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Algebraic and combinatorial codimension 1 transversality

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Abstract The Waldhausen construction of Mayer-Vietoris splittings for chain complexes over an injective generalized free product is extended to Seifert-van Kampen splittings for CW complexes with fundamental group an injective generalized free product.

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Dedicated to Andrew Casson

Introduction

The close relationship between the topological properties of codimension 1 submanifolds and the algebraic properties of groups with a generalized free product structure first became apparent with the Seifert-van Kampen Theorem on the fundamental group of a union, the work of Kneser on 3-dimensional manifolds with fundamental group a free product, and the topological proof of Grushko's theorem by Stallings.

This paper describes two abstractions of the geometric codimension 1 transversality properties of manifolds (in all dimensions) :

(1) the algebraic transversality construction of Mayer-Vietoris splittings of chain complexes of free modules over the group ring of an injective generalized free product,

(2) the combinatorial transversality construction of Seifert-van Kampen splittings for CW complexes with fundamental group an injective generalized free product.

By definition, a group G is a *generalized free product* if it has one of the following structures :

(A) $G = G_1 *_H G_2$ is the amalgamated free product determined by group morphisms $i_1 : H \to G_1$, $i_2 : H \to G_2$, so that there is defined a pushout square of groups

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"] & G \end{tikzcd}$$

The amalgamated free product is *injective* if i_1, i_2 are injective, in which case so are j_1, j_2, with

$$G_1 \cap G_2 \;=\; H \subseteq G \;.$$

The amalgamated free product is *finitely presented* if the groups G_1, G_2, H are finitely presented, in which case so is G. (If G is finitely presented, it does not follow that G_1, G_2, H need be finitely presented).

(B) $G = G_1 *_H \{t\}$ is the HNN extension determined by group morphisms $i_1, i_2 : H \to G_1$

$$\begin{tikzcd} H \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & G_1 \arrow[r, "j_1"] & G \end{tikzcd}$$

with $t \in G$ such that

$$j_1 i_1(h) t \;=\; t j_1 i_2(h) \in G \quad (h \in H) \;.$$

The HNN extension is *injective* if i_1, i_2 are injective, in which case so is j_1, with

$$G_1 \cap t G_1 t^{-1} \;=\; i_1(H) \;=\; t i_2(H) t^{-1} \subseteq G \;.$$

The HNN extension is *finitely presented* if the groups G_1, H are finitely presented, in which case so is G. (If G is finitely presented, it does not follow that G_1, H need be finitely presented).

A subgroup $H \subseteq G$ is *2-sided* if G is either an injective amalgamated free product $G = G_1 *_H G_2$ or an injective HNN extension $G = G_1 *_H \{t\}$. (See Stallings [13] and Hausmann [5] for the characterization of 2-sided subgroups in terms of bipolar structures.) If G is an injective generalized free product $\begin{cases} \text{with } i_1, i_2 \text{ not isomorphisms} \\ -- \end{cases}$ then G is an infinite group and the subgroups $\begin{cases} G_1, G_2, H \\ G_1 \end{cases}$ are of infinite index in $G = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}$.

A *CW* pair $(X, Y \subset X)$ is *2-sided* if Y has an open neighbourhood $Y \times \mathbb{R} \subset X$. The pair is *connected* if X and Y are connected. By the Seifert-van Kampen Theorem $\pi_1(X)$ is a generalized free product :

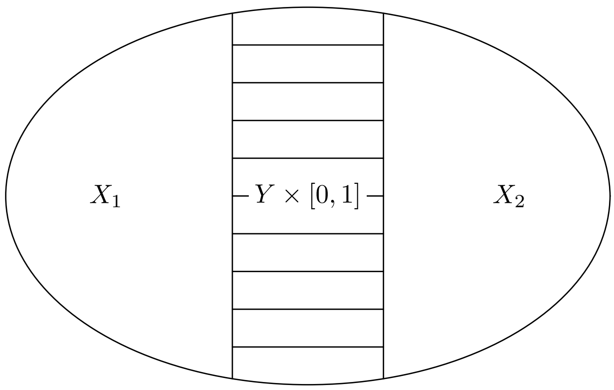
(A) if Y separates X then $X - Y$ has two components, and

$$X = X_1 \cup_Y X_2$$

for connected $X_1, X_2 \subset X$ with

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

the amalgamated free product determined by the morphisms $i_1 : \pi_1(Y) \to \pi_1(X_1)$, $i_2 : \pi_1(Y) \to \pi_1(X_2)$ induced by the inclusions $i_1 : Y \to X_1$, $i_2 : Y \to X_2$.



(B) if Y does not separate X then $X - Y$ is connected and

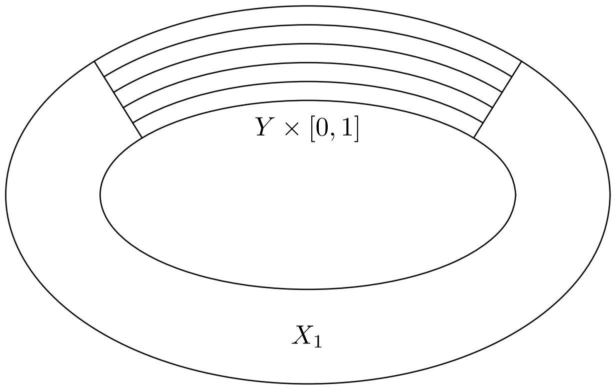
$$X = X_1 \cup_{Y \times \{0,1\}} Y \times [0,1]$$

for connected $X_1 \subset X$, with

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \{t\}$$

the *HNN* extension determined by the morphisms $i_1, i_2 : \pi_1(Y) \to$

$\pi_1(X_1)$ induced by the inclusions $i_1, i_2 : Y \to X_1$.



The generalized free product is injective if and only if the morphism $\pi_1(Y) \to \pi_1(X)$ is injective, in which case $\pi_1(Y)$ is a 2-sided subgroup of $\pi_1(X)$.

A codimension 1 submanifold $N^{n-1} \subset M^n$ is 2-sided if the normal bundle is trivial, in which case (M, N) is a 2-sided CW pair.

For a 2-sided CW pair (X, Y) every map $f : M \to X$ from an n-dimensional manifold M is homotopic to a map (also denoted by f) which is transverse at $Y \subset X$, with

$$N^{n-1} \;=\; f^{-1}(Y) \subset M^n$$

a 2-sided codimension 1 submanifold, by the Sard-Thom theorem.

By definition, a *Seifert-van Kampen splitting* of a connected CW complex W with $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ an injective generalized free product is a connected 2-sided CW pair (X, Y) with a homotopy equivalence $X \to W$ such that

$$\mathrm{im}(\pi_1(Y) \to \pi_1(X)) \;=\; H \subseteq \pi_1(X) \;=\; \pi_1(W) \;=\; G\ .$$

The splitting is *injective* if $\pi_1(Y) \to \pi_1(X)$ is injective, in which case

$$X \;=\; \begin{cases} X_1 \cup_Y X_2 \\ X_1 \cup_{Y \times \{0,1\}} Y \times [0,1] \end{cases}$$

with

$$\begin{cases} \pi_1(X_1) \;=\; G_1\ ,\ \pi_1(X_2) \;=\; G_2 \\ \pi_1(X_1) \;=\; G_1 \end{cases} ,\ \pi_1(Y) \;=\; H\ .$$

The splitting is *finite* if the complexes W, X, Y are finite, and *infinite* otherwise.

A connected CW complex W with $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ an injective generalized free product is a homotopy pushout

$$\begin{cases} \begin{tikzcd} \widetilde{W}/H \arrow[r, "i_1"] \arrow[d, "i_2"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/G_2 \arrow[r, "j_2"] & W \end{tikzcd} \\ \begin{tikzcd} \widetilde{W}/H \times \{0,1\} \arrow[r, "i_1 \cup i_2"] \arrow[d] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/H \times [0,1] \arrow[r] & W \end{tikzcd} \end{cases}$$

with $\widetilde{W}$ the universal cover of W and $\begin{cases} i_1, i_2, j_1, j_2 \\ i_1, i_2, j_1 \end{cases}$ the covering projections. (See Proposition $\begin{cases} 2.6 \\ 2.14 \end{cases}$ for proofs). Thus W has a canonical infinite injective Seifert-van Kampen splitting $(X(\infty), Y(\infty))$ with

$$\begin{cases} Y(\infty) \ = \ \widetilde{W}/H \times \{1/2\} \subset X(\infty) \ = \ \widetilde{W}/G_1 \cup_{i_1} \widetilde{W}/H \times [0,1] \cup_{i_2} \widetilde{W}/G_2 \\ Y(\infty) \ = \ \widetilde{W}/H \times \{1/2\} \subset X(\infty) \ = \ \widetilde{W}/G_1 \cup_{i_1 \cup i_2} \widetilde{W}/H \times [0,1] \ . \end{cases}$$

For finite W it is easy to obtain finite injective Seifert-van Kampen splittings by codimension 1 manifold transversality. In fact, there are two ways of doing so :

(i) Consider a regular neighbourhood $(M, \partial M)$ of $W \subset S^N$ (N large), apply codimension 1 manifold transversality to a map

$$\begin{cases} f \ : \ M \to BG \ = \ BG_1 \cup_{BH \times \{0\}} BH \times [0,1] \cup_{BH \times \{1\}} BG_2 \\ f \ : \ M \to BG \ = \ BG_1 \cup_{BH \times \{0,1\}} BH \times [0,1] \end{cases}$$

inducing the identification $\pi_1(M) = G$ to obtain a finite Seifert-van Kampen splitting $(M, f^{-1}(BH \times \{1/2\}))$, and then make the splitting injective by low-dimensional handle exchanges.

(ii) Assume inductively that the n-skeleton $W^{(n)}$ already has a Seifert-van Kampen splitting (X, Y). For each $(n+1)$-cell $D^{n+1} \subset W^{(n+1)}$ make the attaching map $S^n \to W^{(n)} \simeq X$ transverse at $Y \subset X$, and make the composite $f : D^{n+1} \to W^{(n+1)} \to BG$ transverse at $BH \subset BG$. The transversality gives D^{n+1} a CW structure in which $f^{-1}(BH) \subseteq D^{n+1}$ is a subcomplex, and

$$(X',Y') \;=\; \Big(X \cup \bigcup_{D^{n+1}\subset W^{(n+1)}} D^{n+1}\,,\, Y \cup \bigcup_{D^{n+1}\subset W^{(n+1)}} f^{-1}(BH)\Big)$$

is an extension of the Seifert-van Kampen splitting to the $(n+1)$-skeleton $W^{(n+1)}$. Again, the finite splitting can be made injective by low-dimensional handle exchanges.

However, the geometric nature of manifold transversality does not give any insight into the CW structures of the splittings (X,Y) of W, let alone into the algebraic analogue of transversality for $\mathbb{Z}[G]$-module chain complexes. Here, we obtain Seifert-van Kampen splittings combinatorially, in the following converse of the Seifert-van Kampen Theorem.

Combinatorial Transversality Theorem *Let W be a finite connected CW complex with $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ an injective generalized free product.*

(i) *The canonical infinite Seifert-van Kampen splitting $(X(\infty),Y(\infty))$ of W is a union of finite Seifert-van Kampen splittings $(X,Y) \subset (X(\infty),Y(\infty))$*

$$(X(\infty),Y(\infty)) \;=\; \bigcup (X,Y)\ .$$

(ii) *If $\pi_1(W)$ is a finitely presented generalized free product then for any finite Seifert-van Kampen splitting (X,Y) of W it is possible to attach a finite number of 2- and 3-cells to obtain a finite injective Seifert-van Kampen splitting (X',Y') such that the inclusion $X \to X'$ is a homotopy equivalence and the inclusion $Y \to Y'$ is a $\mathbb{Z}[H]$-coefficient homology equivalence.* $\square$

The Theorem is proved in §2. The main ingredient of the proof is the construction of a finite Seifert-van Kampen splitting of W from a finite *domain* of the universal cover $\widetilde{W}$, as given by finite subcomplexes $\begin{cases} W_1, W_2 \subseteq \widetilde{W} \\ W_1 \subseteq \widetilde{W} \end{cases}$ such that $\begin{cases} G_1W_1 \cup G_2W_2 = \widetilde{W} \\ G_1W_1 = \widetilde{W}\ . \end{cases}$

Algebraic transversality makes much use of the induction and restriction functors associated to a ring morphism $i : A \to B$

$$\begin{aligned} i_! &: \{A\text{-modules}\} \to \{B\text{-modules}\}\ ;\ M \mapsto i_!M \;=\; B \otimes_A M\ , \\ i^! &: \{B\text{-modules}\} \to \{A\text{-modules}\}\ ;\ N \mapsto i^!N \;=\; N\ . \end{aligned}$$

Let $G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ be a generalized free product. By definition, a *Mayer-Vietoris splitting* (or *presentation*) $\mathcal{E}$ of a $\mathbb{Z}[G]$-module chain complex C is :

(A) an exact sequence of $\mathbb{Z}[G]$-module chain complexes

$$\mathcal{E} \ : \ 0 \to k_!D \xrightarrow{\begin{pmatrix} 1\otimes e_1 \\ 1\otimes e_2 \end{pmatrix}} (j_1)_!C_1 \oplus (j_2)_!C_2 \to C \to 0$$

with C_1 a $\mathbb{Z}[G_1]$-module chain complex, C_2 a $\mathbb{Z}[G_2]$-module chain complex, D a $\mathbb{Z}[H]$-module chain complex, $e_1 : (i_1)_!D \to C_1$ a $\mathbb{Z}[G_1]$-module chain map and $e_2 : (i_2)_!D \to C_2$ a $\mathbb{Z}[G_2]$-module chain map,

(B) an exact sequence of $\mathbb{Z}[G]$-module chain complexes

$$\mathcal{E} \ : \ 0 \to (j_1i_1)_!D \xrightarrow{1\otimes e_1 - t\otimes e_2} (j_1)_!C_1 \to C \to 0$$

with C_1 a $\mathbb{Z}[G_1]$-module chain complex, D a $\mathbb{Z}[H]$-module chain complex, and $e_1 : (i_1)_!D \to C_1$, $e_2 : (i_2)_!D \to C_1$ $\mathbb{Z}[G_1]$-module chain maps.

A Mayer-Vietoris splitting $\mathcal{E}$ is *finite* if every chain complex in $\mathcal{E}$ is finite f.g. free, and *infinite* otherwise.

Let X be a connected CW complex with fundamental group $\pi_1(X) = G$ and universal covering projection $p : \widetilde{X} \to X$, with G acting on the left of $\widetilde{X}$. Let $C(\widetilde{X})$ be the cellular free (left) $\mathbb{Z}[G]$-module chain complex. For any subgroup $H \subseteq G$ the covering $Z = \widetilde{X}/H$ of X has universal cover $\widetilde{Z} = \widetilde{X}$ with cellular $\mathbb{Z}[H]$-module chain complex

$$C(\widetilde{Z}) \ = \ k^!C(\widetilde{X})$$

with $k : \mathbb{Z}[H] \to \mathbb{Z}[G]$ the inclusion. If $H = \pi_1(Y)$ for a connected subcomplex $Y \subseteq X$ then $p^{-1}(Y) \subseteq \widetilde{X}$ is a disjoint union of copies of the universal cover $\widetilde{Y}$ of Y

$$p^{-1}(Y) \ = \ \bigcup_{g\in[G;H]} g\widetilde{Y} \subset \widetilde{X}$$

with $[G;H]$ the set of right H-cosets $g = xH \subseteq G$ $(x \in G)$. The cellular $\mathbb{Z}[G]$-module chain complex of $p^{-1}(Y)$ is induced from the cellular $\mathbb{Z}[H]$-module chain complex of $\widetilde{Y}$

$$C(p^{-1}(Y)) \ = \ k_!C(\widetilde{Y}) \ = \ \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C(\widetilde{Y}) \subseteq C(\widetilde{X}) \ .$$

Furthermore, if (X, Y) is a connected 2-sided CW pair then $C(\widetilde{X})$ has a Mayer-Vietoris splitting with respect to the injective generalized free product structure on $\pi_1(X)$:

(A) By the Mayer-Vietoris Theorem applied to

$$\begin{aligned}\widetilde{X} &= p^{-1}(X_1) \cup_{p^{-1}(Y)} p^{-1}(X_2) \\ &= \bigcup_{g_1 \in [G;G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h\in[G;H]} h\widetilde{Y} \bigcup_{g_2\in[G;G_2]} g_2\widetilde{X}_2\end{aligned}$$

$C(\widetilde{X})$ has a Mayer-Vietoris splitting

$$0 \to k_!C(\widetilde{Y}) \xrightarrow{\begin{pmatrix}1\otimes e_1 \\ 1\otimes e_2\end{pmatrix}} (j_1)_!C(\widetilde{X}_1) \oplus (j_2)_!C(\widetilde{X}_2) \xrightarrow{(f_1 \ \ -f_2)} C(\widetilde{X}) \to 0$$

with $\widetilde{X}_1, \widetilde{X}_2$ the universal covers of X_1, X_2 and $e_1 : Y \to X_1$, $e_2 : Y \to X_2$, $f_1 : X_1 \to X$, $f_2 : X_2 \to X$ the inclusions.

(B) By the Mayer-Vietoris Theorem applied to

$$\begin{aligned}\widetilde{X} &= p^{-1}(X_1) \cup_{p^{-1}(Y\times\{0,1\})} p^{-1}(Y\times[0,1]) \\ &= \bigcup_{g_1\in[G;G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h\in[G;H]} (h\widetilde{Y},0)\cup(th\widetilde{Y},1) \bigcup_{h\in[G;H]} h\widetilde{Y}\times[0,1]\end{aligned}$$

(with $H = i_1(H) \subseteq G$) there is defined a short exact sequence of $\mathbb{Z}[G]$-module chain complexes

$$0 \to k_!C(\widetilde{Y}) \oplus k_!C(\widetilde{Y}) \xrightarrow{\begin{pmatrix}1\otimes e_1 & t\otimes e_2 \\ 1 & 1\end{pmatrix}} (j_1)_!C(\widetilde{X}_1) \oplus k_!C(\widetilde{Y}) \xrightarrow{(f_1 \ \ -f_1(1\otimes e_1))} C(\widetilde{X}) \to 0$$

with $\widetilde{X}_1$ the universal cover of X_1, and $e_1, e_2 : Y \to X_1$, $f_1 : X_1 \to X$ the inclusions, so that $C(\widetilde{X})$ has a Mayer-Vietoris splitting

$$\mathcal{E} \ : \ 0 \to k_!C(\widetilde{Y}) \xrightarrow{1\otimes e_1 - t\otimes e_2} (j_1)_!C(\widetilde{X}_1) \xrightarrow{f_1} C(\widetilde{X}) \to 0 \ .$$

If (X,Y) is a finite CW pair the above Mayer-Vietoris splittings are finite.

For any injective generalized free product $G = \begin{cases} G_1 *_H G_2 \\ G *_H \{t\} \end{cases}$ every free $\mathbb{Z}[G]$-module chain complex C has a canonical infinite Mayer-Vietoris splitting

(A) $\mathcal{E}(\infty) \ : \ 0 \to k_!k^!C \to (j_1)_!j_1^!C \oplus (j_2)_!j_2^!C \to C \to 0$

(B) $\mathcal{E}(\infty) \ : \ 0 \to k_!k^!C \to (j_1)_!j_1^!C \to C \to 0 \ .$

For finite C we shall obtain finite Mayer-Vietoris splittings in the following converse of the Mayer-Vietoris Theorem.

Algebraic Transversality Theorem *Let* $G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ *be an injective generalized free product. For a finite f.g. free* $\mathbb{Z}[G]$*-module chain complex* C *the canonical infinite Mayer-Vietoris splitting* $\mathcal{E}(\infty)$ *of* C *is a union of finite Mayer-Vietoris splittings* $\mathcal{E} \subset \mathcal{E}(\infty)$

$$\mathcal{E}(\infty) \;=\; \bigcup \mathcal{E} \ . \qquad \square$$

The existence of finite Mayer-Vietoris splittings was first proved by Waldhausen [14], [15]. The proof of the Theorem in §1 is a simplification of the original argument, using chain complex analogues of the CW domains.

Suppose now that (X,Y) is the finite 2-sided CW pair defined by a (compact) connected n-dimensional manifold X^n together with a connected codimension 1 submanifold $Y^{n-1} \subset X$ with trivial normal bundle. By definition, a homotopy equivalence $f : M^n \to X$ from an n-dimensional manifold *splits* at $Y \subset X$ if f is homotopic to a map (also denoted by f) which is transverse at Y, such that the restriction $f| : N^{n-1} = f^{-1}(Y) \to Y$ is also a homotopy equivalence. In general, homotopy equivalences do not split. For (X,Y) with injective $\pi_1(Y) \to \pi_1(X)$ there are algebraic K- and L-theory obstructions to splitting, involving the Nil-groups of Waldhausen [14], [15] and the UNil-groups of Cappell [2], and for $n \geqslant 6$ these are the complete obstructions to splitting. As outlined in Ranicki [8, §8], algebraic transversality for chain complexes is an essential ingredient for a systematic treatment of both the algebraic K- and L-theory obstructions. Although the algebraic K- and L-theory of generalized free products will not actually be considered here, it is worth noting that the early results of Higman [6], Bass, Heller and Swan [1] and Stallings [12] on the Whitehead groups of polynomial extensions and free products were followed by the work of the dedicatee on the Whitehead group of amalgamated free products (Casson [4]) prior to the general results of Waldhausen [14], [15] on the algebraic K-theory of generalized free products.

1 Algebraic transversality

We shall deal separately with the amalgamated free and HNN cases.

1.1 Algebraic transversality for amalgamated free products

Given an injective amalgamated free product $G = G_1 *_H G_2$ with $G \neq \{1\}$ let T be the infinite tree defined by

$$T^{(0)} = [G;G_1] \cup [G;G_2] \ , \ T^{(1)} = [G;H]$$

(Serre [11], Waldhausen [15]). The edge $h \in [G;H]$ joins the unique vertices $g_1 \in [G;G_1]$, $g_2 \in [G;G_2]$ with

$$g_1 \cap g_2 = h \subset G \ .$$

The group G acts on T by

$$G \times T \to T \ ; \ (g,x) \mapsto gx$$

with $T/G = [0,1]$, $G_i \subseteq G$ the isotropy subgroup of $G_i \in T^{(0)}$ and $H \subseteq G$ the isotropy subgroup of $H \in T^{(1)}$. Conversely, if a group G acts on a tree T with $T/G = [0,1]$ then $G = G_1 *_H G_2$ is an injective amalgamated free product with tree T.

As before write the injections as

$$\begin{aligned} & i_1 : H \to G_1 \ , \ i_2 : H \to G_2 \ , \ j_1 : G_1 \to G \ , \ j_2 : G_2 \to G \ , \\ & k = j_1 i_1 = j_2 i_2 : H \to G \ . \end{aligned}$$

Definition 1.1 (i) A *domain* (C_1, C_2) of a $\mathbb{Z}[G]$-module chain complex C is a pair of subcomplexes $(C_1 \subseteq j_1^! C, C_2 \subseteq j_2^! C)$ such that the chain maps

$$\begin{aligned} e_1 &: (i_1)_!(C_1 \cap C_2) \to C_1 \ ; \ b_1 \otimes y_1 \mapsto b_1 y_1 \ , \\ e_2 &: (i_2)_!(C_1 \cap C_2) \to C_2 \ ; \ b_2 \otimes y_2 \mapsto b_2 y_2 \ , \\ f_1 &: (j_1)_! C_1 \to C \ ; \ a_1 \otimes x_1 \mapsto a_1 x_1 \ , \\ f_2 &: (j_1)_! C_2 \to C \ ; \ a_2 \otimes x_2 \mapsto a_2 x_2 \end{aligned}$$

fit into a Mayer-Vietoris splitting of C

$$\mathcal{E}(C_1, C_2) : 0 \to k_!(C_1 \cap C_2) \xrightarrow{e} (j_1)_! C_1 \oplus (j_2)_! C_2 \xrightarrow{f} C \to 0$$

with $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, $f = (f_1 \ \ -f_2)$.

(ii) A domain (C_1, C_2) is *finite* if C_i $(i = 1,2)$ is a finite f.g. free $\mathbb{Z}[G_i]$-module chain complex, $C_1 \cap C_2$ is a finite f.g. free $\mathbb{Z}[H]$-module chain complex, and *infinite* otherwise. □

Proposition 1.2 *Every free $\mathbb{Z}[G]$-module chain complex C has a canonical infinite domain $(C_1,C_2)=(j_1^!C,j_2^!C)$ with*

$$C_1\cap C_2 \;=\; k^!C\ ,$$

so that C has a canonical infinite Mayer-Vietoris splitting

$$\mathcal{E}(\infty)\;=\;\mathcal{E}(j_1^!C,j_2^!C)\;:\;0\to k_!k^!C\to (j_1)_!j_1^!C\oplus(j_2)_!j_2^!C\to C\to 0\ .$$

Proof It is enough to consider the special case $C=\mathbb{Z}[G]$, concentrated in degree 0. The pair

$$(C_1,C_2)\;=\;(j_1^!\mathbb{Z}[G],j_2^!\mathbb{Z}[G])\;=\;(\bigoplus_{[G;G_1]}\mathbb{Z}[G_1],\bigoplus_{[G;G_2]}\mathbb{Z}[G_2])$$

is a canonical infinite domain for C, with

$$\mathcal{E}(\infty)\;=\;\mathcal{E}(C_1,C_2)\;:\;0\to k_!k^!\mathbb{Z}[G]\to (j_1)_!j_1^!\mathbb{Z}[G]\oplus(j_2)_!j_2^!\mathbb{Z}[G]\to\mathbb{Z}[G]\to 0$$

the chain complex of $C(T\times G)=C(T)\otimes_{\mathbb{Z}}\mathbb{Z}[G]$, along with its augmentation to $H_0(T\times G)=\mathbb{Z}[G]$. $\square$

Definition 1.3 (i) For a based f.g. free $\mathbb{Z}[G]$-module $B=\mathbb{Z}[G]^b$ and a subtree $U\subseteq T$ define a domain for B (regarded as a chain complex concentrated in degree 0)

$$(B(U)_1,B(U)_2)\;=\;(\sum_{U_1^{(0)}}\mathbb{Z}[G_1]^b,\sum_{U_2^{(0)}}\mathbb{Z}[G_2]^b)$$

with

$$\begin{aligned}U_1^{(0)}\;&=\;U^{(0)}\cap[G;G_1]\ ,\ U_2^{(0)}\;=\;U^{(0)}\cap[G;G_2]\ ,\\ B(U)_1\cap B(U)_2\;&=\;\sum_{U^{(1)}}\mathbb{Z}[H]^b\ .\end{aligned}$$

The associated Mayer-Vietoris splitting of B is the subobject $\mathcal{E}(U)\subseteq\mathcal{E}(\infty)$ with

$$\mathcal{E}(U):0\to k_!\sum_{U^{(1)}}\mathbb{Z}[H]^b\to(j_1)_!\sum_{U_1^{(0)}}\mathbb{Z}[G_1]^b\oplus(j_2)_!\sum_{U_2^{(0)}}\mathbb{Z}[G_2]^b\to B\to 0$$

the chain complex of $C(U\times G)^b=C(U)\otimes_{\mathbb{Z}}B$, along with its augmentation to $H_0(U\times G)^b=B$. If $U\subset T$ is finite then $(B(U)_1,B(U)_2)$ is a finite domain.

(ii) Let C be an n-dimensional based f.g. free $\mathbb{Z}[G]$-module chain complex, with $C_r=\mathbb{Z}[G]^{c_r}$. A sequence $U=\{U_n,U_{n-1},\dots,U_1,U_0\}$ of subtrees $U_r\subseteq T$ is *realized* by C if the differentials $d:C_r\to C_{r-1}$ are such that

$$d(C_r(U_r)_i)\subseteq C_{r-1}(U_{r-1})_i\quad(1\leqslant r\leqslant n,\ i=1,2)\ ,$$

so that there is defined a Mayer-Vietoris splitting of C

$$\mathcal{E}(U)\ :\ 0\to k_!\sum_{U^{(1)}}C(U)_1\cap C(U)_2\to (j_1)_!\sum_{U_1^{(0)}}C(U)_1\oplus (j_2)_!\sum_{U_2^{(0)}}C(U)_2\to C\to 0$$

with $C(U)_i$ the free $\mathbb{Z}[G_i]$-module chain complex defined by

$$(C(U)_i)_r\ =\ C_r(U_r)_i\quad (i=1,2)\ .$$

The sequence U is *finite* if each subtree $U_r\subseteq T$ is finite, in which case $\mathcal{E}(U)$ is finite. □

Proposition 1.4 *For a finite based f.g. free $\mathbb{Z}[G]$-module chain complex C the canonical infinite domain is a union of finite domains*

$$(j_1^!C,j_2^!C)\ =\ \bigcup_U (C(U)_1,C(U)_2)\ ,$$

with U running over all the finite sequences which are realized by C. The canonical infinite Mayer-Vietoris splitting of C is thus a union of finite Mayer-Vietoris splittings

$$\mathcal{E}(\infty)\ =\ \bigcup_U \mathcal{E}(U)\ .$$

Proof The proof is based on the following observations :

(a) for any subtrees $V\subseteq U\subseteq T$

$$\mathcal{E}(V)\subseteq\mathcal{E}(U)\subseteq\mathcal{E}(T)\ =\ \mathcal{E}(\infty)$$

(b) the infinite tree T is a union

$$T\ =\ \bigcup U$$

of the finite subtrees $U\subset T$,

(c) for any finite subtrees $U,U'\subset T$ there exists a finite subtree $U''\subset T$ such that $U\subseteq U''$ and $U'\subseteq U''$,

(d) for any finite subtree $U\subset T$ and any element $d\in\mathbb{Z}[G]$ there exists a finite subtree $U'\subset T$ such that the $\mathbb{Z}[G]$-module morphism $d:\mathbb{Z}[G]\to\mathbb{Z}[G]$ is resolved by a morphism $d:\mathcal{E}(U)\to\mathcal{E}(U')$ of finite Mayer-Vietoris splittings.

Assume C is n-dimensional, with $C_r=\mathbb{Z}[G]^{c_r}$. Starting with any finite subtree $U_n\subseteq T$ let

$$U\ =\ \{U_n,U_{n-1},\dots,U_1,U_0\}$$

be a sequence of finite subtrees $U_r \subset T$ such that the f.g. free submodules

$$\begin{aligned} C_r(U)_1 &= \sum_{U^{(0)}_{r,1}} \mathbb{Z}[G_1]^{c_r} \subset j_1^! C_r = \sum_{T^{(0)}_1} \mathbb{Z}[G_1]^{c_r} , \\ C_r(U)_2 &= \sum_{U^{(0)}_{r,2}} \mathbb{Z}[G_2]^{c_r} \subset j_2^! C_r = \sum_{T^{(0)}_2} \mathbb{Z}[G_2]^{c_r} , \\ D(U)_r &= \sum_{U^{(1)}_r} \mathbb{Z}[H]^{c_r} \subset k^! C_r = \sum_{T^{(1)}} \mathbb{Z}[H]^{c_r} \end{aligned}$$

define subcomplexes $C(U)_1 \subset j_1^! C$, $C(U)_2 \subset j_2^! C$, $D(U) \subset k^! C$. Then $(C(U)_1, C(U)_2)$ is a domain of C with $C(U)_1 \cap C(U)_2 = D(U)$, and U is realized by C. □

Remark 1.5 (i) The existence of finite Mayer-Vietoris splittings was first proved by Waldhausen [14],[15], using essentially the same method. See Quinn [7] for a proof using controlled algebra. The construction of generalized free products by noncommutative localization (cf. Ranicki [10]) can be used to provide a different proof.

(ii) The construction of the finite Mayer-Vietoris splittings $\mathcal{E}(U)$ in 1.4 as subobjects of the universal Mayer-Vietoris splitting $\mathcal{E}(T) = \mathcal{E}(\infty)$ is taken from Remark 8.7 of Ranicki [8]. □

This completes the proof of the Algebraic Transversality Theorem for amalgamated free products.

1.2 Algebraic transversality for HNN extensions

The proof of algebraic transversality for HNN extensions proceeds exactly as for amalgamated free products, so only the statements will be given.

Given an injective HNN extension $G = G_1 *_H \{t\}$ with $G \neq \{1\}$ let T be the infinite tree defined by

$$T^{(0)} = [G; G_1] , \; T^{(1)} = [G; H] ,$$

identifying $H = i_1(H) \subseteq G$. The edge $h \in [G; H]$ joins the unique vertices $g_1, g_2 \in [G; G_1]$ with

$$g_1 \cap g_2 t^{-1} = h \subset G .$$

The group G acts on T by

$$G \times T \to T \; ; \; (g, x) \mapsto gx$$

with $T/G = S^1$, $G_1 \subseteq G$ the isotropy subgroup of $G_1 \in T^{(0)}$ and $H \subseteq G$ the isotropy subgroup of $H \in T^{(1)}$. Conversely, if a group G acts on a tree T with $T/G = S^1$ then $G = G_1 *_H \{t\}$ is an injective HNN extension with tree T. As before, write the injections as

$$i_1 \ , \ i_2 \ : \ H \to G_1 \ , \ j \ : \ G_1 \to G \ , \ k \ = \ j_1 i_1 \ = \ j_1 i_2 \ : \ G_1 \to G \ .$$

Definition 1.6 (i) A *domain* C_1 of a $\mathbb{Z}[G]$-module chain complex C is a subcomplex $C_1 \subseteq j_1^! C$ such that the chain maps

$$\begin{aligned} e_1 \ &: \ (i_1)_!(C_1 \cap tC_1) \to C_1 \ ; \ b_1 \otimes y_1 \mapsto b_1 y_1 \ , \\ e_2 \ &: \ (i_2)_!(C_1 \cap tC_1) \to C_1 \ ; \ b_2 \otimes y_2 \mapsto b_2 t^{-1} y_2 \ , \\ f \ &: \ (j_1)_! C_1 \to C \ ; \ a \otimes x \mapsto ax \end{aligned}$$

fit into a Mayer-Vietoris splitting of C

$$\mathcal{E}(C_1) \ : \ 0 \to k_!(C_1 \cap tC_1) \xrightarrow{1 \otimes e_1 - t \otimes e_2} (j_1)_! C_1 \xrightarrow{f} C \to 0 \ .$$

(ii) A domain C_1 is *finite* if C_1 is a finite f.g. free $\mathbb{Z}[G_1]$-module chain complex and $C_1 \cap tC_1$ is a finite f.g. free $\mathbb{Z}[H]$-module chain complex. □

Proposition 1.7 *Every free $\mathbb{Z}[G]$-module chain complex C has a canonical infinite domain $C_1 = j_1^! C$ with*

$$C_1 \cap tC_1 \ = \ k^! C_1 \ ,$$

so that C has a canonical infinite Mayer-Vietoris splitting

$$\mathcal{E}(\infty) \ = \ \mathcal{E}(j_1^! C) \ : \ 0 \to k_! k^! C \to (j_1)_! j_1^! C \to C \to 0 \ .$$

□

Definition 1.8 For any subtree $U \subseteq T$ define a domain for $\mathbb{Z}[G]$

$$C(U)_1 \ = \ \sum_{U^{(0)}} \mathbb{Z}[G_1]$$

with

$$C(U)_1 \cap tC(U)_1 \ = \ \sum_{U^{(1)}} \mathbb{Z}[H] \ .$$

The associated Mayer-Vietoris splitting of $\mathbb{Z}[G]$ is the subobject $\mathcal{E}(U) \subseteq \mathcal{E}(\infty)$ with

$$\mathcal{E}(U) : 0 \to k_! \sum_{U^{(1)}} \mathbb{Z}[H] \to (j_1)_! \sum_{U^{(0)}} \mathbb{Z}[G_1] \to \mathbb{Z}[G] \to 0 \ .$$

If $U \subset T$ is finite then $C(U)_1$ is finite. □

Proposition 1.9 *For a finite f.g. free $\mathbb{Z}[G]$-module chain complex C the canonical infinite domain is a union of finite domains*

$$j_1^! C \;=\; \bigcup C_1 \;.$$

The canonical infinite Mayer-Vietoris splitting of C is thus a union of finite Mayer-Vietoris splittings

$$\mathcal{E}(\infty) \;=\; \bigcup \mathcal{E}(C_1) \;.$$

$\square$

This completes the proof of the Algebraic Transversality Theorem for HNN extensions.

2 Combinatorial transversality

The Combinatorial Transversality Theorem stated in the Introduction will now be proved, treating the cases of an amalgamated free product and an HNN extension separately.

2.1 Mapping cylinders

We review some basic mapping cylinder constructions.

The *mapping cylinder* of a map $e : V \to W$ is the identification space

$$\mathcal{M}(e) \;=\; (V \times [0,1] \cup W)/\{(x,0) \sim e(x) \,|\, x \in V\}$$

such that $V = V \times \{1\} \subset \mathcal{M}(e)$, and the projection

$$p \;:\; \mathcal{M}(e) \to W \;;\; \begin{cases} (x,s) \mapsto e(x) & \text{for } x \in V,\ s \in [0,1] \\ y \mapsto x & \text{for } y \in W \end{cases}$$

is a homotopy equivalence.

If e is a cellular map of CW complexes then $\mathcal{M}(e)$ is a CW complex, such that cellular chain complex $C(\mathcal{M}(e))$ is the algebraic mapping cylinder of $e : C(V) \to C(W)$ with

$$\begin{aligned} d_{C(\mathcal{M}(e))} \;&=\; \begin{pmatrix} d_{C(W)} & (-1)^r e & 0 \\ 0 & d_{C(V)} & 0 \\ 0 & (-1)^{r-1} & d_{C(V)} \end{pmatrix} \;: \\ C(\mathcal{M}(e))_r \;&=\; C(W)_r \oplus C(V)_{r-1} \oplus C(V)_r \\ &\qquad \to C(\mathcal{M}(e))_{r-1} \;=\; C(W)_{r-1} \oplus C(V)_{r-2} \oplus C(V)_{r-1} \;. \end{aligned}$$

The chain equivalence $p: C(\mathcal{M}(e)) \to C(W)$ is given by

$$p \;=\; (1\ 0\ e) \;:\; C(\mathcal{M}(e))_r \;=\; C(W)_r \oplus C(V)_{r-1} \oplus C(V)_r \to C(W)_r \ .$$

The *double mapping cylinder* $\mathcal{M}(e_1, e_2)$ of maps $e_1 : V \to W_1,\ e_2 : V \to W_2$ is the identification space

$$\begin{aligned}\mathcal{M}(e_1,e_2) &= \mathcal{M}(e_1) \cup_V \mathcal{M}(e_2)\\ &= W_1 \cup_{e_1} V \times [0,1] \cup_{e_2} W_2\\ &= (W_1 \cup V \times [0,1] \cup W_2)/\{(x,0) \sim e_1(x), (x,1) \sim e_2(x) \,|\, x \in V\}\ .\end{aligned}$$

Given a commutative square of spaces and maps

$$\begin{tikzcd} V \arrow[r, "e_1"] \arrow[d, "e_2"'] & W_1 \arrow[d, "f_1"] \\ W_2 \arrow[r, "f_2"] & W \end{tikzcd}$$

define the map

$$f_1 \cup f_2 \;:\; \mathcal{M}(e_1,e_2) \to W \ ;\ \begin{cases} (x,s) \mapsto f_1e_1(x) \;=\; f_2e_2(x) \quad (x \in V,\ s \in [0,1]) \\ y_i \mapsto f_i(y_i) \quad (y_i \in W_i\ ,\ i = 1,2)\ . \end{cases}$$

The square is a *homotopy pushout* if $f_1 \cup f_2 : \mathcal{M}(e_1, e_2) \to W$ is a homotopy equivalence.

If $e_1 : V \to W_1,\ e_2 : V \to W_2$ are cellular maps of CW complexes then $\mathcal{M}(e_1, e_2)$ is a CW complex, such that cellular chain complex $C(\mathcal{M}(e_1, e_2))$ is the algebraic mapping cone of the chain map

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \;:\; C(V) \to C(W_1) \oplus C(W_2)$$

with

$$\begin{aligned} d_{C(\mathcal{M}(e_1,e_2))} \;=\; &\begin{pmatrix} d_{C(W_1)} & (-1)^r e_1 & 0 \\ 0 & d_{C(V)} & 0 \\ 0 & (-1)^r e_2 & d_{C(W_2)} \end{pmatrix} \;: \\ C(\mathcal{M}(e_1,e_2))_r \;=\; &C(W_1)_r \oplus C(V)_{r-1} \oplus C(W_2)_r \\ &\to C(\mathcal{M}(e_1,e_2))_{r-1} \;=\; C(W_1)_{r-1} \oplus C(V)_{r-2} \oplus C(W_2)_{r-1}\ .\end{aligned}$$

2.2 Combinatorial transversality for amalgamated free products

In this section W is a connected CW complex with fundamental group an injective amalgamated free product

$$\pi_1(W) \;=\; G \;=\; G_1 *_H G_2$$

with tree T. Let $\widetilde{W}$ be the universal cover of W, and let

$$\begin{tikzcd} \widetilde{W}/H \arrow[r, "i_1"] \arrow[d, "i_2"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/G_2 \arrow[r, "j_2"] & W \end{tikzcd}$$

be the commutative square of covering projections.

Definition 2.1 (i) Suppose given subcomplexes $W_1, W_2 \subseteq \widetilde{W}$ such that

$$G_1W_1 \;=\; W_1 \;,\; G_2W_2 \;=\; W_2$$

so that

$$H(W_1 \cap W_2) \;=\; W_1 \cap W_2 \subseteq \widetilde{W} \;.$$

Define a commutative square of CW complexes and cellular maps

$$\begin{tikzcd} (W_1 \cap W_2)/H \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\ W_2/G_2 \arrow[r, "f_2"] & W \end{tikzcd}$$

with

$$(W_1 \cap W_2)/H \subseteq \widetilde{W}/H \;,\; W_1/G_1 \subseteq \widetilde{W}/G_1 \;,\; W_2/G_2 \subseteq \widetilde{W}/G_2 \;,$$
$$e_1 \;=\; i_1| \;:\; (W_1 \cap W_2)/H \to W_1/G_1 \;,\; e_2 \;=\; i_2| \;:\; (W_1 \cap W_2)/H \to W_2/G_2 \;,$$
$$f_1 \;=\; j_1| \;:\; W_1/G_1 \to W \;,\; f_2 \;=\; j_2| \;:\; W_2/G_2 \to W \;.$$

(ii) A *domain* (W_1, W_2) for the universal cover $\widetilde{W}$ of W consists of connected subcomplexes $W_1, W_2 \subseteq \widetilde{W}$ such that $W_1 \cap W_2$ is connected, and such that for each cell $D \subseteq \widetilde{W}$ the subgraph $U(D) \subseteq T$ defined by

$$\begin{aligned} U(D)^{(0)} \;&=\; \{g_1 \in [G;G_1] \,|\, g_1D \subseteq W_1\} \cup \{g_2 \in [G;G_2] \,|\, g_2D \subseteq W_2\} \\ U(D)^{(1)} \;&=\; \{h \in [G;H] \,|\, hD \subseteq W_1 \cap W_2\} \end{aligned}$$

is a tree.

(iii) A domain (W_1, W_2) for $\widetilde{W}$ is *fundamental* if the subtrees $U(D) \subseteq T$ are either single vertices or single edges, so that

$$\begin{aligned} g_1W_1 \cap g_2W_2 &= \begin{cases} h(W_1 \cap W_2) & \text{if } g_1 \cap g_2 = h \in [G;H] \\ \emptyset & \text{if } g_1 \cap g_2 = \emptyset \ , \end{cases} \\ W &= (W_1/G_1) \cup_{(W_1 \cap W_2)/H} (W_2/G_2) \ . \end{aligned}$$

$\square$

Proposition 2.2 *For a domain (W_1, W_2) of $\widetilde{W}$ the pair of cellular chain complexes $(C(W_1), C(W_2))$ is a domain of the cellular chain complex $C(\widetilde{W})$.*

Proof The union of $GW_1, GW_2 \subseteq \widetilde{W}$ is

$$GW_1 \cup GW_2 = \widetilde{W}$$

since for any cell $D \subseteq \widetilde{W}$ there either exists $g_1 \in [G;G_1]$ such that $g_1D \subseteq W_1$ or $g_2 \in [G;G_2]$ such that $g_2D \subseteq W_2$. The intersection of $GW_1, GW_2 \subseteq \widetilde{W}$ is

$$GW_1 \cap GW_2 = G(W_1 \cap W_2) \subseteq \widetilde{W} \ .$$

The Mayer-Vietoris exact sequence of cellular $\mathbb{Z}[G]$-module chain complexes

$$0 \to C(GW_1 \cap GW_2) \to C(GW_1) \oplus C(GW_2) \to C(\widetilde{W}) \to 0$$

is the Mayer-Vietoris splitting of $C(\widetilde{W})$ associated to $(C(W_1), C(W_2))$

$$0 \to k_!C(W_1 \cap W_2) \to (j_1)_!C(W_1) \oplus (j_2)_!C(W_2) \to C(\widetilde{W}) \to 0$$

with $C(W_1 \cap W_2) = C(W_1) \cap C(W_2)$. $\square$

Example 2.3 W has a canonical infinite domain $(W_1, W_2) = (\widetilde{W}, \widetilde{W})$ with $(W_1 \cap W_2)/H = \widetilde{W}/H$, and $U(D) = T$ for each cell $D \subseteq \widetilde{W}$. $\square$

Example 2.4 (i) Suppose that $W = X_1 \cup_Y X_2$, with $X_1, X_2, Y \subseteq W$ connected subcomplexes such that the isomorphism

$$\pi_1(W) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \xrightarrow{\cong} G = G_1 *_H G_2$$

preserves the amalgamated free structures. Thus (X, Y) is a Seifert-van Kampen splitting of W, and the morphisms

$$\pi_1(X_1) \to G_1 \ , \ \pi_1(X_2) \to G_2 \ , \ \pi_1(Y) \to H$$

are surjective. (If $\pi_1(Y) \to \pi_1(X_1)$ and $\pi_1(Y) \to \pi_1(X_2)$ are injective these morphisms are isomorphisms, and the splitting is injective). The universal cover of W is

$$\widetilde{W} \;=\; \bigcup_{g_1 \in [G;G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h \in [G;H]} h\widetilde{Y} \bigcup_{g_2 \in [G;G_2]} g_2\widetilde{X}_2$$

with $\widetilde{X}_i$ the regular cover of X_i corresponding to $\ker(\pi_1(X_i) \to G_i)$ $(i = 1,2)$ and $\widetilde{Y}$ the regular cover of Y corresponding to $\ker(\pi_1(Y) \to H)$ (which are the universal covers of X_1, X_2, Y in the case $\pi_1(X_1) = G_1$, $\pi_1(X_2) = G_2$, $\pi_1(Y) = H$). The pair

$$(W_1, W_2) \;=\; (\widetilde{X}_1, \widetilde{X}_2)$$

is a fundamental domain of $\widetilde{W}$ such that

$$\begin{aligned} &(W_1 \cap W_2)/H \;=\; Y \;, \\ &g_1W_1 \cap g_2W_2 \;=\; (g_1 \cap g_2)\widetilde{Y} \subseteq \widetilde{W} \quad (g_1 \in [G;G_1], g_2 \in [G;G_2]) \;. \end{aligned}$$

For any cell $D \subseteq \widetilde{W}$

$$U(D) \;=\; \begin{cases} \{g_1\} & \text{if } g_1D \subseteq \widetilde{X}_1 - \bigcup\limits_{h_1 \in [G_1;H]} h_1\widetilde{Y} \text{ for some } g_1 \in [G;G_1] \\ \{g_2\} & \text{if } g_2D \subseteq \widetilde{X}_2 - \bigcup\limits_{h_2 \in [G_2;H]} h_2\widetilde{Y} \text{ for some } g_2 \in [G;G_1] \\ \{g_1, g_2, h\} & \text{if } hD \subseteq \widetilde{Y} \text{ for some } h = g_1 \cap g_2 \in [G;H]. \end{cases}$$

(ii) If (W_1, W_2) is a fundamental domain for any connected CW complex W with $\pi_1(W) = G = G_1 *_H G_2$ then $W = X_1 \cup_Y X_2$ as in (i), with

$$X_1 \;=\; W_1/G_1 \;,\; X_2 \;=\; W_2/G_2 \;,\; Y \;=\; (W_1 \cap W_2)/H \;.$$

□

Definition 2.5 Suppose that W is n-dimensional. Lift each cell $D^r \subseteq W$ to a cell $\widetilde{D}^r \subseteq \widetilde{W}$. A sequence $U = \{U_n, U_{n-1}, \ldots, U_1, U_0\}$ of subtrees $U_r \subseteq T$ is *realized* by W if the subspaces

$$W(U)_1 \;=\; \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{g_1 \in U^{(0)}_{r,1}} g_1\widetilde{D}^r \;,\; W(U)_2 \;=\; \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{g_2 \in U^{(0)}_{r,2}} g_2\widetilde{D}^r \subseteq \widetilde{W}$$

are connected subcomplexes, in which case $(W(U)_1, W(U)_2)$ is a domain for $\widetilde{W}$ with

$$W(U)_1 \cap W(U)_2 \;=\; \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{h \in U^{(1)}_r} h\widetilde{D}^r \subseteq \widetilde{W}$$

a connected subcomplex. Thus U is realized by $C(\widetilde{W})$ and

$$(C(W(U)_1), C(W(U)_2)) \ = \ (C(\widetilde{W})(U)_1, C(\widetilde{W})(U)_2) \subseteq (C(\widetilde{W}), C(\widetilde{W}))$$

is the domain for $C(\widetilde{W})$ given by $(C_r(\widetilde{W})_1(U_r), C_r(\widetilde{W})(U)_2)$ in degree r. $\square$

If a sequence $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$ realized by W is finite (i.e. if each $U_r \subseteq T$ is a finite subtree) then $(W(U)_1, W(U)_2)$ is a finite domain for $\widetilde{W}$.

Proposition 2.6 (i) *For any domain (W_1, W_2) there is defined a homotopy pushout*

$$\begin{tikzcd} (W_1 \cap W_2)/H \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\ W_2/G_2 \arrow[r, "f_2"] & W \end{tikzcd}$$

with $e_1 = i_1|$, $e_2 = i_2|$, $f_1 = j_1|$, $f_2 = j_2|$. The connected 2-sided CW pair

$$(X, Y) \ = \ (\mathcal{M}(e_1, e_2), (W_1 \cap W_2)/H \times \{1/2\})$$

is a Seifert-van Kampen splitting of W, with a homotopy equivalence

$$f \ = \ f_1 \cup f_2 \ : \ X \ = \ \mathcal{M}(e_1, e_2) \xrightarrow{\ \simeq\ } W \ .$$

(ii) *The commutative square of covering projections*

$$\begin{tikzcd} \widetilde{W}/H \arrow[r, "i_1"] \arrow[d, "i_2"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/G_2 \arrow[r, "j_2"] & W \end{tikzcd}$$

is a homotopy pushout. The connected 2-sided CW pair

$$(X(\infty), Y(\infty)) \ = \ (\mathcal{M}(i_1, i_2), \widetilde{W}/H \times \{1/2\})$$

is a canonical injective infinite Seifert-van Kampen splitting of W, with a homotopy equivalence $j = j_1 \cup j_2 : X(\infty) \to W$ such that

$$\pi_1(Y(\infty)) \ = \ H \subseteq \pi_1(X(\infty)) \ = \ G_1 *_H G_2 \ .$$

(iii) *For any (finite) sequence $U = \{U_n, U_{n-1}, \dots, U_0\}$ of subtrees of T realized by W there is defined a homotopy pushout*

$$\begin{tikzcd} Y(U) \arrow[r, "e_1"] \arrow[d, "e_2"'] & X(U)_1 \arrow[d, "f_1"] \\ X(U)_2 \arrow[r, "f_2"] & W \end{tikzcd}$$

with

$$\begin{aligned} X(U)_1 &= W(U)_1/G_1 \ , \ X(U)_2 \ = \ W(U)_2/G_2 \ , \\ Y(U) &= (W(U)_1 \cap W(U)_2)/H \ , \\ e_1 &= i_1| \ , \ e_2 \ = \ i_2| \ , \ f_1 \ = \ j_1| \ , \ f_2 \ = \ j_2| \ . \end{aligned}$$

Thus

$$(X(U),Y(U)) \ = \ (\mathcal{M}(e_1,e_2),Y(U)\times\{1/2\})$$

is a (finite) Seifert-van Kampen splitting of W*.*

(iv) *The canonical infinite domain of a finite* CW *complex* W *with* $\pi_1(W) = G_1 *_H G_2$ *is a union of finite domains*

$$(\widetilde{W},\widetilde{W}) \ = \ \bigcup_U (W(U)_1,W(U)_2)$$

with U *running over all the finite sequences realized by* W*. The canonical infinite Seifert-van Kampen splitting of* W *is thus a union of finite Seifert-van Kampen splittings*

$$(X(\infty),Y(\infty)) \ = \ \bigcup_U (X(U),Y(U)) \ .$$

Proof (i) Given a cell $D\subseteq W$ let $\widetilde{D}\subseteq\widetilde{W}$ be a lift. The inverse image of the interior $\mathrm{int}(D)\subseteq W$

$$f^{-1}(\mathrm{int}(D)) \ = \ U(\widetilde{D})\times\mathrm{int}(D)\subseteq\mathcal{M}(i_1,i_2) \ = \ T\times_G\widetilde{W}$$

is contractible. In particular, point inverses are contractible, so that $f:X\to W$ is a homotopy equivalence. (Here is a more direct proof that $f:X\to W$ is a $\mathbb{Z}[G]$-coefficient homology equivalence. The Mayer-Vietoris Theorem applied to the union $\widetilde{W}=GW_1\cup GW_2$ expresses $C(\widetilde{W})$ as the cokernel of the $\mathbb{Z}[G]$-module chain map

$$e \ = \ \begin{pmatrix} 1\otimes e_1 \\ 1\otimes e_2\end{pmatrix} \ : \ \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(W_1\cap W_2)\to\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(W_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(W_2)$$

with a Mayer-Vietoris splitting

$$\begin{aligned} 0\to\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(W_1\cap W_2) \xrightarrow{\ e\ } \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(W_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(W_2) \\ \longrightarrow C(\widetilde{W})\to 0 \ . \end{aligned}$$

The decomposition $X=\mathcal{M}(e_1,e_2)=X_1\cup_Y X_2$ with

$$X_i \ = \ \mathcal{M}(e_i) \ (i=1,2) \ , \ Y \ = \ X_1\cap X_2 \ = \ (W_1\cap W_2)/H\times\{1/2\}$$

lifts to a decomposition of the universal cover as

$$\widetilde{X} \;=\; \bigcup_{g_1\in[G;G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h\in[G;H]} h\widetilde{Y} \bigcup_{g_2\in[G;G_2]} g_2\widetilde{X}_2 \ .$$

The Mayer-Vietoris splitting

$$0 \to \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(\widetilde{Y}) \longrightarrow \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(\widetilde{X}_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(\widetilde{X}_2) \to C(\widetilde{X}) \to 0\ ,$$

expresses $C(\widetilde{X})$ as the algebraic mapping cone of the chain map e

$$C(\widetilde{X}) \;=\; \mathcal{C}(e:\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(W_1\cap W_2) \to \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(W_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(W_2))\ .$$

Since e is injective the $\mathbb{Z}[G]$-module chain map

$$\widetilde{f} \;=\; \text{projection} \;:\; C(\widetilde{X}) \;=\; \mathcal{C}(e) \to C(\widetilde{W}) \;=\; \text{coker}(e)$$

induces isomorphisms in homology.)

(ii) Apply (i) to $(W_1,W_2)=(\widetilde{W},\widetilde{W})$.

(iii) Apply (i) to the domain $(W(U)_1,W(U)_2)$.

(iv) Assume that W is n-dimensional. Proceed as for the chain complex case in the proof of Proposition 1.4 for the existence of a domain for $C(\widetilde{W})$, but use only the sequences $U=\{U_n,U_{n-1},\dots,U_0\}$ of finite subtrees $U_r\subset T$ realized by W. An arbitrary finite subtree $U_n\subset T$ extends to a finite sequence U realized by W since for $r\geqslant 2$ each r-cell $\widetilde{D}^r\subset\widetilde{W}$ is attached to an $(r-1)$-dimensional finite connected subcomplex, and every 1-cell $\widetilde{D}^1\subset\widetilde{W}$ is contained in a 1-dimensional finite connected subcomplex. Thus finite sequences U realized by W exist, and can be chosen to contain arbitrary finite collections of cells of $\widetilde{W}$, with

$$(\widetilde{W},\widetilde{W}) \;=\; \bigcup_U (W(U)_1,W(U)_2)\ .$$

$\square$

This completes the proof of part (i) of the Combinatorial Transversality Theorem, the existence of finite Seifert-van Kampen splittings. Part (ii) deals with existence of finite injective Seifert-van Kampen splittings: the adjustment of fundamental groups needed to replace $(X(U),Y(U))$ by a homology-equivalent finite injective Seifert-van Kampen splitting will use the following rudimentary version of the Quillen plus-construction.

Lemma 2.7 *Let K be a connected CW complex with a finitely generated fundamental group $\pi_1(K)$. For any surjection $\phi:\pi_1(K)\to\Pi$ onto a finitely* *presented group* Π *it is possible to attach a finite number* n *of 2- and 3-cells to* K *to obtain a connected CW complex*

$$K' \;=\; K \cup \bigcup_n D^2 \cup \bigcup_n D^3$$

such that the inclusion $K \to K'$ *is a* $\mathbb{Z}[\Pi]$*-coefficient homology equivalence inducing* $\phi : \pi_1(K) \to \pi_1(K') = \Pi$.

Proof The kernel of $\phi : \pi_1(K) \to \Pi$ is the normal closure of a finitely generated subgroup $N \subseteq \pi_1(K)$ by Lemma I.4 of Cappell [3]. (Here is the proof. Choose finite generating sets

$$g \;=\; \{g_1, g_2, \dots, g_r\} \subseteq \pi_1(K) \;,\; h \;=\; \{h_1, h_2, \dots, h_s\} \subseteq \Pi$$

and let $w_k(h_1, h_2, \dots, h_s)$ $(1 \leqslant k \leqslant t)$ be words in h which are relations for Π. As ϕ is surjective, can choose $h'_j \in \pi_1(K)$ with $\phi(h'_j) = h_j$ $(1 \leqslant j \leqslant s)$. As h generates Π $\phi(g_i) = v_i(h_1, h_2, \dots, h_s)$ $(1 \leqslant i \leqslant r)$ for some words v_i in h. The kernel of ϕ is the normal closure $N = \langle N' \rangle \triangleleft \pi_1(K)$ of the subgroup $N' \subseteq \pi_1(K)$ generated by the finite set $\{v_i(h'_1, \dots, h'_s)g_i^{-1}, w_k(h'_1, \dots, h'_s)\}$.) Let $x = \{x_1, x_2, \dots, x_n\} \subseteq \pi_1(K)$ be a finite set of generators of N, and set

$$L \;=\; K \cup_x \bigcup_{i=1}^n D^2 \;.$$

The inclusion $K \to L$ induces

$$\phi \;:\; \pi_1(K) \to \pi_1(L) \;=\; \pi_1(K)/\langle x_1, x_2, \dots, x_n\rangle \;=\; \pi_1(K)/\langle N\rangle \;=\; \Pi \;.$$

Let $\widetilde{L}$ be the universal cover of L, and let $\widetilde{K}$ be the pullback cover of K. Now

$$\pi_1(\widetilde{K}) \;=\; \ker(\phi) \;=\; \langle x_1, x_2, \dots, x_n\rangle \;=\; \langle N\rangle$$

so that the attaching maps $x_i : S^1 \to K$ of the 2-cells in $L - K$ lift to null-homotopic maps $\widetilde{x}_i : S^1 \to \widetilde{K}$. The cellular chain complexes of $\widetilde{K}$ and $\widetilde{L}$ are related by

$$C(\widetilde{L}) \;=\; C(\widetilde{K}) \oplus \bigoplus_n (\mathbb{Z}[\Pi], 2)$$

where $(\mathbb{Z}[\Pi], 2)$ is just $\mathbb{Z}[\Pi]$ concentrated in degree 2. Define

$$x^* \;=\; \{x_1^*, x_2^*, \dots, x_n^*\} \subseteq \pi_2(L)$$

by

$$x_i^* \;=\; (0, (0, \dots, 0, 1, 0, \dots, 0)) \in \pi_2(L) \;=\; H_2(\widetilde{L}) \;=\; H_2(\widetilde{K}) \oplus \mathbb{Z}[\Pi]^n \; (1 \leqslant i \leqslant n) \,,$$

and set

$$K' \; = \; L \cup_{x^*} \bigcup_{i=1}^{n} D^3 \; .$$

The inclusion $K \to K'$ induces $\phi : \pi_1(K) \to \pi_1(K') = \pi_1(L) = \Pi$, and the relative cellular $\mathbb{Z}[\Pi]$-module chain complex is

$$C(\widetilde{K'}, \widetilde{K}) \; : \; \dots \to 0 \to \mathbb{Z}[\Pi]^n \xrightarrow{\;1\;} \mathbb{Z}[\Pi]^n \to 0 \to \dots$$

concentrated in degrees 2,3. In particular, $K \to K'$ is a $\mathbb{Z}[\Pi]$-coefficient homology equivalence. $\square$

Proposition 2.8 *Let (X,Y) be a finite connected 2-sided CW pair with $X = X_1 \cup_Y X_2$ for connected X_1, X_2, Y, together with an isomorphism*

$$\pi_1(X) \; = \; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \xrightarrow{\;\cong\;} G \; = \; G_1 *_H G_2$$

preserving amalgamated free product structures, with the structure on G injective. It is possible to attach a finite number of 2- and 3-cells to (X,Y) to obtain a finite injective Seifert-van Kampen splitting (X',Y') with $X' = X'_1 \cup_{Y'} X'_2$ such that

(i) $\pi_1(X') = G$, $\pi_1(X'_i) = G_i$ $(i = 1,2)$, $\pi_1(Y') = H$,

(ii) *the inclusion $X \to X'$ is a homotopy equivalence,*

(iii) *the inclusion $X_i \to X'_i$ $(i = 1,2)$ is a $\mathbb{Z}[G_i]$-coefficient homology equivalence,*

(iv) *the inclusion $Y \to Y'$ is a $\mathbb{Z}[H]$-coefficient homology equivalence.*

Proof Apply the construction of Lemma 2.7 to the surjections $\pi_1(X_1) \to G_1$, $\pi_1(X_2) \to G_2$, $\pi_1(Y) \to H$, to obtain

$$\begin{aligned} X'_i \; &= \; (X_i \cup_{x_i} \bigcup_{m_i} D^2) \cup_{x_i^*} \bigcup_{m_i} D^3 \; (i = 1,2) \; , \\ Y' \; &= \; (Y \cup_y \bigcup_n D^2) \cup_{y^*} \bigcup_n D^3 \end{aligned}$$

for any $y = \{y_1, y_2, \dots, y_n\} \subseteq \pi_1(Y)$ such that $\ker(\pi_1(Y) \to H)$ is the normal closure of the subgroup of $\pi_1(Y)$ generated by y, and any

$$x_i \; = \; \{x_{i,1}, x_{i,2}, \dots, x_{i,m_i}\} \subseteq \pi_1(X_i)$$

such that $\ker(\pi_1(X_i) \to G_i)$ is the normal closure of the subgroup of $\pi_1(X_i)$ generated by x_i $(i = 1,2)$. Choosing x_1, x_2 to contain the images of y, we obtain the required 2-sided CW pair (X',Y') with $X' = X'_1 \cup_{Y'} X'_2$. $\square$

This completes the proof of the Combinatorial Transversality Theorem for amalgamated free products.

2.3 Combinatorial transversality for HNN extensions

The proof of combinatorial transversality for HNN extensions proceeds exactly as for amalgamated free products, so only the statements will be given.

In this section W is a connected CW complex with fundamental group an injective HNN extension

$$\pi_1(W) \; = \; G \; = \; G_1 *_H \{t\}$$

with tree T. Let $\widetilde{W}$ be the universal cover of W, and let

$$\widetilde{W}/H \underset{i_2}{\overset{i_1}{\rightrightarrows}} \widetilde{W}/G_1 \xrightarrow{j_1} W$$

be the covering projections, and define a commutative square

$$\begin{tikzcd} \widetilde{W}/H \times \{0,1\} \arrow[r, "i_1 \cup i_2"] \arrow[d, "i_3"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/H \times [0,1] \arrow[r, "j_2"] & W \end{tikzcd}$$

where

$$\begin{aligned} i_3 \; &= \; \text{inclusion} \; : \; \widetilde{W}/H \times \{0,1\} \to \widetilde{W}/H \times [0,1] \; , \\ j_2 \; &: \; \widetilde{W}/H \times [0,1] \to W \; ; \; (x,s) \mapsto j_1 i_1(x) = j_1 i_2(x) \; . \end{aligned}$$

Definition 2.9 (i) Suppose given a subcomplex $W_1 \subseteq \widetilde{W}$ with

$$G_1 W_1 \; = \; W_1$$

so that

$$H(W_1 \cap tW_1) \; = \; W_1 \cap tW_1 \subseteq \widetilde{W} \; .$$

Define a commutative square of CW complexes and cellular maps

$$\begin{tikzcd} (W_1 \cap tW_1)/H \times \{0,1\} \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\ (W_1 \cap tW_1)/H \times [0,1] \arrow[r, "f_2"] & W \end{tikzcd}$$

with

$$\begin{aligned} &(W_1 \cap tW_1)/H \subseteq \widetilde{W}/H \; , \; W_1/G_1 \subseteq \widetilde{W}/G_1 \; , \\ &e_1 \; = \; (i_1 \cup i_2)| \; : \; (W_1 \cap tW_1)/H \times \{0,1\} \to W_1/G_1 \; , \\ &e_2 \; = \; i_3| \; : \; (W_1 \cap tW_1)/H \times \{0,1\} \to (W_1 \cap tW_1)/H \times [0,1] \; , \\ &f_1 \; = \; j_1| \; : \; W_1/G_1 \to W \; , \; f_2 \; = \; j_2| \; : \; (W_1 \cap tW_1)/H \times [0,1] \to W \; . \end{aligned}$$

(ii) A *domain* W_1 for the universal cover $\widetilde{W}$ of W is a connected subcomplex $W_1 \subseteq \widetilde{W}$ such that $W_1 \cap tW_1$ is connected, and such that for each cell $D \subseteq \widetilde{W}$ the subgraph $U(D) \subseteq T$ defined by

$$\begin{aligned} U(D)^{(0)} &= \{g_1 \in [G;G_1] \,|\, g_1 D \subseteq W_1\} \\ U(D)^{(1)} &= \{h \in [G_1;H] \,|\, hD \subseteq W_1 \cap tW_1\} \end{aligned}$$

is a tree.

(iii) A domain W_1 for $\widetilde{W}$ is *fundamental* if the subtrees $U(D) \subseteq T$ are either single vertices or single edges, so that

$$g_1 W_1 \cap g_2 W_1 = \begin{cases} h(W_1 \cap tW_1) & \text{if } g_1 \cap g_2 t^{-1} = h \in [G_1;H] \\ g_1 W_1 & \text{if } g_1 = g_2 \\ \emptyset & \text{if } g_1 \neq g_2 \text{ and } g_1 \cap g_2 t^{-1} = \emptyset \ , \end{cases}$$

$$W = (W_1/G_1) \cup_{(W_1 \cap tW_1)/H \times \{0,1\}} (W_1 \cap tW_1)/H \times [0,1] \ .$$

$\square$

Proposition 2.10 *For a domain W_1 of $\widetilde{W}$ the cellular chain complex $C(W_1)$ is a domain of the cellular chain complex $C(\widetilde{W})$.* $\square$

Example 2.11 W has a canonical infinite domain $W_1 = \widetilde{W}$ with

$$(W_1 \cap tW_1)/H = \widetilde{W}/H$$

and $U(D) = T$ for each cell $D \subseteq \widetilde{W}$. $\square$

Example 2.12 (i) Suppose that $W = X_1 \cup_{Y \times \{0,1\}} Y \times [0,1]$, with $X_1, Y \subseteq W$ connected subcomplexes such that the isomorphism

$$\pi_1(W) = \pi_1(X_1) *_{\pi_1(Y)} \{t\} \xrightarrow{\cong} G = G_1 *_H \{t\}$$

preserves the HNN extensions. The morphisms $\pi_1(X_1) \to G_1$, $\pi_1(Y) \to H$ are surjective. (If $i_1, i_2 : \pi_1(Y) \to \pi_1(X_1)$ are injective these morphisms are also injective, allowing identifications $\pi_1(X_1) = G_1$, $\pi_1(Y) = H$). The universal cover of W is

$$\widetilde{W} = \bigcup_{g_1 \in [G:G_1]} g_1 \widetilde{X}_1 \cup_{\bigcup\limits_{h \in [G_1;H]} (h\widetilde{Y} \cup ht\widetilde{Y})} \bigcup_{h \in [G_1;H]} h\widetilde{Y} \times [0,1]$$

with $\widetilde{X}_1$ the regular cover of X_1 corresponding to $\ker(\pi_1(X_1) \to G_1)$ and $\widetilde{Y}$ the regular cover of Y corresponding to $\ker(\pi_1(Y) \to H)$ (which are the universal covers of X_1, Y in the case $\pi_1(X_1) = G_1$, $\pi_1(Y) = H$). Then $W_1 = \widetilde{X}_1$ is a fundamental domain of $\widetilde{W}$ such that

$$\begin{aligned} (W_1 \cap tW_1)/H &= Y \ , \ W_1 \cap tW_1 = \widetilde{Y} \ , \\ g_1W_1 \cap g_2W_1 &= (g_1 \cap g_2t^{-1})\widetilde{Y} \subseteq \widetilde{W} \ \ (g_1 \neq g_2 \in [G:G_1]) \ . \end{aligned}$$

For any cell $D \subseteq \widetilde{W}$

$$U(D) = \begin{cases} \{g_1\} & \text{if } g_1D \subseteq \widetilde{X}_1 - \bigcup\limits_{h\in[G_1;H]} (h\widetilde{Y} \cup ht\widetilde{Y}) \text{ for some } g_1 \in [G:G_1] \\ \{g_1,g_2,h\} & \text{if } hD \subseteq \widetilde{Y} \times [0,1] \text{ for some } h = g_1 \cap g_2t^{-1} \in [G_1;H]. \end{cases}$$

(ii) If W_1 is a fundamental domain for any connected CW complex W with $\pi_1(W) = G = G_1 *_H \{t\}$ then $W = X_1 \cup_{Y\times\{0,1\}} Y \times [0,1]$ as in (i), with

$$X_1 = W_1/G_1 \ , \ Y = (W_1 \cap tW_1)/H \ .$$

□

Definition 2.13 Suppose that W is n-dimensional. Lift each cell $D^r \subseteq W$ to a cell $\widetilde{D}^r \subseteq \widetilde{W}$. A sequence $U = \{U_n, U_{n-1}, \ldots, U_1, U_0\}$ of subtrees $U_r \subseteq T$ is *realized* by W if the subspace

$$W(U)_1 = \bigcup_{r=0}^{n} \bigcup_{D^r\subset W} \bigcup_{g_1\in U_r^{(0)}} g_1\widetilde{D}^r \subseteq \widetilde{W}$$

is a connected subcomplex, in which case $W(U)_1$ is a domain for $\widetilde{W}$ with

$$W(U)_1 \cap tW(U)_1 = \bigcup_{r=0}^{n} \bigcup_{D^r\subset W} \bigcup_{h\in U_r^{(1)}} h\widetilde{D}^r \subseteq \widetilde{W}$$

a connected subcomplex. Thus U is realized by $C(\widetilde{W})$ and

$$C(W(U)_1) = C(\widetilde{W}(U)_1 \subseteq j_1^! C(\widetilde{W})$$

is the domain for $C(\widetilde{W})$ given by $C_r(\widetilde{W})_1(U_r)$ in degree r. □

Proposition 2.14 (i) *For any domain W_1 there is defined a homotopy pushout*

$$\begin{tikzcd} (W_1 \cap tW_1)/H \times \{0,1\} \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\ (W_1 \cap tW_1)/H \times [0,1] \arrow[r, "f_2"] & W \end{tikzcd}$$

with $e_1 = i_1 \cup i_2|$, $e_2 = i_3|$, $f_1 = j_1|$, $f_2 = j_2|$. *The connected 2-sided* CW *pair*

$$(X,Y) \;=\; (\mathcal{M}(e_1,e_2),(W_1\cap tW_1)/H\times\{1/2\})$$

is a Seifert-van Kampen splitting of W, *with a homotopy equivalence*

$$f \;=\; f_1\cup f_2 \;:\; X \;=\; \mathcal{M}(e_1,e_2) \xrightarrow{\simeq} W \;.$$

(ii) *The commutative square of covering projections*

$$\begin{tikzcd} \widetilde{W}/H\times\{0,1\} \arrow[r, "i_1\cup i_2"] \arrow[d, "i_3"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/H\times[0,1] \arrow[r, "j_2"] & W \end{tikzcd}$$

is a homotopy pushout. The connected 2-sided CW *pair*

$$(X(\infty),Y(\infty)) \;=\; (\mathcal{M}(i_1\cup i_2,i_3),\widetilde{W}/H\times\{0\})$$

is a canonical injective infinite Seifert-van Kampen splitting of W, *with a homotopy equivalence* $j = j_1\cup j_2 : X(\infty)\to W$ *such that*

$$\pi_1(Y(\infty)) \;=\; H \subseteq \pi_1(X(\infty)) \;=\; G_1 *_H \{t\} \;.$$

(iii) *For any (finite) sequence* $U = \{U_n, U_{n-1},\dots,U_0\}$ *of subtrees of* T *realized by* W *there is defined a homotopy pushout*

$$\begin{tikzcd} Y(U)\times\{0,1\} \arrow[r, "e_1"] \arrow[d, "e_2"'] & X(U)_1 \arrow[d, "f_1"] \\ Y(U)\times[0,1] \arrow[r, "f_2"] & W \end{tikzcd}$$

with

$$\begin{aligned} Y(U) \;&=\; (W(U)_1\cap tW(U)_1)/H \;,\; X(U)_1 \;=\; W(U)_1/G_1 \;, \\ e_1 \;&=\; i_1\cup i_2| \;,\; e_2 \;=\; i_3| \;,\; f_1 \;=\; j_1| \;,\; f_2 \;=\; j_2| \;. \end{aligned}$$

Thus

$$(X(U),Y(U)) \;=\; (\mathcal{M}(e_1,e_2),Y(U)\times\{1/2\})$$

is a (finite) Seifert-van Kampen splitting of W.

(iv) *The canonical infinite domain of a finite* CW *complex* W *with* $\pi_1(W) = G_1 *_H \{t\}$ *is a union of finite domains* $W(U)_1$

$$\widetilde{W} \;=\; \bigcup_U W(U)_1$$

with U running over all the finite sequences realized by W. The canonical infinite Seifert-van Kampen splitting is thus a union of finite Seifert-van Kampen splittings

$$(X(\infty), Y(\infty)) \;=\; \bigcup_U (X(U), Y(U)) \;.$$

□

Proposition 2.15 *Let (X,Y) be a finite connected 2-sided CW pair with $X = X_1 \cup_{Y\times\{0,1\}} Y \times [0,1]$ for connected X_1, Y, together with an isomorphism*

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \{t\} \xrightarrow{\;\cong\;} G \;=\; G_1 *_H \{t\}$$

preserving the HNN structures, with the structure on G injective. It is possible to attach a finite number of 2- and 3-cells to the finite Seifert-van Kampen splitting (X,Y) of X to obtain a finite injective Seifert-van Kampen splitting (X',Y') with $X' = X_1' \cup_{Y'\times\{0,1\}} Y' \times [0,1]$ such that

(i) $\pi_1(X') = G$, $\pi_1(X_1') = G_1$, $\pi_1(Y') = H$,

(ii) *the inclusion $X \to X'$ is a homotopy equivalence,*

(iii) *the inclusion $X_1 \to X_1'$ is a $\mathbb{Z}[G_1]$-coefficient homology equivalence,*

(iv) *the inclusion $Y \to Y'$ is a $\mathbb{Z}[H]$-coefficient homology equivalence.*

□

This completes the proof of the Combinatorial Transversality Theorem for HNN extensions.

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