T, F) = 0$.
(b) Each $E \in \operatorname{Coh} X$ fits into a (unique and functorial) short exact sequence

$$0 \to T(E) \to E \to F(E) \to 0$$

with $T(E) \in T^\beta$ and $F(E) \in F^\beta$.

Proof. Given a non-zero element $f \in \operatorname{Hom}(T, F)$, we have a surjection $T \twoheadrightarrow \operatorname{im} f$ and therefore $\mu_\beta(\operatorname{im} f) > 0$; but we also have an injection $\operatorname{im} f \hookrightarrow F(E)$ and therefore $\mu_\beta(\operatorname{im} f) \leq 0$. This contradiction proves (a).

As for (b), consider the HN filtration of E, and let i be maximal such that $\mu_\beta(E_i/E_{i-1}) > 0$; then $T(E) := E_i$ satisfies the claim. □

For us, the most important result on derived categories is the following Proposition; thereafter, all our arguments will live completely within the newly constructed abelian category.

Proposition 2.3 ([Bri08, HRS96])**.** *The following (equivalent) characterisations define an abelian subcategory of $\mathrm{D}^b(X)$:*

$$\begin{aligned}\operatorname{Coh}^\beta X &= \left\langle T^\beta, F^\beta[1]\right\rangle \\ &= \left\{E:\ H^0(E) \in T^\beta, H^{-1}(E) \in F^\beta, H^i(E) = 0 \text{ for } i \neq 0, -1\right\} \\ &= \left\{E:\ E \cong F^{-1} \xrightarrow{d} F^0, \ker d \in F^\beta, \operatorname{cok} d \in T^\beta\right\}\end{aligned}$$

Rather than giving a proof, I will try to give as good an intuition about the behaviour of this abelian category as possible. To begin with, short exact sequences in $\operatorname{Coh}^\beta X$ are exactly those exact triangles $A \to E \to B \to A[1]$ in $\mathrm{D}^b(X)$ for which all of A, E, B are in $\operatorname{Coh}^\beta X$; then A is the subobject, and B is the quotient. In particular, every object $E \in \operatorname{Coh}^\beta$ fits into a short exact sequence

$$H^{-1}(E)[1] \hookrightarrow E \twoheadrightarrow H^0(E). \tag{1}$$

The isomorphism class of E is determined by the extension class $\operatorname{Ext}^1(H^0(E), H^{-1}(E)[1]) = \operatorname{Ext}^2(H^0(E), H^{-1}(E))$.

More generally, every short exact sequence in $\operatorname{Coh}^\beta X$ gives a six-term long exact sequence in cohomology (with respect to $\operatorname{Coh} X$)

$$0 \to H^{-1}(A) \to H^{-1}(E) \to H^{-1}(B) \to H^0(A) \to H^0(E) \to H^0(B) \to 0 \tag{2}$$

with $H^{-1}(\_) \in F^\beta$ and $H^0(\_) \in T^\beta$.

The following observation already illustrates how closely the abelian category $\operatorname{Coh}^\beta X$ is related to classical vector bundle techniques.

Proposition 2.4. *Let $E \in T^\beta$, considered as an object of $\operatorname{Coh}^\beta X$. To give a subobject $A \hookrightarrow E$ of E (with respect to the abelian category $\operatorname{Coh}^\beta X$) is equivalent of giving a sheaf $A \in T^\beta$ with a map $f\colon A \to E$ whose kernel (as a map of coherent sheaves) satisfies* $\ker f \in F^\beta$.

Proof. Given a subobject $A \hookrightarrow E$, consider the associated long exact cohomology sequence (2). We immediately see that $H^{-1}(A) = 0$, and therefore $A = H^0(A)$ is a sheaf. The map $f\colon A \to E$, considered as a map of coherent sheaves, has kernel $\ker f = H^{-1}(B) \in F^\beta$.

Conversely, assume we are given a map $f\colon A \to E$ as specified. Let B be the cone of f, which is the two-term complex with $B^{-1} = A, B^0 = E$, and the differential given by f. Then there is an exact triangle $A \xrightarrow{f} E \to B$. By assumption, $H^{-1}(B) = \ker f \in F^\beta$; on the other hand, $H^0(B)$ is a quotient of $A \in T^\beta$, and therefore is also in T^β. This shows that $B \in \operatorname{Coh}^\beta X$; hence $A \to E \to B$ is a short exact sequence and f is injective as a map in $\operatorname{Coh}^\beta X$. □

We conclude this section with a tangential observation on $\operatorname{Coh}^\beta X$. One of the features of the derived category is that cohomology classes of coherent sheaves become morphisms:

$$\gamma \in H^k(X, G) = \operatorname{Hom}(\mathcal{O}_X, G[k]).$$

However, this feature is only useful with additional structures on $\mathrm{D}^b(X)$; the abelian category $\operatorname{Coh}^\beta X$ can precisely play this role. For example, if $k = 1$, $\beta < 0$ (hence $\mathcal{O}_X \in T^\beta \subset \operatorname{Coh}^\beta X$) and $G \in F^\beta$, then γ becomes a morphism $\mathcal{O}_X \to G[1]$ in the abelian category $\operatorname{Coh}^\beta X$. This immediately gives additional methods: one can consider the image of γ, or one can try to deduce its vanishing from stability; see [AB11] for an example of this type of argument, in this case reproving Reider's theorem for adjoint bundles on surfaces. If instead $k = 2$, then γ becomes an *extension* $\operatorname{Ext}^1(\mathcal{O}_X, G[1])$ between two objects within the same abelian category, and thus produces a corresponding object in $\operatorname{Coh}^\beta X$. One can, for example, try to determine stability of this object (or study its HN filtration when it is unstable); see [BBMT14] for a conjectural application of this idea towards Fujita's conjecture for threefolds.

3. Geometric stability

The goal of this section is to fully explain the meaning of the following result:

Theorem 3.1 ([Bri08])**.** *Let (X, H) be a polarised K3 surface. For each $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, consider the pair $\sigma_{\alpha,\beta} := \left(\operatorname{Coh}^\beta X, Z_{\alpha,\beta}\right)$ with $\operatorname{Coh}^\beta X$ as constructed in Proposition 2.3, and with $Z_{\alpha,\beta}\colon K(\mathrm{D}^b(X)) \to \mathbb{C}$ defined by*

$$Z_{\alpha,\beta}(E) = \left\langle e^{\sqrt{-1}\alpha H + \beta H}, v(E)\right\rangle.$$

This pair defines a Bridgeland stability condition on $\mathrm{D}^b(X)$ if $Re Z_{\alpha,\beta}(\delta) > 0$ for all roots $\delta \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ of the form $(r, r\beta, s)$ with $r > 0$ and $s \in \mathbb{Z}$ arbitrary; in particular this holds for $\alpha^2 H^2 \geq 2$.

Moreover, the family of stability conditions $\sigma_{\alpha,\beta}$ varies continuously as α, β vary in $\mathbb{R}_{>0} \times \mathbb{R}$.

We begin by explaining the notation. The Mukai vector of an object $E \in \mathrm{D}^b(X)$ given by

$$v(E) = (v_0(E), v_1(E), v_2(E)) := \operatorname{ch}(E) \cdot \sqrt{\operatorname{td}_X} = (\operatorname{ch}_0(E), \operatorname{ch}_1(E), \operatorname{ch}_2(E) + \operatorname{ch}_0(E))$$

lies in the algebraic cohomology $H^*_{\mathrm{alg}}(X, \mathbb{Z})$. The pairing $\langle \_, \_ \rangle$ is the Mukai pairing

$$\begin{aligned}\langle v(E), v(F)\rangle = -\chi(E,F) &= \sum_i (-1)^i \dim \mathrm{Hom}(E, F[i]) \\ &= v_1(E)v_1(F) - v_0(E)v_2(F) - v_2(E)v_0(F).\end{aligned}$$

It equips $H^*_{\mathrm{alg}}(X, \mathbb{Z})$ with the structure of an even lattice of signature $(2, \rho(X))$, where $\rho(X)$ is the Picard rank of X. Roots in this lattice are classes δ with $\delta^2 = -2$.

Explicitly, the central charge $Z_{\alpha,\beta}$ is given by

$$\begin{aligned}Z_{\alpha,\beta}(E) =& \sqrt{-1}\alpha H\big(v_1(E) - \beta H \operatorname{rk}(E)\big) \\ & - v_2(E) + \beta H v_1(E) + \frac{\alpha^2 H^2 - \beta^2 H^2}{2} v_0(E)\end{aligned} \tag{3}$$

For a sheaf E, we have $\Im Z_{\alpha,\beta}(E) \geq 0$ if and only if $\mu_\beta(E) \geq 0$. Using the short exact sequence (1) and $Z_{\alpha,\beta}(F[1]) = -Z_{\alpha,\beta}(F)$, one can immediately conclude

Lemma 3.2. *If $E \in \operatorname{Coh}^\beta X$, then $\Im Z_{\alpha,\beta}(E) \geq 0$.*

In other words, $\Im Z_{\alpha,\beta}$ behaves like a rank function on the abelian category $\operatorname{Coh}^\beta X$: it is a non-negative function on its set of objects that is additive on short exact sequences. We want to define a notion of slope by using the real part $\Re Z_{\alpha,\beta}$ as a degree:

$$E \in \operatorname{Coh}^\beta X \mapsto \nu_{\alpha,\beta}(E) = \frac{-\Re Z_{\alpha,\beta}(E)}{\Im Z_{\alpha,\beta}(E)}. \tag{4}$$

To make this well-behaved, we need one further observation:

Lemma 3.3. *Assume α, β satisfy the assumptions of Theorem 3.1. If $0 \neq E \in \operatorname{Coh}^\beta X$ satisfies $\Im Z_{\alpha,\beta}(E) = 0$, then $\Re Z_{\alpha,\beta}(E) < 0$.*

Proof. The short exact sequence (1) shows that

$$\Im Z_{\alpha,\beta}\left(H^{-1}(E)\right) = 0 = \Im Z_{\alpha,\beta}\left(H^0(E)\right).$$

It follows that if $H^0(E)$ is non-trivial, then it is a zero-dimensional torsion sheaf, in which case $\Re Z_{\alpha,\beta}(H^0(E)) = -\operatorname{ch}_2(E) < 0$. If $H^{-1}(E) \neq 0$, then it must be a slope-semistable sheaf with $\mu_\beta\left(H^{-1}(E)\right) = 0$. It is enough to consider the case that it is *stable*. Then with $\mathbf{v} := \mathbf{v}\left(H^{-1}(E)\right)$ we have $\mathbf{v}^2 \geq -2$ by Hirzebruch-Riemann-Roch and Serre duality. If $\mathbf{v}^2 = -2$, the claim follows from our assumptions on α, β. Otherwise, if $\mathbf{v}^2 \geq 0$, then

$$\Re Z_{\alpha,\beta}(\mathbf{v}) = \left\langle \Re e^{\sqrt{-1}\alpha H}, e^{-\beta H}\mathbf{v}\right\rangle = \left\langle \left(1, 0, -\frac{\alpha^2 H^2}{2}\right), (r, 0, s)\right\rangle = -s + \frac{\alpha^2 H^2 \cdot r}{2}$$

Since $r > 0$ and $-2rs = \left(e^{-\beta H}\mathbf{v}\right)^2 = \mathbf{v}^2 \geq 0$ we have

$$\Re Z_{\alpha,\beta}\left(H^{-1}(E)\right) = -\Re Z_{\alpha,\beta}(\mathbf{v}) < 0$$

proving the claim. $\square$

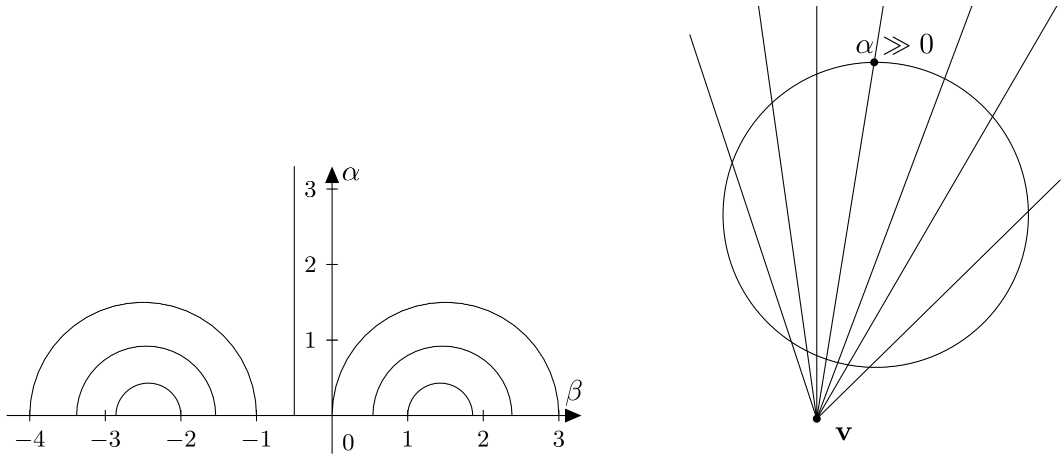

FIGURE 1. Walls as semi-circles in the upper half plane, or as lines in the projective plane

This finally leads to a notion of stability for objects in $\mathrm{D}^b(X)$: we say that $E \in \mathrm{D}^b(X)$ is $\sigma_{\alpha,\beta}$-semistable if some shift $E[k]$ is contained in the abelian category $\mathrm{Coh}^\beta X$, and if that object $E[k]$ is slope-semistable with respect to the slope-function $\nu_{\alpha,\beta}$. We need one more result to conclude that $(\mathrm{Coh}^\beta X, Z_{\alpha,\beta})$ is a stability condition (which we state without proof):

Proposition 3.4. *Any $E \in \mathrm{Coh}^\beta X$ admits a HN filtration: a filtration whose quotients are $\nu_{\alpha,\beta}$-semistable objects of decreasing slopes.*

Finally, we need to explain what we mean by a *continuous* family of stability conditions. The technical underlying notion here is the *support property*; it implies that for small variations of the central charge Z, the variation of the phases $\phi(\_) = \frac{1}{\pi} \cot^{-1} \nu(\_)$ of all semistable objects can be bounded simultaneously. What we need is the following consequence:

Corollary 3.5. *Given a class $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$, there is a chamber decomposition induced by a locally finite set of walls in $\mathbb{R} \times \mathbb{R}_{>0}$ with the following property: for objects of Mukai vector $\mathbf{v}$, being $\sigma_{\alpha,\beta}$-stable (or semistable) is independent on the choice of (β, α) in any given chamber.*

Remark 3.6. Locally, such walls are given by the condition that $Z(A)$ and $Z(E)$ are aligned, where E is of class $\mathbf{v}$ and $A \hookrightarrow E$ is a semistable subobject. In the (β, α)-plane, this condition is given by a semi-circle. It is sometimes easier to visualise the walls if we think of central charges as being characterised by their kernel. For example, if $\rho(X) = 1$, then the kernel of $Z_{\alpha,\beta}$ is a line inside the negative cone in $\mathbb{R}^3 \cong H^*_{\mathrm{alg}}(X, \mathbb{Z}) \otimes \mathbb{R}$; our formula for $Z_{\alpha,\beta}$ is the natural identification of the upper half plane with the projectivization of the negative cone inside $\mathbb{P}^2_{\mathbb{R}}$. The condition that $Z(A)$ and $Z(E)$ are aligned is equivalent to the condition that the kernel is contained in the rank two sublattice spanned by $v(A)$ and $\mathbf{v} = v(E)$; thus all walls become lines in (the image of the negative cone inside) $\mathbb{P}^2_{\mathbb{R}}$ going through the point corresponding to $\mathbb{R} \cdot \mathbf{v}$, see fig. 1.

4. MODULI SPACES OF STABLE OBJECTS

The most important result on moduli spaces of stable sheaves on K3 surfaces is that they have expected dimension, and that they are non-empty whenever this dimension is non-negative and the Mukai vector $\mathbf{v}$ satisfies some obvious assumptions. The same holds for moduli spaces of stable objects in the derived category (in which case we can in fact also drop all assumptions on $\mathbf{v}$):

Theorem 4.1 (Mukai, Yoshioka, Toda, ...)**.** *Consider a primitive vector $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$, and let $\sigma = \sigma_{\alpha,\beta}$ be a stability condition that is generic*[1] *with respect to $\mathbf{v}$. Then the coarse moduli space $M_\sigma(\mathbf{v})$ of $\sigma_{\alpha,\beta}$-stable objects of Mukai vector $\mathbf{v}$ exists as a smooth projective irreducible holomorphic symplectic variety.*

It is non-empty iff $\mathbf{v}^2 \geq -2$, and its dimension is given by $\dim M_\sigma(\mathbf{v}) = \mathbf{v}^2 + 2$.

This is the deepest ingredient in our arguments, and it comes from various sources. The existence of the moduli space as an algebraic space was proved in [Tod08]. An crucial observation in [MYY11a] (generalised to our situation in [BM14b]) uses a Fourier-Mukai transform to reduce all the statements above to the case of Gieseker-stable sheaves. It follows immediately from Serre duality and Hirzebruch-Riemann-Roch that for $\mathbf{v}^2 < -2$, the moduli space is empty. If it is non-empty, Mukai's arguments in [Muk84] shows that the moduli space is smooth and symplectic of the given dimension. The most difficult statement is the non-emptiness for $\mathbf{v}^2 \geq -2$; its proof uses deformation to elliptic K3 surfaces followed by Fourier-Mukai transforms to reduce to the case of Hilbert schemes, see [Yos01b, Theorem 8.1] as well as [KLS06, Section 2.4].

Assume for simplicity that $M_\sigma(\mathbf{v})$ is a fine moduli space, i.e. that it admits a universal family $\mathcal{E} \in \mathrm{D}^b(M_\sigma(\mathbf{v}) \times X)$; let p, q denote the projections from the product to $M_\sigma(\mathbf{v})$ and X, respectively. Let $F \in \mathrm{D}^b(X)$ be an object with $(\mathbf{v}, v(F)) = 0$. Then the determinant line bundle construction $\det$ of [KM76] produces a line bundle on $M_\sigma(\mathbf{v})$ via

$$\det\left(p_* \mathbf{R}\mathcal{H}om(\mathcal{E}, p^* F)\right).$$

Theorem 4.2 ([Yos01b, Sections 7 and 8])**.** *Assume that $\mathbf{v}$ is primitive with $\mathbf{v}^2 > 0$, and that σ is generic with respect to $\mathbf{v}$. Then the determinant line bundle construction induces an isomorphism*

$$\theta_v \colon \mathbf{v}^\perp \to \mathrm{NS}\, M_\sigma(\mathbf{v}) \tag{5}$$

where $\mathbf{v}^\perp$ denotes the orthogonal complement of $\mathbf{v}$ inside the algebraic cohomology $H^*_{\mathrm{alg}}(X, \mathbb{Z})$.

We will call θ_v the *Mukai isomorphism*.

Remark 4.3. In addition, θ_v identifies the restriction of the Mukai pairing in $H^*_{\mathrm{alg}}(X, \mathbb{Z})$ with the *Beauville-Bogomolov-pairing* on the Néron-Severi group of the moduli space; however, we will not need that fact for the proof of Theorem 1.1, only in the concluding sections 8 and 9 on birational geometry of moduli spaces.

Consider equations (3) and (4) for $\alpha \gg 0$; then the slope $\nu_{\alpha,\beta}(E)$ is approximately given by $-\frac{\alpha}{\mu_{\beta(E)}}$. This observation, combined with Proposition 2.4 (as well as the bound on Mukai vectors of stable objects in Theorem 4.1) leads to the following result:

[1] This means that σ is not on any of the walls for the wall-and-chamber decomposition described in Corollary 3.5.

Theorem 4.4. *Let $\mathbf{v} = (v_0, v_1, v_2)$ be a primitive class in $H^*_{\mathrm{alg}}(X, \mathbb{Z})$ having either positive rank $v_0 > 0$, or satisfying $v_0 = 0$ with v_1 being effective. Then there exists α_0 such that for all $\alpha \geq \alpha_0$ and all $\beta > \frac{H.v_1}{H^2 v_0}$ (or $\beta > \frac{v_2}{H.v_1}$ in case $v_0 = 0$), the moduli space $M_{\sigma_{\alpha,\beta}}(\mathbf{v})$ is equal to the moduli space $M_H(\mathbf{v})$ of H-Gieseker-stable sheaves of class $\mathbf{v}$. More precisely, an object $E \in \mathrm{D}^b(X)$ with $v(E) = \mathbf{v}$ is $\sigma_{\alpha,\beta}$-stable if and only if it is the shift of a Gieseker-stable sheaf.*

5. Brill-Noether and the moduli space of torsion sheaves

From now on, let (X, H) be a polarised K3 surfaces satisfying Assumption (*), and let $d \in \mathbb{Z}$ be a degree. The natural moduli space related to Brill-Noether is $M_H(\mathbf{v})$ for $\mathbf{v} = (0, H, d+1-g)$: it parameterises purely one-dimensional sheaves F of Euler characteristic $d + 1 - g$ whose support $|F|$ is a curve in $|H|$. By [Bea91], the map

$$\pi \colon M_H(\mathbf{v}) \to |H| \cong \mathbb{P}^g, \quad F \mapsto |F| \tag{6}$$

is a Lagrangian fibration, called the *Beauville integrable system*. The fibre over a smooth curve $C \subset |H|$ is the Picard variety $\mathrm{Pic}^d(C)$, and the restriction of the symplectic form to any fibre vanishes.

We will make all our definitions in the context of $M_H(\mathbf{v})$. In particular, let $T_d(C) = \pi^{-1}(C)$ be the moduli space of pure torsion sheaves supported on C and with Euler characteristic $d + 1 - g$.

Definition 5.1. We define the following constructible subsets of $T_d(C)$.

- $W^r_d(C)$ is the set of globally generated sheaves with at least $r + 1$ global sections;
- $\overline{W}^r_d(C)$ is as above, but without the assumption of being globally generated;
- $V^r_d(C)$ is the set of sheaves with exactly $r + 1$ sections.

The expected dimension for each of them is given by the Brill-Noether number

$$\rho(r, d, g) = g - (r + 1)(g - d + r).$$

Our wall-crossing methods most naturally deal with $V^r_d(C)$; we will prove:

Theorem 5.2. *Assume (X, H) satisfies Assumption (*), and that $C \in |H|$ is an arbitrary curve (possibly singular). If r, d satisfy $0 < d \leq g - 1$ and $r \geq 0$, then $\dim V^r_d(C) = \rho(r, d, g)$.*

We will briefly explain how Theorem 1.1 implies 5.2. Since $\rho(r, d, g)$ is a strictly decreasing function of r in our range $d \leq g - 1$, and since

$$\overline{W}^r_d(C) = V^r_d(C) \setminus \bigcup_{r' > r} V^{r'}_d(C),$$

we conclude $\dim \overline{W}^r_d(C) = \rho(r, d, g)$ for all $d \leq g - 1$. Similarly,

$$\overline{W}^r_d(C) = W^r_d(C) \cup \bigcup_{d' < d} B_{d'}$$

where $B_{d'}$ parametrises the sheaves whose global sections generate a subsheaf of Euler characteristic $d' + 1 - g$. Since C is assumed to smooth, we have $B_{d'} \subset \overline{W}^r_{d'}(C) \times \mathrm{Sym}^{d-d'}(C)$, and thus

$$\dim B_{d'} \leq \rho(r, d', g) + d - d' < \rho(r, d, g),$$

where the last inequality used the assumption $r \geq 1$ of Theorem 1.1. This proves $\dim W^r_d(C) = \rho(r, d, g)$ as claimed. Finally, the case $d > g - 1$ follows via Serre duality on C.

6. Hitting the wall

We now consider wall-crossing for the moduli space $M_{\sigma_{\alpha,\beta}}(\mathbf{v})$, with $\mathbf{v} = (0, H, d + 1 - g)$ as above. By Theorem 4.4, we have $M_{\sigma_{\alpha,\beta}}(\mathbf{v}) = M_H(\mathbf{v})$ for $\alpha \gg 0$, and we want to find the wall bounding this Gieseker-chamber.

Consider $\beta = 0$. In this case $\Im Z_{\alpha,0}(\mathcal{O}_X) = 0$; by Theorem 3.1, this means we have stability conditions for

$$\alpha > \alpha_0 := \sqrt{\frac{2}{H^2}}.$$

For these stability conditions, note that $\mathcal{O}_X[1]$ is an object of $\operatorname{Coh}^0 X$ with $\Im Z_{\alpha,0}(\mathcal{O}_X[1]) = 0$, i.e. of slope $+\infty$; therefore it is automatically semistable. Applying Proposition 2.4 in fact shows that it has no subobjects in $\operatorname{Coh}^0 X$, and so $\mathcal{O}_X[1]$ is *stable* for $\beta = 0$. (This also shows that the bound of Theorem 3.1 is sharp: we have $Z_{\alpha,0}(\mathcal{O}_X[1]) \to 0$ as $\alpha \to \alpha_0$, and the central charge of stable objects can never become zero.)

Lemma 6.1. *For $\alpha > \alpha_0$ and $\beta = 0$, we have an isomorphism $M_{\sigma_{\alpha,0}}(\mathbf{v}) = M_H(\mathbf{v})$ identifying the stable objects with stable sheaves.*

In other words, there is no wall intersecting the line segment $\beta = 0, \alpha = (\frac{2}{H^2}, +\infty)$.

Proof. This is a direct consequence of Assumption (*): the objects in $M_H(\mathbf{v})$ have "rank one" in $\operatorname{Coh}^\beta X$, and thus can never be destabilised.

To elaborate, consider equation (3). We have $\Im Z_{\alpha,\beta=0}(E) = \alpha H.c_1(E) \in \mathbb{Z}_{\geq 0}\alpha H^2$ for all $E \in \operatorname{Coh}^0 X$. Any $L \in M_H(\mathbf{v})$ has $\Im Z_{\alpha,\beta=0}(L) = \alpha H^2$. If L were semistable, each of its Jordan-Hölder factors A_i would have to have $\Im Z_{\alpha,\beta=0}(A_i) > 0$ (otherwise it could not have the same slope as L), and thus $\Im Z_{\alpha,\beta=0}(A_i) \geq \alpha H^2$. This is a contradiction. Combined with Corollary 3.5, this means they remain stable along the entire path. □

The key observation linking Brill-Noether to wall-crossing is the following Lemma; the case $d = g - 1$ is one of the first wall-crossings studied in the literature, see [AB13].

Lemma 6.2. *There is a wall bounding the Gieseker-chamber where $Z_{\alpha,\beta}(\mathcal{O}_X)$ aligns with $Z_{\alpha,\beta}(\mathbf{v})$. The sheaves $L \in M_{\sigma_{\alpha,\beta}}(\mathbf{v})$ getting destabilised are exactly those with $h^0(L) > 0$, and the destabilising short exact sequences are given by*

$$\mathcal{O}_X^{\oplus h^0(L)} \hookrightarrow L \twoheadrightarrow W \tag{7}$$

for some object W that remains stable at the wall.

Proof. This is perhaps most easily explained using the visualisation of walls as lines in the projective plane discussed in Remark 3.6. The locus where the central charges of all objects in (7) are aligned is the line segment between $\mathbf{v}$ and $v(\mathcal{O}_X)$; in the upper half-plane picture, it is the arc of a circle ending at $(0, \alpha_0)$. Now consider the path in the upper half plane as in fig. 2 that starts at $\beta = 0, \alpha \gg 0$, goes straight to a point $(0, \alpha_0 + \epsilon)$ just slightly above $(0, \alpha_0)$, and then turns left

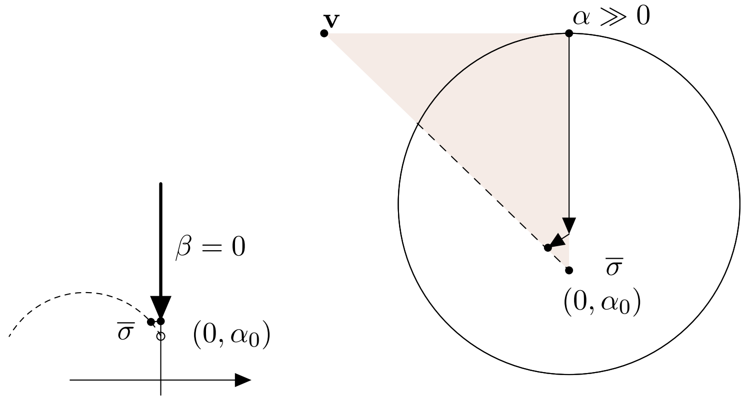

FIGURE 2. From the large volume limit to $\overline{\sigma}$. No walls in the shaded region!

until it hits the above semi-circle. The visualisation of walls via lines shows immediately that if this path would hit any other wall beforehand, then that wall would also intersect the straight line segment $\beta = 0, \alpha \in (\alpha_0, +\infty)$ in contradiction to Lemma 6.1. Also, $\mathcal{O}_X$ cannot be destabilised along this path: for (β, α) near $(0, \alpha_0)$, we have $|Z_{\alpha,\beta}(\mathcal{O}_X)| \ll 1$, and it is the only stable object with that property.

Let $\overline{\sigma} = (\mathrm{Coh}^{\overline{\beta}} X, \overline{Z})$ be the stability condition at the wall. In the abelian category of $\overline{\sigma}$-semistable objects with central charge aligned with $\overline{Z}(\mathbf{v})$, the object $\mathcal{O}_X$ is a simple object; hence the natural map $\mathcal{O}_X^{\oplus h^0(L)} \to L$ must necessarily be an injective map, and the quotient W must be semistable.

It remains to prove that W is stable. Note the $\mathrm{Hom}(W, \mathcal{O}_X) = 0$ as W is a quotient of L. Moreover, $\mathrm{Hom}(\mathcal{O}_X, W) = 0$ follows by applying $\mathrm{Hom}(\mathcal{O}_X, \_)$ to the short exact sequence defining W. Hence stability of W follows from the following Lemma. $\square$

Lemma 6.3. *Let $\overline{\sigma}$ be a stability condition on the wall constructed above. Let W be an object of class $\mathbf{v} - tv(\mathcal{O}_X)$ for some $t \in \mathbb{Z}$, and assume that W is $\overline{\sigma}$-semistable. Then W is stable if and only if* $\mathrm{Hom}(\mathcal{O}_X, W) = \mathrm{Hom}(W, \mathcal{O}_X) = 0$.

Proof. Assume first that $\rho(X) = 1$, and consider the Mukai vector $\mathbf{a}$ of any Jordan-Hölder factor A of W. It must be contained in the rank two sublattice generated by $\mathbf{v}$ and $\mathbf{s} := v(\mathcal{O}_X)$, otherwise its central charge would not be aligned with $\overline{Z}(\mathbf{v})$. Further, since $\overline{Z}(\mathbf{a})$ must be on the same *ray* as $\overline{Z}(\mathbf{v})$, there is a half-plane in this rank two sublattice containing $\mathbf{s}, \mathbf{v}$ and $\mathbf{a}$, see fig. 3

On the other hand, if $A \neq \mathcal{O}_X$, then $(\mathbf{s}, \mathbf{a}) = -\chi(\mathcal{O}_X, A) \geq 0$; since $\mathbf{s}^2 = -2$ and $(\mathbf{s}, \mathbf{v}) > 0$ this cuts out a second half-plane with configuration as in fig. 3: $\mathbf{a}$ must lie in the shaded area of the figure.

It follows that either $\mathbf{a} = \mathbf{s}$, or $\mathbf{a} = a\mathbf{v} + b\mathbf{s}$ with $a > 0$. But since $\mathbf{s}$ and $\mathbf{v}$ are a basis for this rank two lattice we must have $a \geq 1$; it follows that either $a = 1$ or $A = \mathcal{O}_X$. Hence all but one of the Jordan-Hölder factors of W are isomorphic to $\mathcal{O}_X$; so $\mathcal{O}_X$ is either a subobject or a quotient of W, a contradiction.

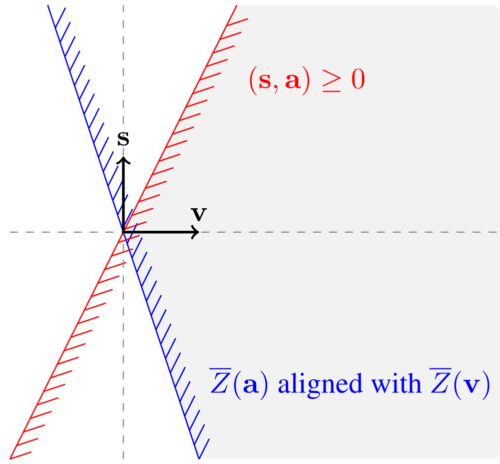

FIGURE 3. Jordan-Hölder factors of W

When $\rho(X) > 1$, the same arguments apply if we replace all Mukai vectors $\mathbf{a} = v(A)$ with the vector $\left(\mathrm{rk}(A), \frac{1}{H^2} H.c_1(A), v_2(A)\right) \in \mathbb{Z}^3$; note again that Assumption (*)is essential here. $\square$

Let $\mathbf{w}_r = \mathbf{v} - (r+1)v(\mathcal{O}_X) = (-(r+1), H, d-g-r))$ be the Mukai vector of W. Note that $\mathbf{w}_r^2 = 2\rho(r,d,g) - 2$. As in [Laz86], this immediately leads to the first conclusion:

Corollary 6.4. *If $\rho(r,d,g) < 0$, then $V_d^r(C) = \emptyset$ for all $C \in |H|$.*

Proof. If $V_d^r(C)$ is non-empty, then by Lemma 6.2 there exists a $\bar{\sigma}$-stable object of class $\mathbf{w}_r$; by Theorem 4.1 this implies $\mathbf{w}_r^2 \geq -2$. $\square$

Let us now write σ_+ for a stability condition on the path of fig. 2 just before hitting the wall at $\overline{\sigma}$. We will prove a converse to Lemma 6.2:

Lemma 6.5. *Let $W \in M_{\overline{\sigma}}^{\mathrm{stable}}(\mathbf{w}_r)$ be an object is* $\overline{\sigma}$-stable*. Consider any extension of the form*

$$\mathcal{O}_X^{r+1} \to E \to W$$

induced by an $(r+1)$-dimensional subspace of $\mathrm{Ext}^1(W, \mathcal{O}_X)$. Then E is σ_+-stable.

Proof. Evidently, E is $\overline{\sigma}$-semistable. If E is not σ_+-stable, then the destabilising subobject $A \hookrightarrow E$ would necessarily be in the abelian category $\overline{\sigma}$-semistable of the same slope as E. But since $\mathcal{O}_X$ and W are simple objects in that category, we can determine all subobjects of E: they are all of the form $\mathcal{O}_X^d$ for some $d < r+1$. But $\mathcal{O}_X$ has smaller slope than E, a contradiction. $\square$

However, note that by the previous Lemmas, σ_+ is in the Gieseker-chamber: $M_{\sigma_+}(\mathbf{v}) = M_H(\mathbf{v})$. Hence such an E is automatically a torsion sheaf in $M_H(\mathbf{v})$, with $h^0(E) = r+1$, i.e. $E \in V_d^r(|H|)$! To confirm the existence of such E, we need one more result:

Lemma 6.6. *If $\rho(r,d,g) \geq 0$, then the set of $\bar{\sigma}$-stable objects in $M_{\sigma_+}(\mathbf{w}_r)$ is open and non-empty.*

Proof. By Theorem 4.1, the moduli space $M_{\sigma_+}(\mathbf{w}_r)$ is non-empty of dimension $\mathbf{w}_r^2+2=2\rho\geq 0$; each of its objects are $\overline{\sigma}$-semistable of class $\mathbf{w}_r$. Consider the Jordan-Hölder factors of such an object W. It cannot have $\mathcal{O}_X$ as a quotient - otherwise W would not be σ_+-stable. By Lemma 6.3, it is either stable, or has $\mathcal{O}_X$ as a subobject.

By induction, it follows just as in Lemma 6.2 that the Jordan-Hölder filtration of W is of the form $\mathcal{O}_X^d \hookrightarrow W \twoheadrightarrow W'$, where W' is $\overline{\sigma}$-stable. We want to compute the dimension of the space of such extension for all such d (if it is non-empty). We write $\mathbf{s} := v(\mathcal{O}_X)$ as before, and set $e := (\mathbf{w}_r, v(\mathcal{O}_X))$; note $e > 0$. Since $\mathcal{O}_X$ and W' are *$\overline{\sigma}$-stable* of the same phase, we have $\operatorname{Hom}(W',\mathcal{O}_X)=0=\operatorname{Hom}(\mathcal{O}_X,W')$ and therefore $\dim\operatorname{Ext}^1(W',\mathcal{O}_X)=(\mathbf{s},v(W'))$. From this, we compute the dimension of the space of extensions as

$$\begin{aligned}\dim M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r-d\mathbf{s})&+\dim\bigl(\operatorname{Gr}(d,\operatorname{Ext}^1(W',\mathcal{O}_X))\bigr)\\&=\mathbf{w}_r^2-2de-2d^2+2+\dim\bigl(\operatorname{Gr}(d,(\mathbf{s},\mathbf{w}_r-d\mathbf{s}))\bigr)\\&=\mathbf{w}_r^2-2de-2d^2+2+d(e+d)=\mathbf{w}_r^2+2-de-d^2<\dim M_{\sigma_+}(\mathbf{w}_r).\end{aligned}$$

Therefore, there is an open subset of $M_{\sigma_+}(\mathbf{w}_r)$ not contained in any of these loci. □

Corollary 6.7. *The set $V^r_d(|H|)$ is a Grassmannian-bundle*[2] *over $M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r)$, and its dimension is*

$$\dim V^r_d(|H|)=\rho(r,d,g)+g.$$

Proof. As we already hinted at above, the first statement follows from Lemma 6.2, Lemma 6.5 (observe that by the long exact cohomology sequence, E as in that Lemma automatically satisfies $h^0(E)=r+1$), and the identification $M_{\sigma_+}(\mathbf{v})=M_H(\mathbf{v})$.

By Lemma 6.6, this bundle is non-empty. As in the previous Lemma, we can use stability with respect to $\overline{\sigma}$ to compute

$$\dim\operatorname{Ext}^1(W,\mathcal{O}_X)=-\chi(W,\mathcal{O}_X)=(\mathbf{w}_r,v(\mathcal{O}_X))=2r+1+g-d$$

for all $W\in M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r)$; the dimension of the Grassmannian-bundle is therefore

$$\begin{aligned}\dim V^r_d(|H|)&=\dim M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r)+\dim\operatorname{Gr}(r+1,2r+1+g-d)\\&=\mathbf{w}_r^2+2+(r+1)(r+g-d)=\rho(r,g,d)+g.\end{aligned}$$

□

Comparison. When the line bundle L is globally generated, then the object W is the shift $M_L[1]$ of the kernel M_L of the evaluation map $\mathcal{O}_X^{h^0(L)}\twoheadrightarrow L$ (which is surjective as a map of sheaves, but injective in our abelian category $\operatorname{Coh}^\beta X$). The Lazarsfeld-Mukai bundle is of course central to Lazarsfeld's approach in [Laz86], and has been studied extensively since. One new ingredient coming from stability conditions is that even without Assumption (*), it is completely automatic that the object W is $\overline{\sigma}$-semistable, see the argument in the proof of Lemma 6.2.

[2]When $M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r)$ is a fine moduli space, i.e. it admits a universal family, then this bundle will be Zariski-locally trivial; in general it will be locally trivial in the étale topology.

The other difference is that wall-crossing immediately gives a global description of $V^r_d(|H|)$, whereas the approach in [Laz86] is based on an infinitesimal analysis. (For us, the only infinitesimal argument will be Lemma 7.2 in the following section.) This is also the reason that we can prove existence of special divisor (i.e., reprove [KL72]) and the bound on the dimension (i.e., reprove [GH80, Laz86]) at the same time.

7. Conclusion

Corollary 6.7 is a family version of the Brill-Noether theorem in the form of Theorem 5.2. To make conclusions about each individual curve, we will use additional input from the restriction of the Beauville integrable system (6). It gives a map $\overline{\pi}\colon V^r_d(|H|) \to \mathbb{P}^g$; and it remains to prove that all its fibres have the same dimension $\rho(r,d,g) = \dim(V^r_d(|H|)) - g$. We will prove this using fairly standard arguments for maps between holomorphic symplectic varieties, as well as one more categorical ingredient.

Consider the following diagram of maps:

$$\begin{tikzcd} V^r_d(|H|) \arrow[r, hook] \arrow[d, "\phi"] \arrow[dr, "\overline{\pi}"] & M_H(\mathbf{v}) \arrow[d, "\pi"] \\ M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r) & \mathbb{P}^g \end{tikzcd} \tag{8}$$

Lemma 7.1. *There is no compact curve $D \subset V^r_d(|H|)$ that is contracted by both $\overline{\pi}$ and by ϕ.*

Proof (sketch). Since the Grassmannian has Picard rank one, all curves contracted by ϕ are proportional (in the group of curves in $M_H(\mathbf{v})$ modulo numerical equivalence) to the line in the Grassmannian given as one of the fibres. If $\overline{\pi}$ were to contract any such curve, it would contract all of them, and so $\overline{\pi}$ would factor via ϕ.

There are various ways to see that this is not possible. For example, using the description of the Néron-Severi group of $M_H(\mathbf{v})$ in Theorem 4.2 one can compute both the class of $L := \pi^*(\mathcal{O}_{\mathbb{P}^g}(1)) \in \mathrm{NS}(M_H(\mathbf{v}))$, and the class of the line l in one of the fibres of ϕ; then one sees easily that $L.l \neq 0$. Alternatively, the moduli space $M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r)$ contains objects of the form $\mathcal{V}[1]$ where $\mathcal{V}$ is a vector bundle whose dual $\mathcal{V}^\vee$ is globally generated; that means that varying the extension subspace in $\mathrm{Ext}^1(\mathcal{V}[1], \mathcal{O}_X) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{V}^\vee)$ will result in varying the support of the line bundle in $V^r_d(|H|)$. □

Let ω denote the symplectic form on $M_H(\mathbf{v})$. Recall that π is a Lagrangian fibration; in particular, the restriction of ω to any fibre $V^r_d(C)$ of $\overline{\pi}$ vanishes. On the other hand:

Lemma 7.2. *The restriction $\omega|_{V^r_d(|H|)}$ is the pull-back $\phi^*\overline{\omega}$ of the symplectic form on $M^{\mathrm{stable}}_{\overline{\sigma}}(\mathbf{w}_r)$.*

Proof. The proof is very similar to arguments in [Muk84].

Consider $L \in V^r_d(|H|)$, and write $\mathcal{O}_X^{r+1} \xrightarrow{\alpha} L \xrightarrow{\beta} W$ for the associated short exact sequence (7). Recall that the tangent space of $M_H(\mathbf{v})$ at L is $\mathrm{Hom}(L, L[1])$. The subspace tangent to $V^r_d(|H|)$ are all $f \in \mathrm{Hom}(L, L[1])$ that provide no obstructions to lifting global sections to the associated extensions. This means $f \circ \alpha = 0$, or, equivalently, $f = g \circ \beta$ for some $g \in \mathrm{Hom}(W, L[1])$. Let $f_W = \beta[1] \circ g \in \mathrm{Hom}(W, W[1])$ denote the associated deformation class of W.

A choice of symplectic form on X makes the Serre duality $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, A[2]) \to \mathbb{C}$ on $\mathrm{D}^b(X)$ canonical and bi-functorial in both arguments. That choice determines the symplectic form on $M_H(\mathbf{v})$ using the Serre duality pairing

$$\mathrm{Hom}(L, L[1]) \times \mathrm{Hom}(L[1], L[2]) \to \mathbb{C}$$

via $\omega(f, f') = \langle f, f'[1]\rangle$; analogously for $M_{\bar{\sigma}}^{\mathrm{stable}}(\mathbf{w}_r)$. Now assume we are given f, g, f_W as above, and f', g', f'_W analogously, see diagram (9) below for illustration. We can compute

$$\begin{aligned}\omega(f, f') &= \langle f, f'[1]\rangle = \langle g \circ \beta, g'[1] \circ \beta[1]\rangle = \langle \beta[1] \circ g \circ \beta, g'[1]\rangle = \langle \beta[1] \circ g, \beta[2] \circ g'[1]\rangle \\ &= \langle f_W, f'_W\rangle = \omega(f_W, f'_W),\end{aligned}$$

which is precisely the claim.

$$\begin{tikzcd} \mathcal{O}_X^{r+1} \arrow[r, "\alpha"] \arrow[d, "0"] & L \arrow[r, "\beta"] \arrow[d, "f"] & W \arrow[d, "f_W"] \arrow[dl, "g"'] \\ \mathcal{O}_X^{r+1}[1] \arrow[r, "{\alpha[1]}"] \arrow[d, "0"] & L[1] \arrow[r, "{\beta[1]}"] \arrow[d, "f'"] & W[1] \arrow[d, "f'_W"] \arrow[dl, "g'"'] \\ \mathcal{O}_X^{r+1}[2] \arrow[r, "{\alpha[2]}"] & L[2] \arrow[r, "{\beta[2]}"] & W[2] \end{tikzcd} \tag{9}$$

$\square$

Proof of Theorem 5.2. By Lemma 7.1, we have

$$\dim V_d^r(C) = \dim \overline{\pi}^{-1}(C) = \dim \phi\left(\overline{\pi}^{-1}(C)\right).$$

On the other hand, we know that $V_d^r(C) \subset M_H(\mathbf{v})$ is isotropic, as it is a subset of the Lagrangian subvariety $\pi^{-1}(C)$; by Lemma 7.2, the same holds true for $\phi\left(V_d^r(C)\right) \subset M_{\bar{\sigma}}^{\mathrm{stable}}(\mathbf{w}_r)$.

Therefore,

$$\dim V_d^r(C) = \dim \phi\left(V_d^r(C)\right) \le \frac{1}{2} \dim M_{\bar{\sigma}}(\mathbf{w}_r) = \rho(r, d, g). \tag{10}$$

Equality, including the non-emptiness of $V_d^r(C)$, follows from classical results [KL72], but it can also be deduced in our context. Recall that

$$\overline{W}_d^r(|H|) = \bigcup_{r' \ge r} V_d^{r'}(|H|)$$

is a closed subvariety of $M_H(\mathbf{v})$, and thus projective. Combining Corollary 6.7 with (10) (for all $r' \ge r$) we see that in the map $\overline{W}_d^r(|H|) \to |H| \cong \mathbb{P}^g$, all fibres have at most expected dimension $\rho(r, g, d) = \dim \overline{W}_d^r(|H|) - \dim \mathbb{P}^g$. Therefore, all fibres have exactly expected dimension. Again applying the inequality (10), this time for all $r' > r$, it follows that we must have equality. $\square$

8. Geometry of the Brill-Noether locus and birational geometry of the moduli space

Our proof of Theorem 1.1 in fact provides a geometric description of the Brill-Noether locus

$$\mathrm{BN}_d(|H|) = \left\{ L \in M_H(\mathbf{v}) : \ h^0(L) > 0 \right\}.$$

We have shown that in the natural stratification

$$\mathrm{BN}_d(|H|) = \bigcup_{r \geq 0} \left\{ L \in M_H(\mathbf{v}) : \ h^0(L) = r+1 \right\} = \bigcup_r V_d^r(|H|), \tag{11}$$

each stratum is a Grassmannian-bundle of $(r+1)$-dimensional subspaces in a $2r+1+g-d$-dimensional space over a holomorphic symplectic variety of dimension $2\rho(r,g,d)$. This recovers a result by Markman [Mar01] and Yoshioka ([Yos99, Lemma 2.4 and Theorem 2.5] and [Yos01a, Theorem 4.17]). One advantage in our description is that we need not distinguish between Gieseker-stable sheaves in $M_H(\mathbf{w}_r)$ that are locally free versus those that are just torsion-free: our discussion in the previous sections shows that instead, $M_{\overline{\sigma}}(\mathbf{w}_r)$ is the right moduli space to consider. One can show that $M_{\sigma_+}(\mathbf{w}_r)$ consists of shifts $W = V^\vee[1]$ of derived duals of Gieseker-stable sheaves V of appropriate class. Such a derived dual can be a locally free sheaf (when V is locally free), or a non-trivial complex W with $H^0(W)$ being a 0-dimensional torsion sheaf. The support of W is simultaneously the locus where the corresponding line bundle in $V_d^r(|H|)$ is not globally generated, and where V is not locally free.

Moduli spaces of Gieseker-stable sheaves come equipped with ample line bundles constructed via GIT. The closest analogue for moduli spaces of Bridgeland-stable objects comes from the following result:

Positivity Lemma 8.1 ([BM14b]). *Let σ be a stability condition on $\mathrm{D}^b(X)$ for a smooth projective*[3] *variety X, and assume we are given a family $\mathcal{E}$ of σ-semistable objects parameterised by a variety S. Then this induces a real nef divisor class*[4] *$l_\sigma \in \mathrm{NS}(S) \otimes \mathbb{R}$ on S. Moreover, for a curve $C \subset S$ we have $l_\sigma.C = 0$ if and only if the objects parameterised by C are* S-equivalent *to each other.*

Any σ-semistable objects has a *Jordan-Hölder filtration*: a filtration whose factors are *σ-stable* of the same slope. Two semistable objects are called S-equivalent if their Jordan-Hölder filtrations have the same stable quotients. In practice, this often means that the Positivity Lemma not only produces nef divisors, but also dually extremal curves describing a boundary facet of the nef cone.

The line bundle can be constructed as follows: we can always normalise the central charge to satisfy $Z(\mathbf{v}) = -1$. Via the Mukai pairing, the imaginary part $\Im Z$ of the central charge can be identified with an element of $\mathbf{v}^\perp \otimes \mathbb{R} \subset H^*_{\mathrm{alg}}(X, \mathbb{R})$. Then $l_\sigma = \theta_v(\Im Z)$, where θ_v is the Mukai isomorphism of Theorem 4.2.

We now apply the Positivity Lemma in our situation. Let us again fix $\mathbf{v} = (0, H, d+1-g)$, and assume for simplicity that the moduli space $M_H(\mathbf{v})$ of torsion sheaves has a universal family. (This assumption is satisfied when H^2 and $d+1-g$ are coprime; otherwise one can descend the line

[3]See [BCZ16] for a generalisation to singular quasi-projective varieties and moduli spaces of objects with compact support.

[4]To be precise, when S is singular we obtain a numerical Cartier divisor class.

bundle constructed in the following from an étale cover of the moduli space that admits a universal family.) We now consider its universal family as a family of σ_0-*semistable* objects. The Positivity Lemma produces a nef line bundle l_{σ_0} on $M_H(\mathbf{v})$. Using Remark 4.3, one can additionally show that its volume is positive, and hence that l_{σ_0} is big. Since M is K-trivial, the base point free theorem says that l_{σ_0} is globally generated, and so it produces a birational contraction

$$\phi_0\colon M_H(\mathbf{v}) \to \overline{M}.$$

To understand the contracted locus, we have to understand S-equivalence for objects in $M_H(\mathbf{v})$ with respect to σ_0. By Lemma 6.2, the Jordan-Hölder filtration of $L \in M_H(\mathbf{v})$ is trivial when $h^0(L) = 0$; otherwise, its filtration quotients are given by $\mathcal{O}_X$ with multiplicity $h^0(L)$, and by the quotient W in the short exact sequence (7). In other words, two objects L, L' are S-equivalent if and only if $h^0(L) = h^0(L')$, i.e. $L, L' \in V_d^r(|H|)$ for some r, and if they are in the same Grassmannian fibre of the map ϕ in (8). In summary, we have proved (see also [Yos01a, Section 4]):

Theorem 8.2. *Assume that (X, H) satisfy assumption (*), that $\mathbf{v} = (0, H, d + 1 - g)$ for some $0 < d \leq g - 1$. Then the moduli space $M_H(\mathbf{v})$ of torsion sheaves admits a birational contraction $\overline{\phi}\colon M_H(\mathbf{v}) \to \overline{M}$, whose exceptional locus is the Brill-Noether locus $\mathrm{BN}_d(|H|)$. The natural stratification of $\mathrm{BN}_d(|H|)$ by the number of global section corresponds to the stratification induced by $\overline{\phi}$: each stratum is a Grassmannian-bundle over its image in $\overline{M}$.*

Note that in the context of this Theorem, Lemma 7.2 becomes a well-known statement, see e.g. [Kal06, Lemma 2.9].

9. BIRATIONAL GEOMETRY OF MODULI SPACES OF SHEAVES: A QUICK SURVEY

Many of the statements we have shown so far can be proved in much bigger generality: given a K3 surface X and a primitive class $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$, once can describe the location of all walls for $\mathbf{v}$ in the entire space of stability conditions, and then in turn use that to completely describe the birational geometry of the moduli space $M_H(\mathbf{v})$ of Gieseker-stable sheaves.

The idea is simple. Let E be an object of Mukai vector $\mathbf{v}$ that is strictly semistable with respect to a stability condition σ on a general point of a given wall. We consider its Jordan-Hölder factors. If $\mathbf{a}_1, \dots, \mathbf{a}_m$ are their Mukai vectors, then

- $\mathbf{v} = \mathbf{a}_1 + \cdots + \mathbf{a}_m$;
- the wall is locally described by the condition that the central charges $Z(\mathbf{a}_i)$ all lie on the same ray;
- by Theorem 4.1 we have $\mathbf{a}_i^2 \geq -2$ for all i; and
- all $\mathbf{a}_i$ are contained in a common rank two sublattice of $H^*_{\mathrm{alg}}(X, \mathbb{Z})$.

One can in fact prove the converse: if all four conditions above are satisfies, then the stability conditions lies on a wall for $\mathbf{v}$. (The main complication comes from "totally semistable walls": there might not exist any object of class $\mathbf{a}_i$ that is *stable* on the wall; in this case, we have to use a different decomposition of $\mathbf{v}$ within the same rank two sublattice.) Further, one can determine when there exist curves of S-equivalent objects, and thus whether the wall induces a birational contraction.

This analysis is the main content of [BM14a]. It leads, for example, to a complete description of the nef cones of all birational models of $M_H(\mathbf{v})$ inside the $\mathrm{NS}\,(M_H(\mathbf{v})) \otimes \mathbb{R}$. To explain that description, we first need to recall a few basic facts about birational geometry and the Beauville-Bogomolov form on irreducible holomorphic symplectic varieties. It is a quadratic form on $\mathrm{NS}(M_H(\mathbf{v}))$ of signature $(1, \rho - 1)$. The cone defined by $(D, D) > 0$ thus has two components; one of them contains the ample cone, and we will call this component the *positive cone*. The volume of a divisor D is, up to a constant factor, given by $(D, D)^n$ where $2n = \dim M_H(\mathbf{v})$, and thus the cone of movable divisors is contained in the positive cone. The cone of movable divisors admits a chamber decomposition whose chambers correspond one-to-one to smooth, K-trivial birational models $g\colon M_H(\mathbf{v}) \dashrightarrow N$ of $M_H(\mathbf{v})$: the chamber is given as $g^*\mathrm{Nef}(N)$; see [HT09].

Theorem 9.1 ([BM14a, Theorem 12.1])**.** *Inside the positive cone of $M_H(\mathbf{v})$, each chamber of the movable cone is cut out by hyperplanes of the form $\theta_{\mathbf{v}}(\mathbf{v}^\perp \cap \mathbf{a}^\perp)$ for all $\mathbf{a} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ satisfying $\mathbf{a}^2 \geq -2$ and $|(\mathbf{v}, \mathbf{a})| \leq \frac{\mathbf{v}^2}{2}$.*

In other words, given the arrangement of hyperplanes of the form $\theta_{\mathbf{v}}(\mathbf{v}^\perp \cap \mathbf{a}^\perp)$ for all $\mathbf{a}$ as above, each such chamber is a connected component of the complement. Combined with a similar description of the movable cone (which is due to Markman [Mar11], but can also be reproved with the methods discussed here) this leads to a complete list of all birational of $M_H(\mathbf{v})$; the only necessary ingredient is the Picard lattice of X.

In any given example, one can also attempt to study the birational geometry of the contraction in order to obtain a result analogous to Theorem 8.2; this has been done systematically up to dimension 10 in [HT15] (along with other applications).

For an analogue of Theorem 9.1 for the singular O'Grady spaces of dimension 10, see [MZ14].

Deformations. Using either twistor deformations [Mon13] or deformation theory of rational curves in families of irreducible holomorphic symplectic manifolds (IHSM) [BHT15] one can deform Theorem 9.1 to an analogue for all IHSM deformation-equivalent to Hilbert schemes on K3 surfaces; this concludes a programme started in [HT01]. Thus, indirectly, the methods discussed here lead to a description of the birational geometry of varieties that (currently) have no interpretation as a moduli space.

Other surfaces. In the case of abelian surfaces, or K3 surfaces of Picard rank one, the Positivity Lemma was first proved in [MYY11a, MYY11b] using Fourier-Mukai transforms. Yoshioka then deduced in [Yos12] a description of nef cones of (Kummer varieties associated to) moduli spaces of sheaves on abelian surfaces, obtaining a result completely analogous to Theorem 9.1.

Extending this result to other surfaces is, to some extent[5], much more difficult. Even for Gieseker-stable sheaves, it is in general unknown for which Chern classes there exist Gieseker-stable sheaves, i.e. there is no analogue of Theorem 4.1. Even when it exists, as in the case

[5]The main difficulty specific to K3 surfaces is essentially due to the large group of autoequivalences of $\mathrm{D}^b(X)$: they produce many walls where *every* objects in a given moduli space becomes strictly semistable. The easiest example is the analogue of our situation for $d \geq g$: the wall corresponding to Lemma 6.2 now destabilises all torsion sheaves. The wall-crossing still induces a birational transformation of the moduli spaces, but on the common open subset each stable object gets replaced via its image under the auto-equivalence given by the spherical twist at $\mathcal{O}_X$.

of $\mathbb{P}^2$, the answer [DLP85] is quite intricate. Moreover, the answer changes as we move from Gieseker-stability to Bridgeland stability conditions, making the wall-crossing analysis much more of a moving target.

For an Enriques surfaces S, one can circumvent some of these difficulties by using the pull-back map π^* where $\pi\colon X \to S$ is the associated 2:1-covering by a K3 surfaces; this induces a finite map between corresponding moduli spaces, and can be used to show that the nef divisors produced by the Positivity Lemma are actually ample. The results are especially powerful for *unnodal* Enriques surfaces (i.e., not containing a smooth rational curve); see [Nue14].

Projective plane. The entire story had in fact started with the case of $\mathbb{P}^2$: in [ABCH13], the authors observed the correspondence between walls for stability conditions and birational transformations of the Hilbert scheme of n points on $\mathbb{P}^2$ in many examples—for example, including all walls for all $n \leq 9$; they conjectured the correspondence between stable base loci and destabilised objects in general. This paper was the original motivation behind all the developments discussed here, and in particular directly motivated the Positivity Lemma above.

The correspondence of [ABCH13] was generalised to all Gieseker-moduli spaces and proved in [BMW13]; a different argument in [CH14b] treated the case of torus fixed points in the Hilbert scheme. It was upgraded to a birational correspondence (by proving that all Bridgeland moduli spaces appearing in the wall-crossing for the Hilbert scheme are irreducible) in [LZ13]. The authors also extended their results to commutative deformations of $\mathrm{Hilb}^n(\mathbb{P}^2)$ using stability conditions on the derived category of non-commutative deformations of $\mathbb{P}^2$.

From this correspondence, one can deduce a description of the nef cone of Gieseker-moduli spaces, see [CC15, Woo13] for torsion sheaves, and [CH14a] for small rank or large discriminant: again the idea is to apply the Positivity Lemma at a wall, producing a nef divisor *and*, dually, a contracted extremal curve of S-equivalent objects.

But due to the difficulties hinted at above, it took additional effort to understand the entire picture, including the nef cones of birational models. One needed to understand for which wall a moduli space of stable objects of given Chern character becomes empty. This turns out to be closely related to another classical problem:

Heuristic 9.2. *For any class $\mathbf{v} \in H^*(\mathbb{P}^2)$, determining the "last wall", i.e. the wall after which $M_\sigma(\mathbf{v})$ becomes empty, is equivalent to determining the boundary of the effective cone of $M_H(\mathbf{v})$.*

The reasoning behind this heuristic goes as follows. Consider the nef divisor $l_{\overline{\sigma}}$ given by the Positivity Lemma for $\overline{\sigma}$ lying on this "last wall"; in particular, this means every object becomes strictly semi-stable with respect $\overline{\sigma}$. Then one can *expect* every point in the moduli space to lie on a curve of objects that are S-equivalent with respect to $\overline{\sigma}$; in other words, $l_{\overline{\sigma}}$ is dual to a moving curve in the Mori cone. This implies that $l_{\overline{\sigma}}$ is on the boundary of the effective cone.[6]

[6]This is a heuristic argument only for two reasons: even if every object is strictly semistable, some or all of them could be the unique non-trivial extensions in their S-equivalence class. Moreover when all objects become strictly semistable, that does not a priori preclude the existence of new stable objects on the other side of the wall; in that case, the wall corresponds to the boundary of the effective cone, but is not the "last wall".

The problem of determining the effective cone was solved in [Hui13, Hui12] for the Hilbert scheme, in [CC15, Woo13] for one-dimensional torsion sheaves, and in [CHW14] for all Gieseker-moduli spaces; see [CH15a] for a survey of the results and the arguments, and the relation to the interpolation problem. The recent preprint [LZ16] then made the above heuristic reasoning precise, and used it to give a complete description of the decomposition of the movable cone into chambers corresponding to nef cones of birational models. I would like to explain one more consequence of their results:

Proposition 9.3 ([LZ13, Theorem 0.1], [LZ16, Corollary 0.3], building on essentially all the other results mentioned in this section)**.** *Let* $\mathbf{v} \in H^*(\mathbb{P}^2)$ *be a primitive class, let* $M(\mathbf{v})$ *be the moduli space of Gieseker-stable sheaves of Chern character* $\mathbf{v}$*, and let* $M \dashrightarrow M(\mathbf{v})$ *be a birational model corresponding to an open chamber in the movable cone of* $M(\mathbf{v})$*. Then* M *is smooth.*

To explain the argument, let us briefly recall why $M(\mathbf{v})$ is smooth. For $F \in M(\mathbf{v})$, we have to show $\mathrm{Ext}^2(F, F) = 0$; by Serre duality, $\mathrm{Ext}^2(F, F) = \mathrm{Hom}(F, F(-3))^\vee$; since $F, F(-3)$ are both slope-semistable with $\mu(F) > \mu(F(-3))$, the claim follows.

To generalise this to birational models of $M(\mathbf{v})$, we first use their interpretation as moduli spaces. As indicated previously, we know that $M \cong M_\sigma(\mathbf{v})$ where $\sigma = \sigma_{\alpha,\beta}$ lies in an open chamber of the space of stability conditions. As above, for $E \in M_\sigma(\mathbf{v})$, we have $\mathrm{Ext}^2(E, E) = \mathrm{Hom}(E, E(-3))^\vee$. However, Bridgeland stability is not invariant under $\_ \otimes \mathcal{O}(-3)$; instead, all we know a priori is that $E(-3)$ is $\sigma_{\alpha,\beta-3}$-stable. The key argument of [LZ13, LZ16] now shows that as we follow the natural path from $\sigma_{\alpha,\beta-3}$ to $\sigma_{\alpha,\beta}$, we can control the phases of the semistable factors appearing in the Harder-Narasimhan filtration of $E(-3)$, and conclude that they all have smaller phase than that of E; then the Hom-vanishing thus follows again from stability.

General surfaces. Similar results for the the Hilbert scheme on other rational surfaces were obtained in [BC13], for example including nef cones of all Hilbert schemes points on Hirzebruch surfaces. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, the effective cone of many moduli spaces of sheaves have been determined in [Rya16], and in all cases where c_1 is symmetric in [Abe16].

Two recent articles show that one can make at least some of the arguments simultaneously for all surfaces. For example, one of the main results of [BHL$^+$15] shows that for a surface of Picard rank one and $n \gg 0$, one can determine the nef cone of $\mathrm{Hilb}^n(X)$. The assumption of $n \gg 0$ is needed to ensure that an effective curve C of minimal degree has non-empty $W^1_n(C)$; the associated map $C \to \mathbb{P}^1$ produces the curve of S-equivalent objects dual to the nef divisor class coming from the Positivity Lemma. Similarly, in [CH15b] the authors show that if one fixes the rank r and the first Chern character c, then for $s \ll 0$ one can determine the nef cone of the moduli space of Gieseker-stable sheaves on X of Chern character (r, c, s) if one knows the set of Chern classes of semistable bundles on X. (In other words, the assumption $s \ll 0$ circumvents the problem of knowing when moduli spaces σ-stable objects become empty.)

Other applications. We list a few more relations between stability conditions and classical questions that have appeared in the literature, and may lead to more applications in the future:

- The contraction from the Gieseker-moduli space to the Uhlenbeck space of slope-semistable vector bundles can be induced by wall-crossing [LQ11, Lo12] (i.e., there is a wall for which the associated line bundle induces this contraction).

- Similarly, the Thaddeus-flips constructed in [MW97] relating Gieseker-moduli spaces for different polarisations are induced by a sequence of walls [Yos14, BM15].
- Flips of secant varieties can be shown to arise naturally in the wall-crossing for moduli spaces of torsion sheaves on $\mathbb{P}^2$ [Mar13].
- One can induce the minimal model programme of a surface X via wall-crossing in $\mathrm{D}^b(X)$ [Tod12]; yet the moduli space becomes reducible if one tries to contract other curves of self-intersection less than -2 [Tra15].
- There is a a close relation between the location of the wall where a given ideal sheaf in $\mathrm{Hilb}^n(\mathbb{P}^2)$ gets destabilised and its Castelnuovo-Mumford regularity [CHP16].

Some recent developments have already lead to new results.

In [AM14], the authors combine stability conditions with Fourier-Mukai techniques to determine precisely which line bundles on an abelian surface of Picard rank one are k*-very ample*.

Finally, returning to a topic closely related to the main content of this survey, consider a globally generated line bundle $L \in V_d^r(|H|)$, and its Mukai-Lazarsfeld bundle M_L (where $M_L \cong W[-1]$ with W as given in Lemma 6.2). In [Fey16], the author uses stability conditions in order to prove ordinary slope-stability of the restriction of M_L to any curve in $|H|$. This leads to many new counter-examples to Mercat's conjecture, which was a proposed bound for the analogue of the Clifford index for slope-stable vector bundles on curves in terms of the Clifford index for line bundles.

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